Unit : 01 First order and first degree ordinary differential equations :-

* Differential Equation :-

→ $v = \frac{ds}{dt}$ ← dependent ← independent

→ $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ } z → dependent ; x, y independent

* Order :- highest derivative occurring.

* Degree :- degree of the highest order derivative which occurs in it.

↳ Ex :- $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = e^x$

→ Order : 3 , Degree : 1.

* Solutions of first order and first degree differential equations :-

1. Variables Separable Method :-

$M(x) \cdot dx + N(y) \cdot dy = 0.$

→ The solution of Variables Separable differential equation can be derived by taking integration on both sides. by

$\int M(x) \cdot dx + \int N(y) \cdot dy = c$, c = arbitrary constant

2. Linear Differential Equations :-
[Leibnitz differential equations]

i] $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

Integrating factor (I.F.) = $e^{\int P(x) \cdot dx}$

→ Solution :-

$$y = \frac{1}{(I.F.)} \left[\int Q(x) \cdot (I.F.) \cdot dx + c \right]$$

c = arbitrary constant

ii] $\frac{dx}{dy} + P(y) \cdot x = Q(y)$.

Integrating factor (I.F.) = $e^{\int P(y) \cdot dy}$

→ Solution :-

$$x = \frac{1}{(I.F.)} \left[\int Q(y) \cdot (I.F.) \cdot dy + c \right]$$

c = arbitrary constant

3. Bernoulli's Differential Equations :-

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$$

where, P and Q are the functions of x or constants

n is neither 0 nor 1.

Ex:- Solve the differential equation $\frac{dy}{dx} + 3\cdot\frac{y}{x} = 2\cdot x^4 y^4$.

→ Here, we are given, $\frac{dy}{dx} + 3\cdot\frac{y}{x} = 2\cdot x^4 y^4$ —— (1)

which is Bernoulli's d. eq^n with $n = 4$.

∴ dividing by y^4 both the sides,

$$\frac{dy}{dx}\frac{1}{y^4} + 3\cdot\frac{1}{y^3}\cdot\frac{1}{x} = 2\cdot x^4$$

→ Now, let $\frac{1}{y^3} = v$ → always let only positive. DO NOT consider negative value.

$$\therefore -\frac{3}{y^4}\cdot\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{1}{y^4}\cdot\frac{dy}{dx} = -\frac{1}{3}\cdot\frac{dv}{dx}$$

$$\therefore -\frac{1}{3}\cdot\frac{dv}{dx} + 3\cdot v\cdot\frac{1}{x} = 2\cdot x^4$$

$$\therefore \frac{dv}{dx} - \frac{9}{x}\cdot v = -6\cdot x^4$$

∵ multiplying by -3 both the sides.

which is linear diff. eq^n.

where, $p(x) = \frac{-9}{x}$ and $g(x) = -6 \cdot x^4$

$$\therefore \text{I. F.} = e^{\int p(x) \cdot dx}$$

$$= e^{-9\int 1/x \cdot dx}$$

$$= e^{-9 \log x}$$

$$= e^{\log x^{-9}} = x^{-9}$$

$\therefore$ The g. solⁿ of (1) is given by,

$$v = \frac{1}{(I.F.)} \left[\int g(x) \cdot (I.F.) \cdot dx + c \right]$$

$$= \frac{1}{x^{-9}} \left[\int (-6)\, x^4 \cdot x^{-9} \cdot dx + c \right]$$

$$= \frac{1}{x^{-9}} \left[-6 \int x^{-5} \cdot dx + c \right]$$

$$= \frac{1}{x^{-9}} \left[-6 \frac{x^{-4}}{(-4)} + c \right]$$

$$\therefore v = \frac{1}{x^{-9}} \left[\frac{3}{2} x^{-4} + c \right]$$

$$\therefore \frac{1}{y^3} \cdot x^{-9} = \frac{3}{2} \cdot x^{-4} + c$$

$$\therefore y^{-3} \cdot x^{-9} = \frac{3}{2} \cdot x^{-4} + c \quad ; c = \text{arbitrary constant}$$

4. Exact and non-exact differential equations :-

* $M \cdot dx + N \cdot dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

$\therefore$ The g. solⁿ is given by,

$$\int_{y\ \text{Const}} M \cdot dx + \int \begin{pmatrix}\text{terms of } N \\ \text{containing } y \text{ only}\end{pmatrix} \cdot dy = C$$

Case : 1 : $M \cdot dx + N \cdot dy = 0$ —— (1)

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{NON Exact}$$

$$\therefore \text{I.F.} = \frac{1}{M \cdot x + N \cdot y}$$

→ Now, multiplying (1) with I.F., we get,

$$\text{-----}\ dx + \text{-----}\ dy = 0 \quad \text{——} \ (2)$$

→ Comparing (2) with $M^* \cdot dx + N^* \cdot dy = 0$, we get,

$M^* =$ ----- and $N^* =$ -----

$\therefore \frac{\partial M^*}{\partial y} =$ ---- $\therefore \frac{\partial N^*}{\partial x} =$ -----

[Now, it will be exact d. eqⁿ.]

