

**matrix**

Constructs energy and helicity matrices that represent the Beltrami linear system.

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**1.1 gauge conditions**

1. In the  $v$ -th annulus, bounded by the  $(v-1)$ -th and  $v$ -th interfaces, a general covariant representation of the magnetic vector-potential is written

$$\bar{\mathbf{A}} = \bar{A}_s \nabla s + \bar{A}_\theta \nabla \theta + \bar{A}_\zeta \nabla \zeta. \quad (1)$$

2. To this add  $\nabla g(s, \theta, \zeta)$ , where  $g$  satisfies

$$\begin{aligned} \partial_s g(s, \theta, \zeta) &= - \bar{A}_s(s, \theta, \zeta) \\ \partial_\theta g(-1, \theta, \zeta) &= - \bar{A}_\theta(-1, \theta, \zeta) \\ \partial_\zeta g(-1, 0, \zeta) &= - \bar{A}_\zeta(-1, 0, \zeta). \end{aligned} \quad (2)$$

3. Then  $\mathbf{A} = \bar{\mathbf{A}} + \nabla g$  is given by  $\mathbf{A} = A_\theta \nabla \theta + A_\zeta \nabla \zeta$  with

$$A_\theta(-1, \theta, \zeta) = 0 \quad (3)$$

$$A_\zeta(-1, 0, \zeta) = 0 \quad (4)$$

4. This specifies the gauge: to see this, notice that no gauge term can be added without violating the conditions in Eqn.(3) or Eqn.(4).

5. Note that the gauge employed in each volume is distinct.

**1.2 boundary conditions**

1. The magnetic field is  $\sqrt{g} \mathbf{B} = (\partial_\theta A_\zeta - \partial_\zeta A_\theta) \mathbf{e}_s - \partial_s A_\zeta \mathbf{e}_\theta + \partial_s A_\theta \mathbf{e}_\zeta$ .
2. In the annular volumes, the condition that the field is tangential to the inner interface,  $\sqrt{g} \mathbf{B} \cdot \nabla s = 0$  at  $s = -1$ , gives  $\partial_\theta A_\zeta - \partial_\zeta A_\theta = 0$ . With the above condition on  $A_\theta$  given in Eqn.(3), this gives  $\partial_\theta A_\zeta = 0$ , which with Eqn.(4) gives

$$A_\zeta(-1, \theta, \zeta) = 0. \quad (5)$$

3. The condition at the outer interface,  $s = +1$ , is that the field is  $\sqrt{g} \mathbf{B} \cdot \nabla s = \partial_\theta A_\zeta - \partial_\zeta A_\theta = b$ , where  $b$  is supplied by the user. For each of the plasma regions,  $b = 0$ . For the vacuum region, generally  $b \neq 0$ .

**1.3 enclosed fluxes**

1. In the plasma regions, the enclosed fluxes must be constrained.
2. The toroidal and poloidal fluxes enclosed in each volume are determined using

$$\int_S \mathbf{B} \cdot d\mathbf{s} = \int_{\partial S} \mathbf{A} \cdot d\mathbf{l}. \quad (6)$$

## 1.4 Fourier-Chebyshev representation

1. The components of the vector potential,  $\mathbf{A} = A_\theta \nabla + A_\zeta \nabla \zeta$ , are

$$A_\theta(s, \theta, \zeta) = \sum_{i,l} \mathcal{A}_{\theta,e,i,l} \bar{T}_{l,i}(s) \cos \alpha_i + \sum_{i,l} \mathcal{A}_{\theta,o,i,l} \bar{T}_{l,i}(s) \sin \alpha_i, \quad (7)$$

$$A_\zeta(s, \theta, \zeta) = \sum_{i,l} \mathcal{A}_{\zeta,e,i,l} \bar{T}_{l,i}(s) \cos \alpha_i + \sum_{i,l} \mathcal{A}_{\zeta,o,i,l} \bar{T}_{l,i}(s) \sin \alpha_i, \quad (8)$$

where  $\bar{T}_{l,i}(s) \equiv \bar{s}^{m_i/2} T_l(s)$ ,  $T_l(s)$  is the Chebyshev polynomial, and  $\alpha_j \equiv m_j \theta - n_j \zeta$ . The regularity factor,  $\bar{s}^{m_i/2}$ , where  $\bar{s} \equiv (1+s)/2$ , is only included if there is a coordinate singularity in the domain (i.e. only in the innermost volume, and only in cylindrical and toroidal geometry.)

