

$P(\|x_1 - x_2\|_\infty < \epsilon | y_1, y_2)$ as a metric

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1 Definitions

$$\begin{aligned} Y_1 &= H_1 X_1 + N_1 \\ Y_2 &= H_2 X_2 + N_2 \\ X_1, X_2 &\sim N(\mu, \Sigma) \\ N_1, N_2 &\sim N(0, \sigma^2 I_n) \end{aligned}$$

Y_1, Y_2 are independent Gaussian random vectors.

$$\begin{aligned} E(Y_1) &= H_1 \mu \\ Cov(Y_1) &= H_1 \Sigma H_1^T + \sigma^2 I_n = K_1 \\ E(Y_2) &= H_2 \mu \\ Cov(Y_2) &= H_2 \Sigma H_2^T + \sigma^2 I_n = K_2 \end{aligned}$$

The pdf's are thus

$$f_{X_1}(x_1) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x_1 - \mu)^T \Sigma^{-1}(x_1 - \mu)\right\} \quad (1)$$

$$f_{X_2}(x_2) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x_2 - \mu)^T \Sigma^{-1}(x_2 - \mu)\right\} \quad (2)$$

$$f_N(y_1 - H_1 x_1) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \exp\left\{-\frac{1}{2}(y_1 - H_1 x_1)^T \frac{1}{\sigma^2}(y_1 - H_1 x_1)\right\} \quad (3)$$

$$f_N(y_2 - H_2 x_2) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \exp\left\{-\frac{1}{2}(y_2 - H_2 x_2)^T \frac{1}{\sigma^2}(y_2 - H_2 x_2)\right\} \quad (4)$$

$$f_{Y_1}(y_1) = \frac{1}{(2\pi)^{\frac{n}{2}} |K_1|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y_1 - H_1 \mu)^T K_1^{-1}(y_1 - H_1 \mu)\right\} \quad (5)$$

$$f_{Y_2}(y_2) = \frac{1}{(2\pi)^{\frac{n}{2}} |K_2|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y_2 - H_2 \mu)^T K_2^{-1}(y_2 - H_2 \mu)\right\} \quad (6)$$

2 Calculation

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ H_1 & I \end{bmatrix} \times \begin{bmatrix} X_1 \\ N_1 \end{bmatrix} \quad (7)$$

$$\sim N \left[\begin{bmatrix} \mu \\ H_1 \mu \end{bmatrix}, \begin{bmatrix} \Sigma & \Sigma H_1^T \\ H_1 \Sigma & H_1 \Sigma H_1^T + \sigma^2 I \end{bmatrix} \right] \quad (8)$$

Using conditional distributions (Wiki MND)

$$f_{X_1|Y_1}(x_1|y_1) \sim N(\alpha, L) \quad (9)$$

$$f_{X_2|Y_2}(x_2|y_2) \sim N(\beta, M) \quad (10)$$

where

$$\alpha = \mu + \Sigma H_1^T (H_1 \Sigma H_1^T + \sigma^2 I)^{-1} (y_1 - H_1 \mu)$$

$$L = \Sigma - \Sigma H_1^T (H_1 \Sigma H_1^T + \sigma^2 I)^{-1} H_1 \Sigma$$

$$\beta = \mu + \Sigma H_2^T (H_2 \Sigma H_2^T + \sigma^2 I)^{-1} (y_2 - H_2 \mu)$$

$$M = \Sigma - \Sigma H_2^T (H_2 \Sigma H_2^T + \sigma^2 I)^{-1} H_2 \Sigma$$

So

$$P(x_1 - x_2 | y_1, y_2) \sim N(\alpha - \beta, L + M) \quad (11)$$

Let $X_1 - X_2 = X_3$. Then

$$P(\|x_3\|_\infty < \epsilon | y_1, y_2) = \int_{-\epsilon}^{\epsilon} \frac{1}{(2\pi)^{\frac{n}{2}} |L + M|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x_3 - (\alpha - \beta))^T (L + M)^{-1} (x_3 - (\alpha - \beta))\right\} dx_3 \quad (12)$$

For small ϵ this is

$$= \frac{(2\epsilon)^n}{(2\pi)^{\frac{n}{2}} |L + M|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\alpha - \beta)^T (L + M)^{-1} (\alpha - \beta)\right\} \quad (13)$$

So we can define our metric after taking log and dropping out the constant term

$$-\frac{1}{2} \log(|L + M|) - \frac{1}{2} (\alpha - \beta)^T (L + M)^{-1} (\alpha - \beta) \quad (14)$$