Unit : 04 : Recurrence Relations

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* Recurrence Relations :

→ A recurrence relations for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is non-negative integer.

* Solution :

→ A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Ex Consider a sequence $\overset{a_0}{1}, \overset{a_1}{1}, \overset{a_2}{2}, \overset{a_3}{3}, \overset{a_4}{5}, \overset{a_5}{8}, 13, \ldots$ (Fibonacci sequence)

→ Here, we can write $a_2 = 1 + 1$

$$\Rightarrow a_2 = a_0 + a_1$$

$$\Rightarrow a_n = a_{n-2} + a_{n-1} \; ; \; n \geq 2$$

$$\Rightarrow a_{n+2} = a_n + a_{n+1} \quad (\because \text{replacing } n \text{ by } n+2)$$

$$\Rightarrow a_n = a_{n+2} - a_{n+1} \; ; \; n \geq 0.$$

Ex Consider a general expression. $1, 3, 9, 27, 81, \ldots$

$a_0 = 3^0 = 1$

$a_1 = 3 \cdot a_0 = 3$

$a_2 = 3 \cdot a_1 = 9$

$a_3 = 3 \cdot a_2 = 27$

⋮

→ Here, we can observe that,

$$a_n = 3\, a_{n-1} \; ; \; n \geq 1$$

Hence, $\boxed{3^n}$ is a solution of the recurrence relation $\boxed{a_n = a_{n-1} \cdot 3}$ with $a_0 = 1$

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Ex:- Consider the recurrence relation
$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$
Is the sequence $\{a_n\}$ with $a_n = 3n$ a solution of this recurrence relation ?

→ For $n \geq 2$, we can see that

$$2a_{n-1} - a_{n-2} = 2[3(n-1)] - 3(n-2)$$
$$= 6n - 6 - 3n + 6$$
$$= 3n$$
$$= a_n.$$

→ Therefore, $\{a_n\}$ with $a_n = 3n$ is a solution of the given recurrence relation.

Ex:- Consider the recurrence relation
$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$
Is the sequence $\{a_n\}$ with $a_n = 5$ a solution of this recurrence relation ?

→ For $n \geq 2$, we can see that

$$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5$$
$$= a_n.$$

→ Therefore, $\{a_n\}$ with $a_n = 5$ is a solution of the given recurrence relation.

Extra: For any constant k, $a_n = k$ is a solution of the given recurrence relation.

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* Remarks :

→ In other words, a recurrence relation is like a recursively defined sequence, but without specifying any initial values [initial conditions.]

→ Therefore, the same recurrence relation can have (and usually has) multiple solutions.

→ If both the initial conditions and the recurrence relations are specified, then the sequence is uniquely determined.

* Solution of Recurrence Relations :-

→ In general, we would prefer to have an explicit formula to compute the value of a_n rather than conducting n iterations.

→ For one class of recurrence relations, we can obtain such formulas in a systematic way.

→ Those are the recurrence relations that express the terms of a sequence as linear combinations of previous terms.

* Linear Recurrences :-

→ A recurrence relation of the form
$c_0 a_n + c_1 a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k \cdot a_{n-k} = F(n)$
where, $c_0, c_1, \dots$ are constants

is called a linear recurrence relation with constant co-efficient.

→ It is of k^{th} order (or of degree k), provided that both C_0 and C_k are non-zero.

For example,

$(\because n-(n-1)=1)$

$$2a_n + 2a_{n-1} = 2^n \quad \text{(First order)}$$

$$3a_n - 5a_{n-1} + 2a_{n-2} = n^2+5 \quad \text{(Second order)}$$

$(\because n-(n-2)=2)$

⇒ Higehest − lowest = order

→ There are two types of linear recurrence relations with constant co-efficients :

1) Linear homogeneous recurrences ($F(n)=0$)
2) Linear non-homogeneous recurrences ($F(n) \neq 0$)

→ There are three methods of solving recurrence relation :

1) Iteration
2) Characteristic roots
3) Generating functions.

* Linear homogeneous recurrence relation :-

→ Definition : A linear homogeneous recurrence relation of degree k with constant co-efficient is a recurrence relation of the form :

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \ldots + c_k \cdot a_{n-k}$$

where, $c_1, c_2, \ldots, c_k$ are ~~constant~~ real numbers and $c_k \neq 0$.

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A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions.

$$a_0 = C_0, \ a_1 = C_1, \ a_2 = C_2, \ldots, a_{k-1} = C_{k-1}.$$

→ $C_0 a_n + C_1 \cdot a_{n-1} + C_2 \cdot a_{n-2} + \ldots + C_k \cdot a_{n-k} = F(n)$

$\therefore C_0 a_n + C_1 \cdot a_{n-1} + C_2 \cdot a_{n-2} + \ldots + C_k \cdot a_{n-k} = 0.$

($\because$ homogeneous)

$\therefore a_n + \frac{C_1}{C_0} \cdot a_{n-1} + \frac{C_2}{C_0} \cdot a_{n-2} + \ldots + \frac{C_k}{C_0} \cdot a_{n-k} = 0.$

$\therefore a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \ldots + c_k \cdot a_{n-k}.$

where, $c_1, c_2, \ldots, c_k$ are real numbers $\boxed{c_k \neq 0}$

** Examples :- (MCQ)

→ $P_n = (1.11) P_{n-1}$ L.H.R.R. of degree one.

