Unit 04 Matrix Algebra : II

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* Find eigen values and eigen vectors for the matrix :-

1. $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

/*

Do $\boxed{\text{NOT}}$ use this method.

→ Here, $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ — (1)

→ The characteristic eq^n of (1) is given by,

$$\det(A - \lambda I) = 0$$

The good & great method waiting for You.

→ Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$[A - \lambda I]X = 0$

$\therefore \left\{ \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — (★)

→ Now, $\det(A - \lambda I) = 0$

$\therefore \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\therefore (-5-\lambda)(-2-\lambda) - 4 = 0$

$\therefore 10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$

$\therefore \lambda^2 + 7\lambda + 6 = 0$ — (2) (characteristic eq^n of (1))

$\therefore (\lambda+6)(\lambda+1) = 0$

$\therefore \lambda = -6, \lambda = -1$

which are eigen values of A.

→ Now, for eigen vectors,

For $\lambda = -1$, from eq^n (★),

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

*/

/*

$\Rightarrow -4x_1 + 2x_2 = 0 \qquad \Rightarrow 2x_1 - x_2 = 0$

$\Rightarrow 2x_1 - x_2 = 0 \qquad \Rightarrow 2x_1 - x_2 = 0$

$\therefore$ The augumented matrix is,

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \qquad (\because \tfrac{1}{2}R_1 ,\ R_2 \to R_2 - R_1)$$

→ Let, $x_2 = k$; $k \in R$

$\therefore 2x_1 - x_2 = 0$

$\therefore 2x_1 = k$

$\therefore x_1 = k/2$

$$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k/2 \\ k \end{bmatrix} = k \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} , k \in R - \{0\}$$

→ Now, for $\lambda = -6$, from eq^n (*),

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 = 0 \qquad \Rightarrow 2x_1 + 4x_2 = 0$

$\Rightarrow x_1 + 2x_2 = 0 \qquad \Rightarrow x_1 + 2x_2 = 0$

$\therefore$ The augumented matrix is,

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad (R_2 \to R_2 - R_1)$$

*/

→ Let, $x_2 = K$, $K \in R$.

$\therefore x_1 + 2K = 0$

$\therefore x_1 = -2K.$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2K \\ K \end{bmatrix} = K\begin{bmatrix} -2 \\ 1 \end{bmatrix}, K \in R - \{0\}$$

2. $\begin{bmatrix} 1/4 & -1/4 \\ -1/12 & 5/12 \end{bmatrix}$

→ Here, Let $A = \begin{bmatrix} 1/4 & -1/4 \\ -1/12 & 5/12 \end{bmatrix}$ — ①

→ The characteristic eq^n of ① is given by,
det $(A - \lambda I) = 0$.

→ Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and

$[A - \lambda I] X = 0$

$$\therefore \left\{ \begin{bmatrix} 1/4 & -1/4 \\ -1/12 & 5/12 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1/4 - \lambda & -1/4 \\ -1/12 & 5/12 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{— (A)}$$

→ Now, det $(A - \lambda I) = 0$.

$$\therefore \begin{vmatrix} 1/4 - \lambda & -1/4 \\ -1/12 & 5/12 - \lambda \end{vmatrix} = 0.$$

$$\therefore \left(\frac{1}{4} - \lambda\right)\left(\frac{5}{12} - \lambda\right) - \left(\frac{-1}{4}\right)\left(\frac{-1}{12}\right) = 0$$

$$\therefore \frac{5}{48} - \frac{\lambda}{4} - \frac{5\lambda}{12} + \lambda^2 - \frac{1}{48} = 0.$$

$$\therefore 5 - 12\lambda - 20\lambda + 48\lambda^2 - 1 = 0.$$

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$\therefore\ 48\lambda^2 - 32\lambda + 4 = 0.$

$\therefore\ 12\lambda^2 - 8\lambda + 1 = 0$ — ②. characteristic eq^n of A

$\therefore\ 12\lambda^2 - 6\lambda - 2\lambda + 1 = 0$

$\therefore\ 6\lambda(2\lambda - 1) - (2\lambda - 1) = 0.$

$\therefore\ (2\lambda - 1)(6\lambda - 1)$

$\therefore\ \lambda = \frac{1}{2}, \frac{1}{6}$ which are the eigen value of eq^n ①.

