temporal history is modeled by generalizing the impulsive glut rate in equation (4.87) such that

$$\partial_t \boldsymbol{\gamma} = \mathbf{M}\ \delta(\mathbf{x}-\mathbf{x}_{\mathrm{s}})\,\dot{m}(t). \tag{4.99}$$

The *source-time function* $\dot{m}(t)$, where a dot denotes differentiation with respect to time, is normalized such that

$$\int_{t_0}^{t_f} \dot{m}(t)\,dt = 1. \tag{4.100}$$

In teleseismic body-wave studies, the source-time function $\dot{m}(t)$ of large earthquakes is frequently estimated by fitting a sequence of boxcars or overlapping trapezoids to observed body waveforms.

The displacement response (4.81) to such a time-dependent moment-tensor point source

$$\mathbf{f}(\mathbf{x},t) = -\,\mathbf{M}\cdot\boldsymbol{\nabla}\delta(\mathbf{x}-\mathbf{x}_{\mathrm{s}})\,m(t), \tag{4.101}$$

is

$$s_i(\mathbf{x},t) = \nabla_j^{\mathrm{s}} G_{ik}(\mathbf{x},\mathbf{x}_s;t)\,M_{jk} * m(t). \tag{4.102}$$

In the weak implementation, we would add a term

$$\mathbf{M} : \boldsymbol{\nabla}\tilde{\mathbf{s}}(\mathbf{x}_{\mathrm{s}})\,m(t) \tag{4.103}$$

to the right-hand side of the weak equation (4.31); this requires the integrated source-time function $m(t)$.

If the goal of the reader is to study short-period seismology, then all they need to know are the results up to this point, and they may safely skip the remainder of this chapter.

4.9 Seismology with Spin

In section 4.2, we defined the antisymmetric *infinitesimal rotation tensor* in terms of the antisymmetric part of the gradient of the displacement, $\boldsymbol{\nabla}\mathbf{s}$, namely,

$$\boldsymbol{\omega} = -\,\tfrac{1}{2}\left[\boldsymbol{\nabla}\mathbf{s} - (\boldsymbol{\nabla}\mathbf{s})^t\right] = -\,\boldsymbol{\omega}^t. \tag{4.104}$$

We may define the *rotation vector* in terms of the rotation tensor $\boldsymbol{\omega}$ by

$$\tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s} = -\,\tfrac{1}{2}\,\boldsymbol{\epsilon}:\boldsymbol{\omega}, \tag{4.105}$$

where $\boldsymbol{\epsilon}$ denotes the Levi-Civita tensor. To understand the nature of this vector, assume a plane wave with polarization $\hat{\mathbf{s}}$ traveling in the $\hat{\mathbf{x}}$ direction with wavespeed c:

$$\mathbf{s}(\mathbf{r},t) = \hat{\mathbf{s}}\,f(x-ct), \tag{4.106}$$

where the function f prescribes an arbitrary waveform. Then

$$\tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s} = \tfrac{1}{2}\,\hat{\mathbf{x}}\times\hat{\mathbf{s}}\,f'(x-ct) = -\,\frac{1}{2c}\,\hat{\mathbf{x}}\times\partial_t\mathbf{s}. \tag{4.107}$$

We see that the rotation vector is perpendicular to the polarization and travel directions, that is, the orientation $\hat{\mathbf{x}}\times\hat{\mathbf{s}}$, and has an amplitude proportional to the particle velocity divided by the wavespeed, $-\,\partial_t s/c$. Thus, we are presented with the opportunity of observing both the particle displacement $\mathbf{s}$ and the particle rotation $\tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$. This is the realm of *rotational seismology* (see, e.g., Lee, 2021, for a review).

