Geophys. J. Int. (0000) **000**, 000–000

Notes on spin

Jeroen Tromp[1]
[1] *Princeton University, Department of Geosciences and Program in Applied & Computational Mathematics, Princeton, New Jersey, USA.*
E-mail: jtromp@princeton.edu

1 April 2025

1 INTRODUCTION

There are three basic types of crystallographic defects, as illustrated in Figure 1. First, *dislocations* are generated by removing or inserting a plane array of atoms in a perfect lattice. The edge of the plane forms a linear defect referred to as a dislocation. The displacement of atoms leads to translational defects. Second, *disclinations* are generated by removing or inserting a wedge of atoms with a wedge angle equal to any of the rotational symmetries of a perfect crystal lattice. The edge of the wedge forms a linear defect referred to as a disclination. In this case, the displacement of atoms leads to *rotational defects*. Finally, there are three types of *point defects*, namely, 1) a vacancy defect, which is created when an atom is missing from an otherwise perfect lattice, 2) an interstitial defect, which is a constituent or non-constituent atom inserted in the inter-atom space of an otherwise perfect lattice, and 3) a substitutional defect, which is a non-constituent atom inserted in the lattice space of an otherwise perfect lattice.

Defects in continuum mechanics are illustrated in Figure 2.

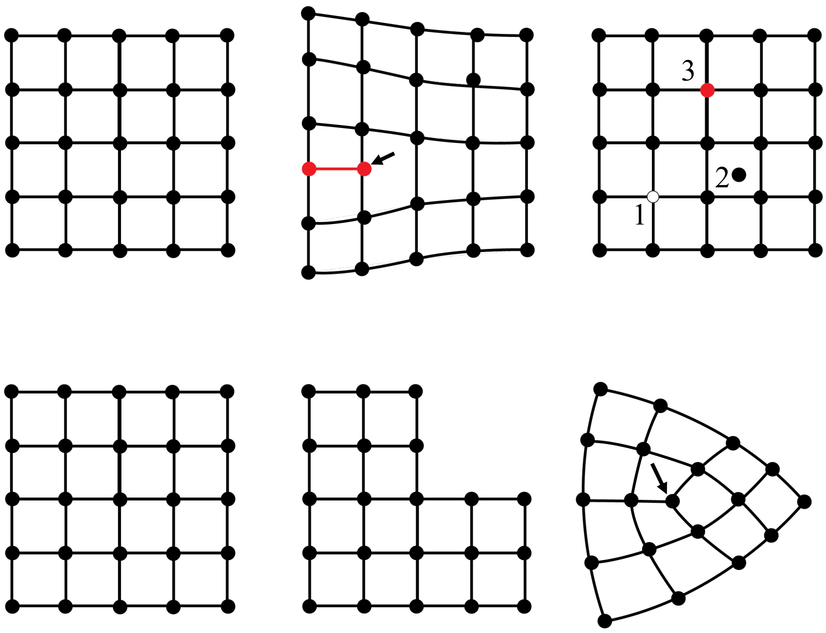

Figure 1. Illustrations of three basic types of crystallographic defects. *Top Left and Middle*: *Dislocations* are generated by removing or inserting a plane array of atoms in a perfect lattice. The edge of the plane, indicated by the arrow, forms a linear defect referred to as a *dislocation line*. *Top Right*: Three types of *point defects*. 1) A *vacancy defect*, which is created when an atom is missing from an otherwise perfect lattice. 2) An *interstitial defect*, which is a constituent or non-constituent atom inserted in the inter-atom space of an otherwise perfect lattice. 3) A *substitutional defect*, which is a non-constituent atom inserted in the lattice of an otherwise perfect lattice. *Bottom*: *Disclinations* are generated by removing or inserting a wedge of atoms with a wedge angle equal to any of the rotational symmetries of a perfect crystal lattice. The edge of the wedge, indicated by the arrow, forms a linear defect referred to as a *disclination line*.

[!ht]

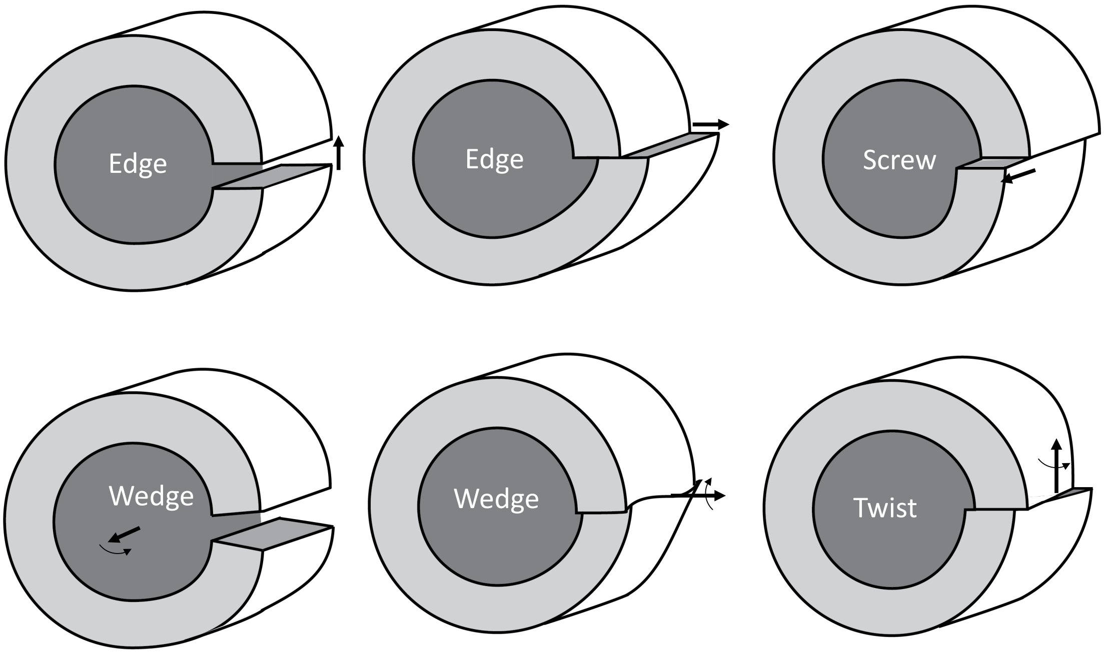

Figure 2. Illustrations of basic defects based on *Volterra's cut-and-weld protocols*. Dislocations correspond to a material torsion, whereas disclinations correspond to a material curvature. The cut parallel to the axis of the cylinder defines the *dislocation line*. *Top Left*: Edge dislocation. The slip/Burgers vector is perpendicular to the dislocation line. *Top Middle*: Edge dislocation. The slip/Burgers vector is perpendicular to the dislocation line. *Top Right*: Screw dislocation. The slip/Burgers vector is tangential to the dislocation line. *Bottom Left*: Wedge disclination. The rotation/Frank vector is tangential to the slip plane. *Bottom Middle*: Wedge disclination. The rotation/Frank vector is tangential to the slip plane. *Bottom Right*: Twist disclination. The rotation/Frank vector is perpendicular to the slip plane.

