matrix

Constructs energy and helicity matrices that represent the Beltrami linear system.

[called by: dforce.] [calls: .]

contents

1.1 gauge conditions

1. In the v-th annulus, bounded by the $(v-1)$-th and v-th interfaces, a general covariant representation of the magnetic vector-potential is written

$$\bar{\mathbf{A}} = \bar{A}_s \nabla s + \bar{A}_\theta \nabla\theta + \bar{A}_\zeta \nabla\zeta. \tag{1}$$

2. To this add $\nabla g(s,\theta,\zeta)$, where g satisfies

$$\begin{array}{rcl} \partial_s g(s,\theta,\zeta) & = & - \quad \bar{A}_s(s,\theta,\zeta) \\ \partial_\theta g(-1,\theta,\zeta) & = & - \quad \bar{A}_\theta(-1,\theta,\zeta) \\ \partial_\zeta g(-1,0,\zeta) & = & - \quad \bar{A}_\zeta(-1,0,\zeta). \end{array} \tag{2}$$

3. Then $\mathbf{A} = \bar{\mathbf{A}} + \nabla g$ is given by $\mathbf{A} = A_\theta \nabla\theta + A_\zeta \nabla\zeta$ with

$$A_\theta(-1,\theta,\zeta) \quad = \quad 0 \tag{3}$$
$$A_\zeta(-1,0,\zeta) \quad = \quad 0 \tag{4}$$

4. This specifies the gauge: to see this, notice that no gauge term can be added without violating the conditions in Eqn.(3) or Eqn.(4).

5. Note that the gauge employed in each volume is distinct.

1.2 boundary conditions

1. The magnetic field is $\sqrt{g}\,\mathbf{B} = (\partial_\theta A_\zeta - \partial_\zeta A_\theta)\,\mathbf{e}_s - \partial_s A_\zeta\,\mathbf{e}_\theta + \partial_s A_\theta\,\mathbf{e}_\zeta$.

2. In the annular volumes, the condition that the field is tangential to the inner interface, $\sqrt{g}\mathbf{B}\cdot\nabla s = 0$ at $s = -1$, gives $\partial_\theta A_\zeta - \partial_\zeta A_\theta = 0$. With the above condition on A_θ given in Eqn.(3), this gives $\partial_\theta A_\zeta = 0$, which with Eqn.(4) gives

$$A_\zeta(-1,\theta,\zeta) = 0. \tag{5}$$

3. The condition at the outer interface, $s = +1$, is that the field is $\sqrt{g}\,\mathbf{B}\cdot\nabla s = \partial_\theta A_\zeta - \partial_\zeta A_\theta = b$, where b is supplied by the user. For each of the plasma regions, $b = 0$. For the vacuum region, generally $b \neq 0$.

1.3 enclosed fluxes

1. In the plasma regions, the enclosed fluxes must be constrained.

2. The toroidal and poloidal fluxes enclosed in each volume are determined using

$$\int_S \mathbf{B}\cdot\mathbf{ds} = \int_{\partial S} \mathbf{A}\cdot\mathbf{dl}. \tag{6}$$

1.4 Fourier-Chebyshev representation

1. The components of the vector potential, $\mathbf{A} = A_\theta \nabla + A_\zeta \nabla\zeta$, are

$$A_\theta(s,\theta,\zeta) = \sum_{i,l} A_{\theta,e,i,l}\,\overline{T}_{l,i}(s)\cos\alpha_i + \sum_{i,l} A_{\theta,o,i,l}\,\overline{T}_{l,i}(s)\sin\alpha_i, \quad (7)$$

$$A_\zeta(s,\theta,\zeta) = \sum_{i,l} A_{\zeta,e,i,l}\,\overline{T}_{l,i}(s)\cos\alpha_i + \sum_{i,l} A_{\zeta,o,i,l}\,\overline{T}_{l,i}(s)\sin\alpha_i, \quad (8)$$

where $\overline{T}_{l,i}(s) \equiv \bar{s}^{m_i/2}\,T_l(s)$, $T_l(s)$ is the Chebyshev polynomial, and $\alpha_j \equiv m_j\theta - n_j\zeta$. The regularity factor, $\bar{s}^{m_i/2}$, where $\bar{s} \equiv (1+s)/2$, is only included if there is a coordinate singularity in the domain (i.e. only in the innermost volume, and only in cylindrical and toroidal geometry.)