$\therefore$ The g. solⁿ of (1) is given by,

$$\int_{y\ const} M^* \cdot dx + \int \begin{pmatrix} \text{terms of } N^* \\ \text{containing } y \text{ only} \end{pmatrix} \cdot dy = c$$

Case : 2 : $\underbrace{f_1(xy) \cdot y \cdot dx}_{M} + \underbrace{f_2(xy) \cdot x \cdot dy}_{N} = 0$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{NON Exact}$$

$$\therefore \text{I.F.} = \frac{1}{Mx - Ny}$$

→ Now, multiplying - - - - same as case : 1

Case : 3 : If eqⁿ does NOT satisfy any of the above given cases, then go for it :

→ $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ or $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$

$\therefore$ I.F. $= e^{\int g(y) \cdot dy}$ $\therefore$ I.F. $= e^{\int f(x) \cdot dx}$

→ Now, multiplying - - - same as case : 1

$\rightarrow \quad z \longrightarrow x, y$

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

* Linear : if the dependent variable and its partial derivatives occur only once in the first degree and not multiplied by the dependent variable.

Ex::

i) $y \cdot \frac{\partial u}{\partial x} + x \cdot \frac{\partial u}{\partial y} = xy$ is linear PDE

ii) $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = u$ is NON linear PDE

* Lagrange's Equation / Lagrange's L.P.D.E :-

→ When we are given the relation of the type

$f(u, v) = 0$ then the partial differential equation is given by,

$$Pp + Qq = R$$

where, $P = \frac{\partial(u,v)}{\partial(y,z)}$, $Q = \frac{\partial(u,v)}{\partial(z,x)}$, $R = \frac{\partial(u,v)}{\partial(x,y)}$

and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

<u>Ex</u>:- $f(z^2 - xy, x/z) = 0$ — (1)

→ let $z^2 - xy = u$ and $\frac{x}{z} = v$

$\therefore \frac{\partial u}{\partial x} = -y$ and $\frac{\partial v}{\partial x} = \frac{1}{z}$

and $\frac{\partial u}{\partial y} = -x$ $\frac{\partial v}{\partial y} = 0$

and $\frac{\partial u}{\partial z} = 2z$ $\frac{\partial v}{\partial z} = \frac{-x}{z^2}$

→ We know that P.D.E. corresponding to (1) is,

$$P_p + Q_q = R \quad — (2).$$

where, $P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} -x & 2z \\ 0 & -x/z^2 \end{vmatrix}$

$= x^2/z^2$

$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & -y \\ -x/z^2 & 1/z \end{vmatrix}$

$= -\frac{xy}{z^2} + 2$

$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -y & -x \\ 1/z & 0 \end{vmatrix}$

$= \frac{x}{z}$

→ Now, substituting all values in (2),

$$\frac{x^2}{z^2}p + \left(-\frac{xy}{z^2} + 2\right)q = \frac{x}{z}$$

$\therefore \quad px^2 + 2qz^2 - xyq = xz$ ——— Ans.

* Lagrange's Equation :-

→ Non linear → i) Method of grouping
→ ii) Method of multipliers

i) Method of grouping :-

Ex: Solve : $\frac{y^2z}{x}p + xzq = y^2$.

→ Given $\frac{y^2z}{x}p + xzq = y^2$ ——— (1)

Comparing (1) with $Pp + Qq = R$, we have

$$P = \frac{y^2z}{x}, \quad Q = xz, \quad R = y^2$$

$\therefore$ The auxiliary eqⁿ are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{——— (2)}$$

→ Taking first two fractions,

$$\frac{dx}{y^2z/x} = \frac{dy}{xz}$$

$\therefore \frac{x}{y^2} dx = \frac{1}{x} \cdot dy$

$\therefore x^2 \cdot dx = y^2 \cdot dy$

$\therefore \int x^2 \cdot dx = \int y^2 \cdot dy$ ($\because$ Integrating both the sides)

$\therefore \frac{x^3}{3} = \frac{y^3}{3} + c_1$

$c_1 \in R$

$\therefore u = \frac{x^3}{3} - \frac{y^3}{3} = c_1$.

