Atma Ram Sanatan Dharma College

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Linear Algebra Assignment

Submitted To –

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ASSIGNMENT

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Q.1 Consider the linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^4$ given by $L(v) = Av$

(i) find $\ker(L)$ & range(L)

(ii) verify dim. theorem

$$A = \begin{bmatrix} 4 & -2 & 8 \\ 7 & 1 & 5 \\ -2 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix}$$

Sol: Consider B = reduced row echelon form of A, i.e. row reducing A

$$\Rightarrow \begin{bmatrix} 4 & -2 & 8 \\ 7 & 1 & 5 \\ -2 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix}.$$

$R_1 \to R_1/4$, $\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 7 & 1 & 5 \\ -2 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix}$

$R_2 \to R_2 - 7R_1$
$R_3 \to R_3 + 2R_1$
$R_4 \to R_4 - 3R_1$
$\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 9/2 & -9 \\ 0 & -2 & 4 \\ 0 & -1/2 & 1 \end{bmatrix}$

$R_2 \to \frac{2}{9}R_2$ $\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \\ 0 & -1/2 & 1 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$
$R_4 \to R_4 + \frac{1}{2}R_2$
$\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \to R_1 + 1/2 R_2$ $\Rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = B$

From B, we see that x_3 is independent, let $x_3 = c \in \mathbb{R}$

we get following for $BX = 0$,

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases}$$

Now, as $x_3 = c$, $c \in R$

$x_1 = -c$

$x_2 = +2c$

i.e. $x = c \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ for some $c \in R$

$$\therefore \operatorname{Ker}(L) = \left\{ c \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \quad (c \in R)$$

$$\Rightarrow \operatorname{Ker}(L) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{—— (1)}$$

Now, range(L) is spanned by columns of A having non-zero pivot entries in B and the first 2 columns of B qualify,

$$\therefore \operatorname{range}(L) = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 7 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\} \quad \text{—— (2)}$$

Now, according to dimention theorem,

$$\dim(\operatorname{Ker}(L)) + \dim(\operatorname{range}(L)) = \dim(V)$$

for a linear transformation $L : V \to W$

$\therefore$ RHS : Here V is R^3

$\Rightarrow \dim(R^3) = 3$

LHS : $\dim(\operatorname{Ker}(L)) = 1$

$+ \dim(\operatorname{range}(L)) = 2$

$$\therefore \dim(\operatorname{Ker}(L)) + \dim(\operatorname{range}(L)) = 3$$

As, LHS = RHS, dimention theorem is verified —— (3)

Q.2 Consider $L: P_2 \rightarrow R^2$ given by $L(P) = [P(1), P'(1)]$

verify dim theorem.

Sol: for $L: P_2 \rightarrow R^2$, consider some p in P_2 as,

$$p = ax^2 + bx + c,$$

Now, $L(P) = \begin{bmatrix} a+b+c \\ 2a+b \end{bmatrix}$

Consider the standard bases for P_2 and R^2 as B and C where, $B = (x^2, x, 1)$; $C = (e_1, e_2)$

Observing effect of L on standard basis B,

$$L(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow L(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow L(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, transformation matrix A_{BC} is given by

$$A_{BC} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Row reducing A_{BC} and let $\beta = rref(A_{BC})$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$R_2 \rightarrow -R_2 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2 \Rightarrow \begin{array}{c} \begin{array}{ccc} x^2 & x & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} = \beta$$

In β, third column is independent consider constant $\alpha \in R$

$\therefore$ for $Bx = 0$,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We get, $\begin{cases} x^2 - 1 = 0 \\ x + 2 = 0 \end{cases}$ Now as constant $= \alpha$, $\alpha \in R$

$$x^2 = \alpha$$
$$x = -2\alpha$$

i.e. $x = \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \middle| \alpha \in R \right\}$

$\therefore \ker(L) = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim(\ker(L)) = 1$

from β, x^2 and x columns has pivot entries.