2. The magnetic field, $\sqrt{g}\,\mathbf{B} = \sqrt{g}B^s\mathbf{e}_s + \sqrt{g}B^\theta\mathbf{e}_\theta + \sqrt{g}B^\zeta\mathbf{e}_\zeta$, is

$$\begin{aligned}\sqrt{g}\,\mathbf{B} &= \mathbf{e}_s \sum_{i,l}[(-m_i A_{\zeta,e,i,l} - n_i A_{\theta,e,i,l})\overline{T}_{l,i}\sin\alpha_i + (+m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\overline{T}_{l,i}\cos\alpha_i] \\ &+ \mathbf{e}_\theta \sum_{i,l}[(- A_{\zeta,e,i,l})\overline{T}'_{l,i}\cos\alpha_i + (- A_{\zeta,o,i,l})\overline{T}'_{l,i}\sin\alpha_i] \\ &+ \mathbf{e}_\zeta \sum_{i,l}[(A_{\theta,e,i,l})\overline{T}'_{l,i}\cos\alpha_i + (A_{\theta,o,i,l})\overline{T}'_{l,i}\sin\alpha_i]\end{aligned} \quad (9)$$

3. The components of the velocity, $\mathbf{v} \equiv v_s\nabla s + v_\theta\nabla\theta + v_\zeta\nabla\zeta$, are

$$v_s(s,\theta,\zeta) = \sum_{i,l} v_{s,e,i,l}\,\overline{T}_{l,i}(s)\cos\alpha_i + \sum_{i,l} v_{s,o,i,l}\,\overline{T}_{l,i}(s)\sin\alpha_i, \quad (10)$$

$$v_\theta(s,\theta,\zeta) = \sum_{i,l} v_{\theta,e,i,l}\,\overline{T}_{l,i}(s)\cos\alpha_i + \sum_{i,l} v_{\theta,o,i,l}\,\overline{T}_{l,i}(s)\sin\alpha_i, \quad (11)$$

$$v_\zeta(s,\theta,\zeta) = \sum_{i,l} v_{\zeta,e,i,l}\,\overline{T}_{l,i}(s)\cos\alpha_i + \sum_{i,l} v_{\zeta,o,i,l}\,\overline{T}_{l,i}(s)\sin\alpha_i. \quad (12)$$

1.5 constrained energy functional

1. The constrained energy functional in each volume depends on the vector potential and the Lagrange multipliers,

$$\mathcal{F} \equiv \mathcal{F}[A_{\theta,e,i,l}, A_{\zeta,e,i,l}, A_{\theta,o,i,l}, A_{\zeta,o,i,l}, v_{s,e,i,l}, v_{s,o,i,l}, v_{\theta,e,i,l}, v_{\theta,o,i,l}, v_{\zeta,e,i,l}, v_{\zeta,o,i,l}, \mu, a_i, b_i, c_i, d_i, e_i, f_i, g_1, h_1], \quad (13)$$

and is given by:

