Unit : 06

Probability and Statistics

* Mean :-

** 1) Mean for ungrouped data :

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

** 2) Mean for grouped data :
(Discrete data and continuous series)

Method : 1 : Direct Method

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad ; \quad \sum_{i=1}^{n} f_i = n$$

Method : 2 : Assumed Mean Method

$$\bar{x} = a + \frac{\sum f_i d_i}{\sum f_i}$$

where, a = assumed mean
$d_i = x_i - a$

Ex:- Calculate the arithmetic mean for the following data :

x	1	2	3	4	5
f	3	5	7	8	4

Date
Page

Sol^n:-

x_i	f_i	$x_i \cdot f_i$
1	3	3
2	5	10
3	7	21
4	8	32
5	4	20
	$\Sigma f_i = 27$	$\Sigma x_i f_i = 86$

discrete frequency distribution

→ Mean $\bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{86}{27}$

Ex:- Find the mean of the female teachers :

continuous frequency distribution

% of female tec	15-25	25-35	35-45	45-55	55-65	65-75	75-85
No. of states	6	11	7	4	4	2	1

Sol^n :-

% of female teachers	No. of states (f_i)	x_i	$d_i = x_i - 50$	$f_i d_i$
15-25	6	20	-30	-180
25-35	11	30	-20	-220
35-45	7	40	-10	-70
45-55	4	50	0	0
55-65	4	60	10	40
65-75	2	70	20	40
75-85	1	80	30	30
	$\Sigma f_i = 35$			$\Sigma f_i d_i = -360$

→ Here, the total no. of observations are 7.
and the middle class is 45 - 55.
Therefore, the assumed mean $a = \frac{45+55}{2} = 50$

→ The deviation $d_i = x_i - a = x_i - 50$
where,

$$x_i = \frac{\text{lower limit of the class} + \text{upper limit of the class}}{2}$$

→ By the assumed mean method,

$$\bar{x} = a + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

$$= 50 + \frac{(-360)}{35} = \boxed{39.71}$$

* Median :-

It is defined as middle value of distribution when the values are arranged in ascending or descending order of magnitude.

→ CASE : 1 : ungrouped data

if the no. of observations N is odd.
then, the median is the middle value,
i.e. $\frac{N+1}{2}$ the value after the values have been arranged in ascending or descending order will be unique.

→ CASE : 2 : ungrouped data

if the no. of observations N is even, then there will be no unique median.

It can be obtained by taking mean of $\left(N/2\right)^{th}$ and $\left(\frac{N+1}{2}\right)^{th}$ value.

→ CASE : 3 : Discrete frequency distribution.

the median is obtained by considering cumulative frequency.

i) Arrange the value of the variable in ascending or descending order of the magnitude.

ii) Find $\frac{N}{2}$, where $N = \Sigma f$.

iii) Find the cumulative frequency just greater than $N/2$ and determine the corresponding value of the variable.

iv) The corresponding value of x is the median.

→ CASE : 4 : Median for continuous frequency distribution

$$\text{Median class} = L + \frac{N/2 - c.f}{f} \times h$$

where, L = lower limit of the median class
c.f. = cumulative frequency of preceding class
f = frequency of median class
h = width of the class interval.

Ex:- Obtain the median for the following distribution :

X	1	2	3	4	5	6	7	8	9
f	8	10	11	16	20	25	15	9	6

x_i	f_i	c.f
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120.
	$\Sigma f_i = 120$	

→ Here, $\Sigma f_i = 120 = N$.

$\therefore \frac{N}{2} = \frac{120}{2} = 60.$

→ The c.f is just greater than 60 which is 65 and corresponding variable value is 5.

$\therefore$ Median = 5.

* Mode :-

It is defined as the value which occur most frequently in the set of observations and around which the other items of the set are heavily distributed.

→ CASE : 1 : ungrouped data
using the definition of mode.

→ CASE : 2 : grouped data
[Continuous frequency distribution]

$$\text{Mode} = L + \frac{f - f_1}{2f - f_1 - f_2} \times h.$$

where, L = lower limit of modal class
h = width of the class
f_1 = frequency of preceding class of modal class.
f_2 = frequency of succeeding class of modal class.
f = frequency of modal class.

Ex:- The wickets taken by a bowler in 10 cricket matches are as follows :
2, 6, 4, 5, 0, 2, 1, 3, 2, 3. Find the mode of data.

No. of wickets	0	1	2	3	4	5	6
No. of matches	1	1	3	2	1	1	1

→ Clearly, 2 is the number of wickets taken by the bowler in the maximum number (i.e., 3) of matches.
So, the mode of this data is 2.

Ex:- Find the mode for the following:

Profit per shop	0-100	100-200	200-300	300-400	400-500	500-600
No. of shop	12	18	27	20	17	6

Preceding successiding

→ Since the highest frequency is 27, the modal class is 200-300.
Here, $L = 200$, $f = 27$, $f_1 = 18$, $f_2 = 20$, $h = 100$

$$\therefore \text{Mode} = L + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

$$= 200 + \frac{27 - 18}{54 - 18 - 20} \times 100$$

$$= \boxed{256.25}$$

* Standard Deviation :-

→ CASE : 1 : ungrouped data

$$\sigma = \sqrt{\frac{\Sigma (x - \bar{x})^2}{N}}$$

→ CASE : 2 : grouped data
→ In case of frequency distribution consisting of N observations $x_1, x_2, \ldots, x_n$

with respective frequencies $f_1, f_2, \ldots, f_n$ then the standard deviation is given by,

$$\sigma = \sqrt{\frac{\Sigma f x^2}{N} - \left(\frac{\Sigma f x}{N}\right)^2}$$

Ex: Calculate the standard deviation of the weights (in kg) of ten persons:
45, 49, 55, 50, 41, 44, 60, 58, 53, 55.