→ $f_n = f_{n-1} + f_{n-2}$ L.H.R.R. of degree two.

→ $a_n = a_{n-1} + a^2_{n-2}$ Not linear.

→ $H_n = 2H_{n-1} + 1$ NOT homogeneous

→ $B_n = n \cdot B_{n-1}$ co-efficient is not constant.

→ $a_n = a_{n-5}$ L.H.R.R. of degree five.

* Solutions of a linear homogeneous recurrence relation using the method of characteristic roots :-

→ We try to find the solutions of the form $a_n = r^n$, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k \cdot a_{n-k}$.

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if and only if,

$$r^n = c_1 . r^{n-1} + c_2 . r^{n-2} + \dots + c_k . r^{n-k}.$$

$$\therefore \quad r^k = c_1 . r^{k-1} + c_2 . r^{k-2} + \dots + c_k .$$

($\because$ dividing by r^{n-k}).

$$\therefore \quad r^k - c_1 . r^{k-1} - c_2 . r^{k-2} - \dots - c_k = 0. \quad \text{—— } (*)$$

→ which is called the characteristic equation of degree k of the recurrence relation.

→ The solutions of this equation are called the characteristic roots of the recurrence relation.

→ There are three types of characteristic roots :

1) Distinct root.
2) Multiple root.
3). Mixed root.

* Distinct roots :-

→ Let us consider linear homogeneous recurrence relations of degree two.

→ Theorem :-

Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ if and only if $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$ for $n = 0, 1, 2, \cdots$ where α_1 and α_2 are constants.

Ex:- What is the solution of the recurrence relation $a_n = a_{n-1} + 2 \cdot a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

→ Here, we are given that, $a_n = a_{n-1} + 2 \cdot a_{n-2}$.
Now, comparing it with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$
we get $c_1 = 1$ and $c_2 = 2$.

→ Now, $r^2 - c_1 r - c_2 = 0$.
$\therefore r^2 - r - 2 = 0$.
$\therefore (r-2)(r+1) = 0$
$\therefore r = 2, -1$, which are distinct roots.

$\therefore$ The sol^n is given by, $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$.

$\therefore a_n = \alpha_1 (2)^n + \alpha_2 \cdot (-1)^n$ —— ⊛

→ we are given that, $a_0 = 2$, $a_1 = 7$

For $n = 0$, $a_0 = \alpha_1 (2)^0 + \alpha_2 \cdot (-1)^0$
$\therefore 2 = \alpha_1 + \alpha_2$ —— (1).

and for $n = 1$, $a_1 = \alpha_1 (2)^1 + \alpha_2 (-1)^1$
$\therefore 7 = 2 \cdot \alpha_1 - \alpha_2$ —— (2).

→ Now, (2) + (1), $2\alpha_1 - \cancel{\alpha_2} + \alpha_1 + \cancel{\alpha_2} = 7 + 2$

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$\therefore \cancel{\boxed{\alpha_1 = 5}}$ $\therefore 3\alpha_1 = 9$ $\therefore \boxed{\alpha_1 = 3}$

From ①, $\cancel{\alpha = 5 - \alpha_2}$ $2 = 3 + \alpha_2$

$\cancel{\therefore \boxed{\alpha_2 = 3}}$ $\therefore \boxed{\alpha_2 = -1}$

→ From ✡, $a_n = 3 \cdot 2^n - 1 \cdot (-1)^n$

→ Therefore, the solⁿ of the recurrence relation and initial conditions is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

Ex:- Find an explicit formula for the fibonacci numbers.

→ The fibonacci numbers satisfy the recurrence relation $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$ with the initial conditions $a_0 = 0$ and $a_1 = 1$.

→ Now comparing the recurrence relation with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$, we get,

$c_1 = 1$ and $c_2 = 1$.

→ Substituting c_1 and c_2 in $r^2 - c_1 r - c_2 = 0$. we get,

$$r^2 - r - 1 = 0.$$

Now, comparing it with $ar^2 + br + c = 0$ we get, $a = 1$, $b = -1$, $c = -1$.

$$\therefore r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore r = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}.$$

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$\therefore r = \dfrac{1 \pm \sqrt{5}}{2}$

$\therefore r = \dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}$, which are real and distinct roots.

$\therefore$ The solⁿ is given by, ~~$\alpha_n \alpha_1 \alpha_2 \alpha_n$~~

$a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$

$\therefore a_n = \alpha_1 \left(\dfrac{1 \pm \sqrt{5}}{2}\right)^n + \alpha_2 \left(\dfrac{1-\sqrt{5}}{2}\right)^n$ — ⊛.

→ We know the initial conditions, $a_0 = 0$, $a_1 = 1$.

$\therefore$ for $n = 0$, $a_0 = \alpha_1 \left(\dfrac{1 \pm \sqrt{5}}{2}\right)^0 + \alpha_2 \left(\dfrac{1-\sqrt{5}}{2}\right)^0$

$\therefore 0 = \alpha_1 + \alpha_2$ — ①.

and for $n = 1$, $a_1 = \alpha_1 \left(\dfrac{1 \pm \sqrt{5}}{2}\right)^1 + \alpha_2 \left(\dfrac{1-\sqrt{5}}{2}\right)^1$

$\therefore 1 = \left(\dfrac{1+\sqrt{5}}{2}\right)\alpha_1 + \left(\dfrac{1-\sqrt{5}}{2}\right)\alpha_2$ — ②.