$$\begin{aligned}\mathcal{F} \equiv & \int \mathbf{B}\cdot\mathbf{B}\,dv + \int \mathbf{v}\cdot\mathbf{v}\,dv - \mu\left[\int \mathbf{A}\cdot\mathbf{B}\,dv - K\right] \\ &+ \sum_{i=1} a_i \left[\sum_l A_{\theta,e,i,l} T_l(-1) - 0\right] \\ &+ \sum_{i=1} b_i \left[\sum_l A_{\zeta,e,i,l} T_l(-1) - 0\right] \\ &+ \sum_{i=2} c_i \left[\sum_l A_{\theta,o,i,l} T_l(-1) - 0\right] \\ &+ \sum_{i=2} d_i \left[\sum_l A_{\zeta,o,i,l} T_l(-1) - 0\right] \\ &+ \sum_{i=2} e_i \left[\sum_l (-m_i A_{\zeta,e,i,l} - n_i A_{\theta,e,i,l})\,T_l(+1) - b_{s,i}\right] \\ &+ \sum_{i=2} f_i \left[\sum_l (+m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\,T_l(+1) - b_{c,i}\right] \\ &+ g_1 \left[\sum_l A_{\theta,e,1,l} T_l(+1) - \Delta\psi_t\right] \\ &+ h_1 \left[\sum_l A_{\zeta,e,1,l} T_l(+1) + \Delta\psi_p\right]\end{aligned} \quad (14)$$

where

i. a_i, b_i, c_i and d_i are Lagrange multipliers used to enforce the combined gauge and interface boundary condition on the inner interface,

ii. e_i and f_i are Lagrange multipliers used to enforce the interface boundary condition on the outer interface, namely $\sqrt{g}\,\mathbf{B}\cdot\nabla s = b$; and

iii. g_1 and h_1 are Lagrange multipliers used to enforce the constraints on the enclosed fluxes.

2. In each plasma volume the boundary condition on the outer interface is $b = 0$.

3. In the vacuum volume (only for free-boundary), we may set $\mu = 0$.

1.6 derivatives of magnetic energy integrals

1. The first derivatives of $\int dv\,\mathbf{B}\cdot\mathbf{B}$ with respect to $A_{\theta,e,i,l}$, $A_{\theta,o,i,l}$, $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$ are

$$\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial A_{\theta,e,i,l}} = 2\int dv\,\mathbf{B}\cdot\left[-n_i\overline{T}_{l,i}\sin\alpha_i\,\mathbf{e}_s + \overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\zeta\right]/\sqrt{g} \tag{15}$$

$$\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial A_{\theta,o,i,l}} = 2\int dv\,\mathbf{B}\cdot\left[+n_i\overline{T}_{l,i}\cos\alpha_i\,\mathbf{e}_s + \overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\zeta\right]/\sqrt{g} \tag{16}$$

$$\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial A_{\zeta,e,i,l}} = 2\int dv\,\mathbf{B}\cdot\left[-m_i\overline{T}_{l,i}\sin\alpha_i\,\mathbf{e}_s - \overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\theta\right]/\sqrt{g} \tag{17}$$

$$\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial A_{\zeta,o,i,l}} = 2\int dv\,\mathbf{B}\cdot\left[+m_i\overline{T}_{l,i}\cos\alpha_i\,\mathbf{e}_s - \overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\theta\right]/\sqrt{g} \tag{18}$$

2. The second derivatives of $\int dv\,\mathbf{B}\cdot\mathbf{B}$ with respect to $A_{\theta,e,i,l}$, $A_{\theta,o,i,l}$, $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$ are