→ Now, for $\lambda = 1/2$, from eq. (A),

$$\begin{bmatrix} -1/4 & -1/4 \\ -1/12 & -1/12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -\frac{1}{4}x_1 - \frac{1}{4}x_2 = 0$; $\Rightarrow -\frac{1}{12}x_1 - \frac{1}{12}x_2 = 0.$

$\Rightarrow x_1 + x_2 = 0.$; $\Rightarrow x_1 + x_2 = 0.$

$\therefore$ The augumented matrix is,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad (R_2 \to R_2 - R_1).$$

→ Let $x_2 = K$, $K \in R$ $\Rightarrow x_1 + x_2 = 0$ $\Rightarrow x_1 = -K$

$$\therefore X_1 = \begin{bmatrix} -K \\ K \end{bmatrix} = K\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad K \in R - \{0\}$$

→ Now, for $\lambda = 1/6$, from eq^n (A)

$$\begin{bmatrix} 1/12 & -1/4 \\ -1/12 & 1/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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/*

$\Rightarrow \frac{x_1}{12} - \frac{x_2}{4} = 0$ ⎫ $\Rightarrow -\frac{1}{12}x_1 + \frac{1}{4}x_2 = 0$

$\Rightarrow x_1 - 3x_2 = 0$ ⎭ $\Rightarrow x_1 - 3x_2 = 0$

$\therefore$ The augumented matrix is, $\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $(R_2 \to R_2 - R_1)$

$\to$ Let, $x_2 = K$, $K \in R$

$\therefore x_1 - 3K = 0$

$\therefore \boxed{x_1 = 3K}$

$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3K \\ K \end{bmatrix} = K\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

*/

3. $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

Use only this for 3×3.

$\to$ Let $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ — (1)

$\to$ The characteristic eq^n of A is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

where, $S_1 = tr(A) = 8.$

$S_2 = (M_{11} + M_{22} + M_{33}) = -5 + 0 + 22 = 17$

$S_3 = 4(3-8) - 2(-5+4) - 2(-20+6)$

$= -20 + 2 + 28$

$= 10.$

→ sum 0 ⇒ $\lambda - 1$.

→ odd power coefficient = even power coefficient ⇒ $\lambda + 1$

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$\therefore \lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$.

which is the characteristic eqⁿ of (1).

→ Eigen values :-

/* Here, the sum of co-efficient of characteristic eqⁿ is 0.

$\therefore$ One of the roots is $(\lambda - 1)$. */

1	1	-8	17	-10
	0	1	-7	10
	1	-7	10	0

$\therefore \lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 7\lambda + 10) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 5)(\lambda - 2) = 0$

$\Rightarrow \lambda = 1, 2, 5$, which are the eigen values of A.

→ Eigen vectors :-

Consider $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

→ for $\lambda = 1$,

$$\begin{bmatrix} 4-1 & 2 & -2 \\ -5 & 3-1 & 2 \\ -2 & 4 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(1)}$$

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→ Now, the argumented matrix is given by,

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2/3 & -2/3 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \quad \left(R_1 \to \frac{R_1}{3} \right)$$

$$\sim \begin{bmatrix} 1 & 2/3 & -2/3 \\ 0 & 16/3 & -4/3 \\ 0 & 16/3 & -4/3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 5R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2/3 & -2/3 \\ 0 & 16/3 & -4/3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_1.$$

$$\sim \begin{bmatrix} 1 & 2/3 & -2/3 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{3}{16} R_2.$$

(Row-echelon form)

By back substitution method,

let $\boxed{x_3 = 4K}$, $K \in R$.