→ Here, $\bar{x} = \frac{\Sigma x}{N} = \frac{510}{10} = 51.$

x	$x - \bar{x}$	$(x-\bar{x})^2$
45	−6	36
49	−2	4
55	4	16
50	−1	1
41	−10	100
44	−7	49
60	9	81
58	7	49
53	2	4
55	4	16
		$\Sigma(x-\bar{x})^2 = 356$

∴ Standard deviation $= \sqrt{\frac{\Sigma(x-\bar{x})^2}{N}}$

$= \sqrt{\frac{356}{10}} = \boxed{\cancel{6.289}}$

$= \boxed{5.967}$

Ex:- Calculate the standard deviation of the following data :

x	10	11	12	13	14	15	16	17	18
f	2	7	10	12	15	11	10	6	3

x	f	fx	x^2	fx^2
10	2	20	100	200
11	7	77	121	847
12	10	120	144	1140
13	12	156	169	2028
14	15	210	196	2940
15	11	165	225	2475
16	10	160	256	2560
17	6	102	289	1734
18	3	54	324	972
	$\Sigma f = 76$	$\Sigma fx = 1064$		$\Sigma fx^2 = 15196$

$\rightarrow \Sigma f = N = 76$

$\therefore$ Standard deviation $= \sqrt{\frac{\Sigma fx^2}{N} - \left(\frac{\Sigma fx}{N}\right)^2}$

$$\therefore \sigma = \sqrt{\frac{15196}{76} - \left(\frac{1064}{76}\right)^2}$$

$$= 1.9867$$

$$\therefore \sigma \simeq 1.987$$

Date
Page

* Random Experiment :

An action or an operation which can produce any result or outcome is called Random Experiment or trail and outcomes are known as events.

* Certain Event or Sure Event :

An event whose occurrence is inevitable when an experiment is performed is called certain events.

* Impossible Event :

An event which can never occur when an experiment is performed is called an impossible event.

* Mutually Exclusive Events :

Two events are said to be mutually exclusive if both can not occur simultaneously. [disjoint events]

* Independent Events :

Two events are said to be independent of each other if occurrence or non-occurrence of one does not affect the occurrence of other.

* Exhaustive Events :

A group of events is said to be exhaustive if it includes all possible outcomes of experiment.

* Sample Space :

A set of all possible outcomes of a random experiment is called a sample space. The sample space of a random experiment is denoted by S.

* Combinatorial Probability :-

* Counting Rules :-

The calculation of probability involves counting of the number of favorable outcomes and the total number of outcomes. Sometimes, the number of possible outcomes may be so large that listing them all ~~with~~ will not be possible so we require rules of counting.

→ Rule 1 :

If any one of k different mutually exclusive and exhaustive events can occur on each of n trails, the number of possible outcomes is equal to k^n.

→ Rule 2 :

If there are k_1 events on the first trail, k_2 events on the second trail. and k_n events on the nth trail, then the number of possible outcomes is $k_1, k_2, k_3, \ldots, k_n$.

→ Rule 3 :

The number of ways that all n objects can be arranged in order is :

$n! = n(n-1)(n-2)\ldots 2 \cdot 1.$

→ Rule 4 :

The number of ways of $\boxed{\text{arranging}}$ r objects selected from n objects in order is given by $\frac{n!}{(n-r)!}$

→ Rule 5 :

The number of ways of $\boxed{\text{selecting}}$ r ~~objects~~ ~~in~~ objects out of n objects ^nC_r irrespective of order is equal to $\frac{n!}{r!(n-r)!}$.

→ Rule 6 :

The number of different permutations of n objects where there are n_1 repeated items, n_2 repeated items, ... n_k repeated items : $\frac{n!}{n_1!\, n_2! \ldots n_k!}$

* <u>Probability</u> :-

Let S be a sample space of an experiment and A be any event of this sample space. The probability P(A) of the event A is defined as the real valued set function which associates a real value

corresponding to a subset A of the sample space. The probability P(A) satisfies the following three axioms :

1) $P(A) \geq 0$
2) $P(S) = 1$
3) If $A_1, A_2, \ldots, A_n$ are finite mutually exclusive events then
$P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

* **Algebra of events :**

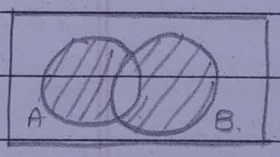
→ If A and B are two events on sample space S, then

$A \cup B$ => Occurrence of A or B
$A \cap B$ => Occurrence of A and B

* **Addition Theorem on Probability :-**

→ Probability that either of A or B will occur is given by,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



For mutually exclusive events :-

$P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

Date
Page

* Joint Probability :-

→ It is simply the likelihood that two events will happen at the same time. Joint probability is calculated by multiplying the probability of event A, expressed as $P(A)$, by the probability of event B, expressed as $P(B)$. It is denoted by $P(A \cap B)$.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

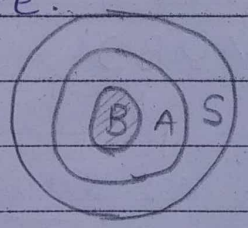
or

$$P(A \cap B) = P(A) \times P(B)$$, where A and B are independent events.

* Conditional probability :-

→ The probability that B will occur with the given condition that A has already occurred is called the conditional probability of B and is denoted by $P(B|A)$.
It can also be interpreted as probability of B with A as sample space.

$$\therefore P(B|A) = \frac{P(A \cap B)}{P(A)} \longrightarrow$$



Similarly, we can define $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

The above definition written in product form give the multiplication theorem for probability.

$$P(A \cap B) = P(A/B) \times P(B)$$

→ **Remark :-**

For $\boxed{\text{independent}}$ events, if probability of A remains constant whether B has occurred or not, then A and B are independent events. i.e. $P(A/B) = P(A)$.

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

$$\therefore P(A \cap B) = P(A) \cdot P(B)$$

is known as multiplication theorem for independent events.

Ex :- A card is drawn from a well shuffled pack of cards. What is the probability that it is either a spade or an ace.

→ Let A and B be the events of getting a spade and an ace respectively.

$$\therefore P(A) = \frac{13}{52} \qquad P(B) = \frac{4}{52}$$

and $P(A \cap B) = \frac{1}{52}$

$\therefore$ Probability of getting either spade or an ace,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \boxed{\frac{4}{13}}$$

Ex:- If the letters of the words 'ATTRACTION' are written down at random, find the probability that (i) all the T's occur together, (ii) no two T's occur together.

