# SPEC ideal MHD limit verification in cylindrical geometry

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# 1 Introduction

SPEC [1] solution can be tested to converge to an ideal MHD solution when the number of volumes grows (analyticaly demonstrated in [2]). We propose here to do so in cylindrical geometry, *i.e.* studying the general screw pinch equilibrium [3]. In the first section, we shortly introduce the equations necessary to derive the ideal MHD equilibrium of the srew pinch, and present the results in the second section.

# 2 Theory

## 2.1 MHD equations

When considering an equilibrium ($\partial/\partial t = 0$) with no plasma flow ($\mathbf{v} = 0$), the ideal MHD model [3] reduces to

$$\mathbf{J}\times\mathbf{B} = \nabla p \tag{1}$$
$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{2}$$
$$\nabla\cdot\mathbf{B} = 0. \tag{3}$$

It is noteworthy to notice that for a parallel current $\mathbf{J}\parallel\mathbf{B}$, Eq.(1) leads to

$$\mathbf{J}\times\mathbf{B} = 0 = \nabla p. \tag{4}$$

This particular case is called a *force-free field*.

### 2.1.1 Screw pinch equilibrium

A screw pinch is a cylindre of radius $a$ and length $L = 2\pi R_0$ (see Figure 1), where a toroidal ($z$) and poloidal ($\theta$) symmetry are assumed. Using the usual cylindrical coordinates $(r, \theta, z)$, the magnetic field can be written as $\mathbf{B} = B_r(r)\hat{e}_r + B_\theta(r)\hat{e}_\theta + B_z(r)\hat{e}_z$ and the current density $\mathbf{J} = J_r(r)\hat{e}_r + J_\theta(r)\hat{e}_\theta + J_z(r)\hat{e}_z$. Eq.(3) leads to the differential equation

$$\frac{1}{r}\frac{\partial}{\partial}(rB_r) = 0, \tag{5}$$

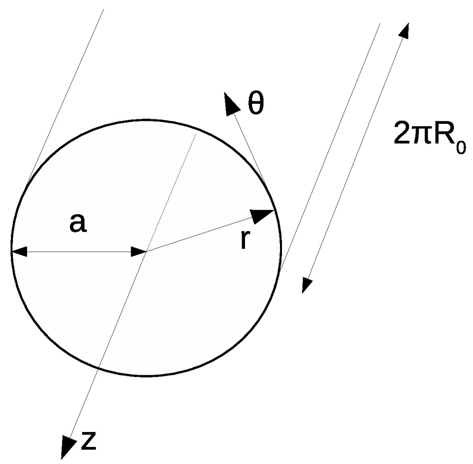



Figure 1: Sketch of a screw pinch with cylindrical coordinates

which general solution is

$$B_r(r) = \frac{c_1}{r}, \tag{6}$$

with $c_1$ an integration constant. This constant is determined by observing that the solution diverges at $r = 0$ if $c_1 \neq 0$. Thus we set $c_1 = 0$ and

$$B_r(r) = 0. \tag{7}$$

Ampere's law (Eq.2) leads to

$$\begin{aligned} J_r(r) &= 0 && (8)\\ J_\theta(r) &= -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r} && (9)\\ J_z(r) &= \frac{1}{\mu_0 r}\frac{\partial}{\partial r}(rB_\theta). && (10) \end{aligned}$$

Combining Eq.(9) and (10) with Eq.(1) finaly leads to the general screw pinch equilibrium equation [3]

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{r\mu_0} = 0. \tag{11}$$

For a given pressure and poloidal magnetic field profile, a solution to Eq.(11) can be obtained numerically.

**Screw pinch equilibrium equation with rotational profile** It may happen that the poloidal magnetic field is unknow, but the rotational transform is. In this case, one wants another form of Eq.(11) which doesn't involve $B_\theta$. The Eq.(11) can be rewritten using the definition of the rotational transform [3]

$$\iota = \frac{1}{2\pi}\int_0^{2\pi R_0} \frac{d\theta}{dz}dz = \frac{1}{2\pi}\int_0^{2\pi R_0} \frac{dB_\theta}{rBz}dz = \frac{R_0 B_\theta}{rB_z}, \tag{12}$$

where the symmetry assumption has been used in the last equality. Expressing $B_\theta$ as a function of $\iota$ and $B_z$ before substituing into Eq(11) leads to

$$\frac{d}{dr}\left[p + \left(1 + \frac{\iota^2 r^2}{R_0^2}\right)\frac{B_z^2}{2\mu_0}\right] + \frac{rB_z^2\iota^2}{R_0^2\mu_0} = 0. \tag{13}$$