$\therefore x_2 - \frac{x_3}{4} = 0.$ and $x_1 + \frac{2x_2}{3} - \frac{2x_3}{3} = 0.$

$\therefore x_2 - K = 0$ $\qquad$ $\therefore x_1 + \frac{2K}{3} - \frac{2(4K)}{3} = 0.$

$\therefore \boxed{x_2 = K}$ $\qquad$ $\therefore x_1 = \frac{8K}{3} - \frac{2K}{3} = \frac{6K}{3}$

$\therefore \boxed{x_1 = 2K}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$K \in R - \{0\}.$

* why we have chosen z as zero quantity ?
Can't we choose x or y ?
→ because there is No leading 1 corresponds to the 'z' variable.

→ for $\lambda = 2$,

$$\begin{bmatrix} 4-2 & 2 & -2 \\ -5 & 3-2 & 2 \\ -2 & 4 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (1)}.$$

→ Now the augumented matrix is,

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \quad \left(R_1 \to \frac{R_1}{2} \right)$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -3 \\ 0 & 6 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 5R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix}. \quad R_3 \to R_3 - R_2$$

x and y will be depedent variable because of leading 1.

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & \boxed{0} \end{bmatrix} \quad R_2 \to R_2\left(\frac{1}{6}\right)$$

where z is INDEPENDENT variable

(Row-echelon form).

By back-substitution method,

let $\boxed{z = 2k}$, $k \in R$

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$\therefore y - \frac{z}{2} = 0$ and $x + y - z = 0$

$\therefore x = z - y = 2k - k$

$\therefore y - k = 0$

$\therefore \boxed{x = k}$

$\therefore \boxed{y = k}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad k \in R - \{0\}$$

→ for $\lambda = 5$,

$$\begin{bmatrix} 4-5 & 2 & -2 \\ -5 & 3-5 & 2 \\ -2 & 4 & 1-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ Now, the augumented matrix is,

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \quad R_1 \to (-1) R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & -12 & 12 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 5R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 \left(\frac{-1}{12}\right)$$

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By back-substitution method,

let $\boxed{z = k}$, $k \in R$.

$\therefore$ $y - z = 0$ and $x - 2y + 2z = 0$

$\therefore$ $\boxed{y = k}$ $\therefore$ $x - 2k + 2k = 0$

$\therefore$ $\boxed{x = 0}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad k \in R - \{0\}.$$

4. $\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$ Use only this for 2x2.

→ Here, let $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$ —— (1)

→ The characteristic eq^n of A is given by,

$\lambda^2 - S_1 \lambda + S_2 = 0$

where, $S_1 = tr(A) = 8$.

$S_2 = |A| = 15 - 3 = 12$

$\therefore$ $\lambda^2 - 8\lambda + 12 = 0$.

which is the characteristic eq^n of eq^n (1)

$\therefore$ $(\lambda - 6)(\lambda - 2) = 0$.

$\therefore$ $\lambda = 2, 6$.

which are the eigen values of A.

→ Eigen vectors :-

→ Let $X = \begin{bmatrix} x \\ y \end{bmatrix} \neq 0 \ni [A - \lambda I]X = 0$ —— (A)

→ For $\lambda = 2$, (A) becomes,

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ Now the augmented matrix is,

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (R_1 \to R_1/3)$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

By back-substitution method,
Let $\boxed{y = K}$, $K \in R$

$\therefore x + y = 0 \Rightarrow \boxed{x = -K}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = K \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad K \in R - \{0\}$$

→ For $\lambda = 6$, (A) becomes,

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ Now, the augmented matrix is,

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \quad R_1 \to R_1(-1)$$

$$\sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$\therefore$ By back-substitution method,
Let $\boxed{y = K}$, $K \in R$ $\quad \therefore x - 3y = 0 \Rightarrow \boxed{x = 3K}$

$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $k \in R - \{0\}$

5. $\begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ $\Rightarrow$ Triangular matrix
$\Rightarrow$ Eigen values are elements of diagonal matrix 3, 6, 2 ∴

→ Since, the given matrix is triangular matrix and we know that from the property of the eigen values that for triangular matrix the eigen values are precisely its diagonal elements.
So, the eigen values of the given matrix is 3, 6, 2.