In this section, we consider a continuum in which particles have a *total rotation*, $\boldsymbol{\phi}$, different from that induced by the motion via the rotation, $\frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$. Thus, a *rotational seismometer* (see, e.g., Jaroszewicz et al., 2016, for a review) might record not only the rotation vector, $\frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}$, induced by the displacement field, but also an additional contribution due to the *intrinsic rotation* of an element of the continuum. Based on equation (3.398), we associate a *total rotation tensor* $\boldsymbol{\Psi}$ with the total rotation vector $\boldsymbol{\phi}$:

$$\boldsymbol{\Psi} = -\,\boldsymbol{\epsilon}\cdot\boldsymbol{\phi} = -\,\boldsymbol{\Psi}^{t}. \tag{4.108}$$

In such a material, the total strain has two contributions: the infinitesimal strain (4.2) induced by the displacement and a second *rotational strain* induced by the difference between the total rotation tensor (4.108) and the infinitesimal rotation tensor (4.104). The total infinitesimal strain is thus the sum of the symmetric infinitesimal strain $\boldsymbol{\varepsilon}$ and the rotational strain $\boldsymbol{\Psi}-\boldsymbol{\omega}$:

$$\boldsymbol{\varepsilon} + (\boldsymbol{\Psi}-\boldsymbol{\omega}) = \boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\epsilon}\cdot\boldsymbol{\phi}. \tag{4.109}$$

Note that the total infinitesimal strain is not symmetric. For a more general description of asymmetric strain in terms of vector-valued forms, see box 1.15. The full complications of intrinsic material rotation are discussed in boxes 2.9, 2.11, and 2.13 and summarized in table 2.5.

The *kinetic energy density* of the material has two contributions: one associated with the kinetic energy density

$$\tfrac{1}{2}\,\rho\,\partial_t\mathbf{s}\cdot\partial_t\mathbf{s}, \tag{4.110}$$

and a second associated with the total *spin energy density*

$$\tfrac{1}{2}\,\partial_t\boldsymbol{\phi}\cdot\mathbf{j}\cdot\partial_t\boldsymbol{\phi}, \tag{4.111}$$

where $\mathbf{j}$ denotes an *inertia-density tensor*. Like the mass density ρ, the inertia-density tensor is a material property.[5] The *intrinsic* or *potential energy density* also has two contributions: one is quadratic in the total strain,

$$\tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s}-\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}):\boldsymbol{\Gamma}:(\boldsymbol{\nabla}\mathbf{s}-\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}), \tag{4.112}$$

and a second is quadratic in rotation gradients,

$$\tfrac{1}{2}\,(\boldsymbol{\nabla}\boldsymbol{\phi}):\boldsymbol{\Gamma}_c:(\boldsymbol{\nabla}\boldsymbol{\phi}). \tag{4.113}$$

Because these two contributions are quadratic, the fourth-order tensors $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_c$ have the symmetries[6]

$$\Gamma_{ijk\ell} = \Gamma_{k\ell ij}, \qquad \Gamma^{c}_{ijk\ell} = \Gamma^{c}_{k\ell ij}, \tag{4.114}$$

but these tensors are not symmetric in the indices i and j or in k and ℓ. Each tensor thus potentially has 45 independent elements, which is significantly more than the 21 elastic parameters in elasticity without rotation, discussed in section 4.3. The internal energy is obtained by combining (4.112) and (4.113):[7]

$$\rho\,U(\,\boldsymbol{\nabla}\mathbf{s}-\boldsymbol{\epsilon}\cdot\boldsymbol{\phi},\boldsymbol{\nabla}\boldsymbol{\phi}\,) = \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s}-\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}):\boldsymbol{\Gamma}:(\boldsymbol{\nabla}\mathbf{s}-\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}) + \tfrac{1}{2}\,(\boldsymbol{\nabla}\boldsymbol{\phi}):\boldsymbol{\Gamma}_c:(\boldsymbol{\nabla}\boldsymbol{\phi}). \tag{4.115}$$

[5] Note that $[\rho] = \mathrm{kg\,m^{-3}}$ and that $[\mathbf{j}] = \mathrm{kg\,m^{-1}}$.
[6] Note that $[\boldsymbol{\Gamma}] = \mathrm{kg\,m^{-1}\,s^{-2}} = \mathrm{Pa}$ and that $[\boldsymbol{\Gamma}_c] = \mathrm{kg\,m\,s^{-2}} = \mathrm{N}$.
[7] See Pujol (2009) for a discussion of the absence of terms involving combinations of $\boldsymbol{\nabla}\mathbf{s}-\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}$ and $\boldsymbol{\nabla}\boldsymbol{\phi}$.