→ Now, taking first and last part,

$$\frac{x \cdot dx}{y^2 \cdot z} = \frac{dz}{y^2}$$

$\therefore x \cdot dx = z \cdot dz$

$\therefore \int x \cdot dx = \int z \cdot dz$ ($\because$ Integrating both the sides)

$\therefore \frac{x^2}{2} = \frac{z^2}{2} + C_2$, $C_2 \in R$

$\therefore v = \frac{x^2}{2} - \frac{z^2}{2} = C_2$

$\therefore$ The general solⁿ is given by,

$$f(u,v) = f\left(\frac{x^3}{3} - \frac{y^3}{3} , \frac{x^2}{2} - \frac{z^2}{2} \right) = 0$$

Ans.

ii) Method of multipliers :-

Ex:- Solve : $(y^2 + z^2) p - xy q - xz = 0$

→ Given, $(y^2 + z^2) p - xy q - xz = 0$ — (1)

→ Comparing ① with $Pp + Qq = R$, we have

$P = y^2 + z^2$, $Q = -xy$, $R = xz$

∴ The auxiliary eqⁿ are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{xz} \quad \text{— ②}$$

→ Taking last two fractions,

$$\frac{dy}{-xy} = \frac{dz}{xz}$$

$$\therefore \quad -\int \frac{1}{y} \cdot dy = \int \frac{1}{z} \cdot dz$$

$$\therefore \quad -\log y = \log z + \log c$$

$$\therefore \quad -(\log yz) = \log c$$

$$\therefore \quad yz = c_1 = u \qquad (\because -\log c = \log c_1)$$

→ Now, from eqⁿ ②, multipliers we have,

$$\therefore \quad \text{Each fraction} = \frac{x \cdot dx + y \cdot dy - z \cdot dz}{x(y^2+z^2) - xy^2 - xz^2}$$ ← Try to make this 0.

$$\left(\therefore \text{Each fraction} = \frac{l \cdot dx + m \cdot dy + n \cdot dz}{\underbrace{lP + mQ + nR}_{0}} \right)$$

$$\therefore \quad \text{Each fraction} = \frac{x \cdot dx + y \cdot dy - z \cdot dz}{0}$$

$$\therefore \quad x \cdot dx + y \cdot dy - z \cdot dz = 0$$

$\therefore \int x \cdot dx + \int y \cdot dy - \int z \cdot dz = 0.$

$\therefore \dfrac{x^2}{2} + \dfrac{y^2}{2} - \dfrac{z^2}{2} = c_2 = v.$

$\therefore$ The general solⁿ is given by,

$$f(u,v) = f\left(yz , \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} \right) = 0$$

Ans.

* Special types of non-linear PDE of first order :-

Case : 1 : Equations involving only p and q that is, $f(p,q) = 0.$

→ Assume that $z = ax + by + c$ is a solution of $f(p,q) = 0.$

$$p = \frac{\partial z}{\partial x} = a = f(b) \qquad q = \frac{\partial z}{\partial y} = b = f(a)$$

If we find 'a' then, $z = f(b)x + by + c$

or

If we find 'b' then, $z = ax + f(a)y + c$

$\Big]$ is solⁿ.

Case : 2 : Equations not involving the independent variables, that is $f(z,p,q) = 0.$

→ Assume that $u = x + ay$,

where a = arbitrary constant

$$z \longrightarrow u \begin{cases} \nearrow x \\ \searrow y \end{cases}$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \qquad \therefore \boxed{p = \frac{dz}{du}}$$

$$\therefore q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \qquad \therefore \boxed{q = a \cdot \frac{dz}{du}}$$

→ Substituting the value of p and q in $f(z, p, q) = 0$, we obtain ordinary differential equation by usual methods and replace u by $x + ay$.

Case : 3 : Separable equations that is,

$$f(x, p) = g(y, q).$$

→ Assume $f(x, p) = g(y, q) = a$ [Constant].

→ From this we can find the value of p and q as $p = f(a, x)$ and $q = g(a, y)$.

→ Substituting the value of p and q in

$$dz = p \cdot dx + q \cdot dy$$

and integrating it, we get the complete solution.

Case : 4 : Clairaut's equation that is,

$$z = p \cdot x + q \cdot y + f(p, q).$$

→ Assume that $p = a$ and $q = b$.