6. $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

→ Here, let $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

→ The characteristic eq^n of A is,

$$\lambda^2 - S_1\lambda + S_2 = 0.$$

where, $S_1 = tr(A) = 1 + 4 = 5$
$S_2 = |A| = 4 - 10 = -6.$

$\therefore \lambda^2 - 5\lambda - 6 = 0.$
which is the characteristic eq^n of A.
$\therefore \lambda^2 - 5\lambda - 6 = 0$
$\therefore (\lambda - 3)(\lambda - 2) = 0$
$\therefore \lambda = 2, 3.$ which are the eigen values of A.

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→ Eigen Vectors :-

Let $X = \begin{bmatrix} x \\ y \end{bmatrix} \neq 0 \ni [A - \lambda I]X = 0$ — ⓐ

→ For, $\lambda = 2$, ⓐ becomes,

$$\begin{bmatrix} -1 & -2 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x - 2y = 0$ and
$-5x + 2y = 0.$

∴ The augmented matrix ~~is~~ of the above system is,

$$\begin{bmatrix} -1 & -2 \\ -5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ -5 & 2 \end{bmatrix} \quad (\because R_1 \to (-1)R_1)$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 12 \end{bmatrix} \quad (R_2 \to R_2 + 5R_1)$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \left(\because R_2 \to \frac{R_2}{12}\right)$$

∴ By back-substitution method,
Let $\boxed{y = K}$, $K \in R$

∴ $x + 2y = 0$
∴ $\boxed{x = -2K}$

∴ $X = \begin{bmatrix} x \\ y \end{bmatrix} = K \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $K \in R - \{0\}$

→ For, $\lambda = 3$, ⓐ becomes,

$$\begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

∴ $-2x - 2y = 0$
& $-5x + y = 0.$

5) Property : A is not invertible if 0 is one of the eigen value of the given square matrix.

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∴ The augmented matrix of the system is $\begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$ $R_1 \to R_1\left(\frac{-1}{2}\right)$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 6 \end{bmatrix}$ $R_2 \to R_2 + 5R_1$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $R_2 \to \frac{R_2}{6}$

∴ By back-substitution method,
let $\boxed{y = k}$, $k \in R$.

∴ $x + y = 0$

∴ $\boxed{x = -k}$

∴ $X = \begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $k \in R - \{0\}$

★ 7. $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

→ Here, let $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$.

→ The characteristic eq^n of A is given by,

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where, $S_1 = tr(A) = 1 + 4 + 2 = 2$
$S_2 = M_{11} + M_{22} + M_{33} = 0 + (-3) + 4 = 1$
$S_3 = |A| = 0$. 5^{th} property is satisfied

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$\therefore \lambda^3 - 2\lambda^2 + \lambda = 0.$

$\therefore \lambda(\lambda^2 - 2\lambda + 1) = 0$

$\therefore \lambda(\lambda-1)^2 = 0.$

$\therefore \underbrace{\lambda = 0}_{\text{one eigen value}}, \underbrace{1, 1}_{\text{Repeated eigen values}}$ which are eigen values of A.

→ Eigen Vectors :-

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0 \ni [A - \lambda I]X = 0$ — (A)

→ For $\lambda = 0$, (A) becomes,

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ The augmented mathix is $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -6 & -4 \\ 0 & 1 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$ $R_2 \to \frac{R_2}{4}$, $R_3 \to R_3\left(\frac{-1}{6}\right)$

$\sim \begin{bmatrix} 1 & -6 & -4 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\therefore$ By back-substitution method,

let $\boxed{z = 2K}$, $K \in R$.

