

$P(\|\mathcal{H}(x_1 - x_2)\|_p < \epsilon | y_1, y_2)$ as a metric

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1 Definitions

$$\begin{aligned} Y_1 &= H_1 X_1 + N_1 \\ Y_2 &= H_2 X_2 + N_2 \\ X_1, X_2 &\sim N(\mu, \Sigma) \\ N_1, N_2 &\sim N(0, \sigma^2 I_n) \end{aligned}$$

Y_1, Y_2 are independent Gaussian random vectors. \mathcal{H} : Filter function

$$\begin{aligned} E(Y_1) &= H_1 \mu \\ Cov(Y_1) &= H_1 \Sigma H_1^T + \sigma^2 I_n = K_1 \\ E(Y_2) &= H_2 \mu \\ Cov(Y_2) &= H_2 \Sigma H_2^T + \sigma^2 I_n = K_2 \end{aligned}$$

The pdf's are thus

$$f_{X_1}(x_1) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x_1 - \mu)^T \Sigma^{-1}(x_1 - \mu)\right\} \quad (1)$$

$$f_{X_2}(x_2) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x_2 - \mu)^T \Sigma^{-1}(x_2 - \mu)\right\} \quad (2)$$

$$f_N(y_1 - H_1 x_1) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \exp\left\{-\frac{1}{2}(y_1 - H_1 x_1)^T \frac{1}{\sigma^2}(y_1 - H_1 x_1)\right\} \quad (3)$$

$$f_N(y_2 - H_2 x_2) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \exp\left\{-\frac{1}{2}(y_2 - H_2 x_2)^T \frac{1}{\sigma^2}(y_2 - H_2 x_2)\right\} \quad (4)$$

$$f_{Y_1}(y_1) = \frac{1}{(2\pi)^{\frac{n}{2}} |K_1|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y_1 - H_1 \mu)^T K_1^{-1}(y_1 - H_1 \mu)\right\} \quad (5)$$

$$f_{Y_2}(y_2) = \frac{1}{(2\pi)^{\frac{n}{2}} |K_2|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y_2 - H_2 \mu)^T K_2^{-1}(y_2 - H_2 \mu)\right\} \quad (6)$$

2 Calculation

Suppose we apply some filter \mathcal{H} to the clean signal, this would mean that we now have

$$\mathcal{H}Y_1 = \mathcal{H}H_1X_1 + \mathcal{H}N_1 \quad (7)$$

$$\begin{bmatrix} \mathcal{H} & 0 \\ 0 & \mathcal{H} \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ H_1 & I \end{bmatrix} \begin{bmatrix} X_1 \\ N_1 \end{bmatrix} \quad (8)$$

$$\sim N \left[\begin{bmatrix} \mathcal{H}\mu \\ \mathcal{H}H_1\mu \end{bmatrix}, \begin{bmatrix} \mathcal{H}\Sigma\mathcal{H}^T & \mathcal{H}\Sigma H_1^T \mathcal{H}^T \\ \mathcal{H}H_1\Sigma\mathcal{H}^T & \mathcal{H}H_1\Sigma H_1^T \mathcal{H}^T + \mathcal{H}\sigma^2\mathcal{H}^T \end{bmatrix} \right] \quad (9)$$

Using conditional distributions (Wiki MND)

$$f_{X_1|Y_1}(x_1|y_1) \sim N(\alpha, L) \quad (10)$$

$$f_{X_2|Y_2}(x_2|y_2) \sim N(\beta, M) \quad (11)$$

where

$$\begin{aligned} \alpha &= \mathcal{H}\mu + \mathcal{H}\Sigma H_1^T \mathcal{H}^T (\mathcal{H}H_1\Sigma H_1^T \mathcal{H}^T + \mathcal{H}\sigma^2\mathcal{H}^T)^{-1} (\mathcal{H}y_1 - \mathcal{H}H_1\mu) \\ L &= \mathcal{H}\Sigma\mathcal{H}^T - \mathcal{H}\Sigma H_1^T \mathcal{H}^T (\mathcal{H}H_1\Sigma H_1^T \mathcal{H}^T + \mathcal{H}\sigma^2\mathcal{H}^T)^{-1} (\mathcal{H}H_1\Sigma\mathcal{H}^T) \\ \beta &= \mathcal{H}\mu + \mathcal{H}\Sigma H_2^T \mathcal{H}^T (\mathcal{H}^T H_2\Sigma H_2^T \mathcal{H}^T + \mathcal{H}\sigma^2\mathcal{H}^T)^{-1} (\mathcal{H}y_2 - \mathcal{H}H_2\mu) \\ M &= \mathcal{H}\Sigma\mathcal{H}^T - \mathcal{H}\Sigma H_2^T \mathcal{H}^T (\mathcal{H}H_2\Sigma H_2^T \mathcal{H}^T + \mathcal{H}\sigma^2\mathcal{H}^T)^{-1} (\mathcal{H}H_2\Sigma\mathcal{H}^T) \end{aligned}$$

So

$$P(x_1 - x_2|y_1, y_2) \sim N(\alpha - \beta, L + M) \quad (12)$$

$$P(\mathcal{H}(x_1 - x_2)|y_1, y_2) \sim N(\mathcal{H}(\alpha - \beta), \mathcal{H}(L + M)\mathcal{H}^T) \quad (13)$$

Let $X_1 - X_2 = X_3$. Then

$$P(\|\mathcal{H}x_3\|_\infty < \epsilon|y_1, y_2) =$$

$$\int_{-\epsilon}^{\epsilon} \frac{1}{(2\pi)^{\frac{n}{2}} |\mathcal{H}(L + M)\mathcal{H}^T|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x_3 - \mathcal{H}(\alpha - \beta))^T (\mathcal{H}(L + M)\mathcal{H}^T)^{-1} (x_3 - \mathcal{H}(\alpha - \beta))\right\} dx_3 \quad (14)$$

For small ϵ this is

$$= \frac{(2\epsilon)^n}{(2\pi)^{\frac{n}{2}} |\mathcal{H}(L + M)\mathcal{H}^T|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mathcal{H}(\alpha - \beta))^T (\mathcal{H}(L + M)\mathcal{H}^T)^{-1} (\mathcal{H}(\alpha - \beta))\right\} \quad (15)$$

So we can define our metric after taking log and dropping out the constant term

$$-\frac{1}{2} \log(|\mathcal{H}(L + M)\mathcal{H}^T|) - \frac{1}{2} (\mathcal{H}(\alpha - \beta))^T (\mathcal{H}(L + M)\mathcal{H}^T)^{-1} (\mathcal{H}(\alpha - \beta)) \quad (16)$$