2 LINEAR ELASTICITY WITHOUT SPIN

In linear elasticity without spin, we consider the particle displacement field $\mathbf{s}$. The (linearized) strain associated with this displacement field is

$$\boldsymbol{\epsilon} = \tfrac{1}{2}[\boldsymbol{\nabla}\mathbf{s} + (\boldsymbol{\nabla}\mathbf{s})^t] \quad , \tag{1}$$

and the (linearized) vorticity associated with this displacement field is

$$\boldsymbol{\omega} = -\tfrac{1}{2}[\boldsymbol{\nabla}\mathbf{s} - (\boldsymbol{\nabla}\mathbf{s})^t] \quad . \tag{2}$$

Note that the strain tensor is symmetric

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^t \quad , \tag{3}$$

that the vorticity tensor is antisymmetric

$$\boldsymbol{\omega} = -\boldsymbol{\omega}^t \quad , \tag{4}$$

and that

$$\boldsymbol{\epsilon} + \boldsymbol{\omega} = (\boldsymbol{\nabla}\mathbf{s})^t \quad . \tag{5}$$

Finally, note that $\mathrm{tr}(\boldsymbol{\epsilon}) = \boldsymbol{\nabla}\cdot\mathbf{s}$, and that $\mathrm{tr}(\boldsymbol{\omega}) = \mathbf{0}$.

We may define the vorticity vector $\mathbf{w}$ in terms of the vorticity tensor $\boldsymbol{\omega}$ by

$$\mathbf{w} = -\tfrac{1}{2}\,\boldsymbol{\varepsilon} : \boldsymbol{\omega} = \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s} \quad , \tag{6}$$

where $\boldsymbol{\varepsilon}$ denotes the Levi-Civita or alternating tensor. The inverse relationship is

$$\boldsymbol{\omega} = -\,\boldsymbol{\varepsilon}\cdot\mathbf{w} \quad . \tag{7}$$

In (Cartesian) matrix notation, we have

$$
\begin{aligned}
(\epsilon_{ij}) &= \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix} \\
&= \begin{pmatrix} \partial_1 s_1 & \frac{1}{2}(\partial_1 s_2 + \partial_2 s_1) & \frac{1}{2}(\partial_1 s_3 + \partial_3 s_1) \\ \frac{1}{2}(\partial_1 s_2 + \partial_2 s_1) & \partial_2 s_2 & \frac{1}{2}(\partial_2 s_3 + \partial_3 s_2) \\ \frac{1}{2}(\partial_1 s_3 + \partial_3 s_1) & \frac{1}{2}(\partial_2 s_3 + \partial_3 s_2) & \partial_3 s_3 \end{pmatrix} \quad ,
\end{aligned} \tag{8}
$$

(six independent elements) and

$$
\begin{aligned}
(\omega_{ij}) &= \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\,\omega_{12} & 0 & \omega_{23} \\ -\,\omega_{13} & -\,\omega_{23} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\,w_3 & w_2 \\ w_3 & 0 & -\,w_1 \\ -\,w_2 & w_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{2}(\partial_2 s_1 - \partial_1 s_2) & \frac{1}{2}(\partial_3 s_1 - \partial_1 s_3) \\ -\,\frac{1}{2}(\partial_2 s_1 - \partial_1 s_2) & 0 & \frac{1}{2}(\partial_3 s_2 - \partial_2 s_3) \\ -\,\frac{1}{2}(\partial_3 s_1 - \partial_1 s_3) & -\,\frac{1}{2}(\partial_3 s_2 - \partial_2 s_3) & 0 \end{pmatrix} \quad ,
\end{aligned} \tag{9}
$$

(three independent elements).

Assume a plane wave with polarization $\hat{\mathbf{s}}$ traveling in the $\hat{\boldsymbol{\xi}}$ direction with wavespeed c:

$$
\mathbf{s}(\mathbf{r}, t) = \hat{\mathbf{s}}\, f(||\boldsymbol{\xi}|| - ct) \quad . \tag{10}
$$

Then

$$
\mathbf{w} = \tfrac{1}{2}\, \boldsymbol{\nabla} \times \mathbf{s} = \tfrac{1}{2}\, \hat{\boldsymbol{\xi}} \times \hat{\mathbf{s}}\, f'(||\boldsymbol{\xi}|| - ct) = -\,\frac{1}{2c}\, \hat{\boldsymbol{\xi}} \times \dot{\mathbf{s}} \quad . \tag{11}
$$

We see that the vorticity vector is perpendicular to the polarization and travel directions, i.e., the orientation $\hat{\boldsymbol{\xi}} \times \hat{\mathbf{s}}$, and has an amplitude proportional to the particle velocity divided by the wavespeed, $-\,\dot{s}/c$.

In linear elasticity without spin, the kinetic energy density of the material is given in terms of the mass density ρ and particle velocity $\dot{\mathbf{s}} = \partial_t \mathbf{s}$ by

$$
K = \tfrac{1}{2}\, \rho\, \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} \quad . \tag{12}
$$

The potential energy density is a quadratic function of the strain, and is written in terms of the strain tensor $\boldsymbol{\epsilon}$ as

$$
V = \tfrac{1}{2}\, \boldsymbol{\epsilon} : \mathbf{c} : \boldsymbol{\epsilon} \quad . \tag{13}
$$

Note that $[\mathbf{c}] = \mathrm{kg\, m^{-1}\, s^{-2}}$. Because the strain tensor is symmetric, and because the energy is quadratic in the strain, the elements of the elastic tensor $\mathbf{c}$ exhibit the symmetries

$$
c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij} \quad . \tag{14}
$$

The most general elastic tensor has 21 independent elements. We may now define the Lagrangian density by

$$
\mathcal{L}(\dot{\mathbf{s}}, \boldsymbol{\nabla}\mathbf{s}) = K - V = \tfrac{1}{2}\, \rho\, \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} - \tfrac{1}{2}\, (\boldsymbol{\nabla}\mathbf{s}) : \mathbf{c} : (\boldsymbol{\nabla}\mathbf{s}) \quad . \tag{15}
$$

The associated action is

$$
I = \iint_\Omega \mathcal{L}(\dot{\mathbf{s}}, \boldsymbol{\nabla}\mathbf{s})\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t \quad , \tag{16}
$$

where Ω denotes the volume occupied by the continuum. Stationarity of the action, $\delta I = 0$, leads to the Euler-Lagrange equations

$$
\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{s}}} + \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\mathbf{s}} = \mathbf{0} \quad , \qquad \mathbf{x} \in \Omega \quad , \tag{17}
$$

subject to the boundary conditions

$$
\hat{\mathbf{n}} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\mathbf{s}} = \mathbf{0} \quad , \qquad \mathbf{x} \in \partial\Omega \quad , \tag{18}
$$

where $\hat{\mathbf{n}}$ denotes the unit outward normal to the surface $\partial\Omega$.

Written out in full these are

$$
\rho\, \partial_t^2 \mathbf{s} = \boldsymbol{\nabla} \cdot (\mathbf{c} : \boldsymbol{\epsilon}) = \boldsymbol{\nabla} \cdot \mathbf{t} \quad , \tag{19}
$$

and

$$
\hat{\mathbf{n}} \cdot (\mathbf{c} : \boldsymbol{\epsilon}) = \hat{\mathbf{n}} \cdot \mathbf{t} = \mathbf{0} \quad . \tag{20}
$$

Note that, in the absence of spin, the stress tensor

$$\mathbf{t} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\mathbf{s}} = \mathbf{c} : \boldsymbol{\epsilon} \tag{21}$$

is symmetric, $\mathbf{t} = \mathbf{t}^t$, as a result of the symmetries of the elastic tensor.