$\therefore y + \frac{z}{2} = 0.$

$\therefore y + 2K/2 = 0 \Rightarrow \boxed{y = -K}$

and $x - 6y - 4z = 0$

$\Rightarrow x - 6(-K) - 4(2K) = 0$

$\Rightarrow x + 6K - 8K = 0. \Rightarrow \boxed{x = 2K}$

→ why $K \in R - \{0\}$

because we are considering eigen vector as a non-zero vector.

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$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = K \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad K \in R - \{0\}$$

→ For $\lambda = 1$, (A) becomes,

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~∴ The augmented matrix is~~ $\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}$

[Don't go for ~~augment~~ this directly]

Write the system first.

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ -6 & 3 & -6 \\ -4 & 2 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} -6 & 3 & -6 \\ 0 & 0 & 0 \\ -4 & 2 & -4 \end{bmatrix} \quad (R_1 \leftrightarrow R_2) \quad \sim \begin{bmatrix} 1 & -1/2 & 1 \\ -4 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{(-6)}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 + 4R_1$$

$$\therefore \left.\begin{aligned} -6y - 4z &= 0 \\ 3y + 2z &= 0 \\ -6y - 4z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 3y + 2z &= 0 \\ 3y + 2z &= 0 \\ 3y + 2z &= 0. \end{aligned}$$

∴ The augmented matrix of the above system is,

$$\begin{bmatrix} 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{pmatrix} \because R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

- Here, only y is a dependent variable where, x and z are independent

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$$N \sim \begin{bmatrix} 0 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \left(\because R_1 \to \frac{R_1}{(3)} \right)$$

$\therefore$ By back - substitution method,

Let $x = k_1$ and $z = 3k_2$; $k_1, k_2 \in R$.

$\therefore \quad y + \frac{2}{3} z = 0$

$\therefore \quad y = -2/3 \, (3k_2) = -2k_2$

$\therefore \quad \boxed{y = -2k_2}$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ -2k_2 \\ 3k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

$(k_1, k_2 \in R - \{0\})$

→ **Eigen space** : $k_1 = k_2 = k = 1$.

$$\therefore \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \right\}$$

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(8) → Here, let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

→ The characteristic eqⁿ of A is given by,

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

where, $S_1 = tr(A) = 2+3+2 = 7$

$S_2 = M_{11} + M_{22} + M_{33}$

$= 6 + 3 + 6 = 15.$

$S_3 = |A| = 12 + (-3) = 9.$

$\therefore \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0.$

which is the characteristic eqⁿ of A.

$\therefore (\lambda-1)[\lambda^2 - 6\lambda + 9] = 0.$

$\therefore (\lambda-1)(\lambda-3)(\lambda-3) = 0.$

$\therefore \lambda = 1, 3, 3.$

which are the eigen value of A.

1	1	-7	15	-9
	0	1	-6	9
	1	-6	9	0

→ Eigen Vectors :-

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0 \ni [A - \lambda I]X = 0$ —— (A)

→ For $\lambda = 1$, (A) becomes

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ The augmented matrix is, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \left(\because \begin{array}{l} R_3 \to R_3 - R_1 \\ R_2 \to \frac{1}{2} R_2 \end{array} \right)$$

$\therefore$ By back substitution method,

let $\boxed{z = K}$, $K \in R$.

$x + z = 0$.

