Verification of the Hessian calculation in SPEC

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This is a simple study aiming at verifying that the Hessian is calculated correctly in SPEC. To this aim, I consider the simplest possible system where a Hessian can be defined. That is, a slab MRxMHD equilibrium with two relaxed volumes separated by an ideal-interface and unperturbed outer boundaries. In this case, the Hessian should be a scalar for which an analytical formula can be derived.

Model

Consider a slab, two-volume MRxMHD equilibrium. That implies a single ideal-inteface separating two Taylor states. The boundary geometry is given by a function $R(\theta,\zeta)$ on each of the two sides. Assume the simplest form: $R_0(\theta,\zeta)=0$ and $R_2(\theta,\zeta)=1$, where the labels refer to the lower (0) and upper (2) boundaries (see Figure 1). The geometry of the ideal-interface, $R_1(\theta,\zeta)$, is to be calculated. Assume zero pressure and given toroidal and poloidal fluxes, as well as Beltrami parameters, in each volume. Namely, $\{\Psi_{t1},\Psi_{t2},\Psi_{p1},\Psi_{p2},\mu_1,\mu_2\}$ are given. Here the label "1" refers to the lower volume parameters, and the label "2" refers to the upper volume parameters.

This completely defines the equilibrium problem. Because of the symmetry of the problem, the solution is obviously independent of the two periodic coordinates and only depends on the "radial" coordinate R, which can be parametrized in each volume via a coordinate $s\in[-1,1]$, such that $R(s)=R_0+(R_1-R_0)(1+s)/2$ in the lower volume and $R(s)=R_1+(R_2-R_1)(1+s)/2$ in the upper volume.

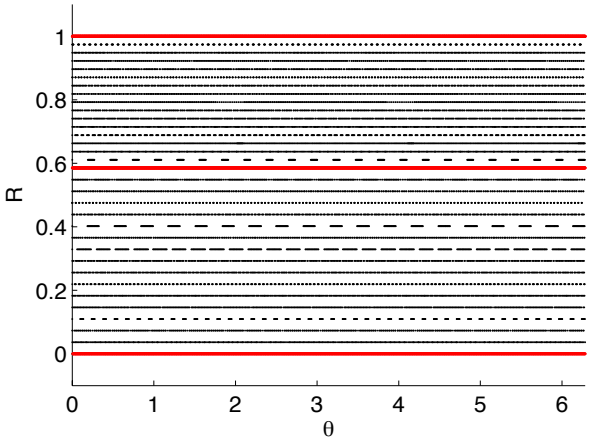

FIG. 1: Poincaré plot for the slab equilibrium. Red lines indicate lower and upper boundaries, and inner ideal interface.

Analytical prediction

In each volume, the field satisfies $\nabla\times\mathbf{B}=\mu\mathbf{B}$. Given the symmetry of the problem, there is an exact analytical solution, see [Loizu et al, Physics of Plasmas 22, 022501, 2015]. One can write the solution in terms of the fluxes and Beltrami parameters,

$$B_s=0 \tag{1}$$

$$B_\theta=\frac{\mu}{4\pi}\left(\Psi_t\frac{\sin(\bar{\mu}s)}{\sin(\bar{\mu})}+\Psi_p\frac{\cos(\bar{\mu}s)}{\sin(\bar{\mu})}\right) \tag{2}$$

$$B_\zeta=\frac{\mu}{4\pi}\left(\Psi_t\frac{\cos(\bar{\mu}s)}{\sin(\bar{\mu})}-\Psi_p\frac{\sin(\bar{\mu}s)}{\sin(\bar{\mu})}\right) \tag{3}$$

where $\bar{\mu} = \mu\Delta/2$ and Δ is the width of each volume, namely $\Delta = R_1 - R_0$ for the lower volume and $\Delta = R_2 - R_1$ for the upper volume. From this solution, we can compute the force imbalance,

$$f = \frac{1}{2}[[B^2]] = \frac{1}{2}\Big(B^2_{upper}(s=-1) - B^2_{lower}(s=+1)\Big) \tag{4}$$

which gives

$$f = \frac{1}{32\pi^2}\Big(\frac{\mu_2^2}{\sin^2(\bar{\mu}_2)}(\Psi_{t2}^2 + \Psi_{p2}^2) - \frac{\mu_1^2}{\sin^2(\bar{\mu}_1)}(\Psi_{t1}^2 + \Psi_{p1}^2)\Big)\ . \tag{5}$$

Finally, the Hessian is thus given by:

$$H = \frac{\partial f}{\partial R_1} = \frac{1}{32\pi^2}\Big(\mu_1^3(\Psi_{t1}^2 + \Psi_{p1}^2)\frac{\cos(\bar{\mu}_1)}{\sin^3(\bar{\mu}_1)} + \mu_2^3(\Psi_{t2}^2 + \Psi_{p2}^2)\frac{\cos(\bar{\mu}_2)}{\sin^3(\bar{\mu}_2)}\Big)\ . \tag{6}$$

In principle, this last expression could be simplified if one knows that the equilibirium condition, $f = 0$, is satisfied. But otherwise this is the general expression for the Hessian of this system and depends on the input parameters and the inner interface position R_1.

SPEC calcuation

SPEC has been run for this system (using $Lconstraint = 0$ to enforce the fluxes and Beltrami coefficients). The input parameters have been chosen as follows:

$$\begin{aligned}
\Psi_{t1} &= 0.6 \\
\Psi_{t2} &= 0.4 \text{ (in the input file the total flux is prescribe here, namely 1)} \\
\Psi_{p1} &= 0.27 \\
\Psi_{p2} &= -0.2632 \text{ (in the input file the total flux is prescribe here, namely 0.0068)} \\
\mu_1 &= -1.006 \\
\mu_2 &= -1.032
\end{aligned}$$

The code is run with $L_{rad} = 8$ and $M_{pol} = N_{tor} = 0$ and converges to $|f| \sim 10^{-16}$. The solution for the geometry of the inner interface is (see Figure 1)

$$R_1 = 0.5855589140059669$$

and the Hessian as calculated from SPEC is

$$H_{SPEC} = 0.135516428941496$$

which is positive, indicating stability of the equilibrium. Inserting the input parameters and the value of R_1 into the analytical prediction for the Hessian gives

$$H_{theory} = 0.136160204352188$$

which is very close to the SPEC value. The question remains: what would make these two numbers converge to each other? Surprisingly to me, increasing L_{rad} does not seem to improve the error.