Lista 4 - Matemática Discreta

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Observação: Esta lista é para uma consolidação dos assuntos trabalhados em aula, ou seja, não é necessário o envio da mesma para o professor.

Nos Exercícios 1 a 10, use a indução matemática para demonstrar que os resultados são válidos para qualquer inteiro positivo n.

Exercício 1. $2 + 6 + 10 + \ldots + (4n - 2) = 2n^2$

Exercício 2. $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

Exercício 3. $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$

Exercício 4. $1 + 3 + 6 + \ldots + \dfrac{n(n + 1)}{2} = \dfrac{n(n + 1)(n + 2)}{6}$

Exercício 5. $4 + 10 + 16 + \ldots + (6n - 2) = n(3n + 1)$

Exercício 6. $5 + 10 + 15 + \ldots + 5n = \dfrac{5n(n + 1)}{2}$

Exercício 7. $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

Exercício 8. $1^3 + 2^3 + \ldots + n^3 = \dfrac{n^2(n + 1)^2}{4}$

Exercício 9. $1 + a + a^2 + \ldots + a^{n-1} = \dfrac{a^n - 1}{a - 1}$ para $a \neq 0$ e $a \neq 1$

Exercício 10. $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$

Gabarito:

1. $P(1) : 4.1 - 2 = 2(1)^2$ ou $2 = 2$, verdadeiro
Assuma $P(k) : 2 + 6 + 10 + \ldots + (4k - 2) = 2k^2$
Mostre $P(k + 1) : 2 + 6 + 10 + \ldots + [4(k + 1) - 2] \stackrel{?}{=} 2(k + 1)^2$
$2 + 6 + 10 + \ldots + [4(k + 1) - 2]$

$= 2 + 6 + 10 + \ldots + (4k-2) + [4(k+1)-2]$
$= 2k^2 + 4(k+1) - 2$
$= 2k^2 + 4k + 2$
$= 2(k^2 + 2k + 1)$
$= 2(k+1)^2$
2. $P(1) : 2.1 = 1(1+1)$, verdadeiro
Assuma $P(k) : 2 + 4 + 6 + \ldots + 2k = k(k+1)$
Mostre $P(k+1) : 2 + 4 + 6 + \ldots + 2(k+1) \stackrel{?}{=} (k+1)[(k+1)+1]$
$2 + 4 + 6 + \ldots + +2(k+1)$
$= 2 + 4 + 6 + \ldots + 2k + 2(k+1)$
$= k(k+1) + 2(k+1)$
$= (k+1)(k+2)$
$= (k+1)[(k+1)+1]$
3. $P(1) : 1 = 1.(2.1 - 1)$, verdadeiro
Assuma $P(k) : 1 + 5 + 9 + \ldots + (4k-3) = k(2k-1)$
Mostre $P(k+1) : 1 + 5 + 9 + \ldots + [4(k+1)-3] \stackrel{?}{=} (k+1)[2(k+1)-1]$
$1 + 5 + 9 + \ldots + [4(k+1)-3]$
$= 1 + 5 + 9 + \ldots + (4k-3) + [4(k+1)-3]$
$= k(2k-1) + 4(k+1) - 3$
$= 2k^2 - k + 4k + 1$
$= 2k^2 + 3k + 1 = 2k^2 + 4k + 2 - (k+1)$
$= 2(k^2 + 2k + 1) - (k+1) = 2(k+1)^2 - (k+1)$
$= (k+1)[2(k+1)-1]$
4. $P(1) : 1 = \dfrac{1(1+1)(1+2)}{6}$, verdadeiro
Assuma $P(k) : 1 + 3 + 6 + \ldots + \dfrac{k(k+1)}{2} = \dfrac{k(k+1)(k+2)}{6}$
Mostre $P(k+1) : 1 + 3 + 6 + \ldots + \dfrac{(k+1)[(k+1)+1]}{2} \stackrel{?}{=} \dfrac{(k+1)[(k+1)+1][(k+1)+2]}{6}$
$1 + 3 + 6 + \ldots + \dfrac{(k+1)[(k+1)+1]}{2}$
$= 1 + 3 + 6 + \ldots + \dfrac{k(k+1)}{2} + \dfrac{(k+1)[(k+1)+1]}{2}$
$= \dfrac{k(k+1)(k+2)}{6} + \dfrac{(k+1)[(k+1)+1]}{2}$
$= \dfrac{k(k+1)(k+2)}{6} + \dfrac{3(k+1)(k+2)}{6}$
$= \dfrac{(k+1)(k+2)(k+3)}{6}$
5. $P(1) : 4 = 1(3.1 + 1)$, verdadeiro
Assuma $P(k) : 4 + 10 + 16 + \ldots + (6k-2) = k(3k+1)$
Mostre $P(k+1) : 4 + 10 + 16 + \ldots + [6(k+1)-2] \stackrel{?}{=} (k+1)[3(k+1)+1]$
$4 + 10 + 16 + \ldots + [6(k+1)-2]$
$= 4 + 10 + 16 + \ldots + (6k-2) + [6(k+1)-2]$
$= k(3k+1) + [6(k+1)-2]$