→ Substituting the values of p and q in

$$z = px + qy + f(p, q)$$

we get the complete integral which is

$$z = ax + by + f(a, b).$$

Unit : 2 : Higher Order Ordinary Linear Differential Equations

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→ Linear differential equations are those differential equations in which the dependent variable and its derivatives occur only in first degree and are not multiplied together.

→ $$\frac{d^n y}{dx^n} + P_1(x)\cdot\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_{n-1}(x)\cdot\frac{dy}{dx} + P_n(x)\cdot y = R(x).$$

→ If $R(x) = 0 \Rightarrow$ Homogeneous lin. diff. eq^n.

1. Real and different

C.F. is $y(x) = c_1 \cdot e^{m_1 x} + c_2 \cdot e^{m_2 x} + \dots + c_n \cdot e^{m_n x}$

2. Real and equal

C.F. is $y(x) = e^{mx}\left[c_1 + c_2 x + \dots + c_n x^{n-1}\right]$

3. Imaginary $[m = p \pm qi]$

C.F. is $y(x) = e^{px}\left[c_1 \cdot \cos qx + c_2 \cdot \sin qx\right]$

4. Imaginary and equal $[m = p \pm qi, \; p \pm qi]$

C.F. is $y(x) = e^{px}\left[(c_1 + c_2 x)\cos qx + (c_3 + c_4 x)\sin qx\right] + c_5 \cdot e^{m_5 x} + \dots + c_n \cdot e^{m_n x}$

where, $m_1 = m_2 = p + qi$, $m_3 = m_4 = p - qi$

$m_5, m_6, \dots, m_n$.

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Ex :- Solve $(D^4 - 16)y = 0$.

→ Here, we are given $(D^4 - 16)y = 0$ — ①
which is in operator form.

∴ A.E. of ① is,

$$m^4 - 16 = 0$$

$$\therefore m^4 = 16$$

$$\therefore m^2 = \pm 4 = +4, -4$$

$$\therefore m = \pm 2, \pm 2i \rightarrow p \pm qi$$

$e^{px}[c_1 \cdot \cos qx + c_2 \cdot \sin qx]$

∴ The general solⁿ is given by,

$$y(x) = c_1 \cdot e^{2x} + c_2 \cdot e^{-2x} + e^{0x}[c_3 \cdot \cos 2x + c_4 \cdot \sin 2x]$$

$$\therefore y(x) = c_1 \cdot e^{2x} + c_2 \cdot e^{-2x} + c_3 \cdot \cos 2x + c_4 \cdot \sin 2x$$

* Linearly Dependent and Independent functions :-

→ The Wronskian of the functions $y_1(x)$ and $y_2(x)$ is,

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

For $y_1(x)$, $y_2(x)$ & $y_3(x)$

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

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$W = 0 \Rightarrow$ Linearly Dependent
$W \neq 0 \Rightarrow$ Linearly Independent

If $R(x) \neq 0 \Rightarrow$ Non-homogeneous L.D.E.

Sol^n : $y = y_c + y_p$

y_p → Variation of parameters
y_p → Undetermined coefficien

y_c: Complementary function (C.F.) Homogeneous.

y_p: Particular integral

* For order two : (i) Variation of parameters

$$y_p(x) = y_1(x) \cdot u_1 + y_2(x) \cdot u_2$$

$$= y_1(x) \cdot \int \frac{R(x) \cdot W_1}{W} \cdot dx + y_2(x) \cdot \int \frac{R(x) \cdot W_2}{W} \cdot dx$$

* For order three :

$$y_p(x) = y_1(x) \cdot u_1 + y_2(x) \cdot u_2 + y_3(x) \cdot u_3$$

$$= y_1(x) \cdot \int \frac{R(x) \cdot W_1}{W} \cdot dx + y_2(x) \cdot \int \frac{R(x) \cdot W_2}{W} \cdot dx + y_3(x) \cdot \int \frac{R(x) \cdot W_3}{W} \cdot dx$$

Ex :- Solve $(D^2+1)y = \operatorname{cosec} x$ using the method of variation of parameters.

