Chapter 6

Diffusion and Transport in Axisymmetric Geometry

6.1 Introduction

In this chapter we consider the time evolution of magnetically confined plasmas over time scales that are very long compared to the Alfvén transit time, and are thus characterized by resistive diffusion and particle and heat transport. This will lead into and provide motivation for a discussion of finite difference methods for parabolic equations in Chapter 7. Because the electron and ion heat fluxes $\mathbf{q}_e$ and $\mathbf{q}_i$ are extremely anisotropic in a highly magnetized plasma, it becomes essential to work in a coordinate system that is aligned with the magnetic field. The derivation of an appropriate set of transport equations to deal with this is presented in Section 6.2 and its subsections. These transport equations need to be supplemented by an equilibrium constraint obtained by solving a particular form of the equilibrium equation as described in Section 6.3. Together, this system of equations provides an accurate description of the long time scale evolution of a MHD stable plasma.

6.2 Basic Equations and Orderings

Here we consider the scalar pressure two-fluid MHD equations of Section 1.2.1, which for our purposes can be written:

$$nm_i\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \mathbf{J}\times\mathbf{B}, \tag{6.1}$$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = S_n, \tag{6.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \tag{6.3}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{R}, \tag{6.4}$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \cdot \left[\mathbf{q} + \frac{3}{2}p\mathbf{u}\right] = -p\nabla \cdot \mathbf{u} + \mathbf{J} \cdot \mathbf{R} + S_e, \tag{6.5}$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \cdot \left[\mathbf{q}_e + \frac{3}{2}p_e\mathbf{u}\right] = -p_e\nabla \cdot \mathbf{u} + \mathbf{J} \cdot \mathbf{R} + Q_{\Delta ei} + S_{ee}. \tag{6.6}$$

We have denoted by $\mathbf{R}$ the inhomogeneous term in the generalized Ohm's law, Eq. (6.4), by $\mathbf{q} = \mathbf{q}_i + \mathbf{q}_e$ the random heat flux vector due to ions and electrons, by $Q_{\Delta ei}$ the electron-ion equipartition term, and by S_n and $S_e = S_{ei} + S_{ee}$ sources of particles and energy. Note that in comparing Eq. (6.4) with Eq. (1.32), we have the relation

$$\mathbf{R} = \frac{1}{ne}[\mathbf{R}_e + \mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e]. \tag{6.7}$$

We are also neglecting (assumed small) terms involving the ion and electron stress tensors π_e, π_i, and the heating due to $\nabla T_e \cdot \mathbf{J}$.

We now apply a *resistive time scale ordering* [110, 111, 112] to these equations to isolate the long time scale behavior. This consists of ordering all the source and transport terms to be the order of the inverse magnetic Lundquist number, Eq. (1.59), $S^{-1} = \epsilon \ll 1$, where ϵ is now some small dimensionless measure of the dissipation. Thus

$$\eta \sim \mathbf{R} \sim S_n \sim S_e \sim \mathbf{q} \sim \epsilon \ll 1. \tag{6.8}$$

We look for solutions in which all time derivatives and velocities are also small, of order ϵ ,

$$\frac{\partial}{\partial t} \sim \mathbf{u} \sim \epsilon \ll 1, \tag{6.9}$$

as is the electric field, $\mathbf{E} \sim \epsilon$.

Applying this ordering to Eqs. (6.1)–(6.6), we find that the last five equations remain unchanged, merely picking up the factor ϵ in every term, which can then be canceled. However, the momentum equation, Eq. (6.1), does change, with a factor of ϵ^2 multiplying only the inertial terms,

$$\epsilon^2 nm_i \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) + \nabla p = \mathbf{J} \times \mathbf{B}. \tag{6.10}$$

Thus in the limit $\epsilon \to 0$ we can neglect the inertial terms, replacing the momentum equation with the equilibrium condition

$$\nabla p = \mathbf{J} \times \mathbf{B}. \tag{6.11}$$

Equation (6.11) is correct to second order in ϵ, and provides significant simplifications since replacing Eq. (6.10) by this removes all the wave propagation characteristics from the system.

We note here that the system of equations given by Eqs. (6.11) and (6.2) through (6.6) involve the plasma velocity $\mathbf{u}$, but there is no longer a time advancement equation for $\mathbf{u}$. We will derive a method to solve this system asymptotically in spite of this apparent difficulty. We restrict consideration here to axisymmetric geometry. The most general form for an axisymmetric magnetic field consistent with Eq. (6.11) was shown in Section 4.2 to be given by

$$\mathbf{B} = \nabla\phi \times \nabla\Psi + g(\Psi)\nabla\phi. \tag{6.12}$$

Let us first consider the poloidal part of the magnetic field evolution equation, Eq. (6.3). Insertion of Eq. (6.12) into Eq. (6.3) and taking the $(\hat{R}, \hat{Z})$ projections gives an equation to evolve the poloidal flux function

$$\frac{\partial\Psi}{\partial t} = R^2\mathbf{E}\cdot\nabla\phi + C(t). \tag{6.13}$$

The integration constant $C(t)$ can be set to zero by adopting the convention that Ψ be proportional to the actual poloidal flux that must vanish at $R = 0$. Setting $C(t) = 0$ and using Eq. (6.4) to eliminate the electric field, we have

$$\frac{\partial\Psi}{\partial t} + \mathbf{u}\cdot\nabla\Psi = R^2\nabla\phi\cdot\mathbf{R}. \tag{6.14}$$

(Note that the symbol R is being used to represent the cylindrical coordinate, while $\mathbf{R}$, defined in Eq. (6.7), is the vector inhomogeneous term in the generalized Ohm's law.) The $\nabla\phi$ projection of Eq. (6.3) gives

$$\frac{\partial g}{\partial t} = R^2\nabla\cdot[\nabla\phi\times\mathbf{E}],$$

or, upon substituting from Eq. (6.4),

$$\frac{\partial g}{\partial t} + R^2\nabla\cdot\left[\frac{g}{R^2}\mathbf{u} - (\nabla\phi\cdot\mathbf{u})\,\nabla\phi\times\nabla\Psi - \nabla\phi\times\mathbf{R}\right] = 0. \tag{6.15}$$

The system of time evolution equations that we are solving is thus reduced to the five scalar equations (6.2), (6.5), (6.6), (6.14), and (6.15) as well as the equilibrium equation (6.11).