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+n_j n_i\overline{T}_{p,j}\overline{T}_{l,i}s_j s_i g_{ss} - n_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j c_i g_{s\zeta} - n_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i c_j g_{s\zeta} + \overline{T}'_{p,j}\overline{T}'_{l,i}c_j c_i g_{\zeta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-n_j n_i\overline{T}_{p,j}\overline{T}_{l,i}c_j s_i g_{ss} + n_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j c_i g_{s\zeta} - n_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i s_j g_{s\zeta} + \overline{T}'_{p,j}\overline{T}'_{l,i}s_j c_i g_{\zeta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+m_j n_i\overline{T}_{p,j}\overline{T}_{l,i}s_j s_i g_{ss} - m_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j c_i g_{s\zeta} + n_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i c_j g_{s\theta} - \overline{T}'_{p,j}\overline{T}'_{l,i}c_j c_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-m_j n_i\overline{T}_{p,j}\overline{T}_{l,i}c_j s_i g_{ss} + m_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j c_i g_{s\zeta} + n_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i s_j g_{s\theta} - \overline{T}'_{p,j}\overline{T}'_{l,i}s_j c_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-n_j n_i\overline{T}_{p,j}\overline{T}_{l,i}s_j c_i g_{ss} - n_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j s_i g_{s\zeta} + n_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i c_j g_{s\zeta} + \overline{T}'_{p,j}\overline{T}'_{l,i}c_j s_i g_{\zeta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+n_j n_i\overline{T}_{p,j}\overline{T}_{l,i}c_j c_i g_{ss} + n_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j s_i g_{s\zeta} + n_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i s_j g_{s\zeta} + \overline{T}'_{p,j}\overline{T}'_{l,i}s_j s_i g_{\zeta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-m_j n_i\overline{T}_{p,j}\overline{T}_{l,i}s_j c_i g_{ss} - m_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j s_i g_{s\zeta} - n_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i c_j g_{s\theta} - \overline{T}'_{p,j}\overline{T}'_{l,i}c_j s_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+m_j n_i\overline{T}_{p,j}\overline{T}_{l,i}c_j c_i g_{ss} + m_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j s_i g_{s\zeta} - n_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i s_j g_{s\theta} - \overline{T}'_{p,j}\overline{T}'_{l,i}s_j s_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+n_j m_i\overline{T}_{p,j}\overline{T}_{l,i}s_j s_i g_{ss} + n_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j c_i g_{s\theta} - m_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i c_j g_{s\zeta} - \overline{T}'_{p,j}\overline{T}'_{l,i}c_j c_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-n_j m_i\overline{T}_{p,j}\overline{T}_{l,i}c_j s_i g_{ss} - n_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j c_i g_{s\theta} - m_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i s_j g_{s\zeta} - \overline{T}'_{p,j}\overline{T}'_{l,i}s_j c_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+m_j m_i\overline{T}_{p,j}\overline{T}_{l,i}s_j s_i g_{ss} + m_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j c_i g_{s\theta} + m_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i c_j g_{s\theta} + \overline{T}'_{p,j}\overline{T}'_{l,i}c_j c_i g_{\theta\theta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-m_j m_i\overline{T}_{p,j}\overline{T}_{l,i}c_j s_i g_{ss} - m_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j c_i g_{s\theta} + m_i\overline{T}_{l,i}\overline{T}'_{p,j}s_i s_j g_{s\theta} + \overline{T}'_{p,j}\overline{T}'_{l,i}s_j c_i g_{\theta\theta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-n_j m_i\overline{T}_{p,j}\overline{T}_{l,i}s_j c_i g_{ss} + n_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j s_i g_{s\theta} + m_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i c_j g_{s\zeta} - \overline{T}'_{p,j}\overline{T}'_{l,i}c_j s_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+n_j m_i\overline{T}_{p,j}\overline{T}_{l,i}c_j c_i g_{ss} - n_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j s_i g_{s\theta} + m_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i s_j g_{s\zeta} - \overline{T}'_{p,j}\overline{T}'_{l,i}s_j s_i g_{\theta\zeta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(-m_j m_i\overline{T}_{p,j}\overline{T}_{l,i}s_j c_i g_{ss} + m_j\overline{T}_{p,j}\overline{T}'_{l,i}s_j s_i g_{s\theta} - m_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i c_j g_{s\theta} + \overline{T}'_{p,j}\overline{T}'_{l,i}c_j s_i g_{\theta\theta})/\sqrt{g}^2$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\,\mathbf{B}\cdot\mathbf{B} = 2\int dv\,(+m_j m_i\overline{T}_{p,j}\overline{T}_{l,i}c_j c_i g_{ss} - m_j\overline{T}_{p,j}\overline{T}'_{l,i}c_j s_i g_{s\theta} - m_i\overline{T}_{l,i}\overline{T}'_{p,j}c_i s_j g_{s\theta} + \overline{T}'_{p,j}\overline{T}'_{l,i}s_j s_i g_{\theta\theta})/\sqrt{g}^2$$