→ Given, $(D^2+y)y = \operatorname{cosec} x$ —— (1)

$\therefore R(x) = \operatorname{cosec} x$

∴ The auxiliary equation (A.E.) of (1) is,

$m^2+1 = 0$

$\therefore m^2 = -1$

$\therefore m = \pm i$

$p \pm qi$

sol^n: $e^{px}[c_1\cos qx + c_2 \sin qx]$

∴ C.F. $y_c = c_1 \cdot \cos x + c_2 \cdot \sin x$ —— (2)

$y_1(x)$ $y_2(x)$

c_1, c_2 = arbitrary constants

→ let $y_1(x) = \cos x$, $y_2(x) = \sin x$

$\therefore W(y_1, y_2)$ or $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= 1 \neq 0$

∴ linearly independent

→ $W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x$

→ $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$

∴ By method of variation of parameters,

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$$y_p(x) = y_1(x) \cdot u_1 + y_2(x) \cdot u_2$$

$$= \cos x \int \frac{R(x) \cdot W_1}{W} \cdot dx + \sin x \int \frac{R(x) \cdot W_2}{W} dx$$

$$= \cos x \int \frac{\operatorname{cosec} x \cdot (-\sin x)}{1} \cdot dx +$$

$$\sin x \int \frac{\operatorname{cosec} x \cdot \cos x}{1} \cdot dx$$

$$= -\cos x \int 1 \cdot dx + \sin x \int \frac{\cos x}{\sin x} \cdot dx$$

$$/^{*} = -\cos x \cdot \int 1 \cdot dx + \sin x \int \frac{\frac{d}{dx} \sin x}{\sin x} \cdot dx \ ^{*}/$$

(અં: છેદ નું વિકલન અંશમાં)

$$\therefore y_p(x) = -x \cdot \cos x + \sin x \cdot \ln(\sin x) \quad \text{--- (3)}$$

→ From (2) and (3),

$$y(x) = y_c(x) + y_p(x)$$

$$\therefore y(x) = c_1 \cdot \cos x + c_2 \cdot \sin x + (-x) \cos x + \sin x \cdot \ln(\sin x)$$

$$\therefore y(x) = c_1 \cos x + c_2 \cdot \sin x - x \cdot \cos x + \sin x \cdot \ln(\sin x)$$

Ans.

For 3^{rd} order

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \text{---}, \quad W_3 = \text{----}$$

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(ii) Method of undetermined co-efficient :-

$$\frac{d^n y}{dx^n} + p_1 \cdot \frac{d^{n-1} y}{dx^{n-1}} + \dots + p_{n-1} \cdot \frac{dy}{dx} + p_n y = R(x).$$

$$R(x) \neq 0.$$

*	Terms of $R(x)$.	Assumption of particular integral (y_p)
i)	$C \cdot e^{ax}$ Exponential function	$A \cdot e^{ax}$
ii)	$C \cdot \sin ax$ OR $C \cdot \cos ax$ Trigonometry function	$A \cdot \sin ax + B \cdot \cos ax$
iii)	$A \cdot x^n$ Polynomial function	$c_1 \cdot x^n + c_2 \cdot x^{n-1} + \dots + c_n$
iv)	$A \cdot e^{ax} \cdot \sin bx$ OR $A \cdot e^{ax} \cdot \cos bx$ Combination of exponential and trigonometry function.	$e^{ax}[A \cdot \sin bx + B \cdot \cos bx]$

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Case : 1 : No terms of $R(x)$ occurs in the C.F.

Ex:- $y'' + 2y' + 10y = 25x^2 + 3$

→ Given, $y'' + 2y' + 10y = 25x^2 + 3$ —— (1)

∴ The operator form of (1) is,

$$(D^2 + 2D + 10)y = 25x^2 + 3.$$

∴ The A.E. is $m^2 + 2m + 10 = 0$

$$\therefore \quad m = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2}$$

$$= \frac{-2 \pm \sqrt{4-40}}{2}$$

$$= \frac{-2 \pm 6i}{2}$$

$$= -1 \pm 3i$$

∴ C.F. $y_c = e^{-x}[c_1 \cdot \cos 3x + c_2 \cdot \sin 3x]$ —— (2)

c_1, c_2 = arbitrary constants.

→ Here, $R(x) = 25x^2 + 3$.

∴ No terms of $R(x)$ occurs in the C.F.

therefore, we need to assume y_p as follows :

$$\therefore y_p = Ax^2 + Bx + C \quad \text{—— (3)}$$

$$\therefore y_p' = 2Ax + B$$

$$\therefore y_p'' = 2A$$

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→ Now, substituting these values in (1),

$$y_p'' + 2y_p' + 10y_p = 25x^2 + 3.$$

$$\therefore \; 2A + 2[2Ax + B] + 10[Ax^2 + Bx + C] = 25x^2 + 3.$$

$$\therefore \; 2A + 4Ax + 2B + 10Ax^2 + 10Bx + 10C = 25x^2 + 3.$$

$$\therefore \; 10Ax^2 + (4A + 10B)x + 2A + 2B + 10C = 25x^2 + 0x + 3$$

$$\Rightarrow 10A = 25 \qquad \