Unit : 03 Partial Differential Equations and Applications

* $z \rightarrow x, y$

→ $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$

* Derive a partial differential equations by eliminating arbitrary constants from the equations.

Ex:- $z = (x^2+a)(y^2+b)$ — ①.

Differentiating ①, w.r.t x and y respectively, partially

$\frac{\partial z}{\partial x} = 2x(y^2+b)$ and $\frac{\partial z}{\partial y} = 2y(x^2+a)$

$\therefore p = 2x(y^2+b)$ $\therefore q = 2y(x^2+a)$

$\therefore y^2+b = \frac{p}{2x}$ — ② $\therefore x^2+a = \frac{q}{2y}$ — ③

→ Substituting ② and ③ in ①, we get...

$z = \frac{pq}{4xy}$

$\therefore \boxed{pq = 4xyz}$ where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

(it is not linear)

Ex:- $z = a(x+y)+b$ — ①

Differentiating ① partially w.r.t. x and y respectively, we get ...

$\frac{\partial z}{\partial x} = a(1)$ and $\frac{\partial z}{\partial y} = a(1)$

$\therefore \ p = a \qquad\qquad \therefore \ q = a$

$$\Rightarrow \boxed{p = q}$$

where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Ex:- $z = ax + a^2y^2 + b$ —— (1)

Differentiating (1), partially w.r.t. x and y respectively, we get...

$\frac{\partial z}{\partial x} = a(1)$ and $\frac{\partial z}{\partial y} = 2y a^2$

$\therefore \ p = a$ —— (2) $\qquad \therefore \ q = 2y a^2$

$\therefore \ q = 2y p^2 \quad (\because (2))$

$\therefore \boxed{q = 2p^2y}$ (it is not linear)

where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Ex:- $2z = (ax + y)^2 + b$ —— (1)

Differentiating (1), partially w.r.t x and y respectively, we get...

$2 \cdot \frac{\partial z}{\partial x} = 2(ax + y)(a)$ and $2\frac{\partial z}{\partial y} = 2(ax + y)(1)$

$\therefore \ q = ax + y$ —— (2)

$\therefore \ a = \frac{q - y}{x}$ —— (3)

$\therefore \ 2p = 2q\left(\frac{q-y}{x}\right)$ ($\because$ From (2) & (3))

$\therefore \ px = q^2 - qy$

$\therefore \boxed{px - q^2 + qy = 0}$ where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Ex :- $x^2 + y^2 + (z-c)^2 = a^2$ — ①

Differentiating ①, partially w.r.t x and y respectively, we get...

$2x + 0 + 2(z-c)\dfrac{\partial z}{\partial x} = 0$ and $2y + 2(z-c)\cdot\dfrac{\partial z}{\partial y} = 0$

$\therefore x + (z-c)\cdot p = 0$ $\quad\therefore y + (z-c)\cdot q = 0$

$\therefore z - c = \dfrac{-x}{p}$ — ② $\quad\therefore z - c = \dfrac{-y}{q}$ — ③

→ From eqⁿ ② and ③,

$-\dfrac{x}{p} = -\dfrac{y}{q}$

$\therefore \boxed{qx - py = 0}$ where, $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$

* Derive a partial differential equation by eliminating arbitrary functions from the following relations :

Ex :- $z = y^2 + 2g\left(\frac{1}{x} + \log y\right)$ — ①

Differentiating ① partially w.r.t. x and y respectively, we get...

$\dfrac{\partial z}{\partial x} = 0 + 2g'\left(\frac{1}{x} + \log y\right)\cdot\left(-\frac{1}{x^2}\right)$

$\therefore p = -\dfrac{2}{x^2}\cdot g'\left(\frac{1}{x} + \log y\right)$

$\therefore g'\left(\frac{1}{x} + \log y\right) = -\dfrac{p\cdot x^2}{2}$ — ②

and $\dfrac{\partial z}{\partial y} = 2y + 2g'(1/x + \log y) \cdot (1/y)$

$\therefore \dfrac{q}{2} = y + g'\left(\dfrac{1}{x} + \log y\right) \cdot \dfrac{1}{y}$

$\therefore g'\left(\dfrac{1}{x} + \log y\right) = \left(\dfrac{q}{2} - y\right) \cdot y$ — (3)

→ From (2) and (3),

$$-\frac{p}{2} x^2 = y\left(\frac{q}{2} - y\right)$$

$\therefore -px^2 = qy - 2y^2$

$\therefore x^2 p + qy - 2y^2 = 0$

where, $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$.

Ex :- $z = f(y/x)$ — (1)

Differentiating (1) partially w.r.t x and y respectively, we get...

$\dfrac{\partial z}{\partial x} = f'(y/x) \cdot \left(\dfrac{-y}{x^2}\right)$ and $\dfrac{\partial z}{\partial y} = f'(y/x) \cdot \left(\dfrac{1}{x}\right)$

$\therefore p = -\dfrac{y}{x^2} \cdot f'(y/x)$ $\therefore q = \dfrac{1}{x} \cdot f'(y/x)$

$\therefore f'(y/x) = \dfrac{-px^2}{y}$ — (2) $\therefore f'(y/x) = qx$ — (3)

→ From (2) and (3), $\dfrac{-px^{\cancel{2}}}{y\cancel{x}} = qx$

where, $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$ $\therefore \boxed{qy + px = 0}$

★ Ex:- $xyz = f(x+y+z)$ ——— (1)

Differentiating (1) partially w.r.t x and y respectively, we get...

$$(1)yz + xy \cdot \frac{\partial z}{\partial x} = f'(x+y+z)\left(1+0+\frac{\partial z}{\partial x}\right)$$

$$\therefore f'(x+y+z) = \frac{y(z+xp)}{1+p} \quad \text{——— (2)}$$

and
$$(1)xz + xy \cdot \frac{\partial z}{\partial y} = f'(x+y+z)\left(0+1+\frac{\partial z}{\partial y}\right)$$

$$\therefore f'(x+y+z) = \frac{x(z+y \cdot q)}{1+q} \quad \text{——— (3)}$$

$$\Rightarrow \frac{y(z+xp)}{1+p} = \frac{x(z+yq)}{1+q} \quad (\because \text{From (2) \& (3)})$$

$$\therefore y(z+xp)(1+q) = x(z+yq)(1+p)$$

$$\therefore (yz+xyp)(1+q) = (xz+xyq)(1+p)$$

$$\therefore yz + xyp + yzq + \cancel{xypq} - xz - xyq - xzp - \cancel{xypq} = 0$$

$$\therefore px(y-z) + qy(z-x) + z(y-x) = 0$$

where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

Ex:- $z = x \cdot f(x+t) + g(x+t)$ ——— (1)

