

**volume**

Computes volume of each region; and, if required, the derivatives of the volume with respect to the interface geometry.

[called by: [dforce](#) and [xspech](#).]

[calls: .]

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**1.1 volume integral**

1. The volume enclosed by the  $v$ -th interface is given by the integral

$$V = \int_V dv = \frac{1}{3} \int_V \nabla \cdot \mathbf{x} dv = \frac{1}{3} \int_S \mathbf{x} \cdot d\mathbf{s} = \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta |\mathbf{x} \cdot \mathbf{x}_\theta \times \mathbf{x}_\zeta|^s \quad (1)$$

where we have used  $\nabla \cdot \mathbf{x} = 3$ , and have assumed that the domain is periodic in the angles.

**1.2 representation of surfaces**

1. The coordinate functions are

$$R(\theta, \zeta) = \sum_i R_{e,i} \cos \alpha_i + \sum_i R_{o,i} \sin \alpha_i \quad (2)$$

$$Z(\theta, \zeta) = \sum_i Z_{e,i} \cos \alpha_i + \sum_i Z_{o,i} \sin \alpha_i, \quad (3)$$

where  $\alpha_i \equiv m_i \theta - n_i \zeta$ .

**1.3 geometry**

1. The geometry is controlled by the input parameter **Igeometry** as follows:

2. **Igeometry.eq.1** : Cartesian :  $\sqrt{g} = R_s$

$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta R \\ &= 2\pi \frac{2\pi}{N} R_{e,1} \end{aligned} \quad (4)$$

3. **Igeometry.eq.2** : cylindrical :  $\sqrt{g} = RR_s = \frac{1}{2} \partial_s (R^2)$

$$\begin{aligned} V &= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta R^2 \\ &= \frac{1}{2} 2\pi \frac{2\pi}{N} \frac{1}{2} \sum_i \sum_j R_{e,i} R_{e,j} [\cos(\alpha_i - \alpha_j) + \cos(\alpha_i + \alpha_j)] \\ &\quad + \frac{1}{2} 2\pi \frac{2\pi}{N} \frac{1}{2} \sum_i \sum_j R_{o,i} R_{o,j} [\cos(\alpha_i - \alpha_j) - \cos(\alpha_i + \alpha_j)] \end{aligned} \quad (5)$$

4. **Igeometry.eq.3** : toroidal :  $\mathbf{x} \cdot \mathbf{e}_\theta \times \mathbf{e}_\zeta = R(ZR_\theta - RZ_\theta)$

$$\begin{aligned} V &= \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta R (ZR_\theta - RZ_\theta) \\ &= \frac{1}{3} \sum_i \sum_j \sum_k R_{e,i} (Z_{e,j} R_{o,k} - R_{e,j} Z_{o,k}) (+m_k) \iint d\theta d\zeta \cos \alpha_i \cos \alpha_j \cos \alpha_k \\ &\quad + \frac{1}{3} \sum_i \sum_j \sum_k R_{e,i} (Z_{o,j} R_{e,k} - R_{o,j} Z_{e,k}) (-m_k) \iint d\theta d\zeta \cos \alpha_i \sin \alpha_j \sin \alpha_k \\ &\quad + \frac{1}{3} \sum_i \sum_j \sum_k R_{o,i} (Z_{e,j} R_{e,k} - R_{e,j} Z_{e,k}) (-m_k) \iint d\theta d\zeta \sin \alpha_i \cos \alpha_j \sin \alpha_k \\ &\quad + \frac{1}{3} \sum_i \sum_j \sum_k R_{o,i} (Z_{o,j} R_{o,k} - R_{o,j} Z_{o,k}) (+m_k) \iint d\theta d\zeta \sin \alpha_i \sin \alpha_j \cos \alpha_k \end{aligned} \quad (6)$$

5. (Recall that the integral over an odd function is zero, so various terms in the above expansion have been ignored.)

6. The trigonometric terms are

$$\begin{aligned}
4 \cos \alpha_i \cos \alpha_j \cos \alpha_k &= + \cos(\alpha_i + \alpha_j + \alpha_k) + \cos(\alpha_i + \alpha_j - \alpha_k) + \cos(\alpha_i - \alpha_j + \alpha_k) + \cos(\alpha_i - \alpha_j - \alpha_k) \\
4 \cos \alpha_i \sin \alpha_j \sin \alpha_k &= - \cos(\alpha_i + \alpha_j + \alpha_k) + \cos(\alpha_i + \alpha_j - \alpha_k) + \cos(\alpha_i - \alpha_j + \alpha_k) - \cos(\alpha_i - \alpha_j - \alpha_k) \\
4 \sin \alpha_i \cos \alpha_j \sin \alpha_k &= - \cos(\alpha_i + \alpha_j + \alpha_k) + \cos(\alpha_i + \alpha_j - \alpha_k) - \cos(\alpha_i - \alpha_j + \alpha_k) + \cos(\alpha_i - \alpha_j - \alpha_k) \\
4 \sin \alpha_i \sin \alpha_j \cos \alpha_k &= - \cos(\alpha_i + \alpha_j + \alpha_k) - \cos(\alpha_i + \alpha_j - \alpha_k) + \cos(\alpha_i - \alpha_j + \alpha_k) + \cos(\alpha_i - \alpha_j - \alpha_k)
\end{aligned} \tag{7}$$

7. The required derivatives are

$$\begin{aligned}
3 \frac{\partial V}{\partial R_{e,i}} &= (+Z_{e,j} R_{o,k} m_k - R_{e,j} Z_{o,k} m_k - R_{e,j} Z_{o,k} m_k) \iint d\theta d\zeta \cos \alpha_i \cos \alpha_j \cos \alpha_k \\
&+ (-Z_{o,j} R_{e,k} m_k + R_{o,j} Z_{e,k} m_k + R_{o,j} Z_{e,k} m_k) \iint d\theta d\zeta \cos \alpha_i \sin \alpha_j \sin \alpha_k \\
&+ (-R_{o,k} Z_{e,j} m_i) \iint d\theta d\zeta \sin \alpha_i \cos \alpha_j \sin \alpha_k \\
&+ (-R_{e,k} Z_{o,j} m_i) \iint d\theta d\zeta \sin \alpha_i \sin \alpha_j \cos \alpha_k
\end{aligned} \tag{8}$$

$$\begin{aligned}
3 \frac{\partial V}{\partial Z_{o,i}} &= (-R_{e,k} R_{e,j} m_i) \iint d\theta d\zeta \cos \alpha_i \cos \alpha_j \cos \alpha_k \\
&+ (-R_{o,k} R_{o,j} m_i) \iint d\theta d\zeta \cos \alpha_i \sin \alpha_j \sin \alpha_k \\
&+ (-R_{e,j} R_{e,k} m_k) \iint d\theta d\zeta \sin \alpha_i \cos \alpha_j \sin \alpha_k \\
&+ (+R_{o,j} R_{o,k} m_k) \iint d\theta d\zeta \sin \alpha_i \sin \alpha_j \cos \alpha_k
\end{aligned} \tag{9}$$