2. The magnetic field,  $\sqrt{g} \mathbf{B} = \sqrt{g} B^s \mathbf{e}_s + \sqrt{g} B^\theta \mathbf{e}_\theta + \sqrt{g} B^\zeta \mathbf{e}_\zeta$ , is

$$\begin{aligned} \sqrt{g} \mathbf{B} = & \mathbf{e}_s \sum_{i,l} [(-m_i \mathcal{A}_{\zeta,e,i,l} - n_i \mathcal{A}_{\theta,e,i,l}) \bar{T}_{l,i} \sin \alpha_i + (+m_i \mathcal{A}_{\zeta,o,i,l} + n_i \mathcal{A}_{\theta,o,i,l}) \bar{T}_{l,i} \cos \alpha_i] \\ & + \mathbf{e}_\theta \sum_{i,l} [(-\mathcal{A}_{\zeta,e,i,l}) \bar{T}'_{l,i} \cos \alpha_i + (-\mathcal{A}_{\zeta,o,i,l}) \bar{T}'_{l,i} \sin \alpha_i] \\ & + \mathbf{e}_\zeta \sum_{i,l} [(\mathcal{A}_{\theta,e,i,l}) \bar{T}'_{l,i} \cos \alpha_i + (\mathcal{A}_{\theta,o,i,l}) \bar{T}'_{l,i} \sin \alpha_i] \end{aligned} \quad (9)$$

3. The components of the velocity,  $\mathbf{v} \equiv v_s \nabla s + v_\theta \nabla \theta + v_\zeta \nabla \zeta$ , are

$$v_s(s, \theta, \zeta) = \sum_{i,l} \mathcal{v}_{s,e,i,l} \bar{T}_{l,i}(s) \cos \alpha_i + \sum_{i,l} \mathcal{v}_{s,o,i,l} \bar{T}_{l,i}(s) \sin \alpha_i, \quad (10)$$

$$v_\theta(s, \theta, \zeta) = \sum_{i,l} \mathcal{v}_{\theta,e,i,l} \bar{T}_{l,i}(s) \cos \alpha_i + \sum_{i,l} \mathcal{v}_{\theta,o,i,l} \bar{T}_{l,i}(s) \sin \alpha_i, \quad (11)$$

$$v_\zeta(s, \theta, \zeta) = \sum_{i,l} \mathcal{v}_{\zeta,e,i,l} \bar{T}_{l,i}(s) \cos \alpha_i + \sum_{i,l} \mathcal{v}_{\zeta,o,i,l} \bar{T}_{l,i}(s) \sin \alpha_i. \quad (12)$$

## 1.5 constrained energy functional

1. The constrained energy functional in each volume depends on the vector potential and the Lagrange multipliers,

$$\mathcal{F} \equiv \mathcal{F}[\mathcal{A}_{\theta,e,i,l}, \mathcal{A}_{\zeta,e,i,l}, \mathcal{A}_{\theta,o,i,l}, \mathcal{A}_{\zeta,o,i,l}, \mathcal{v}_{s,e,i,l}, \mathcal{v}_{s,o,i,l}, \mathcal{v}_{\theta,e,i,l}, \mathcal{v}_{\theta,o,i,l}, \mathcal{v}_{\zeta,e,i,l}, \mathcal{v}_{\zeta,o,i,l}, \mu, a_i, b_i, c_i, d_i, e_i, f_i, g_1, h_1], \quad (13)$$

and is given by:

$$\begin{aligned} \mathcal{F} \equiv & \int \mathbf{B} \cdot \mathbf{B} dv + \int \mathbf{v} \cdot \mathbf{v} dv - \mu \left[ \int \mathbf{A} \cdot \mathbf{B} dv - K \right] \\ & + \sum_{i=1} a_i \left[ \sum_l \mathcal{A}_{\theta,e,i,l} T_l(-1) - 0 \right] \\ & + \sum_{i=1} b_i \left[ \sum_l \mathcal{A}_{\zeta,e,i,l} T_l(-1) - 0 \right] \\ & + \sum_{i=2} c_i \left[ \sum_l \mathcal{A}_{\theta,o,i,l} T_l(-1) - 0 \right] \\ & + \sum_{i=2} d_i \left[ \sum_l \mathcal{A}_{\zeta,o,i,l} T_l(-1) - 0 \right] \\ & + \sum_{i=2} e_i \left[ \sum_l (-m_i \mathcal{A}_{\zeta,e,i,l} - n_i \mathcal{A}_{\theta,e,i,l}) T_l(+1) - b_{s,i} \right] \\ & + \sum_{i=2} f_i \left[ \sum_l (+m_i \mathcal{A}_{\zeta,o,i,l} + n_i \mathcal{A}_{\theta,o,i,l}) T_l(+1) - b_{c,i} \right] \\ & + g_1 \left[ \sum_l \mathcal{A}_{\theta,e,1,l} T_l(+1) - \Delta \psi_t \right] \\ & + h_1 \left[ \sum_l \mathcal{A}_{\zeta,e,1,l} T_l(+1) + \Delta \psi_p \right] \end{aligned} \quad (14)$$

where

- i.  $a_i, b_i, c_i$  and  $d_i$  are Lagrange multipliers used to enforce the combined gauge and interface boundary condition on the inner interface,
- ii.  $e_i$  and  $f_i$  are Lagrange multipliers used to enforce the interface boundary condition on the outer interface, namely  $\sqrt{g} \mathbf{B} \cdot \nabla s = b$ ; and
- iii.  $g_1$  and  $h_1$  are Lagrange multipliers used to enforce the constraints on the enclosed fluxes.

2. In each plasma volume the boundary condition on the outer interface is  $b = 0$ .

3. In the vacuum volume (only for free-boundary), we may set  $\mu = 0$ .

## 1.6 derivatives of magnetic energy integrals

1. The first derivatives of  $\int dv \mathbf{B} \cdot \mathbf{B}$  with respect to  $A_{\theta,e,i,l}$ ,  $A_{\theta,o,i,l}$ ,  $A_{\zeta,e,i,l}$  and  $A_{\zeta,o,i,l}$  are

$$\frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} = 2 \int dv \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial A_{\theta,e,i,l}} = 2 \int dv \mathbf{B} \cdot \left[ -n_i \bar{T}_{l,i} \sin \alpha_i \mathbf{e}_s + \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (15)$$

$$\frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} = 2 \int dv \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial A_{\theta,o,i,l}} = 2 \int dv \mathbf{B} \cdot \left[ +n_i \bar{T}_{l,i} \cos \alpha_i \mathbf{e}_s + \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (16)$$

$$\frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} = 2 \int dv \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial A_{\zeta,e,i,l}} = 2 \int dv \mathbf{B} \cdot \left[ -m_i \bar{T}_{l,i} \sin \alpha_i \mathbf{e}_s - \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\theta \right] / \sqrt{g} \quad (17)$$

$$\frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} = 2 \int dv \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial A_{\zeta,o,i,l}} = 2 \int dv \mathbf{B} \cdot \left[ +m_i \bar{T}_{l,i} \cos \alpha_i \mathbf{e}_s - \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\theta \right] / \sqrt{g} \quad (18)$$

2. The second derivatives of  $\int dv \mathbf{B} \cdot \mathbf{B}$  with respect to  $A_{\theta,e,i,l}$ ,  $A_{\theta,o,i,l}$ ,  $A_{\zeta,e,i,l}$  and  $A_{\zeta,o,i,l}$  are