→ ① × $\left(\dfrac{1+\sqrt{5}}{2}\right)$ − ②.

$\therefore \left(\dfrac{1+\sqrt{5}}{2}\right)\alpha_1 + \left(\dfrac{1-\sqrt{5}}{2}\right)\alpha_2 = 0.$

$-\left(\dfrac{1+\sqrt{5}}{2}\right)\alpha_1 - \left(\dfrac{1+\sqrt{5}}{2}\right)\alpha_2 = -1$

$\alpha_2\left(\cancel{\dfrac{1}{2}} - \dfrac{\sqrt{5}}{2} - \cancel{\dfrac{1}{2}} - \dfrac{\sqrt{5}}{2}\right) = -1$

$\therefore -\sqrt{5}\,\alpha_2 = -1$ $\therefore \boxed{\alpha_2 = \dfrac{1}{\sqrt{5}}}$

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From ①, $\alpha_1 + \alpha_2 = 0$

$\therefore \alpha_1 = -\alpha_2$

$\therefore \boxed{\alpha_1 = \frac{1}{\sqrt{5}}}$

$\therefore$ the sol^n of the recurrence relation and initial conditions is given by,

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

($\because$ from ⊛).

* Multiple roots :-

→ Theorem :-

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0 which is repeated two times.

A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ if and only if $a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot n r_0^n$

$\therefore a_n = (\alpha_1 + n \cdot \alpha_2) r_0^n$, for $n = 0, 1, 2, \ldots$ where α_1 and α_2 are constants

Ex :- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

→ We are given that $a_n = 6 \cdot a_{n-1} - 9 a_{n-2}$.
Now, comparing given eq^n with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$, we get,

$c_1 = 6$ and $c_2 = -9$

∴ The characteristic eq^n is, $r^2 - 6r - (-9) = 0$

$\therefore r^2 - 6r + 9 = 0$

$\therefore r = 3, 3$

which are multiple roots.

∴ The sol^n is : $a_n = (\alpha_1 + \alpha_2 \cdot n) \cdot 3^n$

$\therefore a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$ ——— (1).

where, α_1 and α_2 are constants.

→ For $n = 0$, $a_0 = \alpha_1 + 0$

$\therefore \boxed{\alpha_1 = 1}$

and for $n = 1$, $a_1 = \alpha_1 \cdot 3 + \alpha_2 (1) \cdot 3$

$\therefore 6 = (1)(3) + 3 \cdot \alpha_2$

$\therefore 3\alpha_2 = 6 - 3$

$\therefore \boxed{\alpha_2 = 1}$

∴ The sol^n is given by,

$a_n = 3^n + n \cdot 3^n$. (∵ from (1))

$= [1 + n] 3^n$.

Ex:- Find the solution to the recurrence relation $a_n = -3 \cdot a_{n-1} - 3 a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$, $a_2 = -1$.

→ We are given that, $a_n = -3 a_{n-1} - 3 a_{n-2} - a_{n-3}$.

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Now, comparing the given eq^n with,

$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + c_3 \cdot a_{n-3}$.

we get, $c_1 = -3$, $c_2 = -3$, $c_3 = -1$

$\therefore$ The char. eq^n is, $r^3 - c_1 \cdot r^2 - c_2 \cdot r - c_3 = 0$.

$\therefore r^3 - (-3) \cdot r^2 - (-3) \cdot r - (-1) = 0$

$\therefore r^3 + 3r^2 + 3r + 1 = 0$.

$\therefore (r+1)(r^2 + 2r + 1) = 0$

$\therefore (r+1)(r+1)^2 = 0$

$\therefore r = -1, -1, -1$.

-1	1	3	3	1
	0	1	2	1
	1	2	1	0.

$\therefore$ The sol^n is : $a_n = (\alpha_1 + n \cdot \alpha_2 + n^2 \cdot \alpha_3)(-1)^n$

$\therefore a_n = (-1)^n \cdot \alpha_1 + (-1)^n \cdot n \alpha_2 + (-1)^n n^2 \alpha_3$

where, $\alpha_1, \alpha_2, \alpha_3$ are constants.

→ Now, for $n = 0$, $a_0 = \alpha_1 + 0 + 0$

$\therefore \boxed{\alpha_1 = 1}$

and for $n = 1$, $a_1 = (-1)^1 \cdot \alpha_1 + (-1)^1 \cdot \alpha_2 + (-1)^1 \cdot \alpha_3$

$\therefore a_1 = -\alpha_1 - \alpha_2 - \alpha_3$

$\therefore -2 = -1 - \alpha_2 - \alpha_3$

$\therefore \alpha_2 + \alpha_3 = 1$. —— (1)

and for $n = 2$, $a_2 = \alpha_1 + 2 \cdot \alpha_2 + 4 \cdot \alpha_3$

$\therefore -1 = 1 + 2\alpha_2 + 4\alpha_3$

$\therefore 2\alpha_2 + 4\alpha_3 = -2$ —— (2)

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$$② - ① \times 2, \quad \begin{aligned} 2\cancel{\alpha_2} + 4\alpha_3 &= -2 \\ - \cancel{2\alpha_2} + 2\alpha_3 &= 2 \\ \hline 2\alpha_3 &= -4 \end{aligned}$$

$$\therefore \boxed{\alpha_3 = -2}$$

from ①, $\alpha_2 = 1 - \alpha_3$
$= 1 - (-2)$
$\therefore \boxed{\alpha_2 = 3}$

$\therefore$ The solⁿ is : $a_n = (-1)^n + (-1)^n \cdot n \cdot (3) + (-1)^n \cdot n^2 \cdot (-2)$

$\therefore a_n = [1 + 3n - 2n^2] (-1)^n$.

