casing

Constructs the field created by the plasma currents, at an arbitrary, external location using virtual casing.

[called by: bnorml.]

contents

1.1 theory and numerics

1. Required inputs to this subroutine are the geometry of the plasma boundary,

$$\mathbf{x}(\theta,\zeta) \equiv x(\theta,\zeta)\mathbf{i} + y(\theta,\zeta)\mathbf{j} + z(\theta,\zeta)\mathbf{k}, \tag{1}$$

   and the tangential field on this boundary,

$$\mathbf{B}_s = B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta, \tag{2}$$

   where θ and ζ are arbitrary poloidal and toroidal angles, and $\mathbf{e}_\theta \equiv \partial\mathbf{x}/\partial\theta$, $\mathbf{e}_\zeta \equiv \partial\mathbf{x}/\partial\zeta$. This routine assumes that the plasma boundary is a flux surface, i.e. $\mathbf{B}\cdot\mathbf{e}_\theta\times\mathbf{e}_\zeta = 0$.

2. The virtual casing principle [Shafranov & Zakharov (1972)[1], Lazerson (2012)[2], Hanson (2015)[3]] shows that the field outside/inside the plasma arising from plasma currents inside/outside the boundary is equivalent to the field generated by a surface current,

$$\mathbf{j} = \mathbf{B}_s \times \mathbf{n}, \tag{3}$$

   where $\mathbf{n}$ is normal to the surface.

3. The field at some arbitrary point, $\bar{\mathbf{x}}$, created by this surface current is given by

$$\mathbf{B}(\bar{\mathbf{x}}) = \int_{\mathcal{S}} \frac{(\mathbf{B}_s \times d\mathbf{s}) \times \hat{\mathbf{r}}}{r^2}, \tag{4}$$

   where $d\mathbf{s} \equiv \mathbf{e}_\theta \times \mathbf{e}_\zeta \, d\theta d\zeta$.

4. For ease of notation introduce

$$\mathbf{J} \quad\equiv\quad \mathbf{B}_s \times d\mathbf{s} = \alpha\,\mathbf{e}_\theta - \beta\,\mathbf{e}_\zeta, \tag{5}$$

   where $\alpha \equiv B_\zeta = B^\theta g_{\theta\zeta} + B^\zeta g_{\zeta\zeta}$ and $\beta \equiv B_\theta = B^\theta g_{\theta\theta} + B^\zeta g_{\theta\zeta}$,

5. We may write in Cartesian coordinates $\mathbf{J} = j_x\,\mathbf{i} + j_y\,\mathbf{j} + j_z\,\mathbf{k}$, where

$$\begin{aligned} j_x &= \alpha\,x_\theta - \beta\,x_\zeta \quad (6)\\ j_y &= \alpha\,y_\theta - \beta\,y_\zeta \quad (7)\\ j_z &= \alpha\,z_\theta - \beta\,z_\zeta. \quad (8) \end{aligned}$$

6. Requiring that the current,

$$\mathbf{j} \quad\equiv\quad \nabla\times\mathbf{B} = \sqrt{g}^{-1}(\partial_\theta B_\zeta - \partial_\zeta B_\theta)\,\mathbf{e}_s + \sqrt{g}^{-1}(\partial_\zeta B_s - \partial_s B_\zeta)\,\mathbf{e}_\theta + \sqrt{g}^{-1}(\partial_s B_\theta - \partial_\theta B_s)\,\mathbf{e}_\zeta, \tag{9}$$

   has no normal component to the surface, i.e. $\mathbf{j}\cdot\nabla s = 0$, we obtain the condition $\partial_\theta B_\zeta = \partial_\zeta B_\theta$, or $\partial_\theta\alpha = \partial_\zeta\beta$. In axisymmetric configurations, where $\partial_\zeta\beta = 0$, we must have $\partial_\theta\alpha = 0$.

7. The displacement from an arbitrary point, (X,Y,Z), to a point, (x,y,z), that lies on the surface is given

$$\mathbf{r} \equiv r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k} = (X-x)\,\mathbf{i} + (Y-y)\,\mathbf{j} + (Z-z)\,\mathbf{k}. \tag{10}$$

[1] V.D. Shafranov & L.E. Zakharov, Nucl. Fusion **12**, 599 (1972)
[2] S.A. Lazerson, Plasma Phys. Control. Fusion **54**, 122002 (2012)
[3] J.D. Hanson, Plasma Phys. Control. Fusion **57**, 115006 (2015)

8. The components of the magnetic field produced by the surface current are then

$$
\begin{aligned}
B^x &= \oint\!\!\oint d\theta d\zeta \;\; (j_y r_z - j_z r_y)/r^3, && (11)\\
B^y &= \oint\!\!\oint d\theta d\zeta \;\; (j_z r_x - j_x r_z)/r^3, && (12)\\
B^z &= \oint\!\!\oint d\theta d\zeta \;\; (j_x r_y - j_y r_x)/r^3 && (13)
\end{aligned}
$$

9. The surface integral is performed using NAG: D01EAF, which uses an adaptive subdivision strategy and also computes absolute error estimates. The absolute and relative accuracy required are provided by the input `vcasingtol`. The minimum number of function evaluations is provided by the input `vcasingits`.

10. It may be convenient to have the derivatives:

$$
\begin{aligned}
\frac{\partial B^x}{\partial x} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_y r_z - j_z r_y)(X-x)/r^5 \right], && (14)\\
\frac{\partial B^x}{\partial y} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_y r_z - j_z r_y)(Y-y)/r^5 - j_z/r^3\right], && (15)\\
\frac{\partial B^x}{\partial z} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_y r_z - j_z r_y)(Z-z)/r^5 + j_y/r^3\right], && (16)\\
\frac{\partial B^y}{\partial x} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_z r_x - j_x r_z)(X-x)/r^5 + j_z/r^3\right], && (17)\\
\frac{\partial B^y}{\partial y} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_z r_x - j_x r_z)(Y-y)/r^5 \right], && (18)\\
\frac{\partial B^y}{\partial z} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_z r_x - j_x r_z)(Z-z)/r^5 - j_x/r^3\right], && (19)\\
\frac{\partial B^z}{\partial x} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_x r_y - j_y r_x)(X-x)/r^5 - j_y/r^3\right], && (20)\\
\frac{\partial B^z}{\partial y} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_x r_y - j_y r_x)(Y-y)/r^5 + j_x/r^3\right], && (21)\\
\frac{\partial B^z}{\partial z} &= \oint\!\!\oint d\theta d\zeta \;\; \left[-3(j_x r_y - j_y r_x)(Z-z)/r^5 \right]. && (22)
\end{aligned}
$$