$\therefore \text{range}(L) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim(\text{range}(L)) = 2$

Now, according to dimension theorem,

$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V)$

for linear transformation $L : V \to W$

$\therefore$ RHS, Here V is P_2

$\therefore \dim(P_2) = \underline{\underline{3}}$ $\quad (\because \dim(P_n) = n+1)$

LHS, $\dim(\ker(L)) = 1$

$+ \dim(\text{range}(L)) = 2$

$\Rightarrow \therefore \dim(\ker(L)) + \dim(\text{range}(L)) = \underline{\underline{3}}$

As, LHS = RHS, the dimension theorem is verified! —— 3

Q.3 Consider $L : M_{22} \to M_{32}$ given by

$$L\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} 0 & -a_{12} \\ -a_{21} & 0 \\ 0 & 0 \end{bmatrix}$$

Find $\ker(L)$ and $\text{range}(L)$.

Sol: Consider the standard basis of $M_{22} = B$ and for $M_{32} = C$.

$$B = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Observing effect of L on the elements of B,

$\rightarrow L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \rightarrow L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}$ $\rightarrow L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Therefore, transition matrix A_{BC} is given by,

$$A_{BC} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reducing A_{BC} and let $B = \text{rref}(A_{BC})$,

$R_2 \rightarrow -R_2$
$R_3 \rightarrow -R_3$ $\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \beta$

In β, first and fourth columns are independent,
let $a_{11}, a_{22} \in R$
for $BX = 0$,

$B \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we get $a_{12} = 0, a_{21} = 0,$

i.e. $X = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in R \right\}$ $\quad a = a_{11}, a_{22} = b$

$\therefore \ Ker(L) = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| a, b \in R \right\}$

$\Rightarrow \ Ker(L) = span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ — ①

From B, second and third columns have pivot entries,

$\therefore \ range(L) = span \left\{ \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ — ②

Q.4 Verify that $P_3 \cong R^4$ as $L: P_3 \to R^4$
given by $L(P) = [P(-1), P(0), P(1), \{P(2)]$, $P \in P_3$

Sol: In order to show $P_3 \cong R^4$, we must show that L is an isomorphism from P_3 to R_4.
To show L is an isomorporphism, we must show that,
→ L is a linear transformation
→ L is one-one
→ L is onto.

(I) <u>L is a linear transformation</u>;

Suppose L is a linear transformation, then the following properties must hold:
(1) $L(P_1 + P_2) = L(P_1) + L(P_2)$ for $P_1, P_2 \in P_3$
(2) $L(\alpha P) = \alpha L(P)$ for some $\alpha \in R, P \in P_3$

(II) <u>L is one-one</u>;

As L is one one iff $L(P_1) = L(P_2)$ implies $P_1 = P_2$ for $P_1, P_2 \in P_3$.
we must show the same.

(III) <u>L is onto</u>:

L is onto iff $dim(range(L)) = dim(W)$ for $L; V \to W$

Consider $L: P_3 \to R^4$ given by $L(P) = [P(-1), P(0), P(1), P(2)]$ for $P \in R_3$ & consider some p in P_3 where $P = ax^3 + bx^2 + cx + d$,

then $L(P) = \begin{bmatrix} -a+b-c+d \\ d \\ a+b+c+d \\ 8a+4b+2c+d \end{bmatrix}$

Let $B = (x^3, x^2, x, 1)$ and $C = (e_1, e_2, e_3, e_4)$ be the standard bases for P_3 and R^4 respectively.

Now, observing effect of L on each element of B,

$\rightarrow L(x^3) = [-1, 0, 1, 8]$ $\rightarrow L(x^2) = [1, 0, 1, 4]$

$\rightarrow L(x) = [-1, 0, 1, 2]$ $\rightarrow L(1) = [1, 1, 1, 1]$

Therefore, transformation matrix A_{BC} is given by,

$$A_{BC} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

Also, $|A_{BC}| = 12 \neq 0$

$\Rightarrow A_{BC}$ is invertible

$\Rightarrow A_{BC}^{-1}$ exists

$\Rightarrow L^{-1}$ is possible.