Differentiating (1) partially w.r.t. x and t respectively, we get...

$$\frac{\partial z}{\partial x} = f(x+t) + x\cdot f'(x+t) + g'(x+t) \quad \text{—— (2)}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = f'(x+t) + x\cdot f''(x+t) + (1) f'(x+t) + g'(x+t)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 2f'(x+t) + \underbrace{x\cdot f''(x+t) + g'(x+t)} \quad \text{—— (3)}$$

and $$\frac{\partial z}{\partial t} = x\cdot f'(x+t) + g'(x+t) \quad \text{—— (4)}$$

$$\therefore \frac{\partial^2 z}{\partial t^2} = x\cdot f''(x+t) \cancel{+ f'(x+t)} + g''(x+t) \quad \text{—— (5)}$$

and $$\frac{\partial^2 z}{\partial x \partial t} = f'(x+t) + \underbrace{x\cdot f''(x+t) + g''(x+t)} \quad \text{—— (6)}$$

→ From (3),

$$f'(x+t) = \frac{1}{2}\left[\frac{\partial^2 z}{\partial x^2} - \left(x f''(x+t) + g''(x+t)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2}\right] \quad (\because \text{from (5)}) \quad \text{—— (7)}$$

→ From (6),

$$f'(x+t) = \frac{\partial^2 z}{\partial x \partial t} - \left(x\cdot f''(x+t) + g''(x+t)\right)$$

$$= \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2} \quad (\because \text{from (5)})$$

$$\therefore \frac{1}{2}\cdot\frac{\partial^2 z}{\partial x^2} - \frac{1}{2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2} \quad (\because \text{from (7)})$$

$$\therefore \frac{1}{2}\cdot\frac{\partial^2 z}{\partial x^2} + \frac{1}{2}\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x \partial t} = 0$$

$$\therefore \frac{r}{2} + \frac{t}{2} - s = 0$$

$\therefore$ $\boxed{r + t - 2s = 0.}$

where, $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial t}$, $t = \dfrac{\partial^2 z}{\partial t^2}$.

* Lagrange's linear partial differential eq^n :-

→ When we are given the relation of the type

$f(u, v) = 0$ then the partial differential equation is given by,

$$P p + Q q = R.$$

where, $P = \dfrac{\partial(u, v)}{\partial(y, z)}$ and $p = \dfrac{\partial z}{\partial x}$

$Q = \dfrac{\partial(u, v)}{\partial(x, z)}$ $q = \dfrac{\partial z}{\partial y}$

$R = \dfrac{\partial(u, v)}{\partial(x, y)}$.

Ex :- Form the partial differential equation by eliminating arbitrary function from $f(x^2+y^2, z - xy) = 0$.

→ Given, $f(x^2+y^2, z - xy) = 0$ —— ①

let $x^2+y^2 = u$ and $z - xy = v$

$\therefore$ $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial u}{\partial y} = 2y$, $\dfrac{\partial u}{\partial z} = 0$.

and $\dfrac{\partial v}{\partial x} = -y$, $\dfrac{\partial v}{\partial y} = -x$, $\dfrac{\partial v}{\partial z} = 1$

→ We know that the P.D.E. corresponding to (1) is

$$Pp + Qq = R \quad \text{--- (2)}$$

where, $P = \dfrac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 2y & 0 \\ -x & 1 \end{vmatrix} = 2y$$

$Q = \dfrac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x} \end{vmatrix}$

$$= \begin{vmatrix} 0 & 2x \\ 1 & -y \end{vmatrix} = -2x$$

$R = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -y & -x \end{vmatrix}$

$$= -2x^2 + 2y^2$$

→ Substituting all values in eq^n (2),

$\cancel{2y(2x) + 2x(2y) = -2x^2 + 2y^2}$

$\therefore 2yp - 2xq = 2(y^2 - x^2)$

$\therefore \boxed{py - qx = y^2 - x^2}$

~~where,~~ $\cancel{p = \dfrac{\partial u}{\partial x}}$, $\cancel{q = \dfrac{\partial u}{\partial y}}$

Ex :- $f(z^2 - xy, \ x/z) = 0$ — (1)

→ let $z^2 - xy = u$ and $\frac{x}{z} = v$

$\therefore \frac{\partial u}{\partial x} = -y$, $\frac{\partial u}{\partial y} = -x$, $\frac{\partial u}{\partial z} = 2z$.

and $\frac{\partial v}{\partial x} = \frac{1}{z}$, $\frac{\partial v}{\partial y} = 0$, $\frac{\partial v}{\partial z} = -\frac{x}{z^2}$.

→ We know that the P.D.E. corresponding to (1) is,

$$Pp + Qq = R \quad \text{— (2)}$$

where, $P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} -x & 2z \\ 0 & -x/z^2 \end{vmatrix} = x^2/z^2$$

$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} -y & 2z \\ 1/z & -x/z^2 \end{vmatrix}$

$$= -\frac{xy}{z^2} + 2$$

$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -y & -x \\ 1/z & 0 \end{vmatrix}$

$$= 0 + x/z = \frac{x}{z}$$

→ Substituting all values in (2),

$$\frac{x^2}{z^2} p + \left(-\frac{xy}{z^2} + 2\right) q = \frac{x}{z}$$

$$\therefore \; p\frac{x^2}{z^2} + 2q - \frac{xy}{z^2}q = \frac{x}{z}$$

$$\therefore \; \boxed{p x^2 + 2q z^2 - xy q = xz}$$

Ex:- $f(xy + z^2, \; x + y + z) = 0$ —— (1).

→ let $u = xy + z^2$ and $v = x + y + z$

$$\therefore \frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial u}{\partial z} = 2z$$

and $\dfrac{\partial v}{\partial x} = 1, \quad \dfrac{\partial v}{\partial y} = 1, \quad \dfrac{\partial v}{\partial z} = 1$

→ We know that the P.D.E. corresponding to (1) is,

$$Pp + Qq = R \quad \text{—— (2).}$$

where, $P = \dfrac{\partial(u,v)}{\partial(y,z)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} x & 2z \\ 1 & 1 \end{vmatrix}$$

$$= x - 2z$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} x & y \\ 1 & 1 \end{vmatrix}$$

$$= x - y.$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix}$$

$$= y - x.$$

→ Substituting all value in eq. (2),

$$\boxed{(x-2z)p + (x-y)q = y-x}$$

* Lagrange's Equation :-

→ Non linear → Method of grouping
↳ Method of multipliers

Ex :- Solve the following Lagrange's equation.

