volume

Computes volume of each region; and, if required, the derivatives of the volume with respect to the interface geometry.

[called by: dforce and xspech.]   [calls: .]

contents

1.1 volume integral

1. The volume enclosed by the v-th interface is given by the integral

$$V = \int_{\mathcal{V}} dv = \frac{1}{3} \int_{\mathcal{V}} \nabla \cdot \mathbf{x} \, dv = \frac{1}{3} \int_{\mathcal{S}} \mathbf{x} \cdot d\mathbf{s} = \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta \quad \mathbf{x} \cdot \mathbf{x}_\theta \times \mathbf{x}_\zeta \big|^s \tag{1}$$

where we have used $\nabla \cdot \mathbf{x} = 3$, and have assumed that the domain is periodic in the angles.

1.2 representation of surfaces

1. The coordinate functions are

$$R(\theta, \zeta) = \sum_i R_{e,i} \, \cos \alpha_i + \sum_i R_{o,i} \, \sin \alpha_i \tag{2}$$

$$Z(\theta, \zeta) = \sum_i Z_{e,i} \, \cos \alpha_i + \sum_i Z_{o,i} \, \sin \alpha_i, \tag{3}$$

where $\alpha_i \equiv m_i \theta - n_i \zeta$.

1.3 geometry

1. The geometry is controlled by the input parameter `Igeometry` as follows:

2. `Igeometry.eq.1` : Cartesian : $\sqrt{g} = R_s$

$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta \, R \\ &= 2\pi \frac{2\pi}{N} R_{e,1} \end{aligned} \tag{4}$$

3. `Igeometry.eq.2` : cylindrical : $\sqrt{g} = RR_s = \frac{1}{2}\partial_s(R^2)$

$$\begin{aligned} V &= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta \, R^2 \\ &= \frac{1}{2} \, 2\pi \, \frac{2\pi}{N} \frac{1}{2} \, \sum_i \sum_j R_{e,i} R_{e,j} \left[\cos(\alpha_i - \alpha_j) + \cos(\alpha_i + \alpha_j) \right] \\ &+ \frac{1}{2} \, 2\pi \, \frac{2\pi}{N} \frac{1}{2} \, \sum_i \sum_j R_{o,i} R_{o,j} \left[\cos(\alpha_i - \alpha_j) - \cos(\alpha_i + \alpha_j) \right] \end{aligned} \tag{5}$$

4. `Igeometry.eq.3` : toroidal : $\mathbf{x} \cdot \mathbf{e}_\theta \times \mathbf{e}_\zeta = R(ZR_\theta - RZ_\theta)$

$$\begin{aligned} V &= \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{2\pi/N} d\zeta \, R\,(ZR_\theta - RZ_\theta) \\ &= \frac{1}{3} \sum_i \sum_j \sum_k R_{e,i} \, (Z_{e,j} R_{o,k} - R_{e,j} Z_{o,k}) \, (+m_k) \iint d\theta d\zeta \, \cos\alpha_i \cos\alpha_j \cos\alpha_k \\ &+ \frac{1}{3} \sum_i \sum_j \sum_k R_{e,i} \, (Z_{o,j} R_{e,k} - R_{o,j} Z_{e,k}) \, (-m_k) \iint d\theta d\zeta \, \cos\alpha_i \sin\alpha_j \sin\alpha_k \\ &+ \frac{1}{3} \sum_i \sum_j \sum_k R_{o,i} \, (Z_{e,j} R_{e,k} - R_{e,j} Z_{e,k}) \, (-m_k) \iint d\theta d\zeta \, \sin\alpha_i \cos\alpha_j \sin\alpha_k \\ &+ \frac{1}{3} \sum_i \sum_j \sum_k R_{o,i} \, (Z_{o,j} R_{o,k} - R_{o,j} Z_{o,k}) \, (+m_k) \iint d\theta d\zeta \, \sin\alpha_i \sin\alpha_j \cos\alpha_k \end{aligned} \tag{6}$$

5. (Recall that the integral over an odd function is zero, so various terms in the above expansion have been ignored.)

6. The trigonometric terms are

$$
\begin{aligned}
4 \, \cos\alpha_i \cos\alpha_j \cos\alpha_k &= + \cos(\alpha_i+\alpha_j+\alpha_k) + \cos(\alpha_i+\alpha_j-\alpha_k) + \cos(\alpha_i-\alpha_j+\alpha_k) + \cos(\alpha_i-\alpha_j-\alpha_k) \\
4 \, \cos\alpha_i \sin\alpha_j \sin\alpha_k &= - \cos(\alpha_i+\alpha_j+\alpha_k) + \cos(\alpha_i+\alpha_j-\alpha_k) + \cos(\alpha_i-\alpha_j+\alpha_k) - \cos(\alpha_i-\alpha_j-\alpha_k) \\
4 \, \sin\alpha_i \cos\alpha_j \sin\alpha_k &= - \cos(\alpha_i+\alpha_j+\alpha_k) + \cos(\alpha_i+\alpha_j-\alpha_k) - \cos(\alpha_i-\alpha_j+\alpha_k) + \cos(\alpha_i-\alpha_j-\alpha_k) \\
4 \, \sin\alpha_i \sin\alpha_j \cos\alpha_k &= - \cos(\alpha_i+\alpha_j+\alpha_k) - \cos(\alpha_i+\alpha_j-\alpha_k) + \cos(\alpha_i-\alpha_j+\alpha_k) + \cos(\alpha_i-\alpha_j-\alpha_k)
\end{aligned}
\tag{7}
$$

7. The required derivatives are

$$
\begin{aligned}
3\frac{\partial V}{\partial R_{e,i}} = \;&(+Z_{e,j}R_{o,k}m_k - R_{e,j}Z_{o,k}m_k - R_{e,j}Z_{o,k}m_k) \iint d\theta d\zeta \; \cos\alpha_i \cos\alpha_j \cos\alpha_k \\
+ \;&(-Z_{o,j}R_{e,k}m_k + R_{o,j}Z_{e,k}m_k + R_{o,j}Z_{e,k}m_k) \iint d\theta d\zeta \; \cos\alpha_i \sin\alpha_j \sin\alpha_k \\
+ \;&(-R_{o,k}Z_{e,j}m_i) \qquad\qquad\qquad\qquad\quad\; \iint d\theta d\zeta \; \sin\alpha_i \cos\alpha_j \sin\alpha_k \\
+ \;&(-R_{e,k}Z_{o,j}m_i) \qquad\qquad\qquad\qquad\quad\; \iint d\theta d\zeta \; \sin\alpha_i \sin\alpha_j \cos\alpha_k
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
3\frac{\partial V}{\partial Z_{o,i}} = \;&(-R_{e,k}R_{e,j}m_i) \iint d\theta d\zeta \; \cos\alpha_i \cos\alpha_j \cos\alpha_k \\
+ \;&(-R_{o,k}R_{o,j}m_i) \iint d\theta d\zeta \; \cos\alpha_i \sin\alpha_j \sin\alpha_k \\
+ \;&(-R_{e,j}R_{e,k}m_k) \iint d\theta d\zeta \; \sin\alpha_i \cos\alpha_j \sin\alpha_k \\
+ \;&(+R_{o,j}R_{o,k}m_k) \iint d\theta d\zeta \; \sin\alpha_i \sin\alpha_j \cos\alpha_k
\end{aligned}
\tag{9}
$$