Now, (I) Proving L is a linear transformation:

If L is a linear tranformation,

(1) $L(P_1 + P_2) = L(P_1) + L(P_2)$

Consider $\left.\begin{array}{l} P_1 = a_1x^3 + b_1x^2 + c_1x + d_1 \\ P_2 = a_2x^3 + b_2x^2 + c_2x + d_2 \end{array}\right\} \in P_3$

LHS $L(P_1 + P_2) = \cancel{L(P_1) + L}$

$\Rightarrow L(a_1+a_2)x^3 + (b_1+b_2)^2 + (c_1+c_2)x + (d_1+d_2)$

$$\Rightarrow \begin{bmatrix} -(a_1+a_2) + (b_1+b_2) - (c_1-c_2) + (d_1+d_2) \\ (d_1+d_2) \\ (a_1+a_2) + (b_1+b_2) + (c_1+c_2) + (d_1+d_2) \\ 8(a_1+a_2) + 4(b_1+b_2) + 2(c_1+c_2) + (d_1+d_2) \end{bmatrix}$$

RHS $L(P_1) + L(P_2)$

$\Rightarrow L(a_1x^3 + b_1x^2 + c_1x + d_1) + L(a_2x^3 + b_2x^2 + c_2x + d_2)$

$$\Rightarrow \begin{bmatrix} -a_1 + b_1 - c_1 + d_1 \\ d_1 \\ a_1 + b_1 + c_1 + d_1 \\ 8a_1 + 4b_1 + 2c_1 + d_1 \end{bmatrix} + \begin{bmatrix} -a_2 + b_2 - c_2 + d_2 \\ d_2 \\ a_2 + b_2 + c_2 + d_2 \\ 8a_2 + 4b_2 + 2c_2 + d_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -(a_1+a_2) + (b_1+b_2) - (c_1+c_2) + (d_1+d_2) \\ (d_1+d_2) \\ (a_1+a_2) + (b_1+b_2) + (c_1+c_2) + (d_1+d_2) \\ 8(a_1+a_2) + 4(b_1+b_2) + 2(c_1+c_2) + (d_1+d_2) \end{bmatrix}$$

As, LHS = RHS

$\Rightarrow L(P_1+P_2) = L(P_1) + L(P_2)$

(II) $L(\alpha p) = \alpha L(p)$: Consider $\alpha \in R$, $p = ax^3 + bx^2 + cx + d \in R$

LHS $L(\alpha p) = L(\alpha ax^3 + \alpha bx^2 + \alpha cx + \alpha d)$

$$\Rightarrow \begin{bmatrix} -\alpha a + \alpha b - \alpha c + \alpha d \\ \alpha d \\ \alpha a + \alpha b + \alpha c + \alpha d \\ 8(\alpha a) + 4(\alpha b) + 2(\alpha c) + (\alpha d) \end{bmatrix}$$

RHS $\alpha(L(P)) = \alpha \cdot L(ax^3 + bx^2 + cx + d)$

$$\Rightarrow \alpha \cdot \begin{bmatrix} -a+b-c+d \\ d \\ a+b+c+d \\ 8a+4b+2c+d \end{bmatrix} \Rightarrow \begin{bmatrix} -\alpha a + \alpha b - \alpha c + \alpha d \\ \alpha d \\ \alpha a + \alpha b + \alpha c + \alpha d \\ 8(\alpha a) + 4(\alpha b) + 2(\alpha c) + (\alpha d) \end{bmatrix}$$

As, LHS = RHS,

$\Rightarrow L(\alpha \cdot p) = \alpha \cdot L(P)$

$\therefore$ $L: P_3 \to R^4$ is a linear tranformation!

Now, (II) Proving L is one-one:

Now, $[L(P)]_e = ABC\,[P]_B$ as A_{BC} exists.