Taking a variational approach, details of which may be found in appendix G.11.5, we may now define the Lagrangian density

$$
\begin{aligned}
L(\partial_t\mathbf{s}, \boldsymbol{\nabla}\mathbf{s}, \phi, \partial_t\phi, \boldsymbol{\nabla}\phi) &= \tfrac{1}{2}\,\rho\,\left[\partial_t\mathbf{s}\cdot\partial_t\mathbf{s} + \tfrac{1}{2}\,\partial_t\boldsymbol{\phi}\cdot\mathbf{j}\cdot\partial_t\boldsymbol{\phi} - U(\,\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\epsilon}\cdot\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\phi}\,)\right] \\
&= \tfrac{1}{2}\,\rho\,\partial_t\mathbf{s}\cdot\partial_t\mathbf{s} + \tfrac{1}{2}\,\partial_t\boldsymbol{\phi}\cdot\mathbf{j}\cdot\partial_t\boldsymbol{\phi} \\
&\quad - \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\epsilon}\cdot\boldsymbol{\phi}):\boldsymbol{\Gamma}:(\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\epsilon}\cdot\boldsymbol{\phi}) - \tfrac{1}{2}\,\boldsymbol{\nabla}\boldsymbol{\phi}:\boldsymbol{\Gamma}_c:\boldsymbol{\nabla}\boldsymbol{\phi}.
\end{aligned} \tag{4.116}
$$

This Lagrangian density is a linearized version of the general three-form representation (2.345). The associated action is

$$
I = \iint_V L(\partial_t\mathbf{s}, \boldsymbol{\nabla}\mathbf{s}, \boldsymbol{\phi}, \partial_t\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\phi})\,dV\,dt, \tag{4.117}
$$

where V denotes the volume occupied by the continuum. Stationarity of the action, $\delta I = 0$, leads to the Euler–Lagrange equations[8]

$$
\partial_t\left(\frac{\partial L}{\partial\partial_t\mathbf{s}}\right) + \boldsymbol{\nabla}\cdot\left(\frac{\partial L}{\partial\boldsymbol{\nabla}\mathbf{s}}\right) = \mathbf{0}, \tag{4.118}
$$

$$
\partial_t\left(\frac{\partial L}{\partial\partial_t\boldsymbol{\phi}}\right) + \boldsymbol{\nabla}\cdot\left(\frac{\partial L}{\partial\boldsymbol{\nabla}\boldsymbol{\phi}}\right) - \frac{\partial L}{\partial\boldsymbol{\phi}} = \mathbf{0}, \tag{4.119}
$$

subject to the boundary conditions

$$
\hat{\mathbf{n}}\cdot\left(\frac{\partial L}{\partial\boldsymbol{\nabla}\mathbf{s}}\right) = \mathbf{0}, \tag{4.120}
$$

$$
\hat{\mathbf{n}}\cdot\left(\frac{\partial L}{\partial\boldsymbol{\nabla}\boldsymbol{\phi}}\right) = \mathbf{0}. \tag{4.121}
$$

Written out in full, these are conservation of linear momentum

$$
\rho\,\partial_t^2\mathbf{s} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}, \tag{4.122}
$$

and conservation of angular momentum

$$
\mathbf{j}\cdot\partial_t^2\boldsymbol{\phi} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_c + \boldsymbol{\epsilon}:\boldsymbol{\sigma}, \tag{4.123}
$$

where

$$
\boldsymbol{\sigma} = \frac{\partial L}{\partial\boldsymbol{\nabla}\mathbf{s}} = \boldsymbol{\Gamma}:(\boldsymbol{\nabla}\mathbf{s} - \boldsymbol{\epsilon}\cdot\boldsymbol{\phi}), \tag{4.124}
$$

$$
\boldsymbol{\sigma}_c = \frac{\partial L}{\partial\boldsymbol{\nabla}\boldsymbol{\phi}} = \boldsymbol{\Gamma}_c:\boldsymbol{\nabla}\boldsymbol{\phi}, \tag{4.125}
$$

subject to the boundary conditions

$$
\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} = \mathbf{0}, \tag{4.126}
$$

$$
\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}_c = \mathbf{0}. \tag{4.127}
$$