$\therefore \boxed{x = -K}$ $\boxed{y = 0}$

$$\therefore X = K \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad K \in R - \{0\}$$

→ For $\lambda = 3$, (A) becomes,

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ The augmented matrix is, $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} (\because R_3 \to R_3 + R_1) \\ (\because R_1 \to (-1) R_1). \end{array}$$

$\therefore$ By back substitution method,

let $\boxed{y = K_1}$, $\boxed{z = K_2}$, $K_1, K_2 \in R$

$\therefore x - z = 0$

$\therefore \boxed{x = K_2}$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} K_2 \\ K_1 \\ K_2 \end{bmatrix} = K_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$(K_1, K_2 \in R - \{0\})$

linear polynomial : $ax + b$; $a \neq 0$.
quadratic polynomial : $ax^2 + bx + c$;

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→ Eigen space :- $K_1 = K_2 = K = 1$.

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note :- If the matrix is a symmetric matrix ($A = A^T$) and eigen values are repeated in nature then it's eigen vectors are mutually perpendicular.

$$X_i^T X_j = 0$$

while in ex :- 07, the matrix is NOT a symmetric matrix ∴ the eigen vectors are not mutually perpendicular and that is why $X_i^T X_j \neq 0$ happens.

* Cayley - Hamilton Theorem and it's applications :

(1.) Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley - Hamilton theorem for this matrix.
Find A^{-1} and Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

→ Given matrix is $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic eq^n of A is,

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where $s_1 = tr(A) = 4$

$S_2 = |A| = 3 - 8 = -5$

$\therefore \lambda^2 - 4\lambda - 5 = 0$ —— (1)

$\therefore \cancel{(\lambda - 5)(\lambda + 1) = 0}$

$\therefore \cancel{\lambda = 5, -1}$

→ If cayley-Hemilton theorem is satisfies, we must get,

$A^2 - 4A - 5I = O$ —— (2)

$$L.H.S. = A^2 - 4A - 5I$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2 = R.H.S.$$

$\therefore$ Cayley-Hamilton the^m is satisfied.

→ From (2), $A^{-1}(A^2 - 4A - 5I) = A^{-1}O$

$$\Rightarrow A - 4I - 5A^{-1} = O$$

$$\Rightarrow A^{-1} = \frac{1}{5}[A - 4I]$$

$$\Rightarrow A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Try to convert it into eq (2)

→ Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ is given

$$= A^3[A^2 - 4A - 5I] + 11A^2 - A - 10I - 2A^3$$

$$= A^3(0) + 11A^2 - A - 10I - 2A^3 \quad (\because eq. (2))$$

Use 2nd method for

$= -2A^3 + 11A^2 - A - 10I$

$= -2A(A^2 - 4A - 5I) - 8A^2 - 10A + 11A^2 - A - 10I$

$= -2A(0) + 3A^2 - 11A - 10I \quad (\because \text{eq. } 2)$

$= 3A^2 - 11A - 10I$

$= 3(A^2 - 4A - 5I) + 12A + 15I - 11A - 10I$

$= 3(0) + A + 5I \quad (\because \text{eq. } 2)$

$= A + 5I$

* Second method :-

Ex:-3

Divisor: $A^2 - 4A + 5I$

Quotient: $A^3 + 3A^2 + 10A$

$A^5 - A^4 + 3A^3 - 5A^2 + 2A - 3I$

$A^5 - 4A^4 + 5A^3$

$(-) \quad (+) \quad (-)$

$3A^4 - 2A^3 - 5A^2$

$3A^4 - 12A^3 + 15A^2$

$(-) \quad (+) \quad (-)$

$10A^3 - 20A^2 + 2A$

$10A^3 - 40A^2 + 50A$

$(-) \quad (+) \quad (-)$

Ans :- $\boxed{20A^2 - 48A - 3I}$

Ex:-1

Divisor: $A^2 - 4A - 5I$

Quotient: $A^3 + 0 - 2A + 3$

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$A^5 - 4A^4 - 5A^3$

$(-) \quad (+) \quad (+)$

$-2A^3 + 11A^2 - A$

$-2A^3 + 8A^2 + 10A$

$(+) \quad (-) \quad (-)$

$3A^2 - 11A - 10I$

$3A^2 - 12A - 15I$

$(-) \quad (+) \quad (+)$

Ans :- $\boxed{A + 5I}$

or the

find

d A^2.

by,

$= 0$

$-1 - 2 = -5$

tisfies,

$(\because \text{from } 1)$

$= O_2 = R.H.S.$

(2) Verify Cayley-Hemilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and hence find A^{-1} and A^2.