* Mixed roots :-

Ex:- Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5 and 9. What is the form of the general solution ?

→ We are given the char. roots of the eqⁿ which are 2, 2, 2, 5, 5, 9.

$\therefore$ The solⁿ is :

$$a_n = (\alpha_1 \cdot 2^n + \alpha_2 \cdot n \cdot 2^n + \alpha_3 \cdot n^2 \cdot 2^n) + (\alpha_4 \cdot 5^n + \alpha_5 \cdot n \cdot 5^n) + \alpha_6 \cdot 9^n.$$

$$\therefore a_n = (\alpha_1 + \alpha_2 \cdot n + \alpha_3 \cdot n^2) \cdot 2^n + (\alpha_4 + \alpha_5 \cdot n) \cdot 5^n + \alpha_6 \cdot 9^n$$

where, $\alpha_1, \alpha_2, \ldots, \alpha_6$ are constants.

* Practice Examples :-

1) $a_n + 5a_{n-1} + 6a_{n-2} = 0$; $a_0 = 1$, $a_1 = 2$
2) $a_n - 7a_{n-1} + 10a_{n-2} = 0$; $a_0 = 0$, $a_1 = 3$
3) $a_n - 13a_{n-1} + 36a_{n-2} = 0$; $a_0 = 2$, $a_1 = 1$
4) $a_r - 4a_{r-1} + 4a_{r-2} = 0$; $a_0 = 1$, $a_1 = 6$.
5) $a_n - 10a_{r-1} + 25a_{r-2} = 0$; $a_0 = 2$, $a_1 = 3$.
6) $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$.

* Linear Non-homogeneous recurrence relation with constant co-efficient :-

→ The general form of linear non-homogeneous recurrence relation with constant co-efficient is :

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k \cdot a_{n-k} + F(n) \quad \text{------} \quad (1)$$

→ Examples :-

1) $a_n = 3 \cdot a_{n-1} + 2n$.
2) $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$
3) $a_n = a_{n-1} + 2^n$.
4) $a_n = 3a_{n-1} + n \cdot 3^n$.
5) $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

→ Every solution of a linear non-homogeneous recurrence relation is the sum of,

→ a solution to the associated linear homogeneous recurrence relation and

→ a particular relation.

$$\Rightarrow a_n = a_n^{(h)} + a_n^{(p)}$$

* Solution of Linear Non-homogeneous recurrence relation :-

→ Theorem :-

If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant co-efficients,

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k \cdot a_{n-k} + F(n),$$

then every solution is of the form

$$a_n^{(h)} + a_n^{(p)}$$

where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k \cdot a_{n-k}$$

* Note :

→ There is no general method for solving such relations.

However, we can solve them for special cases.

In particular, $F(n)$ is

→ a polynomial function

→ exponential function

→ the product of a polynomial and exponential functions.

→ There are total four cases to obtain $a_n^{(p)}$.

Case : 1 :

Suppose $F(n)$ is a polynomial of degree 'q' and α is NOT a root of the characteristic equation of homogeneous part of equation (1).
Then $a_n^{(p)}$ is of the form,

$$a_n^{(p)} = A_0 + A_1 \cdot n + A_2 \cdot n^2 + \dots + A_q \cdot n^q.$$

where, $A_0, A_1, \dots, A_q$ are constants, those can be evaluated by using $a_n = a_n^{(p)}$ satisfies equation (1).

Case : 2 :

Suppose $F(n)$ is a polynomial of degree 'q' and α is a root of multiplicity 'm' of the characteristic equation of homogeneous part of equation (1).
Then $a_n^{(p)}$ is of the form,

$$a_n^{(p)} = n^m \left(A_0 + A_1 \cdot n + A_2 \cdot n^2 + \dots + A_q \cdot n^q \right).$$

where, $A_0, A_1, \dots, A_q$ are constants, those can be evaluated by using $a_n = a_n^{(p)}$ satisfies equation (1).

Case : 3 :

Suppose $F(n) = \alpha \cdot b^n$, where α is any constant and b is not a root of the characteristic equation of homogeneous part of equation (1).
Then $a_n^{(p)}$ is of the form,

$$a_n^{(p)} = A_0 \cdot b^n.$$

Case : 4 :

Suppose $F(n) = \alpha \cdot b^n$, where α is any constant and b is a root of multiplicity 'm' of the characteristic equation of homogeneous part of equation (1).
Then $a_n^{(p)}$ is of the form,

$$a_n^{(p)} = A_0 \cdot n^m \cdot b^n.$$

* Note : (for case : 1 and 3).

$F(n)$	$a_n^{(p)}$
Any constant	A_0
n	$A_0 + A_1 n$
n^2	$A_0 + A_1 n + A_2 n^2$
r^n	$A_0 r^n$.

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Ex :- Find all the solutions of the recurrence relation $a_n = 3 \cdot a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?

→ We are given $a_n = 3 \cdot a_{n-1} + 2n$ — (1)
which is a non-homogeneous recurrence relation.