$x^2p + y^2q = (x+y)z$ —— (1)

→ Here, $P = x^2$, $Q = y^2$, $R = (x+y).z$.

∴ The Auxiliary eq^n is, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\therefore \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \quad —— (2)$$

→ From (2), $\frac{dx}{x^2} = \frac{dy}{y^2}$

$$\therefore x^{-2}.dx = y^{-2}\,dy$$

Integrating both the sides,

$$\int x^{-2}.dx = \int y^{-2}.dy$$

$$\therefore \frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$$

$$\therefore -\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\therefore -\frac{1}{x} + \frac{1}{y} = c_1, \quad c_1 \in R$$

→ From ②; $\frac{1/x \cdot dx}{x} = \frac{1/y \cdot dy}{y} = \frac{1/z \cdot dz}{x+y}$.

l, m, n multiplier : $1/x, 1/y, 1/z$.

$\therefore \frac{1/x\, dx + 1/y\, dy - 1/z \cdot dz}{x+y-x-y}$

$\therefore \int \frac{1}{x} \cdot dx + \int \frac{1}{y} \cdot dy - \int \frac{1}{z} \cdot dz = 0$

$\therefore \log x + \log y = \log z + \log c_2$

$\therefore \log xy = \log z c_2$

$\therefore \frac{xy}{z} = c_2, \quad c_2 \in R$.

$\therefore f(u, v) = f\left(-\frac{1}{x} + \frac{1}{y}, \frac{xy}{z}\right) = 0$.

(The g. solⁿ of this eqⁿ.)

Ex:- $\frac{y^2 z}{x} p + xzq = y^2$. ——①.

→ Comparing ① with $Pp + Qq = R$, we have

$P = \frac{y^2 z}{x}, \quad Q = xz, \quad R = y^2$

$\therefore$ The auxiliary eqⁿ are,

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$ ——②

→ Taking first two fractions, we have

$\therefore \dfrac{dx}{\dfrac{y^2 \cancel{z}}{x}} = \dfrac{dy}{x\cancel{z}}$

$\therefore \dfrac{x}{y^2} \cdot dx = \dfrac{1}{x} \cdot dy$

$\therefore x^2 \cdot dx = y^2 \cdot dy$

$\therefore \int x^2 \cdot dx = \int y^2 \cdot dy$ ($\because$ Integrating both the sides)

$\therefore \dfrac{x^3}{3} = \dfrac{y^3}{3} + C_1$

$\therefore u = \dfrac{x^3}{3} - \dfrac{y^3}{3} = C_1$, $C_1 \in R$.

→ From (2), Taking first and last part,

$\dfrac{x \cdot dx}{y^2 z} = \dfrac{dz}{y^2}$

$\therefore x \cdot dx = z \cdot dz$

$\therefore \int x \cdot dx = \int z \cdot dz$ ($\because$ Integrating both the sides)

$\therefore \dfrac{x^2}{2} = \dfrac{z^2}{2} + C_2$

$\therefore \dfrac{x^2}{2} - \dfrac{z^2}{2} = C_2 = v$

$\therefore$ The general solⁿ is given by,

$f(u, v) = f\left(\dfrac{x^3}{3} - \dfrac{y^3}{3} , \dfrac{x^2}{2} - \dfrac{z^2}{2} \right) = 0$

Ex:- $y^2 p - xyq = x(z - 2y)$ —— (1)

→ Comparing (1) with $Pp + Qq = R$, we have,

$P = y^2$, $Q = -xy$, $R = x(z-2y)$

$\therefore$ The auxiliary eq^n are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$\rightarrow$ Taking first two parts/fractions, we have,

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$\therefore \quad x \cdot dx = -y \cdot dy$

$\therefore \quad \int x \cdot dx = -\int y \cdot dy$ ($\because$ Integrating both the sides)

$\therefore \quad \frac{x^2}{2} = -\frac{y^2}{2} + C_1$, $C_1 \in R$

$\therefore \quad \frac{x^2}{2} + \frac{y^2}{2} = C_1 = u$

and Taking ~~first and~~ last two fractions,

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$\therefore \quad z \cdot dy - 2y \cdot dy = -y \cdot dz$

$\therefore \quad -2y \cdot dy + z \cdot dy + y \cdot dz = 0$

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$\therefore \quad d(yz) = 2y \cdot dy$

$\therefore \quad \int d(yz) = \int 2y \cdot dy$

($\because$ Integrating both the sides)

$\therefore \quad yz = y^2 + C_2 = v$. , $C_2 \in R$

$\therefore \quad -y^2 + yz = C_2$

$\therefore$ The general solⁿ is given by,

$$f(u,v) = f\left(\frac{x^2}{2} + \frac{y^2}{2},\ yz - y^2\right) = 0$$

Ex:- $(y^2+z^2)p - xyq - xz = 0.$

$\therefore (y^2+z^2)p - xy \cdot q = xz$ — (1)

→ Comparing (1) with $Pp + Qq = R$, we have,

$P = y^2+z^2$, $Q = -xy$, $R = xz.$

$\therefore$ The auxiliary equations are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{xz} \quad \text{— (2)}$$

→ Taking last two fractions, we have,

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}z}$$

$\therefore -\int \frac{1}{y} \cdot dy = \int \frac{1}{z} \cdot dz$ ($\because$ Integrating both the sides)

$\therefore -\log y = \log z + \log c_1$

$\therefore yz = c_1$

$\therefore c_1 = u = yz$

→ From (2), multipliers we have $x, y, -z$ respectively,

$$\therefore \text{Each fraction} = \frac{x \cdot dx + y \cdot dy - z \cdot dz}{x(y^2+z^2) - xy^2 - xz^2}$$

$$= \frac{x \cdot dx + y \cdot dy - z \cdot dz}{0}$$

$$\therefore \quad x \cdot dx + y \cdot dy - z \cdot dz = 0.$$

→ Integrating, we get,

$$\int x \cdot dx + \int y \cdot dy - \int z \cdot dz = 0$$

$$\therefore \quad \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c_2 = v.$$

∴ The general solution is given by,

$$f(u, v) = f\left(yz, \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} \right) = 0$$

Ex:- $(mz - ny)p + (nx - lz)q = (ly - mx)$ — ①

→ Comparing eqⁿ ① with $Pp + Qq = R$, we have

$P = mz - ny$, $Q = nx - lz$, $R = ly - mx$

∴ The auxiliary equations are,

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \quad \text{— ②}$$

→ Taking multipliers x, y, z respectively, we have

$$\text{Each Fraction} = \frac{x \cdot dx + y \cdot dy + z \cdot dz}{xzm - xyn + xyn - yzl + yzl - xzm}$$

$$\therefore \quad x \cdot dx + y \cdot dy + z \cdot dz = 0$$

$\therefore \int x \cdot dx + \int y \cdot dy + \int z \cdot dz = \int 0$. ($\because$ Integrating)

$\therefore \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1 = u$

$\rightarrow$ Now, taking multipliers l, m, n respectively, we have

Each fraction $= \dfrac{l \cdot dx + m\,dy + n\,dz}{lmz - lny + mnx - mlz + nly - mnx}$

$\therefore l \cdot dx + m \cdot dy + n\,dz = 0$.