→ Given matrix is $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

The cha. eq^n of A is given by,

$$\lambda^2 - S_1\lambda + S_2 = 0.$$

where, $S_1 = tr(A) = 0$

$S_2 = |A| = -1-2_4 = -5$

$\therefore \lambda^2 - 0\lambda - 5 = 0$

$\therefore \lambda^2 - 5 = 0$ —— (1)

→ If cayley. Hemilton theorem is satisfies, we must get $A^2 - 5I = 0$. (∵ from (1))

—— (2).

$\therefore$ L.H.S. $= A^2 - 5I$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2 = \text{R.H.S.}$$

$\therefore$ Cayley-Hemilton th^m is satisfied.

→ From, eq. ②, $A^{-1}(A^2 - 5I) = A^{-1}O$

$$\Rightarrow A - 5A^{-1} = O.$$

$$\Rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

→ From eq. ②, $A^2 - 5I = O$

$$\Rightarrow A^2 = 5I$$

$$\Rightarrow A^2 = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

③ If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Prove that $A^5 - A^4 + 3A^3 - 5A^2 + 2A - 3I$ in terms of A^2, A and I.

→ Here, given matrix is $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

and $A^5 - A^4 + 3A^3 - 5A^2 + 2A - 3I$.

→ The cha. eq^n of A is given by,

$$\lambda^2 - S_1\lambda + S_2 = O.$$

where $S_1 = tr(A) = 4$

$S_2 = |A| = 5$.

/* imaginary characte ← roots મળશે. */

$\therefore \lambda^2 - 4\lambda + 5 = O.$ —— ①

→ If cayley-Hemilton th. is satisfies, we must have $A^2 - 4A + 5I = O.$ —— ②

($\because$ from ①)

$\therefore$ L.H.S. $= A^2 - 4A + 5I$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$A^2 - 4A \times 5I$

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$$= \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ 4 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2 = R.H.S.$$

$\therefore$ C. H. T. is satisfied,

→ Now, we are given

$$A^5 - A^4 + 3A^3 - 5A^2 + 2A - 3I$$

$$= A^3(A^2 - 4A + 5I) + 4A^4 - 5A^3 - A^4 + 3A^3 - 5A^2 + 2A - 3I$$

$$= A^3(0) + 3A^4 - 2A^3 - 5A^2 + 2A - 3I$$

$(\because$ eq. 2 $)$.

$$= 3A^2(A^2 - 4A + 5I) + 12A^3 - 15A^2 - 2A^3 - 5A^2 + 2A - 3I$$

$$= 3A^2(0) + 10A^3 - 20A^2 + 2A - 3I$$

$$= 10A(A^2 - 4A + 5I) + 40A^2 - 50A - 20A^2 + 2A - 3I$$

~~$= 10A(0) - 20A^2 - 8A - 3I$~~

~~$= -20A^2 - 8A - 3I$~~ ~~$(\because$ eq. 2 $)$~~

$$= 10A(0) + 20A^2 - 48A - 3I$$

$$= 20A^2 - 48A - 3I \quad (\because \text{eq. } 2)$$

Use 2nd method for this

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(4) Verify Cayley Hamilton theorem for the given matrix and hence find A^{-1}.