6.2.1 Time-Dependent Coordinate Transformation

We adopt here the axisymmetric magnetic flux coordinate system developed in Chapter 5. Since the magnetic field and flux surfaces evolve and change in time, the coordinate transformation being considered will be a time-dependent one. At any given time we have the flux coordinates (ψ, θ, ϕ) and

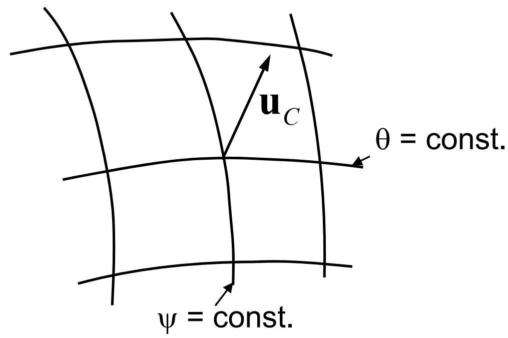

FIGURE 6.1: $\mathbf{u}_C$ is the velocity of a fluid element with a given ψ,θ value relative to a fixed Cartesian frame.

the inverse representation $\mathbf{x}(\psi, \theta, \phi)$, where $\mathbf{x}$ are Cartesian coordinates. We define the *coordinate velocity* at a particular (ψ, θ, ϕ) location as the time rate of change of the Cartesian coordinate at a fixed value of the flux coordinate as shown in Figure 6.1,

$$\mathbf{u}_C = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\psi,\theta,\phi}. \tag{6.16}$$

Consider now a scalar function, α, that may be thought of either as a function of Cartesian coordinates and time, $\alpha(\mathbf{x}, t)$, or of flux coordinates and time, $\alpha(\psi, \theta, \phi, t)$. Time derivatives at a fixed spatial location $\mathbf{x}$ and at fixed coordinates (ψ, θ, ϕ) are related by the chain rule of partial differentiation,

$$\left.\frac{\partial \alpha}{\partial t}\right|_{\psi,\theta,\phi} = \left.\frac{\partial \alpha}{\partial t}\right|_{\mathbf{x}} + \frac{\partial \alpha}{\partial \mathbf{x}} \cdot \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\psi,\theta,\phi}.$$

Using Eq. (6.16), we therefore have the relation

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\psi,\theta,\phi} - \mathbf{u}_C \cdot \nabla. \tag{6.17}$$

We will also make use of the relation for the time derivative of the Jacobian, defined in Eqs. (5.1) and (5.19),

$$\left.\frac{\partial J}{\partial t}\right|_{\psi,\theta,\phi} = J \nabla \cdot \mathbf{u}_C. \tag{6.18}$$

This may be verified directly.

With the introduction of the coordinate velocity in Eq. (6.16), it follows that the fluid velocity appearing in the MHD equations can be thought of as consisting of two parts,

$$\mathbf{u} = \mathbf{u}_C + \mathbf{u}_R, \tag{6.19}$$

where $\mathbf{u}_C$ is the coordinate velocity already discussed, and $\mathbf{u}_R$ is the velocity of the fluid relative to the coordinates. Since the total velocity $\mathbf{u}$ is a physical quantity which must be determined by the MHD equations, constraining either $\mathbf{u}_C$ or $\mathbf{u}_R$ to be of a particular form will determine the other.

The two parts of the velocity field can be thought of as a *Lagrangian* part and an *Eulerian* part. If the total velocity were represented with $\mathbf{u}_C$ only and there were no dissipation, the coordinates would be frozen into the fluid as it moves and distorts. If the total velocity were represented with $\mathbf{u}_R$, the coordinates would be fixed in space, and the fluid would move through them. We will see that an attractive choice is to split the velocity between these two parts so that the coordinates can move just enough to stay flux coordinates, but we allow the fluid to diffuse relative to them.

6.2.2 Evolution Equations in a Moving Frame

We now transform each of the scalar evolution equations, Eqs. (6.14), (6.15), (6.2), (6.5), and (6.6), into the moving flux coordinate frame by using the identities in Eqs. (6.17), (6.18), and (6.19). After some manipulations, we obtain the scalar equations

$$\frac{\partial \Psi}{\partial t} + \mathbf{u}_R \cdot \nabla\Psi = R^2 \nabla\phi \cdot \mathbf{R}, \tag{6.20}$$

$$\frac{\partial}{\partial t}\left(g\frac{J}{R^2}\right) + J\nabla\cdot\left[\frac{g}{R^2}\mathbf{u}_R - (\nabla\phi\cdot\mathbf{u}_R)\,\nabla\phi\times\nabla\Psi - \nabla\phi\times\mathbf{R}\right] = 0, \tag{6.21}$$

$$\frac{\partial}{\partial t}(nJ) + J\nabla\cdot(n\mathbf{u}_R) = JS_n, \tag{6.22}$$

$$\frac{\partial}{\partial t}\left(p^{3/5}J\right) + J\nabla\cdot\left[p^{3/5}\mathbf{u}_R\right] + \frac{2}{5}Jp^{-2/5}[\nabla\cdot q - \mathbf{J}\cdot\mathbf{R} - S_e] = 0, \tag{6.23}$$

$$\frac{\partial}{\partial t}\left(p_e^{3/5}J\right) + J\nabla\cdot\left[p_e^{3/5}\mathbf{u}_R\right] \quad + \quad \frac{2}{5}Jp_e^{-2/5}[\nabla\cdot q_e - \mathbf{J}\cdot\mathbf{R} - Q_{\Delta ei} - S_{ee}] = 0. \tag{6.24}$$

Here, and in what follows, the time derivatives are with ψ and θ held fixed, and are thus in a moving flux coordinate frame. We note that the coordinate velocity $\mathbf{u}_C$ does not appear in Eqs. (6.20)–(6.23), but only the velocity of the fluid relative to the moving coordinate, $\mathbf{u}_R = \mathbf{u} - \mathbf{u}_C$, is present. This is because the equations are of conservation form, and thus valid in a moving frame. We will use this fact to define a coordinate transformation in which the velocity vanishes altogether.

We first derive a reduced set of equations by integrating Eqs. (6.21)–(6.24) over the angle θ at fixed values of the coordinate ψ. This *flux surface averaging* leads to a set of one-dimensional evolution equations that depend only on the coordinate ψ and time t. Furthermore, by using the fact that $g = g(\psi, t)$ and $p = p(\psi, t)$ from the equilibrium constraint, and that $n \simeq n(\psi, t)$ and $p_e \simeq p_e(\psi, t)$ from the fact that the temperatures are nearly constant on flux surfaces because the parallel conductivities are large compared to their perpendicular values, we can obtain a closed set of equations that depend only on these surface averages.

From Eqs. (5.8) and (5.11), and the definition of differential volume and surface average in Eqs. (5.29) and (5.30), it follows that for any vector $\mathbf{A}$, we have the identity

$$2\pi \int_0^{2\pi} J \nabla \cdot \mathbf{A} d\theta = \frac{\partial}{\partial \psi} \left[V' \langle \mathbf{A} \cdot \nabla \psi \rangle \right].$$

Using this, the following set of equations are obtained by integrating Eqs. (6.21)–(6.24) over the angle θ at fixed value of the flux coordinate ψ:

$$\frac{\partial}{\partial t} \left[g V' \left\langle R^{-2} \right\rangle \right] + \frac{\partial}{\partial \psi} \left[g V' \left\langle R^{-2} \mathbf{u}_R \cdot \nabla \psi \right\rangle - V' \left\langle \nabla \phi \times \mathbf{R} \cdot \nabla \psi \right\rangle \right] = 0, \quad (6.25)$$

$$\frac{\partial}{\partial t} \left[n V' \right] + \frac{\partial}{\partial \psi} \left[n V' \left\langle \nabla \psi \cdot \mathbf{u}_R \right\rangle \right] = V' \left\langle S_n \right\rangle, \quad (6.26)$$