$\therefore \int l \cdot dx + \int m \cdot dy + \int n \cdot dz = 0$.

$\therefore lx + my + nz = c_2 = v$.

$\therefore$ The general solⁿ is given by,

$f(u, v) = f\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, \; lx + my + nz \right) = 0$.

* Special types of non-linear PDE of first order :-

* Case : 1 : Equations involving only p and q that is, $f(p, q) = 0$.

$\rightarrow$ Assume that $z = ax + by + c$ is a solution of $f(p, q) = 0$.

$p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$

If we find 'a' then, $z = f(b) \cdot x + by + c$
↳ In terms of b.

OR

If we find 'b' then, $z = ax + f(a)y + c$
↳ In terms of a

Ex:- $pq = 1$. — (1)

→ let $z = ax + by + c$ be the solⁿ of given eqⁿ.

$\therefore p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

$\therefore a \cdot b = 1$ ($\because$ from eqⁿ (1))

$\therefore a = \frac{1}{b} = f(b)$

$\therefore$ The complete solⁿ is $z = f(b)x + by + c$

$= \frac{x}{b} + by + c$

where, b, c arbitrary constants.

Ex:- $pq + p + q = 0$ — (1)

→ let $z = ax + by + c$ be the solⁿ of the given eqⁿ.

$\therefore p = a$ and $q = b$.

→ From eqⁿ (1),

$\therefore ab + a + b = 0$.

$\therefore a(b+1) = -b$

$\therefore a = \frac{-b}{b+1} = f(b)$.

$\therefore$ The complete solⁿ is $z = f(b)x + by + c$

$= \frac{-b}{b+1}x + by + c$

where, b, c arbitrary constants.

Ex :- $\sqrt{p} + \sqrt{q} = 1$ ——— (1).

→ let $z = ax + by + c$ be the solⁿ of (1),

$\therefore p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$.

$\therefore$ From (1), $\sqrt{a} + \sqrt{b} = 1$

$\therefore a^{1/2} = 1 - b^{1/2}$

$\therefore a = (1 - \sqrt{b})^2 = f(b)$.

$\therefore$ The complete solⁿ is given by,

$$z = f(b)x + by + c$$
$$= (1 - \sqrt{b})^2 + by + c$$

where, b, c arbitrary constant

Ex :- $p^2 + q^2 = npq$, where n is fixed number ——— (1).

→ let $z = ax + by + c$ be the solⁿ of (1),

$\therefore p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

$\therefore$ From (1), $a^2 + b^2 = nab$

$\therefore a^2 - nab = -b^2$

$\therefore a^2 - nab + b^2 = 0$. $Aa^2 + Ba + C = 0$

$\therefore A = 1, \quad B = -nb, \quad C = b^2$

$$\therefore A = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-(-nb) \pm \sqrt{(-nb)^2 - 4(1)(b^2)}}{2(1)}$$

$$= \frac{nb \pm b\sqrt{n^2 - 4}}{2}$$

$$\therefore A = b\left(\frac{n \pm \sqrt{n^2 - 4}}{2}\right) = a$$

∴ The complete solⁿ is given by,

$$z = f(b)\,x + by + c$$

$$\therefore z = b\left(\frac{n \pm \sqrt{n^2-4}}{2}\right)x + by + c$$

where, b, c arbitrary constants.

Ex:- $q = e^{-p/\alpha}$, where α is fixed constant. — (1)

→ let $z = ax + by + c$ be the solⁿ of (1),

$$\therefore p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = b$$

∴ From (1), $b = e^{-a/\alpha} = f(a)$.

∴ The complete solⁿ is given by,

$$z = ax + f(a)y + c$$

$$= a + e^{-\frac{a}{\alpha}} \cdot y + c \; ; \; a, c = \text{arbitrary constants.}$$

* **Case : 2 :** Equations not involving the independent variables, that is $f(z, p, q) = 0$.

→ Assume that $u = x + ay$, where a = arbitrary constant.

$$z \rightarrow u \begin{matrix} \nearrow x \\ \searrow y \end{matrix}$$

$\therefore\ p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$

$\therefore\ \boxed{p = \frac{dz}{du}}$ $\quad \left(\because \frac{\partial u}{\partial x} = 1\right)$

and $q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$

$\therefore\ \boxed{q = a \cdot \frac{dz}{du}}$ $\quad \left(\because \frac{\partial u}{\partial y} = a\right)$

→ Substituting the value of p and q in $f(z, p, q) = 0$, we obtain ordinary differential equation by usual methods and replace u by $x + ay$.

Ex :- $q^2 = z^2 p^2 (1 - p^2)$ —— (1)

let $u = x + ay$, $a \in R$

Then, $p = \frac{dz}{du}$, $q = a \cdot \frac{dz}{du}$ $\quad (\because$ Chain rule$)$