Also, L is one-one iff $L(P_1) = L(P_2)$ implies $P_1 = P_2$ for $P_1, P_2 \in P_3$.

Suppose L is one-one,

then, $L(P_1) = L(P_2)$

$\Rightarrow \quad A_{BC} \cdot P_1 = A_{BC} \cdot P_2$

$\Rightarrow \quad A_{BC}^{-1} A_{BC} \cdot P_1 = A_{BC}^{-1} A_{BC} P_2 \quad (\because |A_{BC}| \neq 0)$

$\Rightarrow P_1 = P_2$

As, $L(P_1) = L(P_2) \rightarrow P_1 = P_2$

$\therefore$ L is one-one !,

Now, (III) Proving L is onto : Let B be the reduced row echelon form of A_{BC},

Row reducing A_{BC},

$$\cancel{R_1 \to R} \quad \Rightarrow \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

$$\therefore \; R_1 \to -R_1 \quad \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 8R_1 \end{array} \quad \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & \cancel{2}1 \\ 0 & \cancel{0}2 & 0 & 1 \\ 0 & 12 & -6 & 9 \end{bmatrix}$$

$$R_2 \longleftrightarrow R_3 \quad \Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 12 & -6 & 9 \end{bmatrix}$$

$$R_2 \to \frac{1}{2} R_2 \quad \Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 12 & -6 & 9 \end{bmatrix}$$

$$R_4 \to R_4 - 12R_2 \quad \Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 3 \end{bmatrix}$$

$$R_3 \longleftrightarrow R_4 \quad \Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -6 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{6} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_3 \to R_3 - \frac{R_4}{2} \\ R_2 \to R_2 - R_4 \\ R_1 \to R_1 + R_4 \end{array} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to R_1 - R_3 \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to R_1 + R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$$

In, $B = rref(A)$, all four columns have pivot entries.

$$\Rightarrow range(L) = span\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Now, $\dim(R^4) = 4 \quad (\because \dim(R^n) = n)$

and, $\dim(range(L)) = 4$

$\Rightarrow \dim(range(L)) = \dim(R^4)$

$\therefore$ L is onto !

Hence, as L is a linear transformation is one-one and onto, it is an isomporphism from P_3 to R^4.

$\therefore \boxed{P_3 \cong R^4}$ Proved.

Q.5 Consider linear transformation $L: P_3 \to R^3$

$$L(dx^3 + cx^2 + bx + a) = (a+b,\ 2c,\ d-a)$$

B & C are standard bases for P_3 & R^3 respectively.

① find $A_{BC} = ?$

② find $A_{DE} = ?$ where D & E are ordered basis

$$D = (x^3 + x^2,\ x^2 + x,\ x+1) \in P_3 \ , \quad E = \left(\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} \right) \in R^3.$$

Sol: Standard bases for P_3 and R^3 are,

$$B = (x^3, x^2, x, 1)$$

$$C = (e_1, e_2, e_3)$$

Observing effect of L on P_3,

$$L(x^3) = [0,0,1] \quad ; \quad L(x^2) = [0,2,0]$$

$$L(x) = [1,0,0] \quad ; \quad L(1) = [1,0,-1]$$

Therefore, transformation matrices A_{BC} is given by,

$$\therefore A_{BC} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Now, $A_{DE} = Q_{E \leftarrow C} A_{BC} P^{-1}_{D \leftarrow B}$ where $Q_{E \leftarrow C}$ and $P_{D \leftarrow B}$, are the transition matrices from C to E and B to D.