The most general fourth-order, isotropic tensor is of the form

$$c_{ijk\ell} = \lambda\,\delta_{ij}\,\delta_{k\ell} + \mu(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) + \nu(\delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}) \quad . \tag{22}$$

Since, in the absence of spin, the elastic tensor has the symmetry $c_{ijk\ell} = c_{ij\ell k}$, the isotropic elastic tensor is of the form

$$c_{ijk\ell} = \lambda\,\delta_{ij}\,\delta_{k\ell} + \mu(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) \quad . \tag{23}$$

Thus, Hooke's law in an isotropic material reduces to

$$\mathbf{t} = \lambda\,\mathrm{tr}(\boldsymbol{\epsilon})\,\mathbf{I} + 2\mu\,\boldsymbol{\epsilon} \quad , \tag{24}$$

where $\mathbf{I}$ is the identity tensor.

In an isotropic medium, the stress-free boundary conditions imply that

$$\lambda\,\mathrm{tr}\,(\boldsymbol{\epsilon})\,\hat{\mathbf{n}} + 2\mu\,\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon} = \mathbf{0} \quad . \tag{25}$$

Let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Then

$$2\mu\,\epsilon_{31} = 0 \quad , \tag{26}$$

$$2\mu\,\epsilon_{32} = 0 \quad , \tag{27}$$

$$\lambda\,(\epsilon_{11} + \epsilon_{22}) + (\lambda + 2\mu)\,\epsilon_{33} = 0 \quad . \tag{28}$$

The implication is that on the free surface

$$w_1 = -\,\partial_3 s_2 = \partial_2 s_3 \quad , \tag{29}$$

$$w_2 = \partial_3 s_1 = -\,\partial_1 s_3 \quad . \tag{30}$$

3 LINEAR ELASTICITY WITH SPIN

In this section we consider a continuum in which particles have a displacement, $\mathbf{s}$, as well as an intrinsic spin, $\boldsymbol{\phi}$, distinct from that induced by the motion via the vorticity, $\mathbf{w}$. We have seen that the vorticity vector $\mathbf{w}$ is the dual of the vorticity tensor $\boldsymbol{\omega}$ via the relationships (6) and (7). Similarly, we associate a spin tensor $\boldsymbol{\Phi}$ with the spin vector $\boldsymbol{\phi}$:

$$\boldsymbol{\Phi} = -\,\boldsymbol{\varepsilon}\cdot\boldsymbol{\phi} \quad , \tag{31}$$

where $\boldsymbol{\varepsilon}$ is the alternating tensor. In such a material, the total strain has two contributions: one induced by the displacement (as in the case without spin),

$$\boldsymbol{\epsilon} = \tfrac{1}{2}[\boldsymbol{\nabla}\mathbf{s} + (\boldsymbol{\nabla}\mathbf{s})^t] \quad , \tag{32}$$

and a second induced by the difference between the spin and vorticity,

$$\boldsymbol{\Phi} - \boldsymbol{\omega} = -\,\boldsymbol{\varepsilon}\cdot(\boldsymbol{\phi} - \tfrac{1}{2}\boldsymbol{\nabla}\times\mathbf{s}) \quad , \tag{33}$$

such that additional strain is only induced if the spin of the particle, $\boldsymbol{\phi}$, is not equal to its vorticity, $\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{s}$. The total strain is thus

$$\boldsymbol{\epsilon} + (\boldsymbol{\Phi} - \boldsymbol{\omega}) = \boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi} \quad . \tag{34}$$

The kinetic energy density of the material has two contributions: one associated with the kinetic energy density $\frac{1}{2}\,\rho\,\dot{\mathbf{s}}\cdot\dot{\mathbf{s}}$, and a second associated with the spin energy density $\frac{1}{2}\,\dot{\boldsymbol{\phi}}\cdot\mathbf{j}\cdot\dot{\boldsymbol{\phi}}$, where $\mathbf{j}$ denotes an inertia density tensor. Like the mass density ρ, the inertia density tensor is a material property. Thus, we have

$$K = \tfrac{1}{2}\,\rho\,\dot{\mathbf{s}}\cdot\dot{\mathbf{s}} + \tfrac{1}{2}\,\dot{\boldsymbol{\phi}}\cdot\mathbf{j}\cdot\dot{\boldsymbol{\phi}} \quad . \tag{35}$$

Note that $[\rho] = \mathrm{kg\,m^{-3}}$ and that $[i] = \mathrm{kg\,m^{-1}}$. The potential energy density also has two contributions: one is quadratic in the total strain,

$$\tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}) : \mathbf{c} : (\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}) \quad , \tag{36}$$

and a second is quadratic in spin gradients,

$$\tfrac{1}{2}\,(\boldsymbol{\nabla}\boldsymbol{\phi}) : \mathbf{c}_c : (\boldsymbol{\nabla}\boldsymbol{\phi}) \quad . \tag{37}$$

Note that $[\mathbf{c}] = \mathrm{kg\,m^{-1}\,s^{-2}}$ and that $[\mathbf{c}_c] = \mathrm{kg\,m\,s^{-2}}$. Because these two contributions are quadratic, the fourth-order tensors $\mathbf{c}$ and $\mathbf{c}_c$

have the symmetries

$$c_{ijk\ell} = c_{k\ell ij} \quad , \qquad c^c_{ijk\ell} = c^c_{k\ell ij} \quad . \tag{38}$$

Each tensor thus potentially has 45 independent elements. The potential energy density is given by

$$V = \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}) : \mathbf{c} : (\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}) + \tfrac{1}{2}\,(\boldsymbol{\nabla}\boldsymbol{\phi}) : \mathbf{c}_c : (\boldsymbol{\nabla}\boldsymbol{\phi}) \quad . \tag{39}$$

We may now define the Lagrangian density

$$\mathcal{L}(\dot{\mathbf{s}}, \boldsymbol{\nabla}\mathbf{s}, \boldsymbol{\phi}, \dot{\boldsymbol{\phi}}, \boldsymbol{\nabla}\boldsymbol{\phi}) = \tfrac{1}{2}\,\rho\,\dot{\mathbf{s}}\cdot\dot{\mathbf{s}} + \tfrac{1}{2}\,\dot{\boldsymbol{\phi}}\cdot\mathbf{j}\cdot\dot{\boldsymbol{\phi}} - \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}) : \mathbf{c} : (\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}) - \tfrac{1}{2}\,\boldsymbol{\nabla}\boldsymbol{\phi} : \mathbf{c}_c : \boldsymbol{\nabla}\boldsymbol{\phi} \quad . \tag{40}$$

See Pujol (2009) for a discussion of the absence of terms involving combinations of $\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}$ and $\boldsymbol{\nabla}\boldsymbol{\phi}$. The Euler-Lagrange equations are

$$\partial_t \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{s}}} + \boldsymbol{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\mathbf{s}} = \mathbf{0} \quad , \qquad \mathbf{x}\in\Omega \quad , \tag{41}$$

$$\partial_t \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{\phi}}} + \boldsymbol{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\boldsymbol{\phi}} - \frac{\partial\mathcal{L}}{\partial\boldsymbol{\phi}} = \mathbf{0} \quad , \qquad \mathbf{x}\in\Omega \quad , \tag{42}$$

subject to the boundary conditions

$$\hat{\mathbf{n}}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\mathbf{s}} = \mathbf{0} \quad , \qquad \mathbf{x}\in\partial\Omega \quad , \tag{43}$$

$$\hat{\mathbf{n}}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\boldsymbol{\phi}} = \mathbf{0} \quad , \qquad \mathbf{x}\in\partial\Omega \quad . \tag{44}$$