1.7 derivatives of helicity integrals

1. The first derivatives of $\int dv\ \mathbf{A}\cdot\mathbf{B}$ with respect to $A_{\theta,e,i,l}$, $A_{\theta,o,i,l}$, $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$ are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left(\frac{\partial \mathbf{A}}{\partial A_{\theta,e,i,l}}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial A_{\theta,e,i,l}}\right) = \int dv\ (\overline{T}_{l,i}\cos\alpha_i\nabla\theta\cdot\mathbf{B}+\mathbf{A}\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\zeta/\sqrt{g}) && (19)\\
\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left(\frac{\partial \mathbf{A}}{\partial A_{\theta,o,i,l}}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial A_{\theta,o,i,l}}\right) = \int dv\ (\overline{T}_{l,i}\sin\alpha_i\nabla\theta\cdot\mathbf{B}+\mathbf{A}\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\zeta/\sqrt{g}) && (20)\\
\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left(\frac{\partial \mathbf{A}}{\partial A_{\zeta,e,i,l}}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial A_{\zeta,e,i,l}}\right) = \int dv\ (\overline{T}_{l,i}\cos\alpha_i\nabla\zeta\cdot\mathbf{B}-\mathbf{A}\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\theta/\sqrt{g}) && (21)\\
\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left(\frac{\partial \mathbf{A}}{\partial A_{\zeta,o,i,l}}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial A_{\zeta,o,i,l}}\right) = \int dv\ (\overline{T}_{l,i}\sin\alpha_i\nabla\zeta\cdot\mathbf{B}-\mathbf{A}\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\theta/\sqrt{g}) && (22)
\end{aligned}
$$

2. Note that in the above expressions, $\mathbf{A}\cdot\mathbf{e}_s=0$ has been used.

3. The second derivatives of $\int dv\ \mathbf{A}\cdot\mathbf{B}$ with respect to $A_{\theta,e,i,l}$, $A_{\theta,o,i,l}$, $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$ are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{+\overline{T}_{l,i}\cos\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\zeta}\cancel{+\overline{T}_{p,j}\cos\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\zeta}\right]/\sqrt{g} && (23)\\
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{+\overline{T}_{l,i}\cos\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\zeta}\cancel{+\overline{T}_{p,j}\sin\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\zeta}\right]/\sqrt{g} && (24)\\
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[-\overline{T}_{l,i}\cos\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\theta+\overline{T}_{p,j}\cos\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\zeta\right]/\sqrt{g} && (25)\\
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\theta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[-\overline{T}_{l,i}\cos\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\theta+\overline{T}_{p,j}\sin\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\zeta\right]/\sqrt{g} && (26)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{+\overline{T}_{l,i}\sin\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\zeta}\cancel{+\overline{T}_{p,j}\cos\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\zeta}\right]/\sqrt{g} && (27)\\
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{+\overline{T}_{l,i}\sin\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\zeta}\cancel{+\overline{T}_{p,j}\sin\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\zeta}\right]/\sqrt{g} && (28)\\
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[-\overline{T}_{l,i}\sin\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\theta+\overline{T}_{p,j}\cos\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\zeta\right]/\sqrt{g} && (29)\\
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[-\overline{T}_{l,i}\sin\alpha_i\nabla\theta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\theta+\overline{T}_{p,j}\sin\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\zeta\right]/\sqrt{g} && (30)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[+\overline{T}_{l,i}\cos\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\zeta-\overline{T}_{p,j}\cos\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\theta\right]/\sqrt{g} && (31)\\
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[+\overline{T}_{l,i}\cos\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\zeta-\overline{T}_{p,j}\sin\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\theta\right]/\sqrt{g} && (32)\\
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{-\overline{T}_{l,i}\cos\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\theta}\cancel{-\overline{T}_{p,j}\cos\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\theta}\right]/\sqrt{g} && (33)\\
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{-\overline{T}_{l,i}\cos\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\theta}\cancel{-\overline{T}_{p,j}\sin\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\cos\alpha_i\,\mathbf{e}_\theta}\right]/\sqrt{g} && (34)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[+\overline{T}_{l,i}\sin\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\zeta-\overline{T}_{p,j}\cos\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\theta\right]/\sqrt{g} && (35)\\
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[+\overline{T}_{l,i}\sin\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\zeta-\overline{T}_{p,j}\sin\alpha_j\nabla\theta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\theta\right]/\sqrt{g} && (36)\\
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{-\overline{T}_{l,i}\sin\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\cos\alpha_j\,\mathbf{e}_\theta}\cancel{-\overline{T}_{p,j}\cos\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\theta}\right]/\sqrt{g} && (37)\\
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} &= \int dv\ \left[\cancel{-\overline{T}_{l,i}\sin\alpha_i\nabla\zeta\cdot\overline{T}'_{p,j}\sin\alpha_j\,\mathbf{e}_\theta}\cancel{-\overline{T}_{p,j}\sin\alpha_j\nabla\zeta\cdot\overline{T}'_{l,i}\sin\alpha_i\,\mathbf{e}_\theta}\right]/\sqrt{g} && (38)
\end{aligned}
$$