$$\begin{aligned} &\frac{\partial}{\partial t} \left[p^{\frac{3}{5}} V' \right] + \frac{\partial}{\partial \psi} \left[p^{\frac{3}{5}} V' \left\langle \mathbf{u}_R \cdot \nabla \psi \right\rangle \right] \quad (6.27) \\ &\quad + \frac{2}{5} p^{-\frac{2}{5}} \left[\frac{\partial}{\partial \psi} \left(V' \left\langle \mathbf{q} \cdot \nabla \psi \right\rangle \right) - V' \left\langle \mathbf{J} \cdot \mathbf{R} \right\rangle - V' \left\langle S_e \right\rangle \right] = 0. \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial t} \left[p_e^{\frac{3}{5}} V' \right] + \frac{\partial}{\partial \psi} \left[p_e^{\frac{3}{5}} V' \left\langle \mathbf{u}_R \cdot \nabla \psi \right\rangle \right] \quad (6.28) \\ &\quad + \frac{2}{5} p_e^{-\frac{2}{5}} \left[\frac{\partial}{\partial \psi} \left(V' \left\langle \mathbf{q}_e \cdot \nabla \psi \right\rangle \right) - V' \left(\left\langle \mathbf{J} \cdot \mathbf{R} \right\rangle + \left\langle Q_{\Delta ei} \right\rangle + \left\langle S_{ee} \right\rangle \right) \right] = 0. \end{aligned}$$

As discussed above, an important constraint that must be incorporated is that the coordinates (ψ, θ, ϕ) remain *flux coordinates* as they evolve in time. To this end, we require that in the moving frame, the flux function Ψ evolve in time in such a way that the coordinate ψ remain a flux coordinate, i.e.,

$$\nabla \phi \times \nabla \psi \cdot \frac{\partial \Psi}{\partial t} = 0. \quad (6.29)$$

From Eq. (6.20), this implies

$$\nabla\Psi \cdot \mathbf{u}_R - R^2\mathbf{R} \cdot \nabla\phi = f(\psi). \tag{6.30}$$

Here $f(\psi)$ is a presently undetermined function only of ψ. Equation (6.30) puts an important constraint on the relative velocity $\mathbf{u}_R$ and hence on the coordinate velocity $\mathbf{u}_C$ through Eq. (6.19), but it also leaves some freedom in that we are free to prescribe the function $f(\psi)$. This freedom will be used to identify the flux coordinate ψ with a particular surface function.

The system of equations given by Eqs. (6.20), (6.25)–(6.28), and (6.30) still depend upon the relative velocity $\mathbf{u}_R \cdot \nabla\psi$. This is determined up to a function only of ψ by Eq. (6.30), which follows from the constraint that constant Ψ surfaces align with constant ψ surfaces as they both evolve. We thus have a freedom in the velocity decomposition that we can use to simplify the problem. The remaining function of ψ is determined by specifying which flux function ψ is. Three common choices are the following.

(i) Constant poloidal flux:

$$\mathbf{u}_R \cdot \nabla\Psi = R^2\nabla\phi \cdot \mathbf{R}. \tag{6.31}$$

(ii) Constant toroidal flux:

$$-g\left\langle R^{-2}\mathbf{u}_R \cdot \nabla\psi\right\rangle = \left\langle \nabla\phi \times \nabla\psi \cdot \mathbf{R}\right\rangle. \tag{6.32}$$

(iii) Constant mass:

$$\langle \nabla\psi \cdot \mathbf{u}_R\rangle = 0. \tag{6.33}$$

Here, we choose number (ii), the toroidal magnetic flux, as it is most appropriate for most magnetic fusion applications, particularly for describing tokamaks. The toroidal field in the tokamak is primarily produced by the external field magnets, and is generally much stronger than the poloidal field. This makes it the most immobile, and thus most suitable for use as a coordinate. Also, unlike the poloidal magnetic flux, the toroidal flux at the magnetic axis does not change in time, always remaining zero.

6.2.3 Evolution in Toroidal Flux Coordinates

By combining Eq. (6.32) and the constraint Eq. (6.30), we can eliminate the free function $f(\psi)$ and solve explicitly for the normal relative velocity. This gives

$$f(\psi) = -\frac{\langle \mathbf{B} \cdot \mathbf{R}\rangle}{\langle \mathbf{B} \cdot \nabla\phi\rangle},$$

which when inserted into Eq. (6.30) yields

$$\mathbf{u}_R \cdot \nabla\psi = \frac{1}{\Psi'}\left[R^2\mathbf{R} \cdot \nabla\phi - \frac{\langle \mathbf{B} \cdot \mathbf{R}\rangle}{\langle \mathbf{B} \cdot \nabla\phi\rangle}\right]. \tag{6.34}$$

Using Eq. (6.34) to eliminate the relative velocity $\mathbf{u}_R$ from the surface averaged transport equations allows us to identify the flux coordinate ψ with the toroidal magnetic flux inside a constant flux surface, Φ. This can be verified by calculating directly from the definition of Φ in Eq. (5.31),

$$\left.\frac{\partial \Phi}{\partial t}\right|_{\psi} = \frac{1}{2\pi}\int_0^{\psi}\left[gV'\left\langle R^{-2}\right\rangle\right]_t d\psi = 0,$$

where we used Eqs. (6.25) and (6.34). It is seen that the use of Eq. (6.34) causes Φ and ψ to be stationary with respect to each other, and we may therefore adopt Φ as the flux surface label. Equation (6.25) need no longer be solved as it is intrinsically satisfied by our adoption of Φ as the flux coordinate. Since ψ and Φ are the same, we obtain a useful identity, valid for toroidal flux coordinates, by differentiating each side of Eq. (5.31) by $\psi \equiv \Phi$,

$$V' = \frac{2\pi}{g\left\langle R^{-2}\right\rangle}. \tag{6.35}$$

Using Eq. (6.34) to eliminate the relative velocity from Eqs. (6.20), (6.26), (6.27), and (6.28) then yields the surface-averaged transport equations relative to surfaces of constant toroidal flux. In deriving Eqs. (6.46) and (6.47) that follow, we make use of the equilibrium conditions in Eqs. (4.4)–(4.8) to express the equilibrium current density as

$$\mathbf{J} = -R^2\frac{dp}{d\Psi}\nabla\phi - \frac{1}{\mu_0}\frac{dg}{d\Psi}\mathbf{B}.$$

We also use Eq. (6.34) and the surface average of the inverse equilibrium equation, Eq. (5.53), to obtain the intermediate result

$$V'\left\langle \mathbf{J}\cdot\mathbf{R}\right\rangle = -p'V'\left\langle \mathbf{u}_R\cdot\nabla\Phi\right\rangle + \frac{\left\langle \mathbf{B}\cdot\mathbf{R}\right\rangle}{\left\langle \mathbf{B}\cdot\nabla\phi\right\rangle}\frac{d}{d\Phi}\left[\frac{V'}{2\pi\mu_0 q}\left\langle\frac{|\nabla\Phi|^2}{R^2}\right\rangle\right]. \tag{6.36}$$