Written out in full these are conservation of linear momentum

$$\rho\,\ddot{\mathbf{s}} = \boldsymbol{\nabla}\cdot[\mathbf{c} : (\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi})] = \boldsymbol{\nabla}\cdot\mathbf{t} \quad , \tag{45}$$

and conservation of angular momentum

$$\mathbf{j}\cdot\ddot{\boldsymbol{\phi}} = \boldsymbol{\nabla}\cdot[\mathbf{c}_c : (\boldsymbol{\nabla}\boldsymbol{\phi})] + \boldsymbol{\varepsilon} : [\mathbf{c} : (\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi})] = \boldsymbol{\nabla}\cdot\mathbf{t}_c + \boldsymbol{\varepsilon} : \mathbf{t} \quad , \tag{46}$$

where

$$\mathbf{t} = \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\mathbf{s}} = \mathbf{c} : (\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}) \quad , \tag{47}$$

$$\mathbf{t}_c = \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\boldsymbol{\phi}} = \mathbf{c}_c : \boldsymbol{\nabla}\boldsymbol{\phi} \quad , \tag{48}$$

subject to the boundary conditions

$$\hat{\mathbf{n}}\cdot\mathbf{t} = \mathbf{0} \quad , \tag{49}$$

$$\hat{\mathbf{n}}\cdot\mathbf{t}_c = \mathbf{0} \quad . \tag{50}$$

Note that, in general, the stress tensor, $\mathbf{t}$, and couple stress tensor, $\mathbf{t}_c$, are not symmetric. Note also that $[\mathbf{t}] = \mathrm{Pa} = \mathrm{kg\,m^{-1}\,s^{-2}}$ and that $[\mathbf{t}_c] = \mathrm{Pa\,m} = \mathrm{kg\,s^{-2}}$.

When the medium is isotropic, we have

$$c_{ijk\ell} = \lambda\,\delta_{ij}\,\delta_{k\ell} + \mu(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) + \nu(\delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}) \quad , \tag{51}$$

$$c^c_{ijk\ell} = \lambda_c\,\delta_{ij}\,\delta_{k\ell} + \mu_c(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) + \nu_c(\delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}) \quad . \tag{52}$$

Thus the stress is

$$\begin{aligned} t_{ij} &= c_{ijk\ell}(\nabla_k s_\ell - \varepsilon_{k\ell m}\,\phi_m) \\ &= \lambda\,\delta_{ij}\,\epsilon_{kk} + 2\mu\,\epsilon_{ij} - 2\nu\,\varepsilon_{ijk}(\phi_k - w_k) \\ &= \lambda\,\delta_{ij}\,\epsilon_{kk} + 2\mu\,\epsilon_{ij} + 2\nu\,(\Phi_{ij} - \omega_{ij}) \quad , \end{aligned} \tag{53}$$

where

$$\Phi_{ij} = -\,\varepsilon_{ijk}\,\phi_k \quad , \qquad \epsilon_{ij} = \tfrac{1}{2}(\nabla_i s_j + \nabla_j s_i) \quad , \qquad \omega_{ij} = -\,\tfrac{1}{2}(\nabla_i s_j - \nabla_j s_i) \quad . \tag{54}$$

The couple stress is

$$t^c_{ij} = \lambda_c\,\delta_{ij}\,\nabla_k\phi_k + \mu_c\,(\nabla_i\phi_j + \nabla_j\phi_i) + \nu_c\,(\nabla_i\phi_j - \nabla_j\phi_i) \quad , \tag{55}$$

In an isotropic medium, the stress-free boundary conditions imply that

$$\lambda\,\mathrm{tr}\,(\boldsymbol{\epsilon})\,\hat{\mathbf{n}} + 2\mu\,\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon} + 2\nu\,\hat{\mathbf{n}}\times(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}) = \mathbf{0} \quad , \tag{56}$$

$$\begin{aligned} &\lambda_c\,(\boldsymbol{\nabla}\cdot\boldsymbol{\phi})\,\hat{\mathbf{n}} + (\mu_c + \nu_c)\,\hat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\boldsymbol{\phi}) + (\mu_c - \nu_c)\,(\boldsymbol{\nabla}\boldsymbol{\phi})\cdot\hat{\mathbf{n}} = \\ &\lambda_c\,(\boldsymbol{\nabla}\cdot\boldsymbol{\phi})\,\hat{\mathbf{n}} + \mu_c\,\hat{\mathbf{n}}\cdot[\boldsymbol{\nabla}\boldsymbol{\phi} + (\boldsymbol{\nabla}\boldsymbol{\phi})^t] - \nu_c\,\hat{\mathbf{n}}\times(\boldsymbol{\nabla}\times\boldsymbol{\phi}) = \mathbf{0} \quad . \end{aligned} \tag{57}$$

Let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Then

$$2\mu\,\epsilon_{31} - 2\nu(\phi_2 - w_2) = 0 \quad , \tag{58}$$

$$2\mu\,\epsilon_{32} + 2\nu(\phi_1 - w_1) = 0 \quad , \tag{59}$$

$$\lambda\,(\epsilon_{11} + \epsilon_{22}) + (\lambda + 2\mu)\,\epsilon_{33} = 0 \quad , \tag{60}$$

$$(\mu_c + \nu_c)\,\partial_3\phi_1 + (\mu_c - \nu_c)\,\partial_1\phi_3 = 0 \quad . \tag{61}$$

$$(\mu_c + \nu_c)\,\partial_3\phi_2 + (\mu_c - \nu_c)\,\partial_2\phi_3 = 0 \quad , \tag{62}$$

$$\lambda_c\,(\partial_1\phi_1 + \partial_2\phi_2) + (\lambda_c + 2\mu_c)\,\partial_3\phi_3 = 0 \quad . \tag{63}$$

Conservation of linear momentum becomes

$$\rho\,\ddot{\mathbf{s}} = \boldsymbol{\nabla}\cdot[\lambda\,\mathrm{tr}(\boldsymbol{\epsilon})\,\mathbf{I} + 2\mu\,\boldsymbol{\epsilon}] + \boxed{\boldsymbol{\nabla}\times\left[2\nu(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s})\right]} \quad . \tag{64}$$

We have

$$\varepsilon_{ijk}\,t_{jk} = \,-\,4\nu\,\phi_i + 2\nu\,\varepsilon_{ijk}\nabla_j s_k \quad . \tag{65}$$

The equation for $\boldsymbol{\phi}$, i.e., conservation of angular momentum, is

$$\begin{aligned}\mathbf{j}\cdot\ddot{\boldsymbol{\phi}} &= \boldsymbol{\nabla}\cdot[\lambda_c\,\boldsymbol{\nabla}\cdot\boldsymbol{\phi}\,\mathbf{I} + (\mu_c+\nu_c)\boldsymbol{\nabla}\boldsymbol{\phi} + (\mu_c-\nu_c)(\boldsymbol{\nabla}\boldsymbol{\phi})^t] - 4\nu\left(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}\right)\\ &= \boldsymbol{\nabla}\cdot\{\lambda_c\,\boldsymbol{\nabla}\cdot\boldsymbol{\phi}\,\mathbf{I} + \mu_c[\boldsymbol{\nabla}\boldsymbol{\phi} + (\boldsymbol{\nabla}\boldsymbol{\phi})^t]\} - \boldsymbol{\nabla}\times[\nu_c\,(\boldsymbol{\nabla}\times\boldsymbol{\phi})] - \boxed{4\nu\left(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}\right)} \quad .\end{aligned} \tag{66}$$

Conservation of linear and angular momentum are coupled by the boxed terms involving $\boldsymbol{\phi} - \frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$, i.e., the difference between the spin and half the curl of the displacement. Note that if $\mathbf{j} = \mathbf{0}$ and $\lambda_c = \mu_c = \nu_c = 0$, then $\boldsymbol{\phi} = \frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$, i.e., the spin equals the half curl of the displacement, and we recover classical elasticity with a symmetric stress tensor.