$$\begin{aligned} \frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+n_j n_i \bar{T}_{p,j} \bar{T}_{l,i} s_j s_i g_{ss} - n_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j c_i g_{s\zeta} - n_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i c_j g_{s\zeta} + \bar{T}'_{p,j} \bar{T}'_{l,i} c_j c_i g_{\zeta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-n_j n_i \bar{T}_{p,j} \bar{T}_{l,i} c_j s_i g_{ss} + n_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j c_i g_{s\zeta} - n_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i s_j g_{s\zeta} + \bar{T}'_{p,j} \bar{T}'_{l,i} s_j c_i g_{\zeta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+m_j n_i \bar{T}_{p,j} \bar{T}_{l,i} s_j s_i g_{ss} - m_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j c_i g_{s\zeta} + n_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i c_j g_{s\theta} - \bar{T}'_{p,j} \bar{T}'_{l,i} c_j c_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-m_j n_i \bar{T}_{p,j} \bar{T}_{l,i} c_j s_i g_{ss} + m_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j c_i g_{s\zeta} + n_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i s_j g_{s\theta} - \bar{T}'_{p,j} \bar{T}'_{l,i} s_j c_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-n_j n_i \bar{T}_{p,j} \bar{T}_{l,i} s_j c_i g_{ss} - n_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j s_i g_{s\zeta} + n_i \bar{T}_{l,i} \bar{T}'_{p,j} c_i c_j g_{s\zeta} + \bar{T}'_{p,j} \bar{T}'_{l,i} c_j s_i g_{\zeta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+n_j n_i \bar{T}_{p,j} \bar{T}_{l,i} c_j c_i g_{ss} + n_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j s_i g_{s\zeta} + n_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i s_j g_{s\zeta} + \bar{T}'_{p,j} \bar{T}'_{l,i} s_j s_i g_{\zeta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-m_j n_i \bar{T}_{p,j} \bar{T}_{l,i} s_j c_i g_{ss} - m_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j s_i g_{s\zeta} - n_i \bar{T}_{l,i} \bar{T}'_{p,j} c_i c_j g_{s\theta} - \bar{T}'_{p,j} \bar{T}'_{l,i} c_j s_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+m_j n_i \bar{T}_{p,j} \bar{T}_{l,i} c_j c_i g_{ss} + m_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j s_i g_{s\zeta} - n_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i s_j g_{s\theta} - \bar{T}'_{p,j} \bar{T}'_{l,i} s_j s_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+n_j m_i \bar{T}_{p,j} \bar{T}_{l,i} s_j s_i g_{ss} + n_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j c_i g_{s\theta} - m_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i c_j g_{s\zeta} - \bar{T}'_{p,j} \bar{T}'_{l,i} c_j c_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-n_j m_i \bar{T}_{p,j} \bar{T}_{l,i} c_j s_i g_{ss} - n_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j c_i g_{s\theta} - m_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i s_j g_{s\zeta} - \bar{T}'_{p,j} \bar{T}'_{l,i} s_j c_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+m_j m_i \bar{T}_{p,j} \bar{T}_{l,i} s_j s_i g_{ss} + m_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j c_i g_{s\theta} + m_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i c_j g_{s\theta} + \bar{T}'_{p,j} \bar{T}'_{l,i} c_j c_i g_{\theta\theta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-m_j m_i \bar{T}_{p,j} \bar{T}_{l,i} c_j s_i g_{ss} - m_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j c_i g_{s\theta} + m_i \bar{T}_{l,i} \bar{T}'_{p,j} s_i s_j g_{s\theta} + \bar{T}'_{p,j} \bar{T}'_{l,i} s_j c_i g_{\theta\theta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-n_j m_i \bar{T}_{p,j} \bar{T}_{l,i} s_j c_i g_{ss} + n_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j s_i g_{s\theta} + m_i \bar{T}_{l,i} \bar{T}'_{p,j} c_i c_j g_{s\zeta} - \bar{T}'_{p,j} \bar{T}'_{l,i} c_j s_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+n_j m_i \bar{T}_{p,j} \bar{T}_{l,i} c_j c_i g_{ss} - n_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j s_i g_{s\theta} + m_i \bar{T}_{l,i} \bar{T}'_{p,j} c_i s_j g_{s\zeta} - \bar{T}'_{p,j} \bar{T}'_{l,i} s_j s_i g_{\theta\zeta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (-m_j m_i \bar{T}_{p,j} \bar{T}_{l,i} s_j c_i g_{ss} + m_j \bar{T}_{p,j} \bar{T}'_{l,i} s_j s_i g_{s\theta} - m_i \bar{T}_{l,i} \bar{T}'_{p,j} c_i c_j g_{s\theta} + \bar{T}'_{p,j} \bar{T}'_{l,i} c_j s_i g_{\theta\theta}) / \sqrt{g}^2 \\ \frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{B} \cdot \mathbf{B} &= 2 \int dv (+m_j m_i \bar{T}_{p,j} \bar{T}_{l,i} c_j c_i g_{ss} - m_j \bar{T}_{p,j} \bar{T}'_{l,i} c_j s_i g_{s\theta} - m_i \bar{T}_{l,i} \bar{T}'_{p,j} c_i s_j g_{s\theta} + \bar{T}'_{p,j} \bar{T}'_{l,i} s_j s_i g_{\theta\theta}) / \sqrt{g}^2 \end{aligned}$$

