

jo00aaMeasures error in Beltrami field, $||\nabla \times \mathbf{B} - \mu \mathbf{B}||$.[called by: [xspech](#).][calls: [coords](#).]**contents**

1 jo00aa	1
1.1 overview	1
1.2 construction of current, $\mathbf{j} \equiv \nabla \times \nabla \times \mathbf{A}$	1
1.3 quantification of the error	1
1.4 details of the numerics	2
1.5 comments	2

1.1 overview

This routine is called by [xspech](#) as a post diagnostic and only if [Lcheck](#) = 1.

1.2 construction of current, $\mathbf{j} \equiv \nabla \times \nabla \times \mathbf{A}$

1. The components of the vector potential, $\mathbf{A} = A_\theta \nabla + A_\zeta \nabla \zeta$, are

$$A_\theta(s, \theta, \zeta) = \sum_{i,l} \textcolor{red}{A}_{\theta,e,i,l} \bar{T}_{l,i}(s) \cos \alpha_i + \sum_{i,l} \textcolor{brown}{A}_{\theta,o,i,l} \bar{T}_{l,i}(s) \sin \alpha_i, \quad (1)$$

$$A_\zeta(s, \theta, \zeta) = \sum_{i,l} \textcolor{blue}{A}_{\zeta,e,i,l} \bar{T}_{l,i}(s) \cos \alpha_i + \sum_{i,l} \textcolor{blue}{A}_{\zeta,o,i,l} \bar{T}_{l,i}(s) \sin \alpha_i, \quad (2)$$

where $\bar{T}_{l,i}(s) \equiv \bar{s}^{m_i/2} T_l(s)$, $T_l(s)$ is the Chebyshev polynomial, and $\alpha_j \equiv m_j \theta - n_j \zeta$. The regularity factor, $\bar{s}^{m_i/2}$, where $\bar{s} \equiv (1+s)/2$, is only included if there is a coordinate singularity in the domain (i.e. only in the innermost volume, and only in cylindrical and toroidal geometry.)

2. The magnetic field, $\sqrt{g} \mathbf{B} = \sqrt{g} B^s \mathbf{e}_s + \sqrt{g} B^\theta \mathbf{e}_\theta + \sqrt{g} B^\zeta \mathbf{e}_\zeta$, is

$$\begin{aligned} \sqrt{g} \mathbf{B} &= \mathbf{e}_s \sum_{i,l} [(-m_i \textcolor{blue}{A}_{\zeta,e,i,l} - n_i \textcolor{red}{A}_{\theta,e,i,l}) \bar{T}_{l,i} \sin \alpha_i + (+m_i \textcolor{blue}{A}_{\zeta,o,i,l} + n_i \textcolor{brown}{A}_{\theta,o,i,l}) \bar{T}_{l,i} \cos \alpha_i] \\ &+ \mathbf{e}_\theta \sum_{i,l} [(-\textcolor{blue}{A}_{\zeta,e,i,l}) \bar{T}'_{l,i} \cos \alpha_i + (-\textcolor{blue}{A}_{\zeta,o,i,l}) \bar{T}'_{l,i} \sin \alpha_i] \\ &+ \mathbf{e}_\zeta \sum_{i,l} [(\textcolor{red}{A}_{\theta,e,i,l}) \bar{T}'_{l,i} \cos \alpha_i + (\textcolor{brown}{A}_{\theta,o,i,l}) \bar{T}'_{l,i} \sin \alpha_i] \end{aligned} \quad (3)$$

3. The current is

$$\sqrt{g} \mathbf{j} = (\partial_\theta B_\zeta - \partial_\zeta B_\theta) \mathbf{e}_s + (\partial_\zeta B_s - \partial_s B_\zeta) \mathbf{e}_\theta + (\partial_s B_\theta - \partial_\theta B_s) \mathbf{e}_\zeta, \quad (4)$$

where (for computational convenience) the covariant components of \mathbf{B} are computed as

$$B_s = (\sqrt{g} B^s) g_{ss} / \sqrt{g} + (\sqrt{g} B^\theta) g_{s\theta} / \sqrt{g} + (\sqrt{g} B^\zeta) g_{s\zeta} / \sqrt{g}, \quad (5)$$

$$B_\theta = (\sqrt{g} B^s) g_{s\theta} / \sqrt{g} + (\sqrt{g} B^\theta) g_{\theta\theta} / \sqrt{g} + (\sqrt{g} B^\zeta) g_{\theta\zeta} / \sqrt{g}, \quad (6)$$

$$B_\zeta = (\sqrt{g} B^s) g_{s\zeta} / \sqrt{g} + (\sqrt{g} B^\theta) g_{\theta\zeta} / \sqrt{g} + (\sqrt{g} B^\zeta) g_{\zeta\zeta} / \sqrt{g}. \quad (7)$$

1.3 quantification of the error

1. The measures of the error are

$$||(\mathbf{j} - \mu \mathbf{B}) \cdot \nabla s|| \equiv \int ds \oint \oint d\theta d\zeta \quad |\sqrt{g} \mathbf{j} \cdot \nabla s - \mu \sqrt{g} \mathbf{B} \cdot \nabla s|, \quad (8)$$

$$||(\mathbf{j} - \mu \mathbf{B}) \cdot \nabla \theta|| \equiv \int ds \oint \oint d\theta d\zeta \quad |\sqrt{g} \mathbf{j} \cdot \nabla \theta - \mu \sqrt{g} \mathbf{B} \cdot \nabla \theta|, \quad (9)$$

$$||(\mathbf{j} - \mu \mathbf{B}) \cdot \nabla \zeta|| \equiv \int ds \oint \oint d\theta d\zeta \quad |\sqrt{g} \mathbf{j} \cdot \nabla \zeta - \mu \sqrt{g} \mathbf{B} \cdot \nabla \zeta|. \quad (10)$$