(1) Finding $Q_{E \leftarrow C}$:

Using change of basis and row reducing $[E \mid C]$,

$$\Rightarrow \left[\begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -3 & -6 & 0 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \to -R_1/2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 1 & -3 & -6 & 0 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & -5/2 & -9/2 & 1/2 & 1 & 0 \\ 0 & -3/2 & -5/2 & -3/2 & 0 & 1 \end{array}\right] R_2 \to -2/5 R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 9/5 & -1/5 & -2/5 & 0 \\ 0 & -3/2 & -5/2 & -3/2 & -3/5 & 1 \end{array}\right] R_3 \to R_3 + 3/2 R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 9/5 & -1/5 & -2/5 & 0 \\ 0 & 0 & 1/5 & -9/5 & -3/5 & 1 \end{array}\right] \quad R_3 \to R_3(5)$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 9/5 & -1/5 & -2/5 & 0 \\ 0 & 0 & 1 & -9 & -3 & 5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{9}{5}R_3 \\ R_1 \to R_1 + \frac{3}{2}R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & -14 & -9/2 & 15/2 \\ 0 & 1 & 0 & 16 & 5 & -9 \\ 0 & 0 & 1 & -9 & -3 & 5 \end{array}\right] \quad R_1 \to R_1 + \frac{R_2}{2}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & -2 & 3 \\ 0 & 1 & 0 & 16 & 5 & -9 \\ 0 & 0 & 1 & -9 & -3 & 5 \end{array}\right] \quad \left(\text{in the form } [I_3 \,|\, Q_{E\leftarrow C}]\right)$$

$$\Rightarrow Q_{E\leftarrow C} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}$$

(2) Finding $P^{-1}_{D\leftarrow B} = P_{B\to D}$:

Row reducing $[B|D]$,

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \left(\text{in the form } [I_4 \,|\, P_{B\leftarrow D}]\right)$$

(3) Computing A_{DE}: As $A_{DE} = Q_{E\leftarrow C} A_{BC} P^{-1}_{D\leftarrow B}$

$$\Rightarrow \cancel{AD}\, A_{DE} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A_{DE} = \begin{bmatrix} 3 & -4 & 6 & -9 \\ -9 & 10 & 16 & 25 \\ 5 & -6 & -9 & -14 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A_{DE} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}$$

Q.6 In R^4, $W = \text{span}\left\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\2\\2\end{bmatrix}\right\}$ & $v = \begin{bmatrix}-1\\3\\3\\2\end{bmatrix}$

① find $\text{proj}_W v = ?$

② find $w_1 \in W$ & $w_2 \in W^{\perp}$

③ also find $W^{\perp} = ?$ (as orthogonal basis)

Sol: (i) we need an orthogonal basis to compete $\text{proj}_W v$, we can normalize the orthogonal basis to get the orthogonal basis.

Checking if $W \subset R^4$ has orthogonal basis,

$$\begin{bmatrix}2\\-1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\2\\2\end{bmatrix} = 5 \neq 0$$ $\therefore$ Given basis is not an orthogonal basis.

$\therefore$ finding orthogonal basis: T

Using Gram-Schmidt method,

$\rightarrow$ $v_1 = w_1 = [2, -1, 1, 0]$

$\rightarrow$ $v_2 = w_2 - \text{proj}_{v_1} w_2 = w_2 - \left(\frac{w_2 - v_1}{||v_1||^2}\right) \cdot v_1$

$v_2 = [1, -1, 2, 2] - \left(\frac{5}{6}\right)[2, -1, 1, 0]$

$v_2 = \left[\frac{-4}{6}, \frac{-1}{6}, \frac{7}{6}, \frac{12}{6}\right]$

$v_2 = [-4, -1, 7, 12]$ (ignoring fractions)

Now, $T = \left\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\-1\\7\\12\end{bmatrix}\right\}$ is an orthogonal basis for W and $W = \text{span}(T)$.

Finding orthogonal basis U: normalizing vectors in T,

Now, $U = \left\{\left[\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{6}, 0\right], \left[\frac{-4}{\sqrt{210}}, \frac{-1}{\sqrt{210}}, \frac{7}{\sqrt{210}}, \frac{12}{\sqrt{210}}\right]\right\}$

is an orthogonal basis for W and $W = \text{span}(U)$.