→ To find $a_n^{(h)}$, we need to consider the homogeneous part of eqn (1),

$$\therefore a_n = 3 \cdot a_{n-1}.$$

by comparing it with $a_n = c_1 \cdot a_{n-1}$, we get $c_1 = 3$.

$\therefore$ The characteristic eqn is, $r - c_1 = 0$

$$\therefore r - 3 = 0$$

$$\therefore r = 3$$

$\therefore$ The soln is : $a_n^{(h)} = \alpha \cdot 3^n$ — (2)

→ Here, $F(n) = 2n$. then $a_n^{(p)} = A_0 + A_1 \cdot n$ — (3)

Now, from (1), we have,

$$a_n^{(p)} = 3 \cdot a_{n-1}^{(p)} + 2n$$

$$\therefore A_0 + A_1 n = 3 \cdot [A_0 + A_1(n-1)] + 2n$$

$$\therefore A_0 + A_1 n = 3A_0 + 3A_1 \cdot n - 3A_1 + 2n$$

$$\therefore 2A_0 + 2A_1 \cdot n - 3A_1 + 2n = 0$$

$$\therefore (2A_0 - 3A_1) + (2A_1 + 2)n = 0.$$

→ Comparing the co-efficients of 1 and n
NOT of A_0 and A_1

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$\therefore \quad 2A_1 + 2 = 0$

$\therefore \quad \boxed{A_1 = -1}$

and $\quad 2A_0 - 3A_1 = 0$

$\therefore \quad 2A_0 = 3(-1)$

$\therefore \quad \boxed{A_0 = -3/2}$

from ③, $\quad a_n^{(p)} = \frac{-3}{2} - n.$

$\therefore$ The gen. sol^n of (1) is given by,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\therefore \quad a_n = \alpha \cdot 3^n - \frac{3}{2} - n \quad , \text{ where } \alpha \in R.$$

→ for $n = 1$, we are given $a_1 = 3$.

$$\therefore \quad a_1 = 3 \cdot \alpha - \frac{3}{2} - 1$$

$$\therefore \quad 3 = 3\alpha - \frac{5}{2}$$

$$\therefore \quad 3\alpha = 3 + \frac{5}{2} = \frac{11}{2}$$

$$\therefore \quad \boxed{\alpha = \frac{11}{6}}$$

$\therefore$ The required sol^n is :

$$\boxed{a_n = \frac{11}{6} \cdot 3^n - \frac{3}{2} - n.}$$

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Given
$a_0 = 1$
$a_1 = 2$

Ex :- Solve the recurrence relation $a_n = 5 \cdot a_{n-1} - 6 \cdot a_{n-2} + 7^n$.

→ We are given $a_n = 5 \cdot a_{n-1} - 6 \cdot a_{n-2} + 7^n$ — ①.
which is a non-homogeneous recurrence relation.

→ To find $a_n^{(h)}$, we need to consider the homogeneous part of eqn ①.

$\therefore \quad a_n = 5 \cdot a_{n-1} - 6 \cdot a_{n-2}$.

by comparing it with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$
we get, $c_1 = 5$ and $c_2 = -6$.

$\therefore$ The char. eqn is $r^2 - c_1 r - c_2 = 0$

$\therefore \quad r^2 - 5r + 6 = 0.$

$\therefore \quad (r-2)(r-3) = 0$

$\therefore \quad r = 2, 3.$

$\therefore$ The soln is : $a_n^{(h)} = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$
where, α_1, α_2 are constants

→ Here, $F(n) = 7^n$ then $a_n^{(p)} = A_0 \cdot 7^n$

Now from ①,

$$a_n^{(p)} = 5 \cdot a_{n-1}^{(p)} - 6 \cdot a_{n-2}^{(p)} + 7^n$$

$$\therefore \quad A_0 \cdot 7^n = 5 \cdot A_0 \cdot 7^{n-1} - 6 \cdot A_0 \cdot 7^{n-2} + 7^n$$

by dividing the above eqn. by 7^{n-2},
we get,

$A_0 \cdot 7^2 = 5 \cdot A_0 \cdot 7 - 6 \cdot A_0 + 7^2$

$\therefore 49 A_0 - 35 A_0 + 6 A_0 - 49 = 0$

$\therefore 20 A_0 - 49 = 0.$

$\therefore \boxed{A_0 = \frac{49}{20}}$

$\therefore a_n^{(p)} = A_0 \cdot 7^n = \frac{49}{20} \cdot 7^n$

$\therefore$ The gen. sol^n is : $a_n = a_n^{(h)} + a_n^{(p)}$

$\therefore a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n + \frac{49}{20} \cdot 7^n$;

$\alpha_1, \alpha_2 \in R.$

→ Now, for $n = 0$, $a_0 = 1.$

$\therefore a_0 = \alpha_1 \cdot (1) + \alpha_2 (1) + \frac{49}{20}$

$\therefore \alpha_1 + \alpha_2 = 1 - \frac{49}{20} = \frac{-29}{20}.$ — (2)

and for $n = 1$, $a_1 = 2.$

$\therefore a_1 = 2\alpha_1 + 3\alpha_2 + \frac{49}{20} \cdot 7$

$\therefore 2\alpha_1 + 3\alpha_2 = 2 - \frac{343}{20} = -\frac{303}{20}.$ — (3)

→ Now, (3) − 2 · (2),

$$\begin{aligned} \cancel{2\alpha_1} + 3\alpha_2 &= -303/20. \\ - \cancel{2\alpha_1} + 2\alpha_2 &= {}_{+} -58/20. \\ \hline \alpha_2 &= -\frac{303}{20} + \frac{58}{20}. \end{aligned}$$