→ Total number of arrangements irrespective of order is : $\dfrac{n!}{n_1!\ ,\ n_2!}$

repeated items

A T T R A C T I O N

10 letters

$\therefore \dfrac{10!}{2!\cdot 3!}$

(i) → rule: 6

TTT

T' A R A C I O N

8 letters.

$\therefore \dfrac{8!}{2!}$

event occurr probability

$\therefore P(A) = \dfrac{\text{probability of occurring of event}}{\text{Total number of outcomes}}$

$= \dfrac{\frac{8!}{2!}}{\frac{10!}{2!\ 3!}} = \dfrac{8! \times 3!}{10 \times 9 \times 8!}$

$= \dfrac{3 \times 2}{10 \times 9}$

$= \boxed{\dfrac{1}{15}}$

(ii) → $P(A) = \dfrac{\frac{7!}{2!} \times {}^8C_3}{\frac{10!}{2!\cdot 3!}} = \dfrac{\frac{7!}{\cancel{2!}} \times \frac{\cancel{8!}}{5!\cdot \cancel{3!}}}{\frac{10\times 9 \times \cancel{8!}}{\cancel{2!}\cdot \cancel{3!}}}$

rule:5

$= \dfrac{7\times \cancel{6}^{\cancel{3}}}{{}_5\cancel{10}\times \cancel{9}_3} = \boxed{\dfrac{7}{15}}$

Ex:- A company has estimated that the probability of success for the three products introduced in the market are 1/3, 2/5, 2/3 respectively. Assuming independence find (i) the probability that the three products are successful (ii) the probability that none of the products is successful.

→ A : First product is successful
B : Second " " "
C : Third " " "

→ Given, $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$, $P(C) = \frac{2}{3}$

(i) → Required probability, $P(A \cap B \cap C) = P(A)\cdot P(B)\cdot P(C)$

$= \frac{1}{3}\cdot\frac{2}{5}\cdot\frac{2}{3}$

$= \boxed{\frac{4}{45}}$

(ii) → Required probability, $P(A^c \cap B^c \cap C^c) = P(A^c)\,P(B^c)\,P(C^c)$

$= (1 - P(A))\cdot(1 - P(B))\cdot(1 - P(C))$

$= (1 - 1/3)\cdot(1 - 2/5)\cdot(1 - 2/3)$

$= 2/3 \cdot 3/5 \cdot 1/3 = \boxed{\frac{2}{15}}$

Date
Page

Ex:- In two successive throws of a pair of dice, determine the probability of getting total of 8 each time.

→ Sample space, $6 \times 6 = 36$

(2,6)
(3,5)
(4,4)
(3,5)
(2,6)

→ $P(A)$ = first throw of a dice.
$= \frac{5}{36}$

→ $P(B)$ = Second throw of a dice
$= \frac{5}{36}$

→ Required probability, $P(A \cap B) = P(A) \cdot P(B)$
$= \frac{25}{1296}$

Ex:- In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is a girl.
condition → always અહીંથી આવે

→ Let A denote the event that a student chosen randomly studies in class XII. and
B denot the student is a girl.

∴ Required probability, $P(A/B) = \frac{P(A \cap B)}{P(B)}$
condition

→ 12 th માં બધી છોકરાઓ girl બધી છે

$$\therefore P(A|B) = \frac{\frac{430}{1000}}{\frac{430}{1000}} = \boxed{0.1}$$

Ex:- A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once.

→ B : the sum of numbers appearing is 6.

$\{(1,5), (5,1), (2,4), (4,2), (3,3)\}$

$\therefore P(B) = \frac{5}{36}$

and A : the ~~sum~~ 4 has appeared at least once

$\{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (5,4), (6,4)\}$

$\therefore P(A) = \frac{11}{36}$

and $P(A \cap B) = \frac{2}{36}$

→ $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{2/36}{5/36}$

$= \boxed{\frac{2}{5}}$

Date
Page

* **Bayes Theorem :-** (for two events)

→ Let A and B are mutually exclusive and exhaustive events. If we know the conditional probability $P(B|A)$ and the probability $P(A)$. Then the Bayes' theorem tells that we can compute the conditional probability $P(A|B)$ as follows :

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \; ; \; P(B) \neq 0$$

* **Bayes Theorem :-** (General)

→ Let, $A_1, A_2, \ldots, A_n$ be n mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for $i = 1, 2, 3, \ldots, n$ in a sample space S.

→ Let, B be an event that can occur in combination with one of the events $A_1, A_2, \ldots, A_n$ with $P(B) \neq 0$.
The probability of the events A_i when the event B has actually occurred is given by,

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)}$$

Ex :- A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random

from one of the bags and it is found to be red. Find the probability that the red ball is drawn from bag B.

→ Let, A_1 and A_2 be the events that the ball is drawn from bag A and B respectively. Let, B be the event that ball drawn is red.

$\therefore P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{2}$

→ The probability that the ball is drawn from bag B which has to be red is,

$$P(A_2|B) = \frac{P(B|A_2) \cdot P(A_2)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)} \quad (*)$$

→ Probability that the ball drawn is red given that it is picked up from bag A:

$$P(B|A_1) = \frac{P(B \cap A_1)}{P(A_1)} = \frac{3}{5}$$

and the probability that the ball drawn is red given that it is picked up from bag B:

$$P(B|A_2) = \frac{P(B \cap A_2)}{P(A_2)} = \frac{5}{9}$$

→ Substituting all values in (*),

$$P(A_2|B) = \frac{5/9 \cdot 1/2}{1/2 \cdot 3/5 + 1/2 \cdot 5/9}$$

$$= \frac{5/9 \times 9 \times 5}{27+25} = \boxed{\frac{25}{52}}$$

Ex:- A company has two plants to manufacture hydraulic machines. Plant 1 manufactures 70% of the hydraulic machines and plant 2 manufactures 30%. At plant 1, 80% of hydraulic machines are rated standard quality and at plant 2, 90% of hydraulic machines are rated standard quality.
A machine is picked up at random and is found to be of standard quality.
What is the chance that it come from plant 1 ?

→ Let A_1 and A_2 be the events that the machine is picked up from plant 1 and 2 respectively.

let B be an event that the machine is of standard quality.