4. In these expressions the terms $\nabla\theta\cdot\mathbf{e}_\theta=\nabla\zeta\cdot\mathbf{e}_\zeta=1$, and $\cancel{\nabla\theta\cdot\mathbf{e}_\zeta}=\cancel{\nabla\zeta\cdot\mathbf{e}_\theta}=0$ have been included to show the structure of the derivation.

1.8 derivatives of kinetic energy integrals

1. The first derivatives of $\int dv\; v^2$ with respect to $v_{s,e,i,l}$ etc. are

$$
\begin{aligned}
\frac{\partial}{\partial v_{s,e,i,l}} \int dv\; \mathbf{v}\cdot\mathbf{v} &= 2 \int dv\; \mathbf{v}\cdot\overline{T}_{l,i} \cos\alpha_i \nabla s && (39)\\
\frac{\partial}{\partial v_{s,o,i,l}} \int dv\; \mathbf{v}\cdot\mathbf{v} &= 2 \int dv\; \mathbf{v}\cdot\overline{T}_{l,i} \sin\alpha_i \nabla s && (40)\\
\frac{\partial}{\partial v_{\theta,e,i,l}} \int dv\; \mathbf{v}\cdot\mathbf{v} &= 2 \int dv\; \mathbf{v}\cdot\overline{T}_{l,i} \cos\alpha_i \nabla \theta && (41)\\
\frac{\partial}{\partial v_{\theta,o,i,l}} \int dv\; \mathbf{v}\cdot\mathbf{v} &= 2 \int dv\; \mathbf{v}\cdot\overline{T}_{l,i} \sin\alpha_i \nabla \theta && (42)\\
\frac{\partial}{\partial v_{\zeta,e,i,l}} \int dv\; \mathbf{v}\cdot\mathbf{v} &= 2 \int dv\; \mathbf{v}\cdot\overline{T}_{l,i} \cos\alpha_i \nabla \zeta && (43)\\
\frac{\partial}{\partial v_{\zeta,o,i,l}} \int dv\; \mathbf{v}\cdot\mathbf{v} &= 2 \int dv\; \mathbf{v}\cdot\overline{T}_{l,i} \sin\alpha_i \nabla \zeta && (44)\\
& && (45)
\end{aligned}
$$

1.9 calculation of volume-integrated basis-function-weighted metric information

1. The required geometric information is calculated in ma00aa.