→ Substituting these values in (1), we get,

$$a^2 \cancel{\left(\frac{dz}{du}\right)^2} = z^2 \cdot \cancel{\left(\frac{dz}{du}\right)^2} \left(1 - \left(\frac{dz}{du}\right)^2\right)$$

$$\therefore\ \frac{a^2}{z^2} = 1 - \left(\frac{dz}{du}\right)^2$$

$$\therefore\ \left(\frac{dz}{du}\right)^2 = 1 - \frac{a^2}{z^2} = \frac{z^2 - a^2}{z^2}$$

$$\therefore\ \frac{dz}{du} = \pm \frac{\sqrt{z^2 - a^2}}{z}$$

$\rightarrow \int f'(x)\,[f(x)]^n\,dx$
$\dfrac{[f(x)]^{n+1}}{n+1}$

$$\therefore \frac{z \cdot dz}{\sqrt{z^2-a^2}} = \pm\, du.$$

Integrating both the sides, we get,

$$\frac{1}{2}\int 2z\,(z^2-a^2)^{-\frac{1}{2}}\,dz = \pm\int du + c$$
$$, c \in R$$

$$\Rightarrow \frac{1}{2}\,\frac{(z^2-a^2)^{\frac{1}{2}}}{1/2} = \pm\, u + c$$

$$\Rightarrow (z^2-a^2)^{1/2} = \pm\, u + c$$

$$\Rightarrow z^2 - a^2 = (u+c)^2$$

$$\Rightarrow z^2 = a^2 + (u+c)^2$$

$$\Rightarrow z = \pm\sqrt{a^2+(u+c)^2} \quad (\because u = x+ay)$$

$$\Rightarrow z = \pm\sqrt{a^2+(x+ay+c)^2}\ ; \ a, c \in R$$

Ex :- $z = p^2 + q^2$ —— (1).

→ let $u = x + ay$, $a \in R$

Then $p = \dfrac{dz}{du}$, $q = a \cdot \dfrac{dz}{du}$ ($\because$ Chain rule)

→ Substituting all values in (1), we get,

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2.$$

$$\therefore z = \left(\frac{dz}{du}\right)^2\left[1+a^2\right]$$

$$\therefore \left(\frac{dz}{du}\right)^2 = \frac{z}{1+a^2}$$

$$\therefore \frac{dz}{du} = \frac{\pm\sqrt{z}}{\sqrt{1+a^2}}$$

$$\therefore \frac{dz}{\sqrt{z}} = \pm\frac{1}{\sqrt{1+a^2}}\,du$$

Integrating both the sides, we get,

$$\int \frac{1}{\sqrt{z}} \cdot dz = \pm \int \frac{1}{\sqrt{1+a^2}} \, dy + c' \quad , c' \in R$$

$$\therefore \frac{z^{1/2}}{1/2} = \frac{\pm y}{\sqrt{1+a^2}} + c'$$

$$\therefore 2 \cdot z^{1/2} (1+a^2)^{1/2} = \pm y + c'(1+a^2)^{1/2}$$

$$\therefore 2 z^{1/2} (1+a^2)^{1/2} = \pm y + c$$

$$\therefore 4z(1+a^2) = (y+c)^2$$

$$\therefore z = \frac{(x+ay+c)^2}{4(1+a^2)} \quad , a, c \in R.$$

Ex :- $p(1+q) = qz$ ——— (1)

→ let $y = x + ay$, $a \in R$

Then $p = \frac{dz}{dy}$, $q = a \cdot \frac{dz}{dy}$ ($\because$ Chain rule)

→ Substituting these values in (1), we get,

$$\left(\frac{dz}{dy}\right)\left[1 + a \cdot \frac{dz}{dy}\right] = \left(\frac{a\,dz}{dy}\right) \cdot z$$

$$\therefore \frac{dz}{dy} = \frac{az-1}{a}$$

$$\therefore \frac{1}{az-1} \cdot dz = \frac{1}{a} \cdot dy$$

Integrating both the sides,

$$\int \frac{a}{za-1} \cdot dz = \int 1 \cdot dy + c' \quad , c' \in R$$

$$\Rightarrow \log(az-1) = y + c'$$

$$\Rightarrow az - 1 = e^{y+c'} = e^{y} \cdot e^{c'} = e^{y} \cdot c$$

$$(\because c = e^{c'})$$

$\Rightarrow az = 1 + e^{u} \cdot c$

$\Rightarrow z = \frac{1}{a}\left(1 + c \cdot e^{x+ay}\right)$; $a, c \in R$

Ex :- $pq = z^2$ — (1)

→ let $u = x + ay$; $a \in R$

Then $p = \frac{dz}{du}$, $q = a \cdot \frac{dz}{du}$.

→ Substituting these values in (1),

$$\left(\frac{dz}{du}\right)\left(a \cdot \frac{dz}{du}\right) = z^2$$

$$\therefore \left(\frac{dz}{du}\right)^2 = \frac{z^2}{a}$$

$$\therefore \frac{dz}{du} = \pm \frac{z}{\sqrt{a}}$$

$$\therefore \frac{dz}{z} = \pm \frac{1}{\sqrt{a}} \cdot du$$

Integrating both the sides,

$$\int \frac{1}{z} \cdot dz = \pm \frac{1}{\sqrt{a}} \int 1 \, du + c' \quad , c' \in R$$

$$\therefore \log z = \pm \frac{1}{\sqrt{a}} \cdot u + c'$$

$$\therefore z = e^{\pm u/\sqrt{a} + c'}$$

$$\therefore z = e^{\pm \frac{u}{\sqrt{a}}} \cdot e^{c'} = e^{\pm \frac{u}{\sqrt{a}}} \cdot c \quad (\because c = e^{c'})$$

$$\therefore z = c \cdot e^{\frac{\pm (x+ay)}{\sqrt{a}}} \quad ; a, c \in R$$

Ex :- $p^2 z^2 + q^2 = 1$ ——— (1)

→ let $u = x + ay$, $a \in R$.

Then, $p = \frac{dz}{du}$ and $q = a \cdot \frac{dz}{du}$

→ Substituting these values in (1), we get,

$$\left(\frac{dz}{du}\right)^2 \cdot z^2 + \left(a \cdot \frac{dz}{du}\right)^2 = 1$$

$$\therefore \left(\frac{dz}{du}\right)^2 \left[z^2 + a^2 \right] = 1.$$

$$\therefore \left(\frac{dz}{du}\right)^2 = \frac{1}{z^2 + a^2}$$

$$\therefore \frac{dz}{du} = \frac{\pm 1}{\sqrt{z^2+a^2}}$$

$$\therefore \int \sqrt{z^2+a^2}\, dz = \pm \int du + C \quad , \quad C = \text{constant}$$

$$\therefore \frac{z}{2}\sqrt{z^2+a^2} + \frac{a^2}{2} \log \left| z + \sqrt{z^2+a^2} \right| = \pm u + c'$$

$$\therefore \frac{a^2}{2} \log \left| z + \sqrt{z^2+a^2} \right| = -\frac{z}{2}\sqrt{z^2+a^2} \pm u + c'$$

$$\therefore \log \left| z + \sqrt{z^2+a^2} \right| = \frac{2}{a^2}\left[\pm u - \frac{z}{2}\sqrt{z^2+a^2} + c' \right]$$

$$\therefore \left| z + \sqrt{z^2+a^2} \right| = e^{\left(\pm \frac{2}{a^2} u - \frac{z}{a^2}\sqrt{z^2+a^2} + \frac{2c'}{a^2} \right)}$$

$$\therefore \left| z + \sqrt{z^2+a^2} \right| = e^{\pm \frac{2u}{a^2} - \frac{z}{a^2}\sqrt{z^2+a^2}} \cdot e^{\frac{2c'}{a^2}}$$

$$\therefore \left| z + \sqrt{z^2+a^2} \right| = e^{\pm \frac{2u}{a^2} - \frac{z}{a^2}\sqrt{z^2+a^2}} \cdot C$$

$$\left(\because C = e^{\frac{2c'}{a^2}} \right)$$

(C = constant)