$\therefore P(A_1) = 0.7$, $P(A_2) = 0.3$.

→ The probability that a machine is picked up at random and is found to be of standard

quality and from plant 1 is :

$$P(A_1 | B) = \frac{P(B|A_1) \cdot P(A_1)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)} \quad \text{— (A)}$$

→ At plant 1, 70 % machines are made from which 80 % are of standard quality.

$\therefore P(B|A_1) = 0.8$

and At plant 2, 30 % machines are made from which 90 % are of standard quality, $\therefore P(B|A_2) = 0.9$

→ Substituting these values in (A),

$$P(A_1|B) = \frac{(0.8)(0.7)}{(0.7)(0.8) + (0.3)(0.9)}$$

$$= \boxed{0.6746}$$

Ex :- In a factory which manufacture bolts, machines A, B, C manufacture 25 %, 35 % and 40 % of the bolts respectively. Of their outputs, 5 %, 4 % and 2 % are defective bolts. A bolt is drawn at a random from the product and is found to be defective. What is the probability that it is manufactured by machine B. ?

→ In this question :

$P(A_1/B) = 0.3623$

$P(A_3/B) = 0.2319$

→ Let A_1, A_2 and A_3 be the events that a bolt is picked up from machine A, B and C respectively.

$P(A_1) = 0.25$, $P(A_2) = 0.35$, $P(A_3) = 0.40$.

$\therefore$ ~~$P(A_1) = \frac{1}{3}$~~ , ~~$P(A_2) = \frac{1}{3}$~~ , ~~$P(A_3) = \frac{1}{3}$~~

let B be an event that the bolt is chosen which is defective.

→ The probability that a bolt is drawn from the product and found to be defective which is made by machine B is :

$$P(A_2/B) = \frac{P(B/A_2) \cdot P(A_2)}{P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2) + P(A_3) \cdot P(B/A_3)} \quad (A)$$

→ Machine A manufactures 25% bolts in which 5% are defective,

$\therefore P(B/A_1)$ ~~$= 0.25$~~ $\times 0.05$

~~$= 0.0125$~~.

and machine B manufactures 35% bolts in which 4% are defective,

$\therefore P(B/A_2)$ ~~$= 0.35$~~ $\times 0.04$

~~$= 0.014$~~

Similarly for machine C, we get,

$P(B/A_3)$ ~~$= 0.4$~~ $\times 0.02$ ~~$= 0.008$~~

→ Substituting all these values in (A),

$$P(A_2 | B) = \frac{(0.04)(0.35)}{(0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02)}$$

$$= \boxed{0.4058}$$

Ex :- As you know Covid-19 tests are common nowadays, but some results of tests are not true. Let's discuss example about it: Let's assume a diagonostic test has 99 % accuracy and 60 % of all people have Covid-19. If a patient tests positive, what is the probability that they actually have the disease.

→ Solⁿ :-

100

60 %
covid - 19

40 %
Non-covid-19

True	False	True	False
60 × 0.99	60 × 0.01	40 × 0.99	40 × 0.01
= 59.4	= 0.6	= 39.6	= 0.4

→ Actual positive patients : 59.4 + 0.4
= 59.8 %

→ required probability :

$$P(\text{covid 19} / \text{positive}) = \frac{P(\text{positive} / \text{covid 19})\ P(\text{covid 19})}{P(\text{positive})}$$

and we know that,

$P(\text{Covid19}) = 0.6$
$P(\text{positive}) = 0.598$
$P(\text{positive} / \text{covid19}) = 0.99$

$$\therefore P(\text{covid 19} / \text{positive}) = \frac{0.99 \times 0.6}{0.598}$$

$$= \boxed{0.9933}$$

* Random variables :-

→ A random variable x is a real valued function of the elements of the sample space of a random experiment. In other words, a variable which takes the real values, depending on the outcome of a random experiment is called a random variable.

* Types of Random Variables :-

1) Discrete Random Variable :

→ Let x be a random variable. If the number of values of x is countable then x is called discrete random variable.
E.g. Number of heads in toss of 3 coins, Number of floor of a building.

2) Continuous Random Variable :

→ A random variable x which assume all real value within a given interval is called as a continuous random variable. Thus the possible values of a continuous random variables are unaccountably infinite.
E.g. Age of a person, Height of a person in inches, Temperature.

Date
Page

* Probability Distribution Functions :-

Binomial, Poisson, Exponential, Normal

Discrete

* Binomial Distribution :-

→ Consider n independent trial of a random experiment which results in either success or failure. Let p be the probability of success remaining constant every time and $q = 1 - p$ be the probability of failure.
The probability of x successes and n−x failures is given by,

$$P(X = x) = n(x \cdot p^{x} q^{n-x} = \binom{n}{x} p^{x} q^{n-x}$$

Mean of the Bionomial Distribution : np

Variance of the Bionomial Distribution : npq

Standard Deviation of Bionomial Distribution : $\sqrt{npq}$

Ex :- Two dice are thrown five times. Find the probability of getting the sum as 7.
(i) at least once.
(ii) two times.

→ In a single throw of two dice, the sum 7 can occur in 6 ways out of $6 \times 6 = 36$.

(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).

$\therefore$ The probability of success $p = \frac{6}{36} = \frac{1}{6}$.

and $q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$ and $n = 5$.

→ From bionomial distribution,

$$P(X = x) = \binom{n}{x} p^x \cdot q^{n-x}$$

$$= \binom{5}{x} (1/6)^x (5/6)^{5-x} \quad ; \; x = \underbrace{0, 1, 2, \ldots, 5}_{\text{up to } n}$$

(i) → Probability of getting the sum as 7 at least once in 5 throws,

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - \binom{5}{0} (1/6)^0 (5/6)^{5-0}$$

$$= 1 - (1)(1) \frac{3125}{7776}$$

$$= \frac{4651}{7776} = \boxed{0.598} \simeq \underline{\underline{0.6}}$$

(ii) → Probability of getting the sum as 7 two times in 5 throws of two dice,

$$P(X = 2) = \binom{5}{2} (1/6)^2 (5/6)^{5-2}$$

$$= \frac{5 \times 4}{2 \times 1} \frac{1}{36} \times \frac{125}{216}$$

$$= \boxed{0.1607}$$

Date
Page

Exp :- If 10% of the screws produced by a machine are defective, find the probability that out of 5 screws chosen at random (i) none is defective, (ii) one is defective and (iii) at most two are defective.