1.4 details of the numerics

1. The integration over s is performed using Gaussian integration, e.g., $\int f(s)ds \approx \sum_k \omega_k f(s_k)$; with the abscissae, s_k , and the weights, ω_k , for $k = 1, \text{Iquad}_v$, determined by `CDGQF`. The resolution, $N \equiv \text{Iquad}_v$, is determined by `Nquad` (see `global` and `preset`). A fatal error is enforced by `jo00aa` if `CDGQF` returns an `ifail` $\neq 0$.
2. Inside the Gaussian quadrature loop, i.e. for each s_k ,

- (a) The metric elements, $g_{\mu,\nu} \equiv \text{gij}(1:6,0,1:\text{Ntz})$, and the Jacobian, $\sqrt{g} \equiv \text{sg}(0,1:\text{Ntz})$, are calculated on a regular angular grid, (θ_i, ζ_j) , in `coords`. The derivatives $\partial_i g_{\mu,\nu} \equiv \text{gij}(1:6,i,1:\text{Ntz})$ and $\partial_i \sqrt{g} \equiv \text{sg}(i,1:\text{Ntz})$, with respect to $i \in \{s, \theta, \zeta\}$ are also returned.
- (b) The Fourier components of the vector potential given in Eqn.(1) and Eqn.(2), and their first and second radial derivatives, are summed.
- (c) The quantities $\sqrt{g}B^s$, $\sqrt{g}B^\theta$ and $\sqrt{g}B^\zeta$, and their first and second derivatives with respect to (s, θ, ζ) , are computed on the regular angular grid (using FFTs).
- (d) The following quantities are then computed on the regular angular grid

$$\begin{aligned} \sqrt{g}j^s &= \sum_u [\partial_\theta(\sqrt{g}B^u) g_{u,\zeta} + (\sqrt{g}B^u) \partial_\theta g_{u,\zeta} - (\sqrt{g}B^u) g_{u,\zeta} \partial_\theta \sqrt{g}/\sqrt{g}] / \sqrt{g} \\ &- \sum_u [\partial_\zeta(\sqrt{g}B^u) g_{u,\theta} + (\sqrt{g}B^u) \partial_\zeta g_{u,\theta} - (\sqrt{g}B^u) g_{u,\theta} \partial_\zeta \sqrt{g}/\sqrt{g}] / \sqrt{g}, \end{aligned} \quad (11)$$

$$\begin{aligned} \sqrt{g}j^\theta &= \sum_u [\partial_\zeta(\sqrt{g}B^u) g_{u,s} + (\sqrt{g}B^u) \partial_\zeta g_{u,s} - (\sqrt{g}B^u) g_{u,s} \partial_\zeta \sqrt{g}/\sqrt{g}] / \sqrt{g} \\ &- \sum_u [\partial_s(\sqrt{g}B^u) g_{u,\zeta} + (\sqrt{g}B^u) \partial_s g_{u,\zeta} - (\sqrt{g}B^u) g_{u,\zeta} \partial_s \sqrt{g}/\sqrt{g}] / \sqrt{g}, \end{aligned} \quad (12)$$

$$\begin{aligned} \sqrt{g}j^\zeta &= \sum_u [\partial_s(\sqrt{g}B^u) g_{u,\theta} + (\sqrt{g}B^u) \partial_s g_{u,\theta} - (\sqrt{g}B^u) g_{u,\theta} \partial_s \sqrt{g}/\sqrt{g}] / \sqrt{g} \\ &- \sum_u [\partial_\theta(\sqrt{g}B^u) g_{u,s} + (\sqrt{g}B^u) \partial_\theta g_{u,s} - (\sqrt{g}B^u) g_{u,s} \partial_\theta \sqrt{g}/\sqrt{g}] / \sqrt{g}. \end{aligned} \quad (13)$$

3. The final calculation of the error, which is written to screen, is a sum over the angular grid:

$$E^s \equiv \frac{1}{N} \sum_k \omega_k \sum_{i,j} |\sqrt{g}j^s - \mu \sqrt{g}B^s|, \quad (14)$$

$$E^\theta \equiv \frac{1}{N} \sum_k \omega_k \sum_{i,j} |\sqrt{g}j^\theta - \mu \sqrt{g}B^\theta|, \quad (15)$$

$$E^\zeta \equiv \frac{1}{N} \sum_k \omega_k \sum_{i,j} |\sqrt{g}j^\zeta - \mu \sqrt{g}B^\zeta|, \quad (16)$$

where $N \equiv \sum_{i,j} 1$.

4. The error is stored into an array called `beltramiererror` which is then written to the HDF5 file in `hdfint`.

1.5 comments

1. Is there a better definition and quantification of the error? For example, should we employ an error measure that is dimensionless?
2. If the coordinate singularity is in the domain, then $|\nabla\theta| \rightarrow \infty$ at the coordinate origin. What then happens to $||(\mathbf{j} - \mu\mathbf{B}) \cdot \nabla\theta||$ as defined in Eqn.(9)?
3. What is the predicted scaling of the error in the Chebyshev-Fourier representation scale with numerical resolution? Note that the predicted error scaling for E^s , E^θ and E^ζ may not be standard, as various radial derivatives are taken to compute the components of \mathbf{j} . (See for example the discussion in Sec.IV.C in [Hudson, Dewar et al., Phys. Plasmas **19**, 112502 (2012)], where the expected scaling of the error for a finite-element implementation is confirmed numerically.)
4. Instead of using Gaussian integration to compute the integral over s , an adaptive quadrature algorithm may be preferable.