Note that, in general, the stress tensor, $\boldsymbol{\sigma}$, and couple stress tensor, $\boldsymbol{\sigma}_c$, are not symmetric. Note also that $[\boldsymbol{\sigma}] = \mathrm{kg\,m^{-1}\,s^{-2}} = \mathrm{Pa}$ and that $[\boldsymbol{\sigma}_c] = \mathrm{kg\,s^{-2}} = \mathrm{Pa\,m} = \mathrm{N/m}$.

[8]To be consistent with the convention used in the literature on this subject, in this section only we use the convention that the divergence is taken on the *first* index of a tensor, for example, $(\boldsymbol{\nabla}\cdot\mathbf{T}\,)_i = \partial_j T_{ji}$.

When the medium is isotropic, we have (Hodge, 1961)

$$\Gamma_{ijk\ell} = (\kappa - \tfrac{2}{3}\,\mu)\,\delta_{ij}\,\delta_{k\ell} + \mu\,(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) + \nu\,(\delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}), \tag{4.128}$$

$$\Gamma^c_{ijk\ell} = \lambda_c\,\delta_{ij}\,\delta_{k\ell} + \mu_c\,(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) + \nu_c\,(\delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}). \tag{4.129}$$

In the absence of rotation, an isotropic medium is governed by two elastic parameters, the bulk and shear moduli κ and μ, as discussed in section 4.3. In the case of rotation, we gain one extra parameter for the elastic tensor $\Gamma_{ijk\ell}$, namely, the modulus ν, which breaks the symmetries in i and j and in k and ℓ. We also gain three additional parameters from the couple elastic tensor $\Gamma^c_{ijk\ell}$, namely, κ_c, μ_c, and ν_c. Thus, the stress is

$$\sigma_{ij} = (\kappa - \tfrac{2}{3}\,\mu)\,\delta_{ij}\,\varepsilon_{kk} + 2\mu\,\varepsilon_{ij} + 2\nu\,(\Phi_{ij} - \omega_{ij}), \tag{4.130}$$

and the couple stress is

$$\sigma^c_{ij} = \lambda_c\,\delta_{ij}\,\nabla_k\phi_k + \mu_c\,(\nabla_i\phi_j + \nabla_j\phi_i) + \nu_c\;(\nabla_i\phi_j - \nabla_j\phi_i). \tag{4.131}$$

Using the isotropic constitutive relationships, conservation of linear momentum (4.122) becomes

$$\rho\,\partial_t^2\mathbf{s} = \boldsymbol{\nabla}\cdot\left[(\kappa - \tfrac{2}{3}\,\mu)\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\mathbf{I} + 2\mu\,\boldsymbol{\varepsilon}\right] + \boxed{\boldsymbol{\nabla}\times\left[2\nu\;\left(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}\right)\right]}, \tag{4.132}$$

and conservation of angular momentum (4.123) becomes

$$\mathbf{j}\cdot\partial_t^2\boldsymbol{\phi} = \boldsymbol{\nabla}\cdot\left[\lambda_c\,\boldsymbol{\nabla}\cdot\boldsymbol{\phi}\,\mathbf{I} + (\mu_c+\nu_c)\boldsymbol{\nabla}\boldsymbol{\phi} + (\mu_c-\nu_c)(\boldsymbol{\nabla}\boldsymbol{\phi})^t\right] - \boxed{4\nu\;\left(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}\right)}. \tag{4.133}$$

Conservation of linear momentum is coupled to conservation of angular momentum by the boxed terms in (4.132) and (4.133) and is entirely controlled by the elastic parameter ν.