Eq.(13) can be inverted and implemented to be solved numerically (see MATLAB routine *ScrewPinch_IotaSolver*).

$$\frac{dB_z}{dr} = -\frac{\frac{dp}{dr} + \frac{r\iota^2}{(R_0)^2}\left(2 + \frac{r\frac{d\iota}{dr}}{\iota}\right)B_z^2}{\left[1 + \left(\frac{r\iota}{R_0}\right)^2\right]B_z}. \tag{14}$$

**Particular case** A particular case of the screw pinch is the parallel pinch. In the parallel pinch , a force-free equilibrium is sought, *i.e* $\mathbf{j} = k(r)\mathbf{B}$, with $k(r)$ a scalar function of $r$. Eq.(11) reduces to

$$\frac{1}{rk}\frac{d}{dr}\left[\frac{r}{k}\frac{dB_z}{dr}\right] + B_z = 0. \tag{15}$$

For $k(r) = k_0 = cst$, the solution is

$$B_z = B_0 J_0(k_0 r) \tag{16}$$
$$B_\theta = B_0 J_1(k_0 r), \tag{17}$$

with $J_0$, $J_1$ Bessel function of the first kind. This result will be used as a test for the screw pinch equilibrium solver implemented in MATLAB.

# 3 Results

## 3.1 Verification of the screw pinch MATLAB solver

The MATLAB routine *ScrewPinch_IotaSolver.m* numerically integrates equation (14) to find the screw pinch equilibrium. In the particular case of a parallel pinch, the numerical solution can be compared with the analytical solution (Eq.(16) and (17)). This is presented on Figure.2. Both analytical and numerical solution agrees - we thus validate the screw pinch solver implementation.

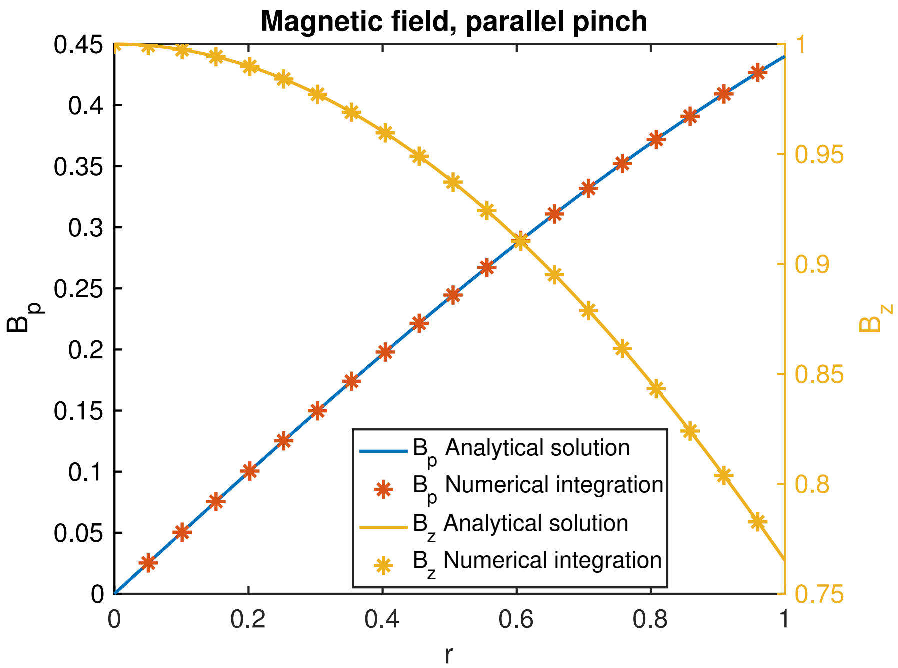
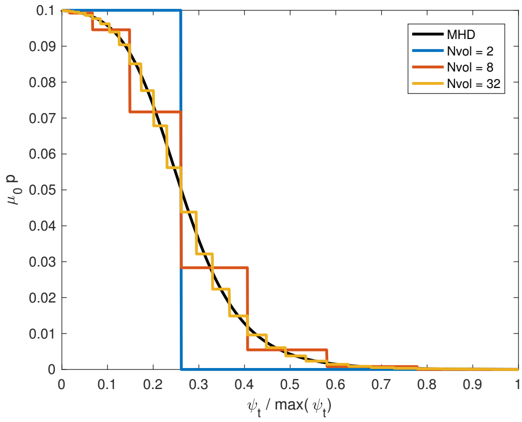



Figure 2: Validation of the screw pinch numerical solver in the case of a parallel pinch