3.1 Homogeneous models

In a homogeneous medium, we have

$$\rho\,\ddot{\mathbf{s}} = (\lambda + 2\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{s}) - \mu\,\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s}) + 2\nu\,\boldsymbol{\nabla}\times(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}) \quad , \tag{67}$$

$$j\,\ddot{\boldsymbol{\phi}} = (\lambda_c + 2\mu_c)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\phi}) - (\mu_c + \nu_c)\,\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\phi}) - 4\nu\left(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}\right) \quad . \tag{68}$$

If we define $\Pi = \boldsymbol{\nabla}\cdot\mathbf{s}$ and $\pi = \boldsymbol{\nabla}\cdot\boldsymbol{\phi}$, then we find upon taking the divergence of the last two equations that

$$\rho\,\ddot{\Pi} = (\lambda + 2\mu)\,\nabla^2\Pi \quad , \tag{69}$$

$$j\,\ddot{\pi} = (\lambda_c + 2\mu_c)\,\nabla^2\pi - j\omega_0^2\,\pi \quad , \tag{70}$$

where

$$\omega_0^2 = 4\,\nu/j \quad . \tag{71}$$

Note that the equations in Π and π are uncoupled. Assuming plane wave propagation in the positive direction of the unit vector $\hat{\mathbf{k}}$, the solutions to these equations are written as

$$\Pi = a\exp[ik(\hat{\mathbf{k}}\cdot\mathbf{r} - ct)]\,, \tag{72}$$

$$\pi = b\exp[ik(\hat{\mathbf{k}}\cdot\mathbf{r} - ct)]\,, \tag{73}$$

where a and b are complex constants, $k = \omega/c$ is the wavenumber, $\mathbf{r}$ is the position vector, and c is the wavespeed. Substituting equations (72) & (73) into equations (69) & (70) gives

$$\rho\,\omega^2 = (\lambda + 2\mu)\,k^2 \quad , \tag{74}$$

$$j\,\omega^2 = (\lambda_c + 2\mu_c)\,k^2 + j\omega_0^2 \quad , \tag{75}$$

which yields

$$\boxed{\alpha^2 = \frac{\lambda + 2\mu}{\rho}}\,, \tag{76}$$

where α is the velocity associated to a longitudinal displacement and

$$\boxed{\alpha_c^2 = \frac{\lambda_c + 2\mu_c}{j(1 - \omega_0^2/\omega^2)}}\,, \tag{77}$$

Table 1. Material parameters

Symbol	Unit	Value
λ	GPa	28
μ	GPa	4
ρ	kg/m^3	10^5
ν	GPa	2
λ_c	N	10^8
μ_c	N	$1.936{\cdot}10^8$
ν_c	N	$3.0464{\cdot}10^9$
j	kg/m	10^4

where α_c is the velocity associated to a longitudinal microrotation. Note that α_c depends on the frequency, and that the wave is therefore dispersive and exists whenever $\omega > \omega_0$ since $\lambda_c + 2\mu_c > 0$.

Next, if we define $\mathbf{\Psi} = \mathbf{\nabla} \times \mathbf{s}$ and $\boldsymbol{\psi} = \mathbf{\nabla} \times \boldsymbol{\phi}$, then we find

$$\rho\ddot{\mathbf{\Psi}} = (\mu + \nu)\,\nabla^2\mathbf{\Psi} + \tfrac{1}{2}j\omega_0^2\,\mathbf{\nabla}\times\boldsymbol{\psi} \quad , \tag{78}$$

$$j\ddot{\boldsymbol{\psi}} = (\mu_c + \nu_c)\,\nabla^2\boldsymbol{\psi} - j\omega_0^2\,\boldsymbol{\psi} + \tfrac{1}{2}j\,\omega_0^2\,\mathbf{\nabla}\times\mathbf{\Psi} \quad . \tag{79}$$

Note that the equations in $\mathbf{\Psi}$ and $\boldsymbol{\psi}$ are coupled. Assuming again plane wave propagation in the positive direction of the unit vector $\hat{\mathbf{k}}$, the solutions to these equations are written as

$$\mathbf{\Psi} = \mathbf{A}\exp[ik(\hat{\mathbf{k}}\cdot\mathbf{r} - ct)]\,, \tag{80}$$

$$\boldsymbol{\psi} = \mathbf{B}\exp[ik(\hat{\mathbf{k}}\cdot\mathbf{r} - ct)]\,, \tag{81}$$

where $\mathbf{A}$ and $\mathbf{B}$ are complex constant vectors. Substituting equations (80) & (81) into equations (78) & (79) gives

$$-\rho\omega^2\mathbf{A} = -(\mu+\nu)k^2\mathbf{A} + \tfrac{1}{2}j\omega_0^2\,ik\,\hat{\mathbf{k}}\times\mathbf{B}\,, \tag{82}$$

$$-j\omega^2\mathbf{B} = -(\mu_c+\nu_c)k^2\,\mathbf{B} - j\omega_0^2\,\mathbf{B} + \tfrac{1}{2}j\omega_0^2\,ik\,\hat{\mathbf{k}}\times\mathbf{A}\,, \tag{83}$$

Eliminating $\mathbf{A}$ and $\mathbf{B}$ yields a quadratic equation in c^2 of the form

$$\begin{aligned} &j\rho\left(1-\omega_0^2/\omega^2\right)\left(c^2\right)^2 + \left[\rho\,(\mu_c+\nu_c) + j\,\left(1-\omega_0^2/\omega^2\right)(\mu+\nu) + j\,(\omega_0/\omega)^2\,\nu\right]c^2 \\ &\quad + (\mu+\nu)(\mu_c+\nu_c) = 0\,. \end{aligned} \tag{84}$$

Equation (84) has two roots

$$\boxed{\begin{aligned} \beta_\pm \;=\; &-\frac{\rho\,(\mu_c+\nu_c)+j\,\left(1-\omega_0^2/\omega^2\right)(\mu+\nu)+j\,(\omega_0/\omega)^2\,\nu}{2\,j\rho\left(1-\omega_0^2/\omega^2\right)} \\ &\pm\frac{\sqrt{\left[\rho\,(\mu_c+\nu_c)+j\,\left(1-\omega_0^2/\omega^2\right)(\mu+\nu)+j\,(\omega_0/\omega)^2\,\nu\right]^2-4\,j\rho\left(1-\omega_0^2/\omega^2\right)(\mu+\nu)(\mu_c+\nu_c)}}{2\,j\rho\left(1-\omega_0^2/\omega^2\right)}\,, \end{aligned}} \tag{85}$$

where $\beta \equiv \beta_+$ corresponds to the transverse displacement wave, whereas $\beta_c \equiv \beta_-$ is associated with the transverse microrotation. Note that both waves are dispersive and exist whenever $\omega > \omega_0$.