4.9.1 Alternative Approach

In section 3.12.2, we discussed the conservation laws of continuum mechanics with spin, which were also summarized in table 2.3. Using those results, we may obtain the linearized equations of motion (4.122) and (4.123) by substituting $v^i = \partial_t s^i$ in (3.378) and $\theta^i = j^{ij}\,\partial_t\phi_j$ in (3.380) and retaining only the linear terms:

$$\rho\,\partial_t^2 s_i = \nabla_j\sigma_i{}^j, \tag{4.134}$$

$$j^{ij}\,\partial_t^2\phi_j = \nabla_j\sigma_c^{ij} + \varepsilon^{ijk}\,\sigma_{jk}. \tag{4.135}$$

4.9.2 Homogeneous Medium

In a homogeneous medium with isotropic inertia density, we have (see, e.g., Kulesh, 2009; Grekova et al., 2009, for a detailed analysis)

$$\rho\,\partial_t^2\mathbf{s} = (\kappa + \tfrac{4}{3}\,\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{s}) - \mu\,\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s}) + 2\nu\,\boldsymbol{\nabla}\times\left(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}\right), \tag{4.136}$$

$$j\,\partial_t^2\boldsymbol{\phi} = (\lambda_c + 2\,\mu_c)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\phi}) - (\mu_c+\nu_c)\,\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\phi}) - 4\nu\;\left(\boldsymbol{\phi} - \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{s}\right). \tag{4.137}$$

Upon taking the divergence of equations (4.136) and (4.137), we find

$$\rho\,\partial_t^2\Pi = (\kappa + \tfrac{4}{3}\,\mu)\,\nabla^2\Pi, \tag{4.138}$$

$$j\,\partial_t^2\pi = (\lambda_c + 2\mu_c)\,\nabla^2\pi - j\omega_0^2\,\pi, \tag{4.139}$$

where $\Pi = \nabla \cdot \mathbf{s}$ and $\pi = \nabla \cdot \phi$, and where

$$\omega_0^2 = 4\,\nu/j. \tag{4.140}$$

The plane-wave solutions to these equations are written as

$$\Pi = a\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega\,t)], \tag{4.141}$$

$$\pi = b\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega\,t)], \tag{4.142}$$

where $\mathbf{k}$ is the wavevector, ω is the angular frequency, and a and b are complex constants. Upon substitution of the plane waves in equations (4.138) and (4.139) we find the requirements

$$\rho\,\omega^2 = (\kappa + \tfrac{4}{3}\,\mu)\,k^2, \tag{4.143}$$

$$j\,\omega^2 = (\lambda_c + 2\,\mu_c)\,k^2 + j\,\omega_0^2\,, \tag{4.144}$$

where $\|\mathbf{k}\| = k = \omega/c$ denotes the wavenumber and c the speed of a plane wave. Thus, we find the wavespeed requirements

$$\alpha^2 = \frac{\kappa + \frac{4}{3}\,\mu}{\rho}, \tag{4.145}$$

where α is the speed associated with a longitudinal displacement, and

$$\alpha_c^2 = \frac{\kappa_c + \frac{4}{3}\,\mu_c}{j(1 - \omega_0^2/\omega^2)}, \tag{4.146}$$

where α_c is the speed associated with a longitudinal microrotation. Note that α_c depends on the frequency and that the wave is, therefore, dispersive and exists whenever $\omega > \omega_0$ since $\kappa_c + \frac{4}{3}\,\mu_c > 0$.

Upon taking the curl of equations (4.136) and (4.137) we find that

$$\rho\,\partial_t^2\boldsymbol{\Psi} = (\mu + \nu)\,\nabla^2\boldsymbol{\Psi} + \tfrac{1}{2}\,j\,\omega_0^2\,\nabla\times\boldsymbol{\psi}, \tag{4.147}$$

$$j\,\partial_t^2\boldsymbol{\psi} = (\mu_c + \nu_c)\,\nabla^2\boldsymbol{\psi} - j\,\omega_0^2\,\boldsymbol{\psi} + \tfrac{1}{2}\,j\,\omega_0^2\,\nabla\times\boldsymbol{\Psi}, \tag{4.148}$$

where $\boldsymbol{\Psi} = \nabla\times\mathbf{s}$ and $\boldsymbol{\psi} = \nabla\times\phi$. Notice that these equations are coupled. Let us seek plane-wave solutions of the form

$$\boldsymbol{\Psi} = \mathbf{A}\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega\,t)], \tag{4.149}$$

$$\boldsymbol{\psi} = \mathbf{B}\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega\,t)]. \tag{4.150}$$