To express the final form of the surface-averaged transport equations, we first define the rotational transform

$$\iota \equiv \frac{1}{q} = 2\pi\frac{d\Psi}{d\Phi}, \tag{6.37}$$

the loop voltage

$$V_L \equiv 2\pi\frac{\left\langle \mathbf{B}\cdot\mathbf{R}\right\rangle}{\left\langle \mathbf{B}\cdot\nabla\phi\right\rangle}, \tag{6.38}$$

the differential particle number, or number of particles in the differential volume between surfaces Φ and $\Phi + d\Phi$,

$$N' \equiv nV', \tag{6.39}$$

the particle flux

$$\Gamma \equiv 2\pi q n \left[\left\langle R^2 \mathbf{R} \cdot \nabla\phi \right\rangle - \frac{\langle \mathbf{B} \cdot \mathbf{R} \rangle}{\langle \mathbf{B} \cdot \nabla\phi \rangle} \right], \tag{6.40}$$

the differential total and electron entropy densities

$$\sigma = p V'^{\frac{5}{3}}, \qquad\qquad \sigma_e = p_e V'^{\frac{5}{3}}, \tag{6.41}$$

the surface integrated current density

$$K \equiv \frac{V'}{(2\pi)^2 \mu_0 q} \left\langle \frac{|\nabla\Phi|^2}{R^2} \right\rangle, \tag{6.42}$$

and the electron and ion heat fluxes

$$\begin{aligned} Q_e &\equiv V' \left[\langle \mathbf{q}_e \cdot \nabla\Phi \rangle + \frac{5}{2} \frac{p_e}{n} \Gamma \right], \\ Q_i &\equiv V' \left[\langle \mathbf{q}_i \cdot \nabla\Phi \rangle + \frac{5}{2} \frac{p_i}{n} \Gamma \right]. \end{aligned} \tag{6.43}$$

With these definitions, the basic transport equations take on the compact form:

$$\frac{\partial\Psi}{\partial t} = \frac{1}{2\pi} V_L, \tag{6.44}$$

$$\frac{\partial N'}{\partial t} + \frac{\partial}{\partial\Phi} V'\Gamma = V' \langle S_n \rangle, \tag{6.45}$$

$$\frac{3}{2} (V')^{-\frac{2}{3}} \frac{\partial\sigma}{\partial t} + \frac{\partial}{\partial\Phi} (Q_e + Q_i) = V_L \frac{\partial K}{\partial\Phi} + V' \langle S_e \rangle, \tag{6.46}$$

$$\frac{3}{2} (V')^{-\frac{2}{3}} \frac{\partial\sigma_e}{\partial t} + \frac{\partial Q_e}{\partial\Phi} + V' \left(\frac{\Gamma}{n} \frac{\partial p_i}{\partial\Phi} - Q_{\Delta ei} \right) = V_L \frac{\partial K}{\partial\Phi} + V' \langle S_{ee} \rangle. \tag{6.47}$$

By differentiating Eq. (6.44) with respect to the toroidal flux Φ, we obtain an evolution equation for the rotational transform, defined in Eq. (6.37),

$$\frac{\partial\iota}{\partial t} = \frac{\partial}{\partial\Phi} V_L. \tag{6.48}$$

In a similar manner, we can obtain an evolution equation for the *toroidal angular momentum density*, $\Omega(\Phi) = m_i N' \left\langle R^2 \right\rangle \omega(\Phi)$, where ω is the toroidal angular velocity as defined in Eq. (4.11):

$$\frac{\partial\Omega}{\partial t} + \frac{\partial}{\partial\Phi} V'\Gamma_\Omega = V' \langle S_\Omega \rangle. \tag{6.49}$$

The physical reason we were able to eliminate the fluid velocity entirely from the system of transport equations derived here is that transport coefficients only determine the *relative transport* of the magnetic field and plasma densities and energies with respect to one another. We cast these equations in a frame moving with the toroidal magnetic flux, and so only the relative motion remains. The absolute motion of the toroidal magnetic surfaces is determined by the equilibrium constraint and the interaction with externally applied magnetic fields. This is addressed in Section 6.3.

6.2.4 Specifying a Transport Model

Specifying a transport model consists of providing the transport fluxes, Γ, $\langle \mathbf{q}_i \cdot \nabla\Phi\rangle$, $\langle \mathbf{q}_e \cdot \nabla\Phi\rangle$, V_L, and Γ_Ω and the equipartition term $Q_{\Delta ei}$ as functions of the thermodynamic and magnetic field variables and the metric quantities. Also required are the source functions for mass, energy, electron energy, and angular momentum, $\langle S_n\rangle$, $\langle S_e\rangle$,$\langle S_{ee}\rangle$,$\langle S_\Omega\rangle$. The radiative loss function $\langle S_{RAD}\rangle$ must be subtracted from the energy and electron energy source functions.

At a given time, when the geometry is fixed, the plasma pressures, densities, and toroidal angular velocities are obtained from the adiabatic variables through the relations, valid for a two-component plasma with charge $Z = 1$ and with $n_e = n_i = n$,

$$\begin{aligned}
p_e(\Phi) &= \sigma_e/V'^{5/3}, \\
p_i(\Phi) &= p - p_e = (\sigma - \sigma_e)/V'^{5/3}, \\
n(\Phi) &= N'/V', \\
k_B T_e(\Phi) &= p_e/n, \\
k_B T_i(\Phi) &= p_i/n, \\
\omega(\Phi) &= \Omega/\left(m_i N' \left\langle R^2\right\rangle\right).
\end{aligned}$$

The electron-ion equipartition term, $Q_{\Delta ei}$, is normally taken to be the classical value given by Eq. (1.45).

It is convenient to define a five-component force vector given by Φ derivatives of the density, total and electron pressures, current density, and toroidal angular velocity relative to the toroidal magnetic flux,

$$\mathbf{F} = \left[n'(\Phi), p'(\Phi), p_e'(\Phi), \left(A(\Phi)\iota(\Phi)\right)', \omega'(\Phi)\right]. \tag{6.50}$$

Here we have defined the geometrical quantity

$$A(\Phi) \equiv \frac{V'}{g}\left\langle \frac{|\nabla\Phi|^2}{R^2}\right\rangle.$$

This is related to the surface-averaged parallel current density by

$$\frac{\langle \mathbf{J}\cdot\mathbf{B}\rangle}{\langle \mathbf{B}\cdot\nabla\phi\rangle} = \frac{g^2}{(2\pi)^2\mu_0}\left[\frac{V'}{g}\left\langle \frac{|\nabla\Phi|^2}{R^2}\right\rangle \iota\right]' = \frac{g^2}{(2\pi)^2\mu_0}\left(A(\Phi)\iota(\Phi)\right)'.$$

The transport fluxes can now be expressed as a matrix of functions multiplying the force vector:

$$\begin{aligned}
\Gamma &= \sum_{j=1}^{5} \Gamma^j F_j, \qquad \langle \mathbf{q}_i \cdot \nabla\Phi \rangle = \sum_{j=1}^{5} q_i^j F_j, \qquad \langle \mathbf{q}_e \cdot \nabla\Phi \rangle = \sum_{j=1}^{5} q_e^j F_j, \\
V_L &= \sum_{j=1}^{5} V_L^j F_j + V_L^6, \qquad \Gamma_\Omega = \sum_{j=1}^{5} \Gamma_\Omega^j F_j.
\end{aligned}$$

With this convention, specification of the 25 scalar functions $\Gamma^1, \Gamma^2, \cdots \Gamma_\Omega^5$ will specify the transport model. The additional term V_L^6 will be used to incorporate a source term for current drive.