Now, given the orthogonal basis $U = \left\{\left[\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0\right],\right.$

$\left.\left[\frac{-4}{\sqrt{210}}, \frac{-1}{\sqrt{210}}, \frac{7}{\sqrt{210}}, \frac{12}{\sqrt{210}}\right]\right\}$

$\{\text{proj}_W v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2\}$ and $v = [-1, 3, 3, 2]$

$$\Rightarrow \text{proj}_W v = \left(\begin{bmatrix}-1\\3\\3\\2\end{bmatrix}\begin{bmatrix}2/\sqrt{6}\\-1/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix}\right)\begin{bmatrix}2/\sqrt{6}\\-1/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix} + \left(\begin{bmatrix}-1\\3\\3\\2\end{bmatrix}\begin{bmatrix}-4/\sqrt{210}\\-1/\sqrt{210}\\7/\sqrt{210}\\12/\sqrt{210}\end{bmatrix}\right)\begin{bmatrix}-4/\sqrt{210}\\-1/\sqrt{210}\\7/\sqrt{210}\\12/\sqrt{210}\end{bmatrix}$$

$$= \left(\frac{-\sqrt{6}}{3}\right)\begin{bmatrix}2/\sqrt{6}\\-1/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix} + \left(\frac{46}{\sqrt{210}}\right)\begin{bmatrix}-4/\sqrt{210}\\-1/\sqrt{210}\\7/\sqrt{210}\\12/\sqrt{210}\end{bmatrix}$$

$$= \begin{bmatrix}-2/3\\1/3\\-1/3\\0\end{bmatrix} + \begin{bmatrix}-92/105\\-23/105\\23/105\\92/35\end{bmatrix}$$

$$\Rightarrow \text{proj}_W v = \left[\frac{-54}{35}, \frac{4}{35}, \frac{42}{35}, \frac{92}{35}\right]$$

(ii) we know that if $W \subset R^n$ and $V \in R^n$ then V can be expressed as $v = w_1 + w_2$ where $w_1 = \text{proj}_W v \in W$ and $w_2 = v - \text{proj}_W v \in W^\perp$.

Therefore, $w_1 = \text{proj}_W v$

$$\therefore w_1 = \left[-\frac{54}{35}, \frac{4}{35}, \frac{42}{35}, \frac{92}{35}\right] \in W \quad \text{—— (a)}$$

and $w_2 \Rightarrow v - \text{proj}_W v$

$$\Rightarrow [-1, 3, 3, 2] - \left[\frac{-54}{35}, \frac{4}{35}, \frac{42}{35}, \frac{92}{35}\right]$$

$$\Rightarrow w_2 = \left[\frac{19}{35}, \frac{101}{35}, \frac{63}{35}, \frac{-22}{35}\right] \in W^\perp \quad \text{—— (b)}$$

Verifying $w_1 + w_2 = v$ with (a) & (b),

LHS : $w_1 + w_2 = \left[\frac{-54+19}{35}, \frac{4+101}{35}, \frac{42+63}{35}, \frac{92-22}{35}\right]$

$$= [-1, 3, 3, 2] = \text{RHS} = v$$

$\therefore$ $v = w_1 + w_2$ for some $w_1 \in W$, $w_2 \in W^\perp$

(iii) finding $W^\perp$: Given $w_1 = [2, -1, 1, 0]$
$w_2 = [1, -1, 2, 2]$

Let $v_i \in W^\perp$ and using gram-Schmidt method,

then $v_1 = w_1 = [2, -1, 1, 0]$

$$v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2}\right) v_1 = [1, -1, 2, 2] - \frac{5}{6}[2, -1, 1, 0]$$