Upon substitution of the plane waves in equations (4.147) and (4.148) we find the requirements

$$-\rho\,\omega^2\,\mathbf{A} = -(\mu + \nu)\,k^2\,\mathbf{A} + \tfrac{1}{2}\,j\,\omega_0^2\,i\,\mathbf{k}\times\mathbf{B}, \tag{4.151}$$

$$-j\,\omega^2\,\mathbf{B} = -(\mu_c + \nu_c)\,k^2\,\mathbf{B} - j\,\omega_0^2\,\mathbf{B} + \tfrac{1}{2}\,j\,\omega_0^2\,i\,\mathbf{k}\times\mathbf{A}. \tag{4.152}$$

Eliminating $\mathbf{A}$ and $\mathbf{B}$ yields a quadratic equation in c^2 of the form

$$\begin{aligned} j\rho\left(1 - \omega_0^2/\omega^2\right)\left(c^2\right)^2 + \left[\rho\,(\mu_c + \nu_c) + j\,\left(1 - \omega_0^2/\omega^2\right)(\mu + \nu) + j\,(\omega_0/\omega)^2\,\nu\right] c^2 \\ + (\mu + \nu)(\mu_c + \nu_c) = 0. \end{aligned} \tag{4.153}$$

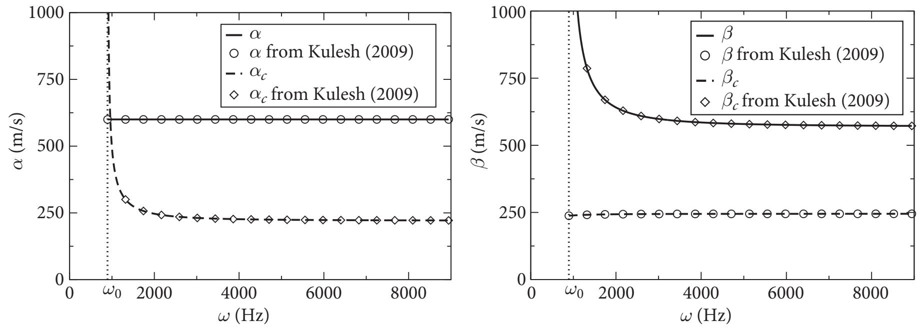

Figure 4.6: Longitudinal (*left*) and transverse (*right*) wavespeeds in a micropolar medium with the properties tabulated in table 4.2, using equations (4.145), (4.146), and (4.154), compared to Kulesh (2009). *Courtesy of Chris Morency.*

SYMBOL	UNIT	VALUE
ρ	kg/m^3	10^5
κ	GPa	22.667
μ	GPa	4
ν	GPa	2
j	kg/m	10^4
λ_c	N	10^8
μ_c	N	$1.936 \cdot 10^8$
ν_c	N	$3.0464 \cdot 10^9$

Table 4.2: Material parameters for figure 4.6.

Equation (4.153) has two roots

$$\begin{aligned}\beta_\pm = &-\frac{\rho\,(\mu_c+\nu_c)+j\left(1-\omega_0^2/\omega^2\right)(\mu+\nu)+j\,(\omega_0/\omega)^2\,\nu}{2\,j\rho\left(1-\omega_0^2/\omega^2\right)}\\ &\pm\frac{\sqrt{\left[\rho\,(\mu_c+\nu_c)+j\left(1-\omega_0^2/\omega^2\right)(\mu+\nu)+j\,(\omega_0/\omega)^2\,\nu\right]^2-4\,j\rho\left(1-\omega_0^2/\omega^2\right)(\mu+\nu)(\mu_c+\nu_c)}}{2\,j\rho\left(1-\omega_0^2/\omega^2\right)},\end{aligned} \tag{4.154}$$

where $\beta \equiv \beta_+$ corresponds to the transverse displacement wave, whereas $\beta_c \equiv \beta_-$ is associated with the transverse microrotation. Note that both waves are dispersive and exist whenever $\omega > \omega_0$. Figure 4.6 shows α, α_c, β, and β_c using the material properties listed in table 4.2. These results are in good agreement with Kulesh (2009).