We have plotted in Figure 3 α, α_c, β, and β_c using the material properties listed in Table 1. Results agree with the study by Kulesh (2009).

3.2 Excitation

In what follows, we borrow heavily from the theory in Chapter 5 of Dahlen & Tromp (1998). The governing equations are

$$\rho\,\ddot{\mathbf{s}} = \mathbf{\nabla}\cdot\mathbf{t} + \mathbf{\nabla}\cdot(\mathbf{t}^{\text{true}} - \mathbf{t}) \quad , \tag{86}$$

$$\mathbf{j}\cdot\ddot{\boldsymbol{\phi}} = \mathbf{\nabla}\cdot\mathbf{t}_c + \boldsymbol{\varepsilon}:\mathbf{t} + \mathbf{\nabla}\cdot(\mathbf{t}_c^{\text{true}} - \mathbf{t}_c) + \boldsymbol{\varepsilon}:(\mathbf{t}^{\text{true}} - \mathbf{t}) \quad , \tag{87}$$

where $\mathbf{t}^{\text{true}}$ and $\mathbf{t}_c^{\text{true}}$ are the true physical stresses. The stress gluts are defined by

$$\mathbf{S} = \mathbf{t} - \mathbf{t}^{\text{true}} \quad , \tag{88}$$

$$\mathbf{S}_c = \mathbf{t}_c - \mathbf{t}_c^{\text{true}} \quad . \tag{89}$$

Let us introduce the moment density tensors

$$\mathbf{m} = \mathbf{c}:\hat{\boldsymbol{\nu}}\,\Delta\mathbf{s} \quad , \tag{90}$$

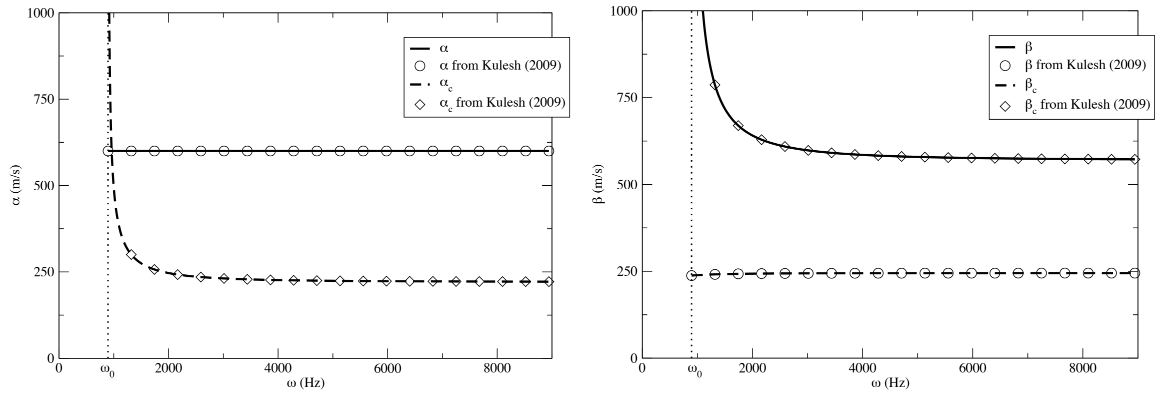

Figure 3. Longitudinal (*Left*) and transverse (*Right*) wavespeeds in a micropolar medium with the properties tabulated in Table 1, using equations (76), (77), and (85), compared to Kulesh (2009).

$$\mathbf{m}_c = \mathbf{c}_c : \hat{\boldsymbol{\nu}}\,\Delta\boldsymbol{\phi} \quad , \tag{91}$$

where $\hat{\boldsymbol{\nu}}$ is the unit normal to the fault plane Σ , $\Delta\mathbf{s}$ denotes the slip across the fault, and $\Delta\boldsymbol{\phi}$ the change in spin across the fault. In an isotropic medium, we have

$$\mathbf{m} \;= (\kappa - \tfrac{2}{3}\,\mu)\,\hat{\mathbf{n}}\cdot\Delta\mathbf{s}\,\mathbf{I} + \mu\,(\hat{\boldsymbol{\nu}}\,\Delta\mathbf{s} + \Delta\mathbf{s}\,\hat{\boldsymbol{\nu}}) + \nu\,(\hat{\boldsymbol{\nu}}\,\Delta\mathbf{s} - \Delta\mathbf{s}\,\hat{\boldsymbol{\nu}}) \quad , \tag{92}$$

$$\mathbf{m}_c \;= \lambda_c\,\hat{\boldsymbol{\nu}}\cdot\Delta\boldsymbol{\phi}\,\mathbf{I} + \mu_c\,(\hat{\boldsymbol{\nu}}\,\Delta\boldsymbol{\phi} + \Delta\boldsymbol{\phi}\,\hat{\boldsymbol{\nu}}) + \nu_c\,(\hat{\boldsymbol{\nu}}\,\Delta\boldsymbol{\phi} - \Delta\boldsymbol{\phi}\,\hat{\boldsymbol{\nu}}) \quad . \tag{93}$$

These expressions tell us how the various Volterra cut-and-weld protocols, as illustrated in Figure 2, excite displacement and spin fields. They generalize the classical expression

$$\mathbf{m} = \mu\,(\hat{\boldsymbol{\nu}}\,\Delta\mathbf{s} + \Delta\mathbf{s}\,\hat{\boldsymbol{\nu}}) \quad , \tag{94}$$

which captures slip on an ideal fault, that is, just edge and screw dislocations. Notably, neither the moment-density tensor nor the couple-moment-density tensor is symmetric. One can imagine designing experiments involving sources specifically designed to excite spin waves, for example by twisting small elements of the medium.

The stress gluts may be written in terms of the moment density tensors as

$$\mathbf{S} = \mathbf{m}\,\delta_\Sigma \quad , \tag{95}$$

$$\mathbf{S}_c = \mathbf{m}_c\,\delta_\Sigma \quad , \tag{96}$$

where δ_Σ denotes the delta function located on the fault surface Σ . The sources driving the motion and spin are

$$\mathbf{f} = \,-\boldsymbol{\nabla}\cdot\mathbf{S} = \,-\mathbf{m}\cdot\boldsymbol{\nabla}\delta_\Sigma \quad , \tag{97}$$

$$\mathbf{f}_c = \,-\boldsymbol{\nabla}\cdot\mathbf{S}_c - \boldsymbol{\varepsilon}:\mathbf{S} = \,-\mathbf{m}_c\cdot\boldsymbol{\nabla}\delta_\Sigma - \boldsymbol{\varepsilon}:\mathbf{m}\,\delta_\Sigma \quad , \tag{98}$$

and thus the equations of motion are

$$\rho\,\ddot{\mathbf{s}} = \boldsymbol{\nabla}\cdot\mathbf{t} - \mathbf{m}\cdot\boldsymbol{\nabla}\delta_\Sigma \quad , \tag{99}$$

$$\mathbf{j}\cdot\ddot{\boldsymbol{\phi}} = \boldsymbol{\nabla}\cdot\mathbf{t}_c + \boldsymbol{\varepsilon}:\mathbf{t} - \mathbf{m}_c\cdot\boldsymbol{\nabla}\delta_\Sigma - \boldsymbol{\varepsilon}:\mathbf{m}\,\delta_\Sigma \quad . \tag{100}$$