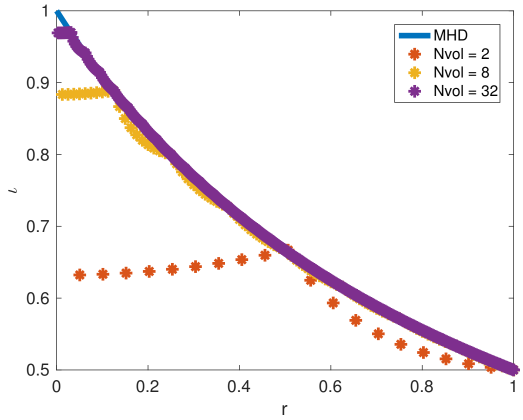
### 3.2 MHD limit

When the number of volume used in MRxMHD is increased to infinity, the model converges to the MHD model [2]. This can be tested by choosing a rotational transform profile and a pressure profile and using them as input to the screw pinch solver and to SPEC and compare their output. In this study we use a quadratic pressure profile $p(r) = 4 \cdot 10^4[1 + \tanh(4 - 8r)]$. On Figure 3a, step pressure profiles used in SPEC as approximation of the analytical pressure are presented.

On Figure 3b the rotational profile is displayed when $N_{vol} = 2, 8, 32$ and compared with the solution from the screw pinch MHD equilibrium. We observe the rotational profile computed by SPEC agrees with the ideal MHD rotational profile when $N_{vol}$ increases. Note that in the first volume, SPEC solution is equal to two Bessel functions [4], and the rotational profile is given by $\iota_1(r) = J_1(\mu r)/(rJ_0(\mu r) \approx cst$ close to 0. To approximate a non-constant rotational profile in the first volume, its size has to be as small as possible.




(a) Pressure profile




(b) Rotational transform profile

Figure 3: Pressure and rotational transform profile for $N_{vol} = 2, 8, 32$ and comparison with the MHD screw pinch solution.

Figures 4 and 5 plots the magnetic field components as a function of $r$, as well as a zoom on one of the discontinuity in the field. The discontinuity occurs due to the discontinuity in the pressure and the interface condition $[[p^2 + B^2/2]] = 0$. As expected both the poloidal and toroidal component of the field increase at the interface to compensate the decrease in pressure. We see that the field is slightly perturbed close to the interface, certainly due to the Chebyshev representation of the solution.

As the number of volumes increases, the discontinuity in the field is getting smaller since the pressure discontinuity is getting smaller as well. For a high enough number of volumes, the solution is converges towards the MHD equilibrium solution, as expected.

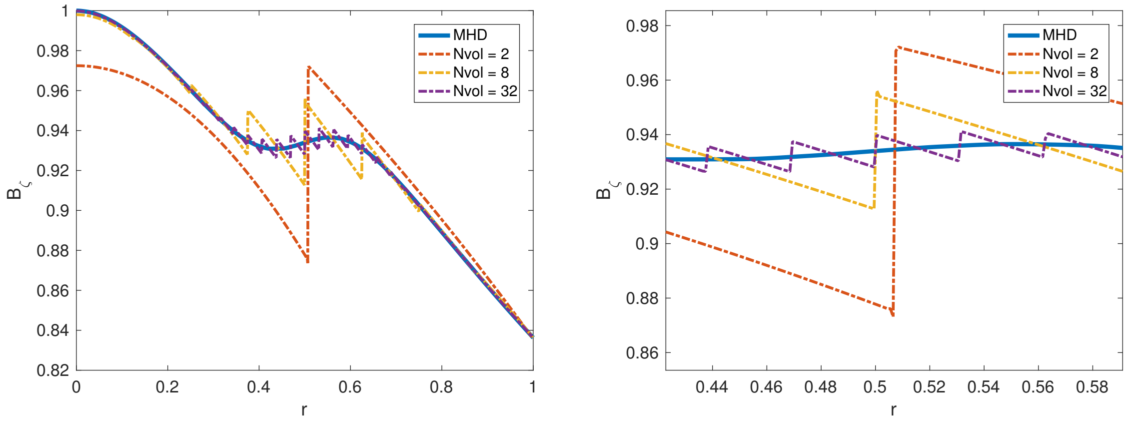
The low $\beta$ approximation, *i.e.* when the pressure is set to 0 is presented on Figure 6. As expected, since the pressure doesn't exhibit any discontinuity, the field is continuous as well. As a side note, we can observe that the pressure has a paramagnetic effect on the plasma, since the toroidal field at non-zero pressure (Figure 4 and 5) is bigger that at zero pressure (Figure 6).