Micropolar Parameter Values

The mass density has units of $[\rho] = \mathrm{kg\,m^{-3}}$. The inertia density has units of $[j] = \mathrm{kg\,m^{-1}}$, and since inertia involves mass times a squared distance, one expects that $j \sim \rho R^2$, where R is the characteristic spin length scale for the problem. This could be the characteristic size of the elements of the continuum that spin together, presumably something on the order of a wavelength λ. From $\boldsymbol{\phi} \sim \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{s}$ one would expect $\|\boldsymbol{\phi}\| \sim \pi\,\|\mathbf{s}\|/\lambda$ in radians. The elastic parameters have units of Pa, for example, $[\mu] = \mathrm{kg\,m^{-1}\,s^{-2}}$, and the couple stress elastic parameters have units of N, for example, $[\mu_c] = \mathrm{kg\,m\,s^{-2}}$, and one expects that $\mu_c \sim \mu R^2$, so that the squared wavespeeds μ/ρ and μ_c/j are comparable. The coupling terms involve factors of $\gamma^2 = 2\nu/\rho$ (linear momentum) and $\omega_0^2 = 4\nu/j \sim 2\gamma^2/R^2$ (angular momentum), or $T_0 = 2\pi/\omega_0 \sim \sqrt{2}\,\pi R/\gamma$.

4.9.3 Couple-Moment Tensor

Excitation of displacement and rotation fields may be accommodated by introducing both a stress glut and a *couple-stress glut*, as discussed in section 3.12.3. To do so, we rewrite the equations of motion (4.122) and (4.123) in the form

$$\rho\,\partial_t^2\mathbf{s} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \mathbf{f}, \tag{4.155}$$

$$\mathbf{j}\cdot\partial_t^2\boldsymbol{\phi} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_c + \boldsymbol{\epsilon}:\boldsymbol{\sigma} + \mathbf{f}_c, \tag{4.156}$$

where the forces are defined in terms of the stress and couple-stress gluts:

$$\mathbf{f} = -\,\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}, \tag{4.157}$$

$$\mathbf{f}_c = -\,\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}_c + \boldsymbol{\epsilon}:\boldsymbol{\gamma}. \tag{4.158}$$

On an ideal fault, as discussed in section 3.12.3, the stress and couple-stress gluts may be expressed in the forms (3.400) and (3.401), namely,

$$\boldsymbol{\gamma} = \mathbf{m}\,\delta_\Sigma, \tag{4.159}$$

$$\boldsymbol{\gamma}_c = \mathbf{m}_c\,\delta_\Sigma, \tag{4.160}$$

where the moment-density tensor is determined in terms of the *fault slip* $\Delta\mathbf{s}$ by

$$\mathbf{m} = \boldsymbol{\Gamma}:\hat{\mathbf{n}}\otimes\Delta\mathbf{s}, \tag{4.161}$$

and the couple-moment-density tensor is determined in terms of the *fault-plane rotation* $\boldsymbol{\Psi}$ by

$$\mathbf{m}_c = \boldsymbol{\Gamma}_c:\hat{\mathbf{n}}\,\boldsymbol{\Psi}. \tag{4.162}$$