### 1.7 derivatives of helicity integrals

1. The first derivatives of  $\int dv \mathbf{A} \cdot \mathbf{B}$  with respect to  $A_{\theta,e,i,l}$ ,  $A_{\theta,o,i,l}$ ,  $A_{\zeta,e,i,l}$  and  $A_{\zeta,o,i,l}$  are

$$\frac{\partial}{\partial \mathbf{A}_{\theta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left( \frac{\partial \mathbf{A}}{\partial \mathbf{A}_{\theta,e,i,l}} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial \mathbf{A}_{\theta,e,i,l}} \right) = \int dv (\bar{T}_{l,i} \cos \alpha_i \nabla \theta \cdot \mathbf{B} + \mathbf{A} \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\zeta / \sqrt{g}) \quad (19)$$

$$\frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left( \frac{\partial \mathbf{A}}{\partial A_{\theta,o,i,l}} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial A_{\theta,o,i,l}} \right) = \int dv (\bar{T}_{l,i} \sin \alpha_i \nabla \theta \cdot \mathbf{B} + \mathbf{A} \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\zeta / \sqrt{g}) \quad (20)$$

$$\frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left( \frac{\partial \mathbf{A}}{\partial A_{\zeta,e,i,l}} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial A_{\zeta,e,i,l}} \right) = \int dv (\bar{T}_{l,i} \cos \alpha_i \nabla \zeta \cdot \mathbf{B} - \mathbf{A} \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\theta / \sqrt{g}) \quad (21)$$

$$\frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left( \frac{\partial \mathbf{A}}{\partial A_{\zeta,o,i,l}} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial A_{\zeta,o,i,l}} \right) = \int dv (\bar{T}_{l,i} \sin \alpha_i \nabla \zeta \cdot \mathbf{B} - \mathbf{A} \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\theta / \sqrt{g}) \quad (22)$$

2. Note that in the above expressions,  $\mathbf{A} \cdot \mathbf{e}_s = 0$  has been used.

3. The second derivatives of  $\int dv \mathbf{A} \cdot \mathbf{B}$  with respect to  $A_{\theta,e,i,l}$ ,  $A_{\theta,o,i,l}$ ,  $A_{\zeta,e,i,l}$  and  $A_{\zeta,o,i,l}$  are

$$\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \cancel{+\bar{T}_{l,i} \cos \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_\zeta} + \bar{T}_{p,j} \cos \alpha_j \nabla \theta \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (23)$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \cancel{+\bar{T}_{l,i} \cos \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_\zeta} + \bar{T}_{p,j} \sin \alpha_j \nabla \theta \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (24)$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial \theta_{\theta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ -\bar{T}_{l,i} \cos \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_\theta + \bar{T}_{p,j} \cos \alpha_j \nabla \zeta \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (25)$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ -\bar{T}_{l,i} \cos \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_\theta + \bar{T}_{p,j} \sin \alpha_j \nabla \zeta \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (26)$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \cancel{+\bar{T}_{l,i} \sin \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_\zeta} + \cancel{\bar{T}_{p,j} \cos \alpha_j \nabla \theta \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\zeta} \right] / \sqrt{g} \quad (27)$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \cancel{+\bar{T}_{l,i} \sin \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_\zeta} + \bar{T}_{p,j} \sin \alpha_j \nabla \theta \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (28)$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ -\bar{T}_{l,i} \sin \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_\theta + \bar{T}_{p,j} \cos \alpha_j \nabla \zeta \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (29)$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ -\bar{T}_{l,i} \sin \alpha_i \nabla \theta \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_\theta + \bar{T}_{p,j} \sin \alpha_j \nabla \zeta \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\zeta \right] / \sqrt{g} \quad (30)$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ +\bar{T}_{l,i} \cos \alpha_i \nabla_{\zeta} \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_{\zeta} - \bar{T}_{p,j} \cos \alpha_j \nabla_{\theta} \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_{\theta} \right] / \sqrt{g} \quad (31)$$