$= 3k^2 + k - 2 + 6k + 6$
$= 3k^2 + 3k + 4k + 4$
$= (k+1)(3k+4)$
$= (k+1)(3k+3+1)$
$= (k+1)[3(k+1)+1]$

6. $P(1) : 5 = \dfrac{5.1(1+1)}{2}$, verdadeiro

Assuma $P(k) : 5 + 10 + 15 + \ldots + 5k = \dfrac{5k(k+1)}{2}$

Mostre $P(k+1) : 5 + 10 + 15 + \ldots + 5(k+1) \stackrel{?}{=} \dfrac{5(k+1)[(k+1)+1]}{2}$

$5 + 10 + 15 + \ldots + 5(k+1)$
$= 5 + 10 + 15 + \ldots + 5k + 5(k+1)$
$= \dfrac{5k(k+1)}{2} + 5(k+1)$
$= \dfrac{5k(k+1)}{2} + \dfrac{10(k+1)}{2}$
$= \dfrac{(5k+10)(k+1)}{2}$
$= \dfrac{5(k+1)(k+2)}{2} = \dfrac{5(k+1)[(k+1)+1]}{2}$

7. $P(1) : 1^2 = \dfrac{1.(1+1).(2.1+1)}{6}$, verdadeiro

Assuma $P(k) : 1^2 + 2^2 + \ldots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$

Mostre $P(k+1) : 1^2 + 2^2 + \ldots + (k+1)^2 \stackrel{?}{=} \dfrac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$

$1^2 + 2^2 + \ldots + (k+1)^2$
$= 1^2 + 2^2 + \ldots + k^2 + (k+1)^2$
$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$
$= \dfrac{k(k+1)(2k+1)}{6} + \dfrac{6(k+1)^2}{6}$
$= \dfrac{k(k+1)(2k+1) + (k+1)(6k+6)}{6} = \dfrac{(k+1)[2k^2 + k + 6k + 6]}{6} = \dfrac{(k+1)[2k^2 + 6k + 4 + (k+2)]}{6}$
$= \dfrac{(k+1)[2(k^2 + 3k + 2) + (k+2)]}{6} = \dfrac{(k+1)[2(k+1)(k+2) + (k+2)]}{6}$
$= \dfrac{(k+1)[(k+2)(2(k+1)+1)]}{6} = \dfrac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$

8. $P(1) : 1^3 = \dfrac{1^2(1+1)^2}{4}$, verdadeiro

Assuma $P(k) : 1^3 + 2^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$

Mostre $P(k+1) : 1^3 + 2^3 + \ldots + (k+1)^3 \stackrel{?}{=} \dfrac{(k+1)^2[(k+1)+1]^2}{4}$

$1^3 + 2^3 + \ldots + (k+1)^3$
$= 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$
$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} = \frac{(k+1)^2(k+2)^2}{4}$$
$$= \frac{(k+1)[(k+1)+1]^2}{4}$$

9. $P(1) : 1 = \dfrac{a^1 - 1}{a - 1}$, verdadeiro

Assuma $P(k) : 1 + a + a^2 + \ldots + a^{k-1} = \dfrac{a^k - 1}{a - 1}$

Mostre $P(k+1) : 1 + a + a^2 + \ldots + a^k \stackrel{?}{=} \dfrac{a^{k+1} - 1}{a - 1}$

$1 + a + a^2 + \ldots + a^k$
$= 1 + a + a^2 + \ldots + a^{k-1} + a^k$
$$= \frac{a^k - 1}{a - 1} + a^k = \frac{a^k - 1}{a - 1} + \frac{a^k(a-1)}{a - 1} = \frac{a^k - 1 + a^{k+1} - a^k}{a - 1}$$
$$= \frac{a^{k+1} - 1}{a - 1}$$

10. $P(1) : \dfrac{1}{1.2} = \dfrac{1}{1+1}$, verdadeiro

Assuma $P(k) : \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{k(k+1)} = \dfrac{k}{k+1}$

Mostre $P(k+1) : \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{(k+1)[(k+1)+1]} \stackrel{?}{=} \dfrac{(k+1)}{(k+1)+1}$

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{(k+1)[(k+1)+1]}$$
$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)[(k+1)+1]}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$
$$= \frac{(k+1)}{[(k+1)+1]}$$