$\therefore \boxed{\alpha_2 = \frac{-245}{20}}$

from (1), $\alpha_1 - \frac{245}{20} = \frac{-29}{20}$

$\therefore \alpha_1 = \frac{245 - 29}{20} = \frac{216}{20}$

$\therefore \boxed{\alpha_1 = \frac{54}{5}}$

$\therefore$ The particular solⁿ is given by,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\therefore a_n = \frac{54}{5} 2^n - \frac{245}{20} 3^n + \frac{49}{20} \cdot 7^n.$$

Ex:- What form does a particular solution of the linear non-homogeneous recurrence relation $a_n = 6 \cdot a_{n-1} - 9 a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n 3^n$, $F(n) = n^2 \cdot 2^n$ and $F(n) = (n^2+1) \cdot 3^n$.

→ Here, we are given $a_n = 6 \cdot a_{n-1} - 9 a_{n-2} + F(n)$, —(1) which is a non-homogeneous recurrence relation.

→ Now, consider the homogeneous part, we get, $a_n = 6 a_{n-1} - 9 a_{n-2}$.
Comparing it with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ we get, $c_1 = 6$ and $c_2 = -9$.

$\therefore$ The char. eq^n is, $r^2 - c_1 r - c_2 = 0.$

$\therefore r^2 - 6r + 9 = 0.$

$\therefore (r-3)^2 = 0$

$\therefore r = 3, 3.$

$\therefore$ The sol^n is : $a_n^{(h)} = (\alpha_1 + \alpha_2 \cdot n) \cdot 3^n.$

$\therefore a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$

where, $\alpha_1, \alpha_2 \in R.$

(1) For $F(n) = 3^n$,

we can observe that 3^n and $n \cdot 3^n$ is there in the sol^n of a homogeneous recurrence relation.

$\therefore$ we have to multiply it by n^2 and modify it.

$\cancel{F(n) =} A_0 \cdot n^2 \cdot 3^n = a_n^{(p)}.$

$\rightarrow$ Now substituting it in eq^n (1), we get,

$a_n = 6 \cdot a_{n-1} - 9 a_{n-2} + \cancel{A_0 \cdot n^2} \cdot 3^n$

$\therefore a_n^{(p)} = 6 \cdot a_{n-1}^{(p)} - 9 \cdot a_{n-2}^{(p)} + \cancel{A_0 \cdot n^2} \cdot 3^n.$

$\therefore A_0 \cdot n^2 \cdot 3^n = 6A_0 \cdot (n-1)^2 \cdot 3^{n-1} - 9 \cdot A_0 \cdot (n-2)^2 \cdot 3^{n-2} + 3^n.$

$\therefore A_0 \cdot n^2 \cdot 3^n = 6 \cdot A_0 [n^2 - 2n + 1] \cdot 3^{n-1} - 9 \cdot A_0 [n^2 - 4n + 4] \cdot 3^{n-2} + 3^n$

$\therefore A_0 \cdot 3^n \cdot n^2 = [6 \cdot A_0 \cdot n^2 - 12 A_0 \cdot n + 6 \cdot A_0] 3^{n-1} - 9 A_0 \cdot n^2 \cdot 3^{n-2} + 36 \cdot A_0 \cdot n \cdot 3^{n-2} - 36 \cdot A_0 \cdot 3^{n-2} + 3^n.$

$\therefore A_0 \cdot 3^n \cdot n^2 = 6 \cdot A_0 \cdot 3^{n-1} \cdot n^2 - 12 \cdot A_0 \cdot 3^{n-1} \cdot n + 6 \cdot A_0 \cdot 3^{n-1} - 9 \cdot A_0 \cdot 3^{n-2} \cdot n^2 + \cancel{36 \cdot A}$

$36 A_0 \cdot n \cdot 3^{n-2} - 36 \cdot A_0 \cdot 3^{n-2} + 3^n$

$\cdot \rightarrow \{\} \times \{\}$
$+/- \rightarrow \{\} \cup \{\}$

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$\therefore A_0 \cdot 3^n \cdot n^2 = [6 \cdot A_0 \cdot 3^{n-1} - 9 \cdot A_0 \cdot 3^{n-2}]n^2$
$+ [36 \cdot A_0 \cdot 3^{n-2} - 12 \cdot A_0 \cdot 3^{n-1}]n$
$+ [6 \cdot A_0 \cdot 3^{n-1} - 36 \cdot A_0 \cdot 3^{n-2} + 3^n]$,

dividing above eq^n by 3^{n-2},

$\therefore A_0 \cdot 3^2 \cdot n^2 = [6 \cdot A_0 \cdot 3 - 9 \cdot A_0 \cdot]n^2 +$
$[36 \cdot A_0 - 12 \cdot A_0 \cdot 3]n +$
$[6 \cdot A_0 \cdot 3 - 36 \cdot A_0 + 3^2]$

$\therefore 9 \cdot A_0 \cdot n^2 = [18 A_0 - 9 A_0] n^2 +$
$[0]n + [-18 A_0 + 9]$

$\therefore \Rightarrow -18 A_0 + 9 = 0$
$\Rightarrow -18 A_0 = -9$
$\Rightarrow \boxed{A_0 = 1/2}$

$\therefore a_n^{(p)} = \frac{1}{2} \cdot n^2 \cdot 3^n.$

$\therefore$ The gen. sol^n is $a_n = a_n^{(h)} + a_n^{(p)}$.

