$P(||\mathscr{H}(x_1 - x_2)||_p < \epsilon | y_1, y_2)$ as a metric

Tejal Bhamre

August 5, 2013

1 Definitions

$Y_1 = H_1 X_1 + N_1$
$Y_2 = H_2 X_2 + N_2$
$X_1, X_2 \sim N(\mu, \Sigma)$
$N_1, N_2 \sim N(0, \sigma^2 I_n)$

Y_1, Y_2 are independent Gaussian random vectors. $\mathscr{H}$: Filter function

$E(Y_1) = H_1 \mu$
$Cov(Y_1) = H_1 \Sigma {H_1}^T + \sigma^2 I_n = K_1$
$E(Y_2) = H_2 \mu$
$Cov(Y_2) = H_2 \Sigma {H_2}^T + \sigma^2 I_n = K_2$

The pdf's are thus

$$f_{X_1}(x_1) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(x_1 - \mu)^T \Sigma^{-1} (x_1 - \mu)\} \tag{1}$$

$$f_{X_2}(x_2) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(x_2 - \mu)^T \Sigma^{-1} (x_2 - \mu)\} \tag{2}$$

$$f_N(y_1 - H_1 x_1) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \exp\{-\frac{1}{2}(y_1 - H_1 x_1)^T \frac{1}{\sigma^2} (y_1 - H_1 x_1)\} \tag{3}$$

$$f_N(y_2 - H_2 x_2) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \exp\{-\frac{1}{2}(y_2 - H_2 x_2)^T \frac{1}{\sigma^2} (y_2 - H_2 x_2)\} \tag{4}$$

$$f_{Y_1}(y_1) = \frac{1}{(2\pi)^{\frac{n}{2}} |K_1|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(y_1 - H_1 \mu)^T {K_1}^{-1} (y_1 - H_1 \mu)\} \tag{5}$$

$$f_{Y_2}(y_2) = \frac{1}{(2\pi)^{\frac{n}{2}} |K_2|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(y_2 - H_2 \mu)^T {K_2}^{-1} (y_2 - H_2 \mu)\} \tag{6}$$

2 Calculation

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ H_1 & I \end{bmatrix} \times \begin{bmatrix} X_1 \\ N_1 \end{bmatrix} \tag{7}$$

$$\sim N\left[\begin{bmatrix} \mu \\ H_1\mu \end{bmatrix}, \begin{bmatrix} \Sigma & \Sigma H_1^T \\ H_1\Sigma & H_1\Sigma H_1^T + \sigma^2 I \end{bmatrix}\right] \tag{8}$$

Using conditional distributions (Wiki MND)

$$f_{X_1|Y_1}(x_1|y_1) \sim N(\alpha, L) \tag{9}$$

$$f_{X_2|Y_2}(x_2|y_2) \sim N(\beta, M) \tag{10}$$

where
$\alpha = \mu + \Sigma H_1^T(H_1\Sigma H_1^T + \sigma^2 I)^{-1}(y_1 - H_1\mu)$
$L = \Sigma - \Sigma H_1^T(H_1\Sigma H_1^T + \sigma^2 I)^{-1}H_1\Sigma$
$\beta = \mu + \Sigma H_2^T(H_2\Sigma H_2^T + \sigma^2 I)^{-1}(y_2 - H_2\mu)$
$M = \Sigma - \Sigma H_2^T(H_2\Sigma H_2^T + \sigma^2 I)^{-1}H_2\Sigma$
So

$$P(x_1 - x_2|y_1, y_2) \sim N(\alpha - \beta, L + M) \tag{11}$$

$$P(\mathscr{H}(x_1 - x_2)|y_1, y_2) \sim N(\mathscr{H}(\alpha - \beta), \mathscr{H}(L + M)\mathscr{H}^T) \tag{12}$$

Let $X_1 - X_2 = X_3$. Then

$$P(||\mathscr{H}x_3||_\infty < \epsilon|y_1, y_2) =$$

$$\int_{-\epsilon}^{\epsilon} \frac{1}{(2\pi)^{\frac{n}{2}}|\mathscr{H}(L + M)\mathscr{H}^T|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(x_3 - \mathscr{H}(\alpha - \beta))^T(\mathscr{H}(L + M)\mathscr{H}^T)^{-1}(x_3 - (\mathscr{H}(\alpha - \beta))\}dx_3 \tag{13}$$

For small ϵ this is

$$= \frac{(2\epsilon)^n}{(2\pi)^{\frac{n}{2}}|\mathscr{H}(L + M)\mathscr{H}^T|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(\mathscr{H}(\alpha - \beta))^T(\mathscr{H}(L + M)\mathscr{H}^T)^{-1}(\mathscr{H}(\alpha - \beta))\} \tag{14}$$

So we can define our metric after taking log and dropping out the constant term

$$-\frac{1}{2}\log(|\mathscr{H}(L + M)\mathscr{H}^T|) - \frac{1}{2}(\mathscr{H}(\alpha - \beta))^T(\mathscr{H}(L + M)\mathscr{H}^T)^{-1}(\mathscr{H}(\alpha - \beta)) \tag{15}$$