In an isotropic medium, we have, using expressions (4.128) and (4.129),

$$\mathbf{m} = (\kappa - \tfrac{2}{3}\mu)\,\hat{\mathbf{n}}\cdot\Delta\mathbf{s}\,\mathbf{I} + \mu\,(\hat{\mathbf{n}}\otimes\Delta\mathbf{s} + \Delta\mathbf{s}\otimes\hat{\mathbf{n}}) + \nu\,(\hat{\mathbf{n}}\otimes\Delta\mathbf{s} - \Delta\mathbf{s}\otimes\hat{\mathbf{n}}), \tag{4.163}$$

$$\mathbf{m}_c = \lambda_c\,\hat{\mathbf{n}}\cdot\boldsymbol{\Psi}\,\mathbf{I} + \mu_c\,(\hat{\mathbf{n}}\otimes\boldsymbol{\Psi} + \boldsymbol{\Psi}\otimes\hat{\mathbf{n}}) + \nu_c\,(\hat{\mathbf{n}}\otimes\boldsymbol{\Psi} - \boldsymbol{\Psi}\otimes\hat{\mathbf{n}}). \tag{4.164}$$

These expressions tell us how the various Volterra cut-and-weld protocols, as illustrated in figure 3.4, excite displacement and rotation fields. They generalize expression (4.61), which captures slip on an ideal fault, that is, just edge and screw dislocations. Notably, neither the moment-density tensor nor the couple-moment-density tensor is symmetric. One can imagine designing experiments involving sources specifically designed to excite rotation waves, for example by twisting small elements of the medium.

In the point-source approximation, the *moment tensor* and *couple-moment tensor* are defined by

$$\mathbf{M} = \int_{\Sigma} \mathbf{m}\, d^2x, \tag{4.165}$$

$$\mathbf{M}_c = \int_{\Sigma} \mathbf{m}_c\, d^2x, \tag{4.166}$$

where Σ is the area of the fault. Generally, the moment tensor and couple-moment tensor are not symmetric.

In the point source approximation, the forces (4.157) and (4.158) become

$$\mathbf{f} = -\mathbf{M} \cdot \boldsymbol{\nabla}\delta(\mathbf{x} - \mathbf{x}_s), \tag{4.167}$$

$$\mathbf{f}_c = -\mathbf{M}_c \cdot \boldsymbol{\nabla}\delta(\mathbf{x} - \mathbf{x}_s) - \boldsymbol{\epsilon} : \mathbf{M}\,\delta(\mathbf{x} - \mathbf{x}_s). \tag{4.168}$$

These indigenous point forces excite displacement and rotation fields in an elastic micropolar medium.

4.10 Equilibrium State

Prior to an earthquake, the Earth is in a state of mechanical equilibrium captured by the referential manifold, as discussed in section 1.1. In this state, the Earth rotates with an angular velocity $\boldsymbol{\Omega}$ about an origin that coincides with the center of mass.

The equilibrium gravitational potential Φ is determined in terms of the equilibrium mass density distribution ρ by Poisson's equation:

$$\nabla^2\Phi = 4\,\pi\, G\,\rho, \tag{4.169}$$

subject to the boundary conditions

$$[\Phi]_-^+ = 0, \qquad [\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\Phi]_-^+ = 0, \tag{4.170}$$

where $\hat{\mathbf{n}}$ denotes the unit outward normal on any discontinuity in density, such as the core-mantle boundary or Earth's surface, and $[\,\cdot\,]_-^+$ denotes the jump in the enclosed quantity across a discontinuity. In outer space, the gravity potential satisfies the *Laplace equation*

$$\nabla^2\Phi = 0. \tag{4.171}$$

The solution to this boundary value problem is[9]

$$\Phi(\mathbf{x}) = -\,G \int_V \frac{\rho(\mathbf{x}')}{\|\mathbf{x} - \mathbf{x}'\|}\, dV', \tag{4.172}$$

where V denotes the volume of the equilibrium Earth model. The associated equilibrium gravitational acceleration is

$$-\,\boldsymbol{\nabla}\Phi = -G \int_V \frac{\rho(\mathbf{x}')\,(\mathbf{x} - \mathbf{x}')}{\|\mathbf{x} - \mathbf{x}'\|^3}\, dV'. \tag{4.173}$$

For future reference, we note that

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\Phi = G \int_V \rho(\mathbf{x}')\,\boldsymbol{\Pi}(\mathbf{x}, \mathbf{x}')\, dV', \tag{4.174}$$

[9]We use $\mathbf{x}$ to denote the Cartesian "position vector" in the Cartesian equilibrium frame.