$\therefore a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n + \frac{1}{2} \cdot n^2 \cdot 3^n$

where, $\alpha_1, \alpha_2 \in R$.

(2) For $F(n) = n \cdot 3^n$.
$=$ Family of n $\times$ Family of 3^n.
$= \{1, n\} \times \{3^n\}$
$= \{3^n, n \cdot 3^n\}$ terms occurs in the homo. sol^n.
$= \{n \cdot 3^n, n^2 \cdot 3^n\}$ "
$= \{n^2 \cdot 3^n, n^3 \cdot 3^n\}$.

$\therefore a_n^{(p)} = A_0 \cdot n^2 \cdot 3^n + A_1 \cdot n^3 \cdot 3^n$

③ For $F(n) = n^2 \cdot 2^n$.

$= \text{Family of } n^2 \times \text{family of } 2^n$

$= \{1, n, n^2\} \times \{2^n\}$

$= \{2^n, n \cdot 2^n, n^2 \cdot 2^n\}$.

$\therefore a_n^{(p)} = A_0 \cdot 2^n + A_1 \cdot n \cdot 2^n + A_2 \cdot n^2 \cdot 2^n$.

④ For $F(n) = n^2 \cdot 3^n$

$= \{1, n, n^2\} \times \{3^n\}$

$= \{3^n, n \cdot 3^n, n^2 \cdot 3^n\}$

$= \{n \cdot 3^n, n^2 \cdot 3^n, n^3 \cdot 3^n\}$

$= \{n^2 \cdot 3^n, n^3 \cdot 3^n, n^4 \cdot 3^n\}$

$\therefore a_n^{(p)} = A_0 \cdot n^2 \cdot 3^n + A_1 \cdot n^3 \cdot 3^n + A_2 \cdot n^4 \cdot 3^n$.

⑤ For $F(n) = (n^2+1) \cdot 3^n$.

$= n^2 \cdot 3^n + 3^n$

$= \text{Family of } (n^2 \cdot 3^n) \cup \text{Family of } 3^n$.

$= \{\{1, n, n^2\} \times \{3^n\}\} \cup \{3^n\}$

$= \{3^n, n \cdot 3^n, n^2 \cdot 3^n\} \cup \{3^n\}$

$= \{3^n, n \cdot 3^n, n^2 \cdot 3^n\}$.

$\therefore a_n^{(p)} = A_0 \cdot 3^n + A_1 \cdot n \cdot 3^n + A_2 \cdot n^2 \cdot 3^n$.

* Practice Examples :-

1. $a_n - 7a_{n-1} + 10a_{n-2} = 3^n$; $a_0 = 0$, $a_1 = 1$
2. $a_n + 6 \cdot a_{n-1} - 9 \cdot a_{n-2} = 3$; $a_0 = 0$, $a_1 = 1$
3. $a_r + 5a_{r-1} + 6 \cdot a_{r-2} = 3r^2 - 2r + 1$
4. $a_n = a_{n-1} + 3$, $a_0 = 1$ (Hint: lec: 06)

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* Generating Functions :-

→ Definition :- The generating function for the sequence $\{a_k\}$, i.e., terms $a_0, a_1, a_2, \ldots,$ of real numbers is the infinite series.

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

* Geometric series :

(1) $1 + r + r^2 + r^3 + \ldots + r^k + \ldots$

$\therefore$ ratio $= \frac{r^2}{r} = r$ and $a = 1$.

$$\therefore \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \; ; \; |r| < 1. \quad \left(r^k = \frac{a}{1-r} \right)$$

(2) $1 - r + r^2 - r^3 + \ldots$

$$\therefore \sum_{k=0}^{\infty} (-1)^k \cdot r^k = \frac{1}{1+r} \; ; \; |r| < 1.$$

$\therefore$ ratio $= \frac{r^2}{-r} = -r$. and $a = 1$

Ex:- Find the generating functions for the sequences $\{a_k\}$ with ...

1) $a_k = 3$

2) $a_k = k+1$

3) $a_k = 2^k$.

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→ 1) $G(x) = \sum_{k=0}^{\infty} a_k \cdot x^k = \sum_{k=0}^{\infty} 3 \cdot x^k$.

2) $G(x) = \sum_{k=0}^{\infty} a_k \cdot x^k = \sum_{k=0}^{\infty} (k+1) \cdot x^k$.

3) $G(x) = \sum_{k=0}^{\infty} a_k \cdot x^k = \sum_{k=0}^{\infty} (2^k) \cdot x^k$.

Ex:- Solve the recurrence relation $a_n = 3 \cdot a_{n-1}$ for $n = 1, 2, 3, \ldots$ and the initial condition $a_0 = 2$ using generating function.

1st method,

→ Here, we are given $a_n = 3 \cdot a_{n-1}$.

∴ The characteristic eqn : $r - 3 = 0$.

∴ $r = 3$.

∴ The soln is : $a_n = \alpha \cdot r^n$

$= \alpha \cdot 3^n$

∴ $a_n = 2 \cdot 3^n$ ($\because \alpha = a_0 = 2$).

2nd method,

→ Another method to solve recurrence relation is using generating function.

→ Let $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k \cdot x^k$ be a generating function for the sequence $\{a_k\}$.