Ex:- $z^2(p^2+q^2+1) = a^2$, where a is fixed constant. — (1)

→ let $u = x + by$, $b \in \mathbb{R}$.

Then $p = \frac{dz}{du}$, $q = b \cdot \frac{dz}{du}$.

→ Substituting these values in (1), we get,

$$z^2\left(\left(\frac{dz}{dy}\right)^2 + b^2\left(\frac{dz}{dy}\right)^2 + 1\right) = a^2$$

$$\therefore \left[\frac{dz}{dy}\right]^2 + b^2\left[\frac{dz}{dy}\right]^2 + 1 = \frac{a^2}{z^2}$$

$$\therefore \left(\frac{dz}{dy}\right)^2\left[1+b^2\right] = \frac{a^2}{z^2} - 1 = \frac{a^2-z^2}{z^2}$$

$$\therefore \left(\frac{dz}{dy}\right)^2 = \frac{a^2-z^2}{z^2}\left(\frac{1}{1+b^2}\right)$$

$$\therefore \frac{dz}{dy} = \pm\frac{\sqrt{a^2-z^2}}{z}\cdot\frac{1}{\sqrt{1+b^2}}$$

$$\therefore \frac{z}{\sqrt{a^2-z^2}}\,dz = \pm\frac{1}{\sqrt{1+b^2}}\cdot dy$$

Integrating both the sides,

$$\therefore \frac{-1}{2}\int\frac{(-2z)}{\sqrt{a^2-z^2}}\cdot dz = \pm\frac{1}{\sqrt{1+b^2}}\int dy + c' \quad (c' \in R)$$

$$\therefore \frac{-1}{2}\int(-2z)(a^2-z^2)^{\frac{-1}{2}}\,dz = \pm\frac{y}{\sqrt{1+b^2}} + c'$$

$$\therefore -\frac{1}{\cancel{2}}\,\frac{(a^2-z^2)^{1/2}}{\cancel{1/2}} = \pm\frac{y}{\sqrt{1+b^2}} + c'$$

$$\therefore (-1)(a^2-z^2)^{1/2} = \pm\frac{y}{\sqrt{1+b^2}} + c'$$

$$\therefore a^2-z^2 = \left(\frac{y}{\sqrt{1+b^2}} + c'\right)^2$$

$$\therefore z^2 = a^2 - \left(\frac{x+by}{\sqrt{1+b^2}} + c'\right)^2 ;\ b, c' \in R$$

* Case : 3 : Separable equations,
that is $f(x, p) = g(y, q)$

→ Assume $f(x, p) = g(y, q) = a$ (constant)
From these we can find the value of p and q as $p = f(a, x)$ and $q = g(a, y)$.
Substituting the value of p and q in $dz = p \cdot dx + q \cdot dy$, and integrating it, we get the complete solution.

Ex:- Solve $p + q = x + y$

→ Given, $p + q = x + y$ —— (1)

$\therefore \quad p - x = y - q = a$ (say)

$\Rightarrow p = a + x$ and $q = y - a$.

→ Substituting these values in $dz = p \cdot dx + q \cdot dy$ we get,

$$dz = (a + x) \cdot dx + (y - a) \cdot dy$$

Integrating both the sides, we get,

$$z = ax + \frac{x^2}{2} + \frac{y^2}{2} - ay + c \quad , c \in R$$

$$\therefore z = a(x - y) + \frac{1}{2}(x^2 + y^2) + c$$

c = constant.

Ex:- Solve $p^2 + q^2 = x + y$.

→ Given, $p^2 + q^2 = x + y$ —— (1)

$\therefore \quad p^2 - x = y - q^2 = a$ (say).

$\Rightarrow p^2 = a + x$ $\qquad$ $\Rightarrow q^2 = y - a$

$\Rightarrow p = \pm\sqrt{a+x}$ $\qquad$ $\Rightarrow q = \pm\sqrt{y-a}$

→ Substituting these values in ~~[illegible]~~ $dz = p \cdot dx + q \cdot dy$, we get,

$\therefore dz = \pm\sqrt{a+x} \cdot dx + (\pm\sqrt{y-a}) \cdot dy$

Integrating both the sides, we get,

$$z = \frac{(a+x)^{3/2}}{3/2} \pm \frac{(y-a)^{3/2}}{3/2} + c, \quad c \in \mathbb{R}$$

$$\therefore z = \frac{2}{3}(a+x)^{\frac{3}{2}} \pm \frac{2}{3}(y-a)^{\frac{3}{2}} + c$$

c = constants

Ex:- Solve $yp = 2yx + \log q$.

→ Given, $yp = 2yx + \log q$ — (1)

$\therefore p \cdot y - 2yx = \log q$

$\therefore y(p - 2x) = \log q$

$$\therefore p - 2x = \frac{\log q}{y} = a \quad \text{(say)}$$

$\Rightarrow p = a + 2x$ $\qquad$ $\Rightarrow \log q = ay$

$\Rightarrow q = e^{ay}$

→ Substituting these values in $dz = p \cdot dx + q \cdot dy$, we get,

$dz = (a + 2x) \cdot dx + e^{ay} \cdot dy$

$\therefore$ Integrating both the sides,

$z = ax + x^2 + e^{ay} \cdot a + c$, $c \in \mathbb{R}$

c = constant

EX :- $p^2 y (1+x^2) = q x^2$

→ Given, $p^2 y (1+x^2) = q \cdot x^2$ — (1)

$\therefore p^2 \frac{(1+x^2)}{x^2} = \frac{1}{y} \cdot q$

$\therefore p^2 \left(\frac{1+x^2}{x^2} \right) = q \cdot \left(\frac{1}{y} \right) = a$