In the point source approximation, the moment tensors are defined by

$$\mathbf{M} \;= \int_\Sigma \mathbf{m}\,\mathrm{d}^2\mathbf{x} \approx (\kappa - \tfrac{2}{3}\,\mu)\,\Sigma\,\hat{\mathbf{n}}\cdot\Delta\mathbf{s}\,\mathbf{I} + \mu\,\Sigma\,(\hat{\boldsymbol{\nu}}\,\Delta\mathbf{s} + \Delta\mathbf{s}\,\hat{\boldsymbol{\nu}}) + \nu\,\Sigma\,(\hat{\boldsymbol{\nu}}\,\Delta\mathbf{s} - \Delta\mathbf{s}\,\hat{\boldsymbol{\nu}}) \quad , \tag{101}$$

$$\mathbf{M}_c \;= \int_\Sigma \mathbf{m}_c\,\mathrm{d}^2\mathbf{x} \approx \;\lambda_c\,\Sigma\,\hat{\boldsymbol{\nu}}\cdot\Delta\boldsymbol{\phi}\,\mathbf{I} + \mu_c\,\Sigma\,(\hat{\boldsymbol{\nu}}\,\Delta\boldsymbol{\phi} + \Delta\boldsymbol{\phi}\,\hat{\boldsymbol{\nu}}) + \nu_c\,\Sigma\,(\hat{\boldsymbol{\nu}}\,\Delta\boldsymbol{\phi} - \Delta\boldsymbol{\phi}\,\hat{\boldsymbol{\nu}}) \quad . \tag{102}$$

where Σ is the area of the fault. In this point source approximation, the complete set of governing equations become

$$\rho\,\ddot{\mathbf{s}} = \boldsymbol{\nabla}\cdot\mathbf{t} - \mathbf{M}\cdot\boldsymbol{\nabla}\delta(\mathbf{r}-\mathbf{r}_s) \quad , \tag{103}$$

$$\mathbf{j}\cdot\ddot{\boldsymbol{\phi}} = \boldsymbol{\nabla}\cdot\mathbf{t}_c + \boldsymbol{\varepsilon}:\mathbf{t} - \mathbf{M}_c\cdot\boldsymbol{\nabla}\delta(\mathbf{r}-\mathbf{r}_s) - \boldsymbol{\varepsilon}:\mathbf{M}\,\delta(\mathbf{r}-\mathbf{r}_s) \quad , \tag{104}$$

$$\begin{aligned}\mathbf{t} \;&= \lambda\,(\boldsymbol{\nabla}\cdot\mathbf{s})\,\mathbf{I} + \mu\,[\boldsymbol{\nabla}\mathbf{s} + (\boldsymbol{\nabla}\mathbf{s})^t] + \nu\,[\boldsymbol{\nabla}\mathbf{s} - (\boldsymbol{\nabla}\mathbf{s})^t] - 2\nu\,\boldsymbol{\varepsilon}\cdot\boldsymbol{\phi}\\ &= \lambda\,(\boldsymbol{\nabla}\cdot\mathbf{s})\,\mathbf{I} + \mu\,[\boldsymbol{\nabla}\mathbf{s} + (\boldsymbol{\nabla}\mathbf{s})^t] - 2\nu\,\boldsymbol{\varepsilon}\cdot(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}) \quad ,\end{aligned} \tag{105}$$

$$
\begin{aligned}
\mathbf{t}_c &= \lambda_c\,(\boldsymbol{\nabla}\cdot\boldsymbol{\phi})\,\mathbf{I} + \mu_c\,[\boldsymbol{\nabla}\boldsymbol{\phi} + (\boldsymbol{\nabla}\boldsymbol{\phi})^t] + \nu_c\,[\boldsymbol{\nabla}\boldsymbol{\phi} - (\boldsymbol{\nabla}\boldsymbol{\phi})^t] \\
&= \lambda_c\,(\boldsymbol{\nabla}\cdot\boldsymbol{\phi})\,\mathbf{I} + \mu_c\,[\boldsymbol{\nabla}\boldsymbol{\phi} + (\boldsymbol{\nabla}\boldsymbol{\phi})^t] + \nu_c\,\boldsymbol{\varepsilon}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\phi}) \quad ,
\end{aligned} \tag{106}
$$

subject to the boundary conditions

$$\hat{\mathbf{n}}\cdot\mathbf{t} = \mathbf{0} \quad , \tag{107}$$

$$\hat{\mathbf{n}}\cdot\mathbf{t}_c = \mathbf{0} \quad . \tag{108}$$

As noted earlier, coupling between conservation of linear and angular momentum is governed by the difference $\boldsymbol{\phi} - \frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$, as indicated by the boxed terms in (64) and (66). In the absence of spin, we have $\boldsymbol{\phi} = \frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$, so in the presence of spin we are expecting $\boldsymbol{\phi}$ to deviate from $\frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$, i.e., one should monitor $\boldsymbol{\phi} - \frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$.

3.3 2D Simulations

Suppose the displacement and spin fields are 2D in the sense

$$\mathbf{s} = s_x(x,y,t)\,\hat{\mathbf{x}} + s_y(x,y,t)\,\hat{\mathbf{y}} + s_z(x,y,t)\,\hat{\mathbf{z}} \quad , \tag{109}$$

$$\boldsymbol{\phi} = \phi_x(x,y,t)\,\hat{\mathbf{x}} + \phi_y(x,y,t)\,\hat{\mathbf{y}} + \phi_z(x,y,t)\,\hat{\mathbf{z}} \quad . \tag{110}$$

Then, e.g.,

$$\boldsymbol{\nabla}\mathbf{s} = \partial_x s_x\,\hat{\mathbf{x}}\,\hat{\mathbf{x}} + \partial_x s_y\,\hat{\mathbf{x}}\,\hat{\mathbf{y}} + \partial_x s_z\,\hat{\mathbf{x}}\,\hat{\mathbf{z}} + \partial_y s_x\,\hat{\mathbf{y}}\,\hat{\mathbf{x}} + \partial_y s_y\,\hat{\mathbf{y}}\,\hat{\mathbf{y}} + \partial_y s_z\,\hat{\mathbf{y}}\,\hat{\mathbf{z}} \quad , \tag{111}$$

and

$$\boldsymbol{\nabla}\cdot\mathbf{s} = \partial_x s_x + \partial_y s_y \quad , \tag{112}$$

$$\boldsymbol{\nabla}\times\mathbf{s} = \partial_y s_z\,\hat{\mathbf{x}} - \partial_x s_z\,\hat{\mathbf{y}} + (\partial_x s_y - \partial_y s_x)\,\hat{\mathbf{z}} \quad , \tag{113}$$

and, furthermore,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{s}) = \partial_x(\partial_x s_x + \partial_y s_y)\,\hat{\mathbf{x}} + \partial_y(\partial_x s_x + \partial_y s_y)\,\hat{\mathbf{y}} \quad , \tag{114}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\times\mathbf{s}) = \partial_x\partial_y s_z\,\hat{\mathbf{x}}\,\hat{\mathbf{x}} - \partial_x\partial_x s_z\,\hat{\mathbf{x}}\,\hat{\mathbf{y}} + \partial_x(\partial_x s_y - \partial_y s_x)\,\hat{\mathbf{x}}\,\hat{\mathbf{z}} + \partial_y\partial_y s_z\,\hat{\mathbf{y}}\,\hat{\mathbf{x}} - \partial_y\partial_x s_z\,\hat{\mathbf{y}}\,\hat{\mathbf{y}} + \partial_y(\partial_x s_y - \partial_y s_x)\,\hat{\mathbf{y}}\,\hat{\mathbf{z}} \quad , \tag{115}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s}) = \partial_y(\partial_x s_y - \partial_y s_x)\,\hat{\mathbf{x}} - \partial_x(\partial_x s_y - \partial_y s_x)\,\hat{\mathbf{y}} - (\partial_x^2 s_z + \partial_y^2 s_z)\,\hat{\mathbf{z}} \quad . \tag{116}$$