A. Pfirsch–Schlüter regime plasma

The non-zero scalar coefficients needed to evaluate the transport fluxes for an electron-ion plasma in the collision-dominated regime are as follows [113, 114]:

$$\begin{aligned}
\Gamma^1 &= -L_{12} p_e, \qquad \Gamma^2 = -L_{11} n, \qquad \Gamma^3 = L_{12} n, \\
\Gamma^4 &= -\frac{\eta_\parallel q g^2 n}{2\pi\mu_0} \left\langle \frac{|\nabla\psi|^2}{R^2} \right\rangle \left\langle B^2 \right\rangle^{-1}, \qquad q_i^1 = L_i p_i^2 / n, \\
q_i^2 &= -L_i p_i, \qquad q_i^3 = L_i p_i, \qquad q_e^1 = L_{22} p_e^2 / n, \\
q_e^2 &= L_{12} p_e, \qquad q_e^3 = -L_{22} p_e, \qquad V_L^4 = \frac{\eta_\parallel g^2}{2\pi\mu_0}.
\end{aligned}$$

The transport coefficients are:

$$\begin{aligned}
L_{11} &= L_0 \left[1 + 2.65(\eta_\parallel/\eta_\perp) q_*^2\right], \\
L_{12} &= (3/2) L_0 \left[1 + 1.47(\eta_\parallel/\eta_\perp) q_*^2\right], \\
L_{22} &= 4.66 L_0 \left[1 + 1.67(\eta_\parallel/\eta_\perp) q_*^2\right], \\
L_i &= \sqrt{2} L_0 (m_i/m_e)^{1/2} (T_e/T_i)^{3/2} \left[1 + 1.60 q_*^2\right], \\
L_0 &= \frac{\eta_\perp}{\mu_0} \left\langle |\nabla\Phi|^2 / B^2 \right\rangle.
\end{aligned}$$

Here, we have introduced the function

$$q_*^2 = \frac{1}{2} \left[\left\langle B^{-2} \right\rangle - \left\langle B^2 \right\rangle^{-1}\right] \frac{g^2}{\langle |\nabla\Psi|^2 / B^2 \rangle}, \tag{6.51}$$

which reduces to the square of the safety factor, q^2, in the low beta, large aspect ratio limit. The resistivity functions $\eta_\parallel$ and $\eta_\perp$ are given by Eqs. (1.43) and (1.44).

If an external source of current drive is present, we can incorporate it into this model with the V_L^6 coefficient by defining

$$V_L^6 = -2\pi\eta_\parallel J_\parallel^{CD}, \tag{6.52}$$

where

$$J_{\|}^{CD} \equiv \frac{\left\langle \mathbf{J}^{CD} \cdot \mathbf{B} \right\rangle}{\langle \mathbf{B} \cdot \nabla \phi \rangle}, \tag{6.53}$$

with $\mathbf{J}^{CD}$ being the external current drive vector.

B. Banana Regime Plasma

The transport coefficients for a low-collisionality plasma including trapped and circulating particles have been computed for an arbitrary aspect ratio and cross section shape [114] with the restriction that $|\mathbf{B}|$ have only a single maximum on each flux surface [112, 114, 115, 116]. The fraction of trapped particles on a flux surface is given by [112]

$$f_t = 1 - (3/4) \left\langle B^2 \right\rangle \int_0^{B_c^{-1}} \lambda d\lambda / \left\langle (1 - \lambda B)^{1/2} \right\rangle, \tag{6.54}$$

where B_c is the maximum value of $|B|$ on a flux surface. Using this, we can express the banana regime transport model as follows:

$$\begin{aligned}
\Gamma^1 &= p_i L_{11}^{bp} y - p_e \left(L_{12}^{bp} + \tilde{L}_{12} + L_E L_{13} L_{23} \right), \\
\Gamma^2 &= n\,(1+y) \left[-L_{11}^{bp} - L_E (L_{13})^2 \right] - n \tilde{L}_{11}, \\
\Gamma^3 &= n \left[L_{11}^{bp} y + L_{12}^{bp} + \tilde{L}_{12} + L_E L_{13} (L_{23} + y L_{13}) \right], \\
\Gamma^4 &= -L_E L_{13} (2\pi)^{-2} g^3 n \left\langle R^{-2} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
q_i^1 &= L_i^{nc} p_i^2 / n^2, \\
q_i^2 &= -L_i^{nc} p_i / n, \\
q_i^3 &= L_i^{nc} p_i / n,
\end{aligned}$$

$$\begin{aligned}
q_e^1 &= \left[-L_{12}^{bp} y p_i + L_{22}^{nc} p_e + L_E L_{23} \left(-L_{13} y p_i + L_{23} p_e \right) \right] p_e / n, \\
q_e^2 &= \left[\left(L_{12}^{bp} + L_E L_{23} L_{13} \right) (1+y) + \tilde{L}_{12} \right] p_e, \\
q_e^3 &= \left[-L_{22}^{nc} - L_{12}^{bp} y - L_E L_{23} (L_{13} y + L_{23}) \right] p_e, \\
q_e^4 &= L_E L_{23} (2\pi)^{-2} g^3 \left\langle R^{-2} \right\rangle,
\end{aligned}$$

$$\begin{aligned}
V_L^1 &= L_E^* L_{13} p / n, \\
V_L^2 &= L_E^* L_{13} n (y+1), \\
V_L^3 &= L_E^* \left(L_{13} - L_{23} \right) n, \\
V_L^4 &= L_E g^2 \left\langle B^2 \right\rangle / 2\pi.
\end{aligned}$$

Here, the transport coefficients are:

$$\begin{aligned}
L_{11}^{bp} &= L_*(1.53 - 0.53 f_t),\\
L_{12}^{bp} &= L_*(2.13 - 0.63 f_t),\\
L_{13} &= (2\pi q) g f_t (1.68 - 0.68 f_t),\\
L_{23} &= 1.25 (2\pi q) g f_t (1 - f_t),
\end{aligned}$$

$$\begin{aligned}
\tilde{L}_{11} &= L_0(1 + 2q_*^2),\\
\tilde{L}_{12} &= (3/2) L_0 (1 + 2q_*^2),
\end{aligned}$$

$$\begin{aligned}
L_{22}^{nc} &= 4.66 \left[L_0(1 + 2q_*^2) + L_* \right],\\
L_i^{nc} &= \sqrt{2}(m_i/m_e)^{1/2}(T_e/T_i)^{3/2} \left[L_0(1 + 2q_*^2) + 0.46 L_* (1 - 0.54 f_t)^{-1} \right],\\
y &= -1.17(1 - f_t)(1 - 0.54 f_t)^{-1},
\end{aligned}$$

$$\begin{aligned}
L_{33} &= -1.26 f_t (1 - 0.18 f_t),\\
L_* &= f_t (2\pi q g)^2 \frac{\eta_\perp}{\mu_0} / \left\langle B^2 \right\rangle,\\
L_E &= \frac{\eta_\parallel (1 + L_{33})^{-1}}{\mu_0 \left\langle B^2 \right\rangle},\\
L_E^* &= L_E \frac{2\pi \left\langle B^2 \right\rangle}{g \left\langle R^{-2} \right\rangle}
\end{aligned}$$

Note that in steady state, when the loop voltage V_L is a spatial constant, the banana regime model implies a current driven by the temperature and density gradients,

$$\left\langle \frac{1}{R^2} \right\rangle \frac{\langle \mathbf{J} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \nabla \phi \rangle} = -f_t \, p \left[\frac{L_{13}^*}{n} \frac{dn}{d\Psi} + \frac{L_{13}^*(1+y)}{T_e + T_i} \frac{dT_i}{d\Psi} + \frac{L_{13}^* - L_{23}^*}{T_e + T_i} \frac{dT_e}{d\Psi} \right]. \tag{6.55}$$

Here $L_{13}^* \equiv L_{13}/(2\pi q g f_t)$, $L_{23}^* \equiv L_{23}/(2\pi q g f_t)$, and y are all dimensionless and of order unity. In deriving Eq. (6.55) we have used the relation $2\pi q d/d\Phi = d/d\Psi$, where Ψ is the poloidal flux function. This current is called the *bootstrap current*. These relations have been extended to multi-charged ions [117] and to arbitrary collisionality [118]. An excellent overview of the results for the transport coefficients in all of the collisionality regimes is given in Helander and Sigmar [119].