$\therefore v_2 = [-4, -1, 7, 12]$

Therefore, orthogonal basis $T = \{[2,-1,1,0], [-4,-1,7,12]\}$

Enlarging $[T \mid I]_1$ (to R^4) and row reducing,

$$\begin{array}{c} (v_1)(v_2)\ (w_3)(w_4) \\ = \left[\begin{array}{cc|cccc} 2 & -4 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 7 & 0 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \quad R_1 \to R_1/2$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 7 & 0 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 9 & -1/2 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 9R_2 \\ R_4 \to R_4 - 12R_2 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 & 1 \end{bmatrix} \quad R_4 \to R_4 - 2R_3$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 1 \end{bmatrix} \quad \begin{array}{l} \cancel{R_3 \to R_3 - 3R_4} \quad R_4 \to -\frac{R_4}{2} \\ \cancel{R_2 \to R_2 + \frac{R_4}{3}} \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1/2 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 3R_4 \\ R_2 \to R_2 + \frac{R_4}{3} \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & 0 & -2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -1/2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_3/6 \\ R_1 \to R_1 - R_3/2 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & -3/4 \\ 0 & 1 & 0 & 0 & 0 & 1/12 \\ 0 & 0 & 1 & 0 & -2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -1/2 \end{bmatrix} \quad R_1 \to R_1 + 2R_2$$

$$= \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & -7/12 \\ 0 & 1 & 0 & 0 & 0 & 1/12 \\ 0 & 0 & 1 & 0 & -2 & 3/12 \\ 0 & 0 & 0 & 1 & 1 & -1/12 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{L.I.}$

Now, $W_3 = [1,0,0,0]$, $W_4 = [0,1,0,0]$

$\Rightarrow$ New orthogonal basis,

$$T^* = \{[2,-1,1,0], [-4,-1,7,12], [1,0,0,0], [0,1,0,0]\}$$

Using Gram-Schmidt for orthogonal basis,

$$\rightarrow V_3 = W_3 - \left(\frac{W_3 \cdot V_1}{\|V_1\|^2}\right)V_1 - \left(\frac{W_3 \cdot V_2}{\|V_2\|^2}\right)V_2$$

$$V_3 = [1,0,0,0] - \frac{2}{6}[2,-1,1,0] + \frac{4}{210}[-4,-1,7,12]$$

$$V_3 = \left[\frac{9}{35}, \frac{11}{35}, \frac{-7}{35}, \frac{8}{35}\right] \text{ ——— (a)}$$

$$\rightarrow V_4 = W_4 - \left(\frac{W_4 \cdot V_1}{\|V_1\|^2}\right)V_1 - \left(\frac{W_4 \cdot V_2}{\|V_2\|^2}\right)V_2 - \left(\frac{W_4 \cdot V_3}{\|V_3\|^2}\right)V_3$$

$$V_4 = [0,1,0,0] + \frac{1}{6}[2,-1,1,0] + \frac{1}{210}[-4,-1,7,12] - \frac{11}{9}\left[\frac{9}{35}, \frac{11}{35}, \frac{-7}{35}, \frac{8}{35}\right]$$

$$V_4 = \left[\frac{11}{35}, \frac{29}{35}, \frac{7}{35}, \frac{2}{35}\right] - \frac{11}{9}\left[\frac{9}{35}, \frac{11}{35}, \frac{-7}{35}, \frac{8}{35}\right]$$

$$V_4 = \left[0, \frac{4}{9}, \frac{4}{9}, \frac{-2}{9}\right] \text{ ——— (b)}$$

As, $V_3, V_4 \in W^\perp$,

$$\therefore W^\perp = \text{span}\left\{\begin{bmatrix} 9/35 \\ 11/35 \\ -7/35 \\ 8/35 \end{bmatrix}, \begin{bmatrix} 0 \\ 4/9 \\ 4/9 \\ -2/9 \end{bmatrix}\right\}$$

where orthogonal basis of $W^\perp$ is $\left(\begin{bmatrix} 9/35 \\ 11/35 \\ -7/35 \\ 8/35 \end{bmatrix}, \begin{bmatrix} 0 \\ 4/9 \\ 4/9 \\ -2/9 \end{bmatrix}\right)$