→ Let (small) p be the probability of $\boxed{\text{defective}}$ screws,

$\therefore p = 0.1 \Rightarrow q = 1-p \Rightarrow q = 0.9$

and $n = 5$.

→ From binomial distribution,

$$P(X=x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{5}{x} (0.1)^x (0.9)^{5-x} \; ; \; n = 0, 1, 2, \ldots, 5$$

(i) None is defective,

$$\therefore P(X=0) = \binom{5}{0} (0.1)^0 (0.9)^{5-0}$$
$$= (1)(1)$$
$$= 0.5904$$

(ii) one is defective,

$$P(X=1) = \binom{5}{1} (0.1)^1 (0.9)^{5-1}$$
$$= (5)(0.1)(0.9)^4$$
$$= 0.3280.$$

(iii) $$p(X \leq 2) = \binom{5}{0} (0.1)^0 (0.9)^{5-0} + \binom{5}{1} (0.1)^1 (0.9)^{5-1} + \binom{5}{2} (0.1)^2 (0.9)^{5-2}$$

$\therefore p(X \leq 2) = 0.5904 + 0.3280 + \frac{5 \times 4}{2 \times 1} (0.1)^2 (0.9)^3$

$= 0.5904 + 0.3281 + 0.0729$

$= 0.99134$

odd : 0.99135 → 0.9914

even : 0.99145 → 0.9914

Ex :- It has been claimed that in 60% of all solar heat installations the utility bill is reduced by at least one third. Accordingly, what are the probabilities that the utility bill will be reduced by at least one third in (i) four of five installations ? (ii) at least four of five installations.

→ Here, let p be the probability of reduced utility bill, $p = 0.6$

$\therefore q = 1 - p = 1 - 0.6 = 0.4$

and $n = 5$.

$\therefore$ From binomial distribution,

$P(X = x) = \binom{n}{x} p^x \cdot q^{n-x}$

$= \binom{5}{x} (0.6)^x (0.4)^{5-x}$; $n = 0, 1, 2, \ldots, 5$

(i) $P(X = 4) = \binom{5}{4} (0.6)^4 (0.4)^{5-4}$

$= 5 \times (0.6)^4 \times (0.4)$

$= 0.2592$

(ii) $P(X \geq 4) = P(X = 4) + P(X = 5)$

$$\therefore P(X \geq 4) = \binom{5}{4}(0.6)^4(0.4)^{5-4} + \binom{5}{5}(0.6)^5(0.4)^0$$
$$= 5(0.6)^4(0.4) + (1)(0.6)^5(1)$$
$$= 0.2592 + 0.07776$$
$$= 0.33696$$
$$= 0.3370$$

* Poisson Distribution :-

Discrete

→ It is a limiting case of binomial distribution under the following conditions :

$n \to \infty$ (i) The number of trials should be infinitely large.

$p \to 0$ (ii) The probability of successes p for each trial should be very small.

(iii) $np = \lambda$ should be finite, mean of poisson distribution where λ is a constant.

$np = \lambda \to$ finite

→ Formula for Poisson distribution if the probability of x is given by,

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad ; \; x = 0, 1, 2, \ldots$$

Ex :- A manufacturer of cotterpins knows that 5% of his products are defective. If he sells cotterpins in boxes of 100 and guarantees that not more that 10 pins will be defective, what is the approximate probability that a box will fail to meet the

guaranteed quality.

→ Let p be the probability of a pin being defective
$p = 5\% = 0.05$ and $n = 100$.

→ Since, p is small and n is large,
Poisson distribution is applicable,

$$\lambda = np = 0.05 \times 100 = 5.$$

→ Let X be the random variable which denotes the number of defective pins,

$$P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

→ Probability that a box ~~fills~~ fails to meet the guaranteed quality,

$$\therefore P(X > 10) = 1 - P(X \leq 10)$$

$$= 1 - \sum_{x=0}^{10} P(X = x)$$

$$= 1 - \sum_{x=0}^{10} \frac{e^{-5} \, 5^{x}}{x!}$$

$$= 1 - \left[e^{-5}\right]\left[1 + \frac{5^1}{1!} + \frac{5^2}{2!} + \ldots + \frac{5^{10}}{10!}\right]$$

$$= 1 - e^{-5}\left[1 + 5 + \frac{25}{2} + \frac{625}{6} + \ldots + \frac{5^{10}}{10!}\right]$$

$$= 0.0137$$

Date
Page

Ex :- If 2 % of light bulbs are defective, find the probability that (i) at least one is defective and (ii) exactly 7 are defective in a sample of 100.

→ Let p be the probability that a light bulb is defective,

$p = 2\% = 0.02$ and $n = 100$

→ Since, p is small and n is large, Poisson distribution is applicable,

$\lambda = np = 100 \times 0.02$
$= 2$

→ Let X be the random variable which denotes the number of defective bulbs

$$P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \; ; \; x = 0, 1, 2, \ldots$$

(i) → Probability that at least one bulb is defective,

$$P(X \geq 1) = 1 - P(X = 0)$$
$$= 1 - \frac{e^{-2} (2)^{0}}{0!}$$
$$= 1 - e^{-2}$$
$$= 1 - (2.718)^{-2}$$
$$= \boxed{0.8647}$$

(ii) → Probability that exactly 7 bulbs are defective,

$$P(X=7) = \frac{e^{-2}\,(2)^7}{7!}$$

$$= \frac{1}{e^2}\,\frac{128}{7\times6\times5\times4\times3\times2\times1} = \boxed{0.0034}$$

* Probability density function :-

→ Let $f(x)$ be a continuous function, then

$$\text{Mean } \bar{x} = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\bar{x})^2 \cdot f(x) \cdot dx \quad ; \bar{x} = \text{mean}$$

→ $f(x)$ continuous function of random variable x is called probability density function if,

(i) $f(x) \geq 0$ for every values of x.