As in the collisional regime case, an external source of current drive can be included by defining the equivalent of Eq. (6.52). In the presence of external current drive, we would add the coefficient

$$V_L^6 = -2\pi \eta_\parallel \left(1 + L_{33}\right)^{-1} J_\parallel^{CD},$$

where $J_\|^{CD}$ is defined in Eq. (6.53).

C. Anomalous Transport Model

Although this is still an area of active research, there are a number of anomalous transport models available that purport to calculate local values of the surface averaged transport coefficients based on local values of the surface averaged profiles of density, the temperatures, angular velocity, and current profile, and their gradients. These profiles are often obtained from a fit to a subsidiary micro-instability calculation [120, 121, 122, 123].

A typical model would return the particle diffusivity D, the electron thermal diffusivity χ_e, the ion thermal diffusivity χ_i, and the toroidal angular velocity diffusivity, χ_ω, all with dimension m^2/s, and a parallel resistivity function $\eta_\|$ with units $\Omega{-}m$. A relatively simple diagonal model would fit into the above formalism as follows [124]:

$$
\begin{aligned}
\Gamma^1 &= -D\left\langle|\nabla\Phi|^2\right\rangle, \\
q_i^1 &= \chi_i\frac{p_i}{n}\left\langle|\nabla\Phi|^2\right\rangle, \qquad q_i^2 = -\chi_i\left\langle|\nabla\Phi|^2\right\rangle, \qquad q_i^3 = \chi_i\left\langle|\nabla\Phi|^2\right\rangle, \\
q_e^1 &= \chi_e\frac{p_e}{n}\left\langle|\nabla\Phi|^2\right\rangle, \qquad q_e^3 = -\chi_e\left\langle|\nabla\Phi|^2\right\rangle, \\
V_L^4 &= \frac{\eta_\| g^2}{2\pi\mu_0}, \\
\Gamma_\Omega^5 &= -\chi_\omega\left\langle|\nabla\Phi|^2\right\rangle m_i n\left\langle R^2\right\rangle.
\end{aligned}
$$

Bootstrap and current drive terms are included in these models by defining the additional V_L^i coefficients as discussed above.

The micro-instability-based models tend to return transport coefficients that have strong dependences on the gradients of the corresponding surface averaged profiles. We discuss in the next chapter special computational techniques for dealing with these.

6.3 Equilibrium Constraint

The variables N', σ, ι, σ_e, and Ω introduced in Section 6.2 are called *adiabatic variables*. If there is no dissipation and no explicit sources of mass or energy, then the time derivatives of these quantities are zero in the toroidal flux coordinate system being used here.

In the presence of dissipation, the surface-averaged transport equations of the last section describe how these adiabatic variables evolve relative to equilibrium magnetic surfaces with fixed values of toroidal magnetic flux. To complete the description, we need to solve a global equation to describe how these surfaces evolve relative to a fixed laboratory frame in which the toroidal field and poloidal field magnetic coils are located. This is the associated equilibrium problem. In the next subsection, we describe the circuit equations that describe how the nearby coil currents evolve in time due to applied and induced voltages. Then, we describe two approaches for incorporating the equilibrium constraint, the Grad–Hogan method and an accelerated form of the Taylor method.

6.3.1 Circuit Equations

The toroidal plasma is coupled electromagnetically to its surroundings; both passive structures and poloidal field coils that are connected to power supplies. We assume here that the external conductors are all axisymmetric, and that their currents and applied voltages are in the toroidal (ϕ) direction. Although the passive structures are continuous, it is normally adequate to subdivide them into discrete elements, each of which obeys a discrete circuit equation. Thus, each of the poloidal field coils and passive structure elements obeys a circuit equation of the form

$$\frac{d}{dt}\Psi_{Pi} + R_i I_i = V_i, \tag{6.56}$$

where the poloidal flux at each coil i is defined by

$$\Psi_{Pi} = L_i I_i + \sum_{i \neq j} M_{ij} I_j + 2\pi \int_P J_\phi(\mathbf{R}') G(\mathbf{R}_i, \mathbf{R}') d\mathbf{R}'. \tag{6.57}$$

Here, R_i and V_i are the resistance and applied voltage at coil i, L_i is the self-inductance of coil i, and M_{ij} is the mutual inductance of coil i with coil j. For a passive conductor, the corresponding $V_i = 0$.

The last term in Eq. (6.57) is an integral over the plasma volume. This represents the mutual inductance between the distributed plasma current and the conductor with index i.

6.3.2 Grad–Hogan Method

The Grad–Hogan method [110, 125] splits every time step into two parts. In the first part, the adiabatic variables, including the poloidal flux at the conductors, are advanced from time t to time $t + \delta t$ by solving Eqs. (6.45)–(6.49) and (6.56) using techniques discussed in Chapter 7. In the second part, these adiabatic variables are held fixed while we solve the appropriate form of the equilibrium equation, where the "free functions" $p'(\Psi)$ and $gg'(\Psi)$ have been expressed in terms of the adiabatic variables $\sigma(\Phi)$ and $\iota(\Phi)$. The individual PF coil and conductor currents will change during this part of the time step in order to keep the poloidal flux fixed at each coil location and at the plasma magnetic axis. This part of the time step effectively determines the absolute motion of the toroidal flux surfaces relative to a fixed frame.

The adiabatic variables may also be expressed in terms of the poloidal flux function Ψ. This is most convenient when solving the Grad–Shafranov equation in the Grad–Hogan method as the poloidal flux function is what is being solved for. Thus, if we define $V_\Psi \equiv dV/d\Psi$ and $\sigma_\Psi \equiv pV_\Psi^{5/3}$, the form of the equilibrium equation that needs to be solved is:

$$\Delta^*\Psi + \mu_0 R^2 \frac{d}{d\Psi}\left[\frac{\sigma_\Psi(\Psi)}{V_\Psi^{5/3}}\right] + \frac{(2\pi)^4 q(\Psi)}{V_\Psi \langle R^{-2}\rangle}\frac{d}{d\Psi}\left[\frac{q(\Psi)}{V_\Psi \langle R^{-2}\rangle}\right] = 0. \tag{6.58}$$

The functions $\sigma_\Psi(\Psi)$ and $q(\Psi)$ must be held fixed while finding the equilibrium solution.