$\Rightarrow p^2 = \frac{x^2}{1+x^2} \cdot a \quad \Rightarrow q = ay.$

$\Rightarrow p = \frac{\pm x}{\sqrt{1+x^2}} \sqrt{a}.$

→ Substituting these values in $dz = p \cdot dx + q \cdot dy$, we get,

$$dz = \frac{\pm x}{\sqrt{1+x^2}} \sqrt{a} \cdot dx + ay \cdot dy.$$

Integrating both the sides, we get,

$$\int dz = \pm \frac{\sqrt{a}}{2} \int 2x \cdot (1+x^2)^{-1/2} \cdot dx + a \int y \cdot dy.$$

$+ c$, c=constant

$\therefore z = \pm \frac{1}{2} \sqrt{a} \frac{(1+x^2)^{1/2}}{1/2} + a \cdot \frac{y^2}{2} + c$

$\therefore z = \pm (1+x^2)^{1/2} \sqrt{a} + a \cdot \frac{y^2}{2} + c$

$\therefore z = \pm \sqrt{a} \, (1+x^2)^{1/2} + \frac{a}{2} \cdot y^2 + c$

c = constant

Ex:- Solve : $x(1+y)p = y(1+x) \cdot q$

→ Given, $x(1+y)p = y(1+x) \cdot q$

$\therefore p \cdot \frac{x}{1+x} = q \cdot \frac{y}{1+y}$

$\therefore p \cdot \frac{x}{1+x} = q \cdot \frac{y}{1+y} = a$

$\Rightarrow p = a \frac{(1+x)}{x}$ $\Rightarrow q = a \cdot \frac{(1+y)}{y}$

→ Substituting these values in $dz = p\,dx + q\,dy$, we get,

$dz = a \cdot \frac{1+x}{x} \cdot dx + a \cdot \frac{1+y}{y} \cdot dy$

Integrating both the sides, we get,

$\int dz = a \int \left(\frac{1}{x} + 1 \right) \cdot dx + a \int \left(\frac{1}{y} + 1 \right) \cdot dy + c$

c = constant

$\therefore z = a \cdot \left(\log|x| + x \right) + a \left(\log|y| + y \right) + c$

$\therefore z = a \cdot \log x + ax + a \cdot \log y + ay + c$

$\therefore z = a \left[\log(xy) + x + y \right] + c$

c = constant

* Case : 4 : Clairaut's equation, that is $z = px + qy + f(p,q)$

→ Assume that $p = a$ and $q = b$.
Substituting the values of p and q in $z = px + qy + f(p,q)$, we get the complete integral is $z = ax + by + f(a,b)$

Ex :- Solve : $(p+q)(z - px - qy) = 1$

→ Given, $(p+q)(z - px - qy) = 1$.

$$\therefore z - px - qy = \frac{1}{p+q}$$

$$\therefore z = px + qy + \frac{1}{p+q} \quad \text{--- (1)}$$

→ (1) is of the form $z = px + qy + f(p,q)$

∴ Assume that $p = a$ and $q = b$.
substituting these in (1), we get,

$$z = ax + by + \frac{1}{a+b} \; ; \; a, b = \text{constants}$$

Ex :- $z = px + qy + \sqrt{1 + p^2 + q^2}$

→ Given, $z = px + qy + \sqrt{1+p^2+q^2}$ --- (1)

(1) is of the form, $z = px + qy + f(p,y)$.

∴ Assume that $p = a$ and $q = b$.
substituting these values in (1),
we get,

$$z = ax + by + \sqrt{1+a^2+b^2}$$

$a, b, =$ constants

Ex :- Solve $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$

$\rightarrow$ Given, $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$

$\therefore z = x \cdot p + y \cdot q + (pq)^{3/2}$ —— (1)

$\therefore$ (1) is of the form $z = px + qy + f(p,q)$

$\therefore$ Assume that $p = a$, $q = b$. and substituting these values in (1), we get,

$z = ax + by + (ab)^{3/2}$; $a, b =$ constants

Ex :- $z = px + qy + \log(pq)$ —— (1).

$\rightarrow$ (1) is of the form $z = px + qy + f(p,q)$

$\therefore$ Assume $p = a$ and $q = b$ and substituting these in (1), we get,

$z = ax + by + \log(ab)$; $a, b =$ constants

Ex :- Solve : $(1-x)p + (2-y)q = (3-z)$

$\rightarrow$ Given, $(1-x)p + (2-y)q = (3-z)$

$\therefore z = 3 - (1-x)p - (2-y)q$

$\therefore z = 3 - p + px - 2q + qy$

$\therefore z = px + qy + (3 - p - 2q)$ —— (1)

$\sqrt{p} + \sqrt{q} = 1.$

order = 1

degree = 2.

$\therefore$ ① is of the form $z = px + qy + f(p, q)$.

$\therefore$ Assume $p = a$ and $q = b$, and substituting these values in ①, we get,

$z = ax + by + (3 - a - 2b)$; a, b = constants

1. Heat Equation

Example: Obtain the solution of the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ using the method of separation of variables.

Solution. Let

$$u(x,t) = X(x)\,T(t) \;---\; (1)$$

be the solution of the given equation.

Then $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$, where $T' = \frac{dT}{dt}$ and $X'' = \frac{d^2X}{dx^2}$.

Substituting these in the given equation, we get

$$XT' = c^2\, X''T$$

Separating the variables, we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} \;---\; (2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, so (2) can hold only when each side equal to some constant, say k.

$$\therefore\; \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k \text{ (say)}$$

It gives two ODEs

$$X'' - kX = 0 \text{ and } T' - kc^2T = 0$$

$\therefore$ $$\frac{d^2X}{dx^2} - kX = 0 \;---\; (3)$$

and $$\frac{dT}{dt} - kc^2T = 0 \;---\; (4)$$

Three cases arise according as k is zero, positive or negative.

<u>Case-1:</u> Let $k = 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{dT}{dt} = 0.$$

Solving these ordinary differential equations, we get

$X(x) = a_1x + a_2$ and $T(t) = a_3$, where a_1, a_2, a_3 are arbitrary constants.

<u>Case-2:</u> Let $k > 0$, say p^2, where $p \neq 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} - p^2X = 0 \text{ and } \frac{dT}{dt} - p^2c^2T = 0\,.$$

Solving these ordinary differential equations, we get

$X(x) = b_1\, e^{px} + b_2e^{-px}$ and $T(t) = b_3e^{p^2c^2t}$, where b_1, b_2, b_3 are arbitrary constants.