$$\frac{\partial}{\partial \theta_{\theta,o,j,p}} \frac{\partial}{\partial \zeta_{\zeta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \bar{T}_{l,i} \cos \alpha_i \nabla \zeta \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_\zeta - \bar{T}_{p,j} \sin \alpha_j \nabla \theta \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\theta \right] / \sqrt{g} \quad (32)$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \cancel{-\bar{T}_{l,i} \cos \alpha_i \nabla \zeta \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_\theta} - \bar{T}_{p,j} \cos \alpha_j \nabla \zeta \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\theta \right] / \sqrt{g} \quad (33)$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \cancel{-\bar{T}_{l,i} \cos \alpha_i \nabla \zeta \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_\theta} - \bar{T}_{p,j} \sin \alpha_j \nabla \zeta \cdot \bar{T}'_{l,i} \cos \alpha_i \mathbf{e}_\theta \right] / \sqrt{g} \quad (34)$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ +\bar{T}_{l,i} \sin \alpha_i \nabla_{\zeta} \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_{\zeta} - \bar{T}_{p,j} \cos \alpha_j \nabla_{\theta} \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_{\theta} \right] / \sqrt{g} \quad (35)$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ +\bar{T}_{l,i} \sin \alpha_i \nabla_{\zeta} \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_{\zeta} - \bar{T}_{p,j} \sin \alpha_j \nabla_{\theta} \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_{\theta} \right] / \sqrt{g} \quad (36)$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \cancel{-\bar{T}_{l,i} \sin \alpha_i \nabla \zeta \cdot \bar{T}'_{p,j} \cos \alpha_j \mathbf{e}_\theta} - \bar{T}_{p,j} \cos \alpha_j \nabla \zeta \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_\theta \right] / \sqrt{g} \quad (37)$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv \mathbf{A} \cdot \mathbf{B} = \int dv \left[ \frac{-\bar{T}_{l,i} \sin \alpha_i \nabla_{\zeta} \cdot \bar{T}'_{p,j} \sin \alpha_j \mathbf{e}_{\theta} - \bar{T}_{p,j} \sin \alpha_j \nabla_{\zeta} \cdot \bar{T}'_{l,i} \sin \alpha_i \mathbf{e}_{\theta}}{\sqrt{g}} \right] \quad (38)$$

4. In these expressions the terms  $\nabla\theta \cdot \mathbf{e}_\theta = \nabla\zeta \cdot \mathbf{e}_\zeta = 1$ , and  $\cancel{\nabla\theta \cdot \mathbf{e}_\zeta} = \cancel{\nabla\zeta \cdot \mathbf{e}_\theta} = 0$  have been included to show the structure of the derivation.

## 1.8 derivatives of kinetic energy integrals

1. The first derivatives of  $\int dv v^2$  with respect to  $v_{s,e,i,l}$  etc. are

$$\frac{\partial}{\partial v_{s,e,i,l}} \int dv \mathbf{v} \cdot \mathbf{v} = 2 \int dv \mathbf{v} \cdot \bar{T}_{l,i} \cos \alpha_i \nabla s \quad (39)$$

$$\frac{\partial}{\partial v_{s,o,i,l}} \int dv \mathbf{v} \cdot \mathbf{v} = 2 \int dv \mathbf{v} \cdot \bar{T}_{l,i} \sin \alpha_i \nabla s \quad (40)$$

$$\frac{\partial}{\partial v_{\theta,e,i,l}} \int dv \mathbf{v} \cdot \mathbf{v} = 2 \int dv \mathbf{v} \cdot \bar{T}_{l,i} \cos \alpha_i \nabla \theta \quad (41)$$

$$\frac{\partial}{\partial v_{\theta,o,i,l}} \int dv \mathbf{v} \cdot \mathbf{v} = 2 \int dv \mathbf{v} \cdot \bar{T}_{l,i} \sin \alpha_i \nabla \theta \quad (42)$$

$$\frac{\partial}{\partial v_{\zeta,e,i,l}} \int dv \mathbf{v} \cdot \mathbf{v} = 2 \int dv \mathbf{v} \cdot \bar{T}_{l,i} \cos \alpha_i \nabla \zeta \quad (43)$$

$$\frac{\partial}{\partial v_{\zeta,o,i,l}} \int dv \mathbf{v} \cdot \mathbf{v} = 2 \int dv \mathbf{v} \cdot \bar{T}_{l,i} \sin \alpha_i \nabla \zeta \quad (44)$$

$$(45)$$

## 1.9 calculation of volume-integrated basis-function-weighted metric information

1. The required geometric information is calculated in [ma00aa](#).