Equation (6.58) for Ψ can be solved using a free boundary analogue to the techniques discussed in Section 5.5.1. Note that the poloidal flux at the magnetic axis, Ψ_{MA}, must be held fixed during the equilibrium solution as well. Since $\nabla\Psi = 0$ at the magnetic axis, from Eq. (4.42), the evolution equation for Ψ at the axis is simply

$$\frac{\partial \Psi_{MA}}{\partial t} = R^2 \nabla\phi \cdot \mathbf{R}. \tag{6.59}$$

(Note that this is equivalent to applying Eq. (6.44) to Ψ at the magnetic axis.) The value of Ψ_{MA} is evolved during the transport part of the time step using Eq. (6.59) and Ψ_{MA} and the adiabatic variables are then held fixed during the equilibrium solution part of the time step.

This completes the formalism needed to describe the transport in an axisymmetric system. The one-dimensional evolution equations, Eqs. (6.45)–(6.49), advance the adiabatic variables in time on the resistive time scale. Equation (6.59) is used to advance the value of Ψ at the magnetic axis, and Eq. (6.56) is used to advance the value of Ψ at the nearby conductors. The equilibrium equation, Eq. (6.58), defines the flux surface geometry consistent with these adiabatic variables and the boundary conditions. It does not introduce any new time scales into the equations.

6.3.3 Taylor Method (Accelerated)

J. B. Taylor [126] suggested an alternative to the Grad–Hogan method that does not require solving the equilibrium equation with the adiabatic constraints, Eq. (6.58). His approach involves solving for the velocity field **u**, which when inserted into the field and pressure evolution equations, Eqs. (6.3), (6.4), and (6.5), will result in the equilibrium equation, Eq. (6.11), continuing to be satisfied as time evolves. It was shown by several authors [111, 112] that an elliptic equation determining this velocity field could be obtained by time differentiating the equilibrium equation and substituting in from the time evolution equations. Taking the time derivative of Eq. (6.11) gives

$$\nabla\dot{p} = \dot{\mathbf{J}} \times \mathbf{B} + \mathbf{J} \times \dot{\mathbf{B}}, \tag{6.60}$$

or, by substituting in from Eqs. (6.3), (6.4), and (6.5),

$$\frac{2}{3}\nabla\left[-\nabla\cdot\left(\mathbf{q}+\frac{5}{2}p\mathbf{u}\right)+\mathbf{J}\cdot(-\mathbf{u}\times\mathbf{B}+\mathbf{R})+S_e\right]$$
$$+\mu_0^{-1}\nabla\times[\nabla\times(\mathbf{u}\times\mathbf{B}-\mathbf{R})]\times\mathbf{B}+\mathbf{J}\times[\nabla\times(\mathbf{u}\times\mathbf{B}-\mathbf{R})]\,. \qquad (6.61)$$

This equation can, in principle, be solved for $\mathbf{u}$, and that velocity field $\mathbf{u}$ can be used to keep the system in equilibrium without repeatedly solving the equilibrium equation. While this approach has been shown to be viable [127], it suffers from "drifting" away from an exact solution of the equilibrium equation. A preferred approach [128, 129] to obtaining this velocity is to use the accelerated steepest descent algorithm which involves obtaining the velocity from the residual equation,

$$\dot{\mathbf{u}}+\frac{1}{\tau}\mathbf{u}=D\left[\mathbf{J}\times\mathbf{B}-\nabla p\right]. \qquad (6.62)$$

By choosing the proportionality and damping factors, D and τ, appropriately, the system can be kept arbitrarily close to an equilibrium state as it evolves. This is equivalent to applying the dynamic relaxation method of Section 3.5.2 to the plasma equilibrium problem.

In using this approach, the magnetic field variables Ψ and g must be evolved in time as two-dimensional functions from Eqs. (6.14) and (6.15) in order to preserve the magnetic flux constraints. Equation (6.48) for ι is then redundant but is useful as a check. The total entropy constraint is preserved by representing the pressure as $p(\psi)=\sigma(\psi)/V'^{5/3}$. The other adiabatic variables, including the poloidal flux at the conductors, are still evolved by solving Eqs. (6.45), (6.46), (6.47), (6.49), and (6.56).

6.4 Time Scales

Here we discuss some of the time scales associated with diffusion of the particles, energy and fields in a collisional regime tokamak plasma. In doing so, we show the relation of the variables and equations introduced here to more common variables used in simpler geometries and in more approximate descriptions.

A. Flux Diffusion

Recall that the rotational transform evolves according to Eq. (6.48) where, for an arbitrary inhomogeneous term $\mathbf{R}$ in Ohm's law, the loop voltage is defined by Eq. (6.38). Here, for simplicity in discussing the basic time scales, we consider the Ohm's law appropriate for a single fluid resistive MHD plasma,

$$\mathbf{R}=\eta_{\parallel}\mathbf{J}_{\parallel}+\eta_{\perp}\mathbf{J}_{\perp}. \qquad (6.63)$$

When inserted into Eq. (6.38), this gives

$$V_L = 2\pi\eta_\parallel \frac{\langle \mathbf{B}\cdot\mathbf{J}\rangle}{\langle \mathbf{B}\cdot\nabla\phi\rangle}. \tag{6.64}$$

Evaluating Eq. (6.64) and inserting into Eq. (6.48) yields a diffusion-like equation for the rotational transform $\iota(\Phi, t)$,

$$\frac{\partial}{\partial t}\iota = \frac{d}{d\Phi}\left[\frac{\eta_\parallel g}{V'\langle R^{-2}\rangle \mu_o}\frac{d}{d\Phi}\left(\frac{V'}{g}\left\langle\frac{|\nabla\Phi|^2}{R^2}\right\rangle\iota\right)\right]. \tag{6.65}$$

We can rewrite this equation in the circular cross section *large aspect ratio limit* in which the major radius and toroidal field strength can be assumed constants and the cylindrical coordinate r is the only independent variable. Thus, we have

$$\begin{aligned} R &\rightarrow R_o(constant), & g &\rightarrow R_o B_z^0, \\ B_z &\rightarrow B_z^o \gg B_\theta, & V' &\rightarrow 2\pi R_0/B_Z^0, \\ \Phi &\rightarrow \pi r^2 B_z^o, & \iota = q^{-1} &\rightarrow \frac{R_0 B_\theta}{r B_z^0}. \end{aligned}$$

In this limit, Eq. (6.65) becomes

$$\frac{\partial}{\partial t}B_\theta = \frac{d}{dr}\left[\frac{\eta_\parallel}{\mu_o}\frac{1}{r}\frac{d}{dr}(rB_\theta)\right]. \tag{6.66}$$

From this, we can evaluate the characteristic time for the solution to decay,

$$t_o = \frac{a^2\mu_o}{\eta_\parallel},$$

where a is the minor radius. This is known as the *skin time* of the device.