Thus the x- and y-components of expressions (67) and (67) are

$$\rho\,\ddot{s}_x = (\lambda+2\mu)\,\partial_x(\partial_x s_x + \partial_y s_y) - \mu\,\partial_y(\partial_x s_y - \partial_y s_x) + 2\nu\,\partial_y[\phi_z - \tfrac{1}{2}\,(\partial_x s_y - \partial_y s_x)] \quad , \tag{117}$$

$$\rho\,\ddot{s}_y = (\lambda+2\mu)\,\partial_y(\partial_x s_x + \partial_y s_y) + \mu\,\partial_x(\partial_x s_y - \partial_y s_x) - 2\nu\,\partial_x[\phi_z - \tfrac{1}{2}\,(\partial_x s_y - \partial_y s_x)] \quad , \tag{118}$$

$$\rho\,\ddot{s}_z = (\mu+\nu)\,(\partial_x^2 s_z + \partial_y^2 s_z) + 2\nu\,(\partial_x\phi_y - \partial_y\phi_x) \quad , \tag{119}$$

$$j\,\ddot{\phi}_x = (\lambda_c+2\mu_c)\,\partial_x(\partial_x\phi_x + \partial_y\phi_y) - (\mu_c+\nu_c)\,\partial_y(\partial_x\phi_y - \partial_y\phi_x) - 4\nu\,(\phi_x - \tfrac{1}{2}\,\partial_y s_z) \quad , \tag{120}$$

$$j\,\ddot{\phi}_y = (\lambda_c+2\mu_c)\,\partial_y(\partial_x\phi_x + \partial_y\phi_y) + (\mu_c+\nu_c)\,\partial_x(\partial_x\phi_y - \partial_y\phi_x) - 4\nu\,(\phi_y + \tfrac{1}{2}\,\partial_x s_z) \quad , \tag{121}$$

$$j\,\ddot{\phi}_z = (\mu_c+\nu_c)\,(\partial_x^2\phi_z + \partial_y^2\phi_z) - 4\nu\,[\phi_z - \tfrac{1}{2}\,(\partial_x s_y - \partial_y s_x)] \quad . \tag{122}$$

We conclude that the PDEs for s_x, s_y (P-SV) and ϕ_z are decoupled from the PDEs for s_z (SH), ϕ_x, and ϕ_y.

$$\rho\,\ddot{s}_x = (\lambda+2\mu)\,\partial_x(\partial_x s_x + \partial_y s_y) - \mu\,\partial_y(\partial_x s_y - \partial_y s_x) + 2\nu\,\partial_y[\phi_z - \tfrac{1}{2}\,(\partial_x s_y - \partial_y s_x)] \quad , \tag{123}$$

$$\rho\,\ddot{s}_y = (\lambda+2\mu)\,\partial_y(\partial_x s_x + \partial_y s_y) + \mu\,\partial_x(\partial_x s_y - \partial_y s_x) - 2\nu\,\partial_x[\phi_z - \tfrac{1}{2}\,(\partial_x s_y - \partial_y s_x)] \quad , \tag{124}$$

$$j\,\ddot{\phi}_z = (\mu_c+\nu_c)\,(\partial_x^2\phi_z + \partial_y^2\phi_z) - 4\nu\,[\phi_z - \tfrac{1}{2}\,(\partial_x s_y - \partial_y s_x)] \quad . \tag{125}$$

3.4 Some notes on parameter values

The mass density has units of $[\rho] = \mathrm{kg\,m^{-3}}$. The inertia density has units of $[j] = \mathrm{kg\,m^{-1}}$, and since inertia involves mass times a squared distance, one expects that $j \sim \rho\,R^2$, where R is the characteristic spin length scale for the problem. This could be the characteristic size of the elements of the continuum that spin together, presumably something on the order of a wavelength λ. From $\boldsymbol{\phi} \sim \frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$ one would expect $||\boldsymbol{\phi}|| \sim \pi\,||\mathbf{s}||/\lambda$ in radians. The elastic parameters have units of Pa, e.g., $[\mu] = \mathrm{kg\,m^{-1}\,s^{-2}}$, and the couple stress elastic parameters have units of N, i.e., $[\mu_c] = \mathrm{kg\,m\,s^{-2}}$, and one expects that $\mu_c \sim \mu\,R^2$, so that the squared wavespeeds μ/ρ and μ_c/j are comparable. The coupling terms involve factors of $\gamma^2 = 2\nu/\rho$ (linear momentum) and $\omega_0^2 = 4\nu/j \sim 2\gamma^2/R^2$ (angular momentum), or $T_0 = 2\pi/\omega_0 \sim \sqrt{2}\,\pi\,R/\gamma$.

Suppose the total spin equals the vorticity:

$$\boldsymbol{\phi} = \tfrac{1}{2}\,\boldsymbol{\nabla} \times \mathbf{s} \quad . \tag{126}$$

Then, in a homogeneous medium, we have from equations (67) and (68)

$$\rho\,\ddot{\mathbf{s}} = (\lambda + 2\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{s}) - \mu\,\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{s}) \quad , \tag{127}$$

$$j\,\ddot{\boldsymbol{\phi}} = (\mu_c + \nu_c)\,\boldsymbol{\nabla}^2\boldsymbol{\phi} \quad , \tag{128}$$

where we have used the fact that $\boldsymbol{\nabla} \cdot \boldsymbol{\phi} = 0$. Upon taking the curl of the momentum equation (127), we find

$$\rho\,\ddot{\boldsymbol{\phi}} = \mu\,\boldsymbol{\nabla}^2\boldsymbol{\phi} \quad . \tag{129}$$

Since equations (129) and (128) must be the same, we find the requirement

$$\frac{\mu_c + \nu_c}{j} = \frac{\mu}{\rho} \quad . \tag{130}$$

REFERENCES

Dahlen, F. A. & Tromp, J., 1998, *Theoretical Global Seismology*, Princeton U. Press, New Jersey.

Kulesh, M., 2009, Waves in linear elastic media with microrotations, Part 1: Isotropic full Cosserat model, *Bull. Seismol. Soc. Am.*, **99**(2B), 1416–1422.

Pujol, J., 2009, Tutorial on rotations in the theories of finite deformation and micropolar (Cosserat) elasticity, *Bull. Seismol. Soc. Am.*, **99**, 1011–1027.