<u>Case-3:</u> Let $k < 0$, say$-p^2$, where $p \neq 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} + p^2X = 0 \text{ and } \frac{dT}{dt} + p^2c^2T = 0\,.$$

Solving these ordinary differential equations, we get

$X(x) = c_1\cos(px) + c_2\sin(px)$ and $T(t) = c_3e^{-p^2c^2t}$, where c_1, c_2, c_3 are arbitrary constants.

Thus the various possible solutions are

$$u(x,t) = A_1x + A_2,$$

$$u(x,t) = (B_1e^{px} + B_2e^{-px})\, e^{p^2c^2t},$$

and $\qquad u(x,t) = (C_1 \cos px + C_2 \sin px)\, e^{-p^2c^2t}$,
where $A_1 = a_1a_3, A_2 = a_2a_3, B_1 = b_1b_3, B_2 = b_2b_3, C_1 = c_1c_3$, and $C_2 = c_2c_3$ are new arbitrary constants.

2. Wave Equation [Equation of Vibrating string]

Example: Obtain the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ using the method of separation of variables.
Solution. Let

$$u(x,t) = X(x)\, T(t) --- (1)$$

be the solution of the given equation.
Then $\frac{\partial^2 u}{\partial t^2} = XT''$ and $\frac{\partial^2 u}{\partial x^2} = X''T$, where $T'' = \frac{d^2T}{dt^2}$ and $X'' = \frac{d^2X}{dx^2}$.
Substituting these in the given equation, we get

$$XT'' = c^2\, X''T$$

Separating the variables, we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} --- (2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, so (2) can hold only when each side equal to some constant, say k.

$$\therefore\ \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k \text{ (say)}$$

It gives two ODEs

$$X'' - kX = 0 \text{ and } T'' - kc^2T = 0$$

$\therefore$
$$\frac{d^2X}{dx^2} - kX = 0 --- (3)$$

and
$$\frac{d^2T}{dt^2} - kc^2T = 0 --- (4)$$

Three cases arise according as k is zero, positive or negative.
Case-1: Let $k = 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{d^2T}{dt^2} = 0.$$

Solving these ordinary differential equations, we get
$X(x) = a_1x + a_2$ and $T(t) = a_3t + a_4$, where a_1, a_2, a_3, a_4 are arbitrary constants.
Case-2: Let $k > 0$, say p^2, where $p \neq 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} - p^2X = 0 \text{ and } \frac{d^2T}{dt^2} - p^2c^2T = 0\,.$$

Solving these ordinary differential equations, we get
$X(x) = b_1\, e^{px} + b_2e^{-px}$ and $T(t) = b_3e^{pct} + b_4\, e^{-pct}$, where b_1, b_2, b_3, b_4 are arbitrary constants.
Case-3: Let $k < 0$, say$-p^2$, where $p \neq 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} + p^2X = 0 \text{ and } \frac{d^2T}{dt^2} + p^2c^2T = 0\,.$$

Solving these ordinary differential equations, we get
$X(x) = c_1 \cos(px) + c_2 \sin(px)$ and

$T(t) = c_3 \cos(pct) + c_4 \sin(pct)$, where c_1, c_2, c_3, c_4 are arbitrary constants.
Thus the various possible solutions are

$$u(x,t) = (a_1 x + a_2)(a_3 t + a_4),$$
$$u(x,t) = (b_1\, e^{px} + b_2 e^{-px})(b_3 e^{pct} + b_4\, e^{-pct}),$$

and $\quad u(x,t) = (c_1 \cos(px) + c_2 \sin(px))(c_3 \cos(pct) + c_4 \sin(pct))$.

3. Laplace Equation

Example: Obtain the solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ using the method of separation of variables.

Solution. Let

$$u(x,y) = X(x)\, Y(y) --- (1)$$

be the solution of the given equation.

Then $\frac{\partial^2 u}{\partial x^2} = X''Y$ and $\frac{\partial^2 u}{\partial y^2} = XY''$, where $X'' = \frac{d^2X}{dx^2}$ and $Y'' = \frac{d^2Y}{dy^2}$.

Substituting these in the given equation, we get

$$X''Y + XY'' = 0$$

Separating the variables, we get

$$\frac{X''}{X} = -\frac{Y''}{Y} --- (2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of y only. Since x and y are independent variables, so (2) can hold only when each side equal to some constant, say k.

$$\therefore\ \frac{X''}{X} = -\frac{Y''}{Y} = k \text{ (say)}$$

It gives two ODEs

$$X'' - kX = 0 \text{ and } Y'' + kY = 0$$

$$\therefore \quad \frac{d^2X}{dx^2} - kX = 0 --- (3)$$

and

$$\frac{d^2Y}{dy^2} + kY = 0 --- (4)$$

Three cases arise according as k is zero, positive or negative.

<u>Case-1:</u> Let $k = 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{d^2Y}{dy^2} = 0.$$

Solving these ordinary differential equations, we get

$X(x) = a_1 x + a_2$ and $Y(y) = a_3 y + a_4$, where a_1, a_2, a_3, a_4 are arbitrary constants.

<u>Case-2:</u> Let $k > 0$, say p^2, where $p \neq 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} - p^2 X = 0 \text{ and } \frac{d^2Y}{dy^2} + p^2 Y = 0\,.$$

Solving these ordinary differential equations, we get

$X(x) = b_1\, e^{px} + b_2 e^{-px}$ and $Y(y) = b_3 \cos(py) + b_4 \sin(py)$, where b_1, b_2, b_3, b_4 are arbitrary constants.

<u>Case-3:</u> Let $k < 0$, say $-p^2$, where $p \neq 0$. Then (3) and (4) become respectively

$$\frac{d^2X}{dx^2} + p^2 X = 0 \text{ and } \frac{d^2Y}{dy^2} - p^2 Y = 0\,.$$

Solving these ordinary differential equations, we get
$X(x) = c_1 \cos(px) + c_2 \sin(px)$ and $Y(y) = c_3 e^{py} + c_4 e^{-py}$,
where c_1, c_2, c_3, c_4 are arbitrary constants.
Thus the various possible solutions are

$$u(x,y) = (a_1 x + a_2)(a_3 y + a_4),$$

$$u(x,y) = (b_1\, e^{px} + b_2 e^{-px})(b_3 \cos(py) + b_4 \sin(py)),$$

and

$$u(x,y) = (c_1 \cos(px) + c_2 \sin(px))(c_3 e^{py} + c_4 e^{-py}).$$