B. Particle Diffusion

The evolution equation for the differential particle number N' is given by Eq. (6.45). In this resistive MHD model, the particle flux is completely defined in terms of the inhomogeneous term $\mathbf{R}$ in Eq. (6.40). By making use of the identity

$$R^2\nabla\phi = \frac{1}{B^2}\left[g\mathbf{B} - \mathbf{B}\times\nabla\Psi\right], \tag{6.67}$$

it can be seen that Γ can be written in the following form

$$\Gamma = 2\pi qn\left[g\left(\left\langle\frac{\mathbf{R}\cdot\mathbf{B}}{B^2}\right\rangle - \frac{\langle\mathbf{R}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}\right) - \left\langle\frac{\mathbf{R}\times\mathbf{B}\cdot\nabla\Psi}{B^2}\right\rangle - \frac{V_L}{2\pi}\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\right]. \tag{6.68}$$

Here $\langle B_p^2\rangle = \langle|\nabla\Psi|^2/R^2\rangle$. The first term in brackets, which only depends on the part of **R** parallel to the magnetic field **B**, corresponds to neo-classical Pfirsch–Schlüter diffusion, the second term to classical diffusion, and the third term to the classical inward pinch. We evaluate this for the simple Ohm's law in Eq. (6.63) to find

$$\frac{\partial N'}{\partial t} = \frac{d}{d\Phi} N' \left[\eta_\perp \left\langle \frac{|\nabla\Phi|^2}{B^2} \right\rangle \left[1 + 2q_*^2 \left(\eta_\parallel / \eta_\perp\right)\right] \frac{dp}{d\Phi} + qV_L \frac{\langle B_p^2 \rangle}{\langle B_T^2 \rangle} \right]. \tag{6.69}$$

Again, we can evaluate Eq. (6.69) in the large aspect ratio, circular limit to find

$$\frac{\partial}{\partial t}(nr) = \frac{d}{dr} n \left[\frac{r}{B^2} (\eta_\perp + 2q^2 \eta_\parallel) \frac{dp}{dr} + r \frac{E_\phi B_\theta}{B^2} \right].$$

The first term in brackets corresponds to diffusion, and the second term to an inward $\mathbf{E} \times \mathbf{B}$ drift. Noting that $p = 2nk_B T$, we can estimate the classical diffusion time in this limit,

$$t_n = \frac{a^2 \mu_o}{\eta \left(1 + 2q^2\right) \beta}, \tag{6.70}$$

where $\beta \equiv \mu_0 p / B^2$. We note that $t_n \gg t_o$ if $\beta \ll 1$.

C. Heat Conduction Equation

The evolution of the differential entropy density is given by Eq. (6.46). Since the inhomogeneous vector **R** does not enter into the heat flux relation, we must rely on an additional kinetic calculation to define this. The heat flux in a collisional plasma has been shown to be [130]

$$Q = -p\eta_\perp \left(\frac{2m_i}{m_e} \right)^{\frac{1}{2}} \left(1 + 2q_*^2\right) \left\langle \frac{|\nabla\Phi|^2}{B^2} \right\rangle nV' \frac{dT}{d\Phi}.$$

By balancing the time derivative term with the heat conduction term, and evaluating in the large aspect ratio limit, we obtain the heat conduction time

$$t_h = \frac{a^2 \mu_o}{\hat{\eta} \left(1 + 2q^2\right) \beta \left(\frac{2m_i}{m_e} \right)^{\frac{1}{2}}}.$$

Normally, for tokamak parameters, we have

$$t_n > t_o > t_h \gg t_a,$$

where t_a is the Alfvén wave transit time of the device.

D. Equilibration

For times longer than each of the characteristic times computed above, one or more of the evolution equations will be in steady state. Thus for $t > t_A$ the system will be in force balance equilibrium with

$$\nabla p = \mathbf{J} \times \mathbf{B},$$

or, in particular, the form of the Grad–Shafranov equation given by (6.58) will be satisfied. For $t > t_h$ the ion heat conduction will balance Joule heating in an ohmically heated tokamak, thus

$$V_L \frac{d}{d\Phi}\left[\frac{V'}{(2\pi)^2 q}\left\langle\frac{|\nabla\Phi|^2}{R^2}\right\rangle\right] + \frac{d}{d\Phi}\left[p\eta_\perp\left(\frac{2m_i}{m_e}\right)^{\frac{1}{2}}(1+2q^2)\left\langle\frac{|\nabla\Phi|^2}{B^2}\right\rangle N'\frac{dT}{d\Phi}\right] = 0. \quad (6.71)$$

This will determine the temperature profile. For longer times $t > t_o$, the toroidal current, or ι, evolves diffusively to obtain the steady-state condition

$$V_L = \text{const.} = \frac{\eta_\parallel \mathrm{g}}{\mu_o \mathrm{V}' \langle \mathrm{R}^{-2}\rangle}\frac{\mathrm{d}}{\mathrm{d}\Phi}\left[\frac{\mathrm{V}'}{\mathrm{g}}\left\langle\frac{|\nabla\Phi|^2}{\mathrm{R}^2}\right\rangle\iota\right].$$

Note that this allows the energy equation, Eq. (6.46), to be integrated so that we have the steady-state relation

$$Q_e + Q_i = V_L K + c, \quad (6.72)$$

where c is an integration constant. Evaluating this for our collisional transport model gives

$$\frac{V_L}{(2\pi)^2 q}\left\langle\frac{|\nabla\Phi|^2}{R^2}\right\rangle + p\eta_\perp\left(\frac{2m_i}{m_e}\right)^{\frac{1}{2}}\left(1+2q^2\right)\left\langle\frac{|\nabla\Phi|^2}{B^2}\right\rangle n\frac{dT}{d\Phi} = c.$$

6.5 Summary

In a magnetic confinement device such as a tokamak, the magnetic field lines form nested surfaces. Because the thermal conductivities and particle diffusivities are so much greater parallel to the field than across it, we can assume that the temperatures and densities are constant on these surfaces when solving for the relatively slow diffusion of particles, energy, magnetic field, and angular momentum across the surfaces. Because this cross-field diffusion is slow compared to the Alfvén time, we can assume that the plasma is always in force balance equilibrium. The toroidal magnetic flux is a good coordinate because it is relatively immobile and is always zero at the magnetic axis. We surface average the evolution equation to get a set of 1D evolution equations for the adiabatic invariants. These are given by Eqs. (6.45), (6.46), (6.48), (6.47), and (6.49). These need to be closed by supplying transport coefficients from a subsidiary transport or kinetic calculation. These 1D evolutions need to be solved together with the 2D equilibrium constraint, which defines the geometry and relates the magnetic fluxes to the coil currents and other boundary conditions. The time scales present in the equations were estimated for a collisional regime plasma and it was found that the the heating time is fastest, followed by the magnetic field penetration time (or the skin time), followed by the particle diffusion time. These times are all much longer than the Alfvén transit time.

Problems

6.1: Derive Eq. (6.18) using the definition of $\mathbf{u}_C$ in Eq. (6.16) and the expression for the Jacobian in Eq. (5.19).

6.2: Derive the evolution equations in a moving frame, Eqs. (6.20)–(6.24).

6.3: Derive the intermediate result, Eq. (6.36).

6.4: Derive a set of evolution equations analogous to Eqs. (6.45), (6.46), and (6.48) using the poloidal flux velocity, defined by Eq. (6.31) as the reference velocity (instead of the toroidal flux velocity).

6.5: Show that the expression for $q*$ in Eq. (6.51) reduces to the normal definition for the safety factor q in the large aspect ratio, low β limit.

6.6: Calculate the time scales as done in Section 6.4, but for the banana regime plasma model introduced in Section 6.2.4.