(ii) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$.

/* (iii) $\int_{a}^{b} f(x) \cdot dx = P(a < x < b)$ */

Ex:- The probability density function $f(x)$ of a continuous random variable x is defined by

$$f(x) = \begin{cases} \frac{4}{x^3}, & 5 \leq x \leq 10 \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of A.

→ Here, $f(x) = \frac{A}{x^3}$; $5 \leq x \leq 10$.

→ Since, $f(x)$ is p.d.f.,

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

$$\therefore \left[\int_{-\infty}^{5} + \int_{5}^{10} + \int_{10}^{\infty} \right] f(x) \cdot dx = 1$$

$$\therefore \int_{5}^{10} f(x) \cdot dx = 1 \quad \left(\because f(x) = 0 \text{ for } x \in (-\infty, 5) \cup (10, \infty) \right)$$

$$\therefore \int_{5}^{10} A/x^3 \cdot dx = 1$$

$$\therefore -\frac{A}{2x^2} \Big|_{5}^{10} = 1$$

$$\therefore -A \left[\frac{1}{200} - \frac{1}{50} \right] = 1$$

$$\therefore \boxed{A = \frac{200}{3}}$$

Ex:- A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 0, & x < 2 \\ 1/18 (2x+3), & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$$

Show that it is a p.d. function.

→ $$\int_{-\infty}^{\infty} f(x) \cdot dx = \int_{-\infty}^{2} f(x) \cdot dx + \int_{2}^{4} f(x) \cdot dx + \int_{4}^{\infty} f(x) \cdot dx$$

$$= 0 + \int_{2}^{4} \frac{1}{18}(2x+3)\cdot dx + 0.$$

$$= 1/18 \left[x^2\Big|_2^4 + 3x\Big|_2^4 \right]$$

$$= \frac{1}{18}\left[(16-4)+(12-6)\right]$$

$$= \frac{1}{18}\left[12+6\right] = \boxed{1}$$

$\therefore$ Here, for p.d.f $\int_{-\infty}^{\infty} f(x)\cdot dx = 1$ and

for $2 \le x \le 4$, $f(x) > 0$.

Hence, the given function is p.d.f.

Ex:- A continuous random variable has p.d.f.

$$f(x) = \begin{cases} K\cdot e^{-x/5}, & x \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

then find the value of K.

$\rightarrow$ For p.d.f. we must get $\int_{-\infty}^{\infty} f(x)\cdot dx = 1$.

$$\therefore \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{\infty} f(x)\cdot dx = 1.$$

$$\therefore 0 + \int_{0}^{\infty} K\cdot e^{-x/5}\, dx = 1$$

$$\therefore K\left[\frac{e^{-x/5}}{-1/5}\Big|_0^{\infty}\right] = 1$$

$$\therefore K(-5)(0-1) = 1$$

$$\therefore 5K = 1$$

$$\therefore \boxed{K = \frac{1}{5}}$$

* Exponential Distribution :

→ The continuous random variable x has exponential distribution if its density function is given by,

$$f_x(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}} & , x > 0 , \beta > 0 \\ 0 & , \text{elsewhere.} \end{cases}$$

Ex:- In a construction site, 3 lorries unload materials per hour, on an average. What is the probability that the time between arrivals of successive lorries will be (i) at least 30 minutes. (ii) less than 10 minutes.

→ Given that, $f_x(x) = \begin{cases} 3e^{-3x} & , \\ 0 & , \end{cases}$ already given in the example

(i) $$\int_{1/2}^{\infty} 3 \cdot e^{-3x} \cdot dx = \left. \frac{3 \cdot e^{-3x}}{-3} \right|_{1/2}^{\infty}$$

$$= -\left(e^{-\infty} - e^{-3/2} \right)$$

$$= e^{-3/2}$$

$$= 0.223.$$

(ii) $\int_0^{1/6} 3 \cdot e^{-3x} \cdot dx = \frac{3}{-3} \left[e^{-3x} \right]_0^{1/6}$

$= -\left[e^{\frac{-3}{6}} - e^0 \right] = \boxed{0.4}$

Ex:- $f_X(x) = \begin{cases} 1/6 \cdot e^{-x/6} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$

What is the probability that the person will talk for (a) more than 8 minutes, (b) less than 4 minutes, (c) between 4 and 8 minutes

(a) $\frac{1}{6} \int_8^{\infty} e^{-x/6} \cdot dx = \frac{1}{6} \left[\frac{e^{-x/6}}{-1/6} \Big|_8^{\infty} \right]$

$= \frac{1}{6} (-6) (0 - e^{-4/3})$

$= e^{-4/3} = \boxed{0.2635}$

(b) $\int_0^4 \frac{1}{6} e^{-x/6} \cdot dx = \frac{1}{6} \int_0^4 e^{-x/6} \cdot dx$

$= \frac{1}{6} \left[\frac{e^{-x/6}}{-1/6} \Big|_0^4 \right]$

$= (-1) \left[e^{-2/3} - 1 \right]$

$= 1 - e^{-2/3} = \boxed{0.4865}$

(c) $\int_4^8 \frac{1}{6} e^{-x/6} \cdot dx = \frac{1}{6} \frac{e^{-x/6}}{-1/6} \Big|_4^8$