→ Here, given matrix is $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

→ The characteristic eq^n of A is given by,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

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where, $S_1 = tr(A) = 5$

$S_2 = M_{11} + M_{22} + M_{33} = 3 + 1 + 5 = 9.$

$S_3 = |A| = 3 + (+1)[2-4]$

$= 3 - 2 = 1$

$\therefore \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0.$ — (1)

→ If the Cayley - Hemilton theorem is satisfies, we must get $A^3 - 5A^2 + 9A - I = 0.$ $(\because (1))$ — (2)

~~$\therefore L.H.S = A^3 - 5A^2 + 9A - I$~~

→ $\therefore A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1-2 & 2+6+4 & -2-2 \\ -1-3 & -2+9 & 2 \\ 2 & -6-2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix},$

and $A^3 = A^2 . A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$

$= \begin{bmatrix} -1-12 & -2+36+8 & 2-4 \\ -4-7 & -8+21-4 & 8+2 \\ 2+8 & 4-24-2 & -4+1 \end{bmatrix}$

$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$

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$\therefore$ L.H.S $= A^3 - 5A^2 + 9A - I$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 16 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 16 & -22 & -3 \end{bmatrix} + \begin{bmatrix} 5 & -60 & 20 \\ 20 & -35 & -10 \\ -10 & 40 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_3$$

$= R.H.S.$

$\therefore$ Cayley-Hamilton theorem satisfied.

$\rightarrow$ From eq. (2), $A^{-1}(A^3 - 5A^2 + 9A - I) = 0$

$\therefore \quad A^2 - 5A + 9I - A^{-1} = 0.$

$\therefore \quad A^{-1} = A^2 - 5A + 9I$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} -5 & -10 & 10 \\ 5 & -15 & 0 \\ 0 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

* LU Decomposition of a Matrix :-

1) Find the LU Decomposition of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

$\Rightarrow$ Let $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

$$\therefore \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & u_{13}l_{21} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$\therefore \boxed{u_{11} = 1}$ $\quad \boxed{u_{12} = 1}$ $\quad \boxed{u_{13} = 3}$

$l_{21} \cdot u_{11} = 1$

$\therefore \boxed{l_{21} = 1}$

$l_{21} u_{12} + u_{22} = 5$

$\therefore (1)(1) + u_{22} = 5$

$\therefore \boxed{u_{22} = 4}$

$u_{13} \cdot l_{21} + u_{23} = 1$

$\therefore (3)(1) + u_{23} = 1$

$\therefore \boxed{u_{23} = -2}$

$l_{31} \cdot u_{11} = 3$

$\therefore \boxed{l_{31} = 3}$

$l_{31} \cdot u_{12} + l_{32} \cdot u_{22} = 1$

$\therefore (3)(1) + l_{32} \cdot (4) = 1$

$\therefore 4 l_{32} = -2$

$\therefore \boxed{l_{32} = -1/2}$

$l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + u_{33} = 1$

$\therefore (3)(3) + (-1/2)(-2) + u_{33} = 1$

$\therefore 9 + 1 + u_{33} = 1$

$\therefore \boxed{u_{33} = -9}$

$$\therefore A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & -9 \end{bmatrix}$$

$\therefore$ One can verify that $A = LU$.

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2nd method

→ $R_i - (b) R_j$

There must be − sign

→ Interchanging rows are NOT allowed in this method

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & -2 \\ 0 & -2 & -8 \end{bmatrix}$$

$R_2 \to R_2 - (1) R_1$

$R_3 \to R_3 - (3) R_1$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & -9 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{-1}{2}\right) R_2$

$\therefore A = LU$

(2). Factorize the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Don't try to make it '1'

Focus only in making these entries '0'.

$$\to A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & -7/2 & 1/2 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1$

$R_3 \to R_3 - \frac{3}{2} R_1$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

$R_3 \to R_3 - (-7) R_2$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

(3). Factorize the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$ into the LU form

$$\rightarrow \quad A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \qquad \cancel{R_2 \rightarrow R_2 - 3R_1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \qquad \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \qquad R_3 \rightarrow R_3 - R_2$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ from these values one can verify $A = LU$.