## 4 Conclusion

This reports presents the verification of SPEC ideal MHD limit in cylindrical geometry. Routines to compare the ideal MHD screwpinch equilibrium to SPEC output have been implemented and used to show that SPEC converges to the ideal MHD solution when the number of volumes increases. Plots of SPEC results with 2, 8 and 32 volumes are presented and compared with the ideal MHD solution.

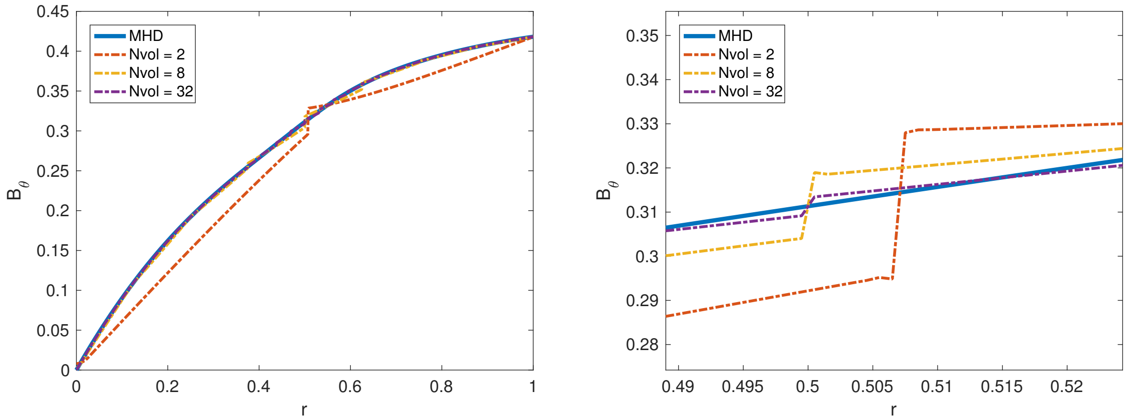
## References


[1] S. R. Hudson, R. L. Dewar, G. Dennis, M. J. Hole, M. Mcgann, G. Von Nessi, and S. Lazerson. Computation of multi-region relaxed magnetohydrodynamic equilibria. *Phys. Plasmas*, 25, Aug 2013.



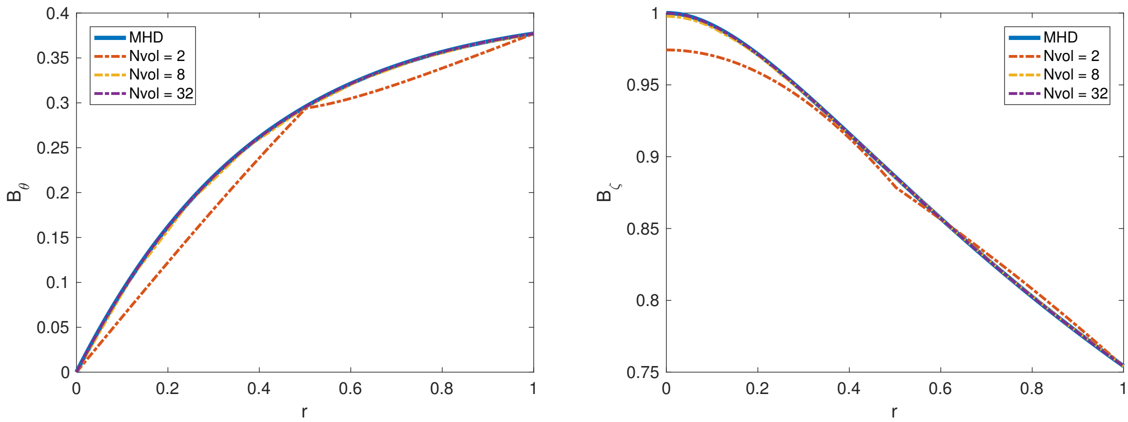

(a) Full solution (b) Zoom on discontinuity

Figure 4: Toroidal magnetic field for $N_{vol} = 2, 8, 32$ and the MHD solution as a function of $r$. The Figure on the right is a zoom on the field discontinuity between the first and second volume when $N_{vol} = 1$.




(a) Full solution (b) Zoom on discontinuity

Figure 5: Poloidal magnetic field for $N_{vol} = 2, 8, 32$ and the MHD solution as a function of $r$. The Figure on the right is a zoom on the field discontinuity between the first and second volume when $N_{vol} = 1$.




(a) Poloidal field (b) Toroidal field

Figure 6: SPEC solution and comparison to the MHD equilibrium at constant (zero) pressure.

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[4] M. J. Hole, S. R. Hudson, and R. L. Dewar. Stepped pressure profile equilibria in cylindrical plasmas via partial Taylor relaxation. *Journal of Plasma Physics*, 72(6):1167–1171, 2006.