→ Given that $a_k = 3 \cdot a_{k-1}$, $k = 1, 2, 3, \ldots$

∴ $a_k \cdot x^k = 3 \cdot a_{k-1} \cdot x^k$

∴ $\sum_{k=1}^{\infty} a_k \cdot x^k = \sum_{k=1}^{\infty} 3 \cdot a_{k-1} \cdot x^k$

$$\therefore \sum_{k=1}^{\infty} a_k \cdot x^k = 3 \cdot \sum_{k=1}^{\infty} a_{k-1} \cdot x^k$$

$$\therefore \sum_{k=1}^{\infty} a_k \cdot x^k = 3 \cdot x \cdot \sum_{\cancel{k=1}}^{\infty} \cdot a_{k-1} \cdot x^{k-1}$$

$$\therefore \sum_{k=1}^{\infty} a_k \cdot x^k = 3 \cdot x \cdot \sum_{k=0}^{k=1} \cdot a_k \cdot x^k$$

(∵ Replacing k by k+1)

it includes a_0

$$\therefore \sum_{k=0}^{\infty} \cdot a_k \cdot x^k - a_0 = 3x \cdot \sum_{k=0}^{\infty} \cdot a_k \cdot x^k$$

$$\therefore G(x) - a_0 = 3x \cdot G(x)$$

$$\therefore G(x) - 2 = 3x \cdot G(x) \quad (\because a_0 = 2)$$

$$\therefore G(x) - 3x \cdot G(x) = 2$$

$$\therefore G(x)[1 - 3x] = 2$$

$$\therefore G(x) = \frac{2}{1-3x} = 2 \cdot \frac{1}{1-3x}$$

$$= 2 \cdot \sum_{k=0}^{\infty} \cdot a(3x)^k$$

$$\therefore G(x) = 2 \cdot \sum_{k=0}^{\infty} 3^k \cdot x^k$$

→ Hence, $a_n = 2 \cdot 3^n$.

Ex:- Solve $a_n = 8 \cdot a_{n-1} + 10^{n-1}$ for $n = 1, 2, 3, \ldots$ and initial conditions $a_0 = 1$ and $a_1 = 9$

→ Let $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k \cdot x^k$

be a generating function $\{a_k\}$.

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→ Here, we are given that $a_k = 8 \cdot a_{k-1} + 10^{k-1}$;

$k = 1, 2, 3, \ldots$

$\therefore a_k \cdot x^k = 8 \cdot a_{k-1} \cdot x^k + 10^{k-1} \cdot x^k$

$\therefore \sum_{k=1}^{\infty} a_k \cdot x^k = 8 \cdot \sum_{k=1}^{\infty} a_{k-1} \cdot x^k + \sum_{k=1}^{\infty} 10^{k-1} \cdot x^k$

$\therefore \sum_{k=1}^{\infty} a_k \cdot x^k = 8x \cdot \sum_{k=1}^{\infty} \cdot a_{k-1} \cdot x^{k-1} +$

$x \cdot \sum_{k=1}^{\infty} 10^{k-1} \cdot x^{k-1}$

$\therefore \sum_{k=1}^{\infty} \cdot a_k \cdot x^k = 8x \cdot \sum_{k+1=1}^{\infty} a_{(k+1)-1} \cdot x^{(k+1)-1} +$

$x \cdot \frac{1}{1-(10x)}$

$\therefore \sum_{k=0}^{\infty} a_k \cdot x^k - a_0 = 8x \cdot \sum_{k=0}^{\infty} \cdot a_k \cdot x^k + \frac{x}{1-10x}$

$\therefore G(x) - a_0 = 8x \cdot G(x) + \frac{x}{1-10x}$

$\therefore G(x) - 1 = 8x \cdot G(x) + \frac{x}{1-10x}$

$\therefore G(x)[1-8x] = 1 + \frac{x}{1-10x} = \frac{1-10x+x}{1-10x}$

$\therefore G(x)[1-8x] = \frac{1-9x}{1-10x}$

$\therefore G(x) = \frac{1-9x}{(1-8x)(1-10x)}$

→ Now, by method of partial fraction,

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$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}.$$

$$\therefore G(x). = \frac{A}{1-8x} + \frac{B}{1-10x}$$

$$\therefore \quad 1-9x = A(1-10x) + B(1-8x) \quad \text{— (1)}.$$

→ when $x = \frac{1}{8}$, $1 - \frac{9}{8} = A\left(1 - \frac{10}{8}\right) + 0.$

$$\therefore -\frac{1}{\cancel{8}} = -\frac{2}{\cancel{8}}A$$

$$\therefore \boxed{A = \frac{1}{2}}$$

and when $x = \frac{1}{10}$, $1 - \frac{9}{10} = 0 + B\left(1 - \frac{8}{10}\right).$

$$\therefore \frac{1}{10} = B.\left(\frac{2}{10}\right)$$

$$\therefore \boxed{B = \frac{1}{2}}$$

$$\therefore G(x) = \frac{1/2}{1-8x} + \frac{1/2}{1-10x}$$

$$= \frac{1}{2}\left[\frac{1}{1-8x} + \frac{1}{1-10x}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} 8^k . x^k + \sum_{k=0}^{\infty} 10^k . x^k\right]$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} (10^k + 8^k) . x^k.$$

→ Hence, $a(n) = \frac{1}{2}(10^n + 8^n).$

—— X ——