$= (-1) \left[e^{-4/3} - e^{-2/3} \right]$

$= \boxed{0.2499}$

* Normal Distribution :-

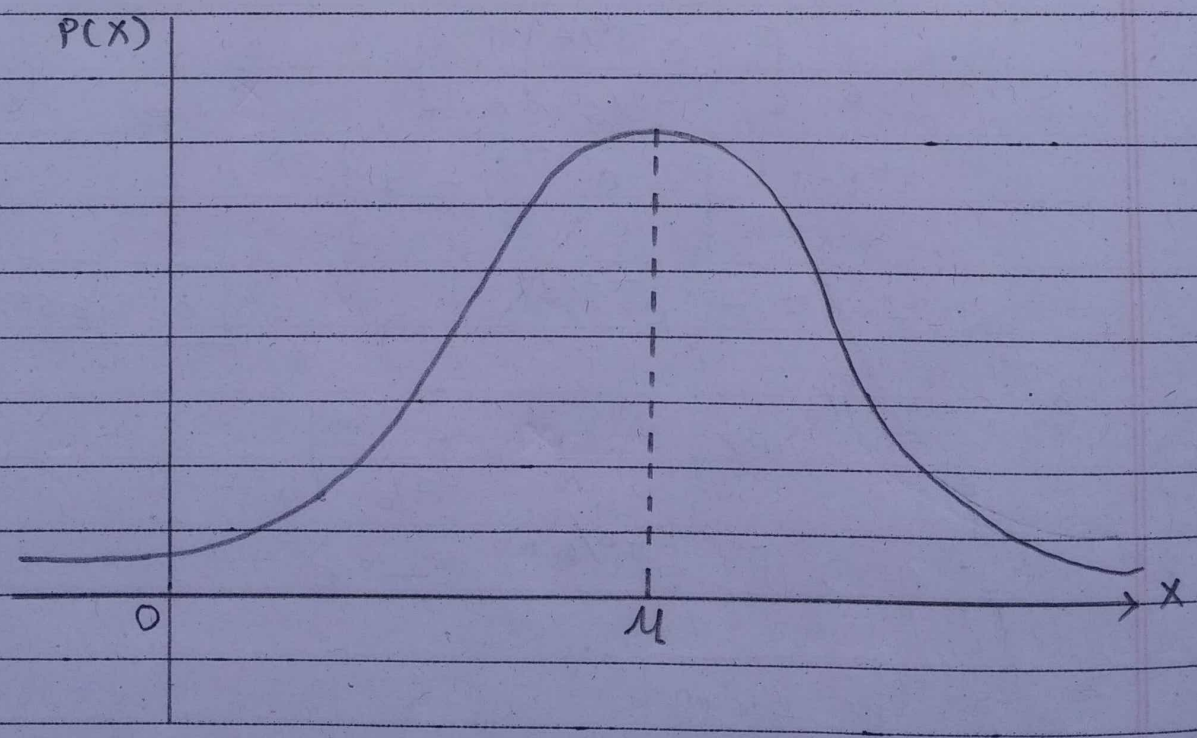
→ A continuous random variable x is said to follow normal distribution with mean μ and variance σ^2, if its probability function is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$-\infty < x < \infty \quad , \sigma > 0$$
$$-\infty < \mu < \infty$$

where, μ and σ are called parameters of the normal distribution.

→ The curve representing the normal distribution is called the normal curve.



* Properties of the Normal Distribution :-

1. It is a bell-shaped symmetrical curve about the ordinate $X = \mu$. The ordinate is maximum at $x = \mu$.

2. It is a unimodal curve and its tails extend infinitely in both the directions. i.e. the curve is asymptotic to x-axis in both the directions.

3. All the three measures of central tendency coincide. i.e., mean = median = mode

4. The total area under the curve gives the total probability of the random variable X taking values between $-\infty$ to ∞. Mathematically,

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1.$$

5. The ordinate at $X = \mu$ divides the area under the normal curve into two equal parts. i.e.

$$\int_{-\infty}^{\mu} f(x) \cdot dx = \int_{\mu}^{\infty} f(x) \cdot dx = \frac{1}{2}$$

6. The value of $f(x)$ is always non-negative for all values of x. i.e. the whole curve lies above the x-axis.

7. The points of inflexion (the point at which curvature changes) of the curve are at $X = \mu + \sigma$ and the curve changes from concave to convex at $x = \mu + \sigma$ to $X = \mu - \sigma$.

8. The area under the normal curve is distributed as follows :

→ The area between ordinates ...

$\mu - \sigma$	and	$\mu + \sigma$	is	68.27 %
$\mu - 2\sigma$	and	$\mu + 2\sigma$	is	95.45 %
$\mu - 3\sigma$	and	$\mu + 3\sigma$	is	99.74 %

* Let $z = \frac{X - \mu}{\sigma}$ be a new random variable.

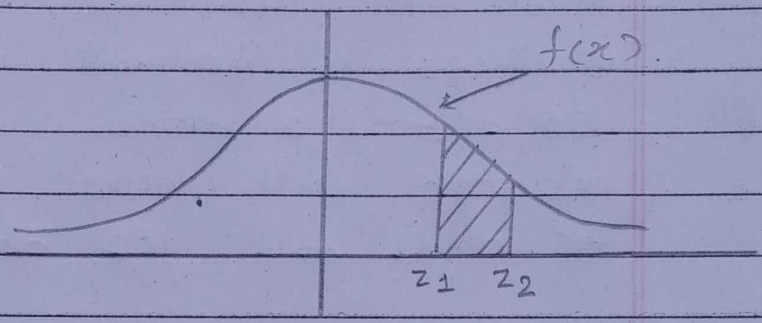
→ The distribution of z is also normal.
Thus if X be a normal random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$ then $z = \frac{X - \mu}{\sigma}$ is also normal random variable. Since the parameters of the distribution of z are fixed, it is known as standard normal distribution.

$$P(x_1 \le X \le x_2) = P\left[\left(\frac{x_1-\mu}{\sigma}\right) \le X \le \left(\frac{x_2-\mu}{\sigma}\right)\right]$$
$$= P(z_1 \le X \le z_2)$$

This ~~distribution~~ probability is equal to the area under the standard normal curve between the ordinates $z = z_1$ and $Z = z_2$ respectively.

* Cases :

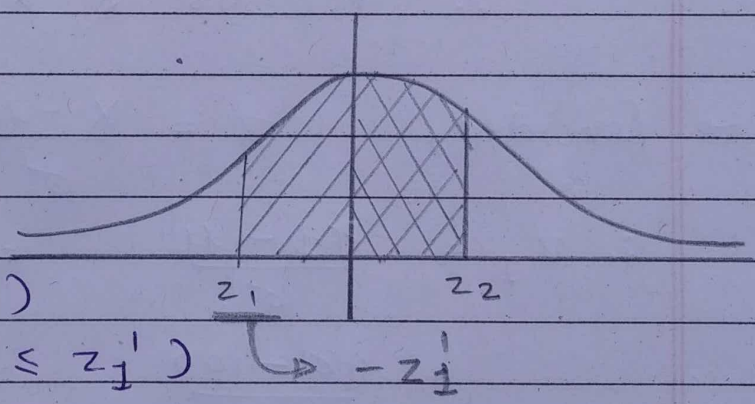
→ Case : 1 :



$P(z_1 \le Z \le z_2)$

$= P(0 \le Z \le z_2)$
$\quad - P(0 \le z \le z_1)$

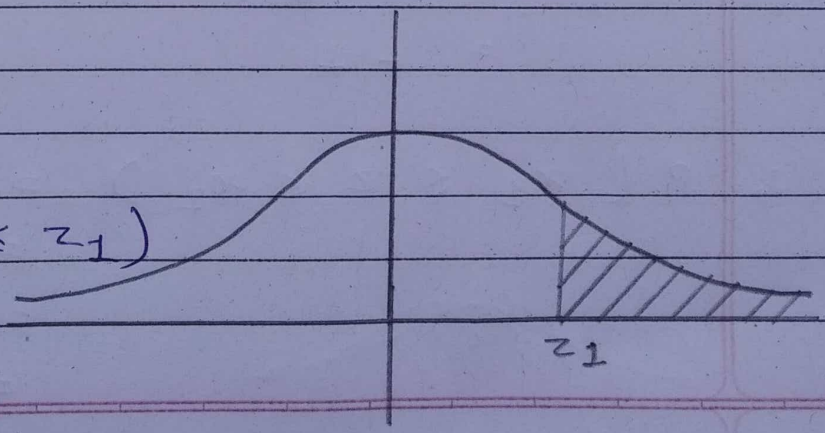
→ Case : 2 :



$P(z_1 \le Z \le z_2)$

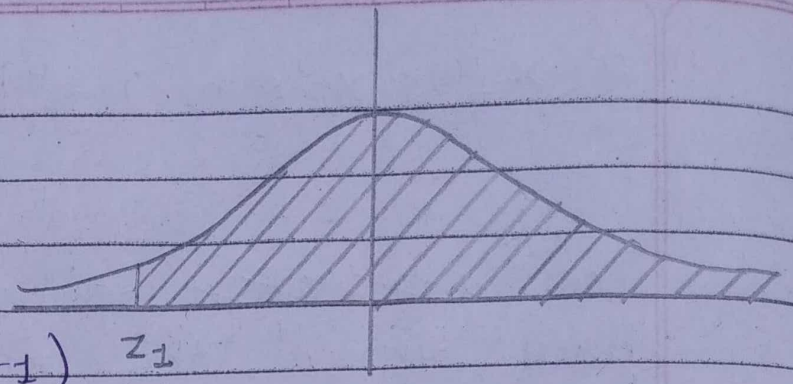
$= P(0 \le z \le z_2)$
$\quad + (0 \le z \le z_1')$

→ Case : 3 :



$P(z_1 \le Z \le \infty)$
$= \frac{1}{2} - P(0 \le Z \le z_1)$

→ Case : 4 :



$P(-z_1 \leq z \leq \infty)$

$= \frac{1}{2} + P(0 \leq z \leq z_1)$ z_1

take positive value

Ex:- The manufacturer knows from his experience that the resistance of the resistor he produces is normal with $\mu = 100$ ohms and SD $= \sigma = 2$ ohms. What percentage of resistor will have resistance between 98 ohms and 102 ohms ?

→ Given, $\mu = 100\,\Omega$, $\sigma = 2\,\Omega$

$\therefore P(98 \leq X \leq 102)$

→ Consider, $z = \frac{X - \mu}{\sigma}$

$\therefore z_1 = \frac{x_1 - \mu}{\sigma}$ and $z_2 = \frac{x_2 - \mu}{\sigma}$

$= \frac{98 - 100}{2}$ $\qquad = \frac{102 - 100}{2}$

$= -1$ $\qquad = 1.$

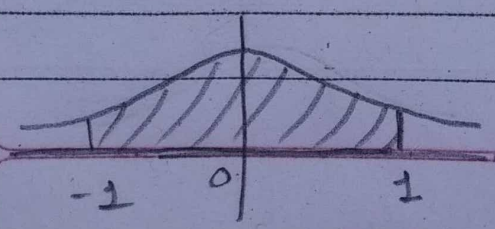
$\therefore P(x_1 \leq z \leq x_2) = P(z_1 \leq z \leq z_2)$

$= P(-1 \leq z \leq 1)$

$= 2\,P(0 \leq z \leq 1)$

$= 2(0.3413)$ ($\because$ By symmetry)

$= 0.6826$



$\therefore$ 68.26 % resistor will have resistance between 98 ohms and 102 ohms.

Ex:- The average seasonal rainfall in a place is 16 inches with an SD of 4 inches. What is the probability that the rainfall in that place will be between 20 and 24 inches in a year?

→ Given, $\mu = 16$ inches, $\sigma = 4$ inches

→ Consider, $z = \dfrac{x - \mu}{\sigma}$

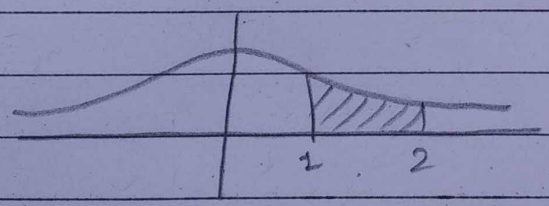
$\therefore z_1 = \dfrac{x_1 - \mu}{\sigma}$ and $z_2 = \dfrac{x_2 - \mu}{\sigma}$

$= \dfrac{20 - 16}{4}$ $\qquad = \dfrac{24 - 16}{4}$

$= 1$ $\qquad = 2$

$\therefore P(x_1 \leq z \leq x_2) = P(z_1 \leq z \leq z_2)$

$= P(1 \leq z \leq 2)$



$= P(0 \leq z \leq 2) - P(0 \leq z \leq 1)$

$= 0.4772 - 0.3413$

$= 0.1359.$

$\therefore$ 13.59 % is the probability that the rainfall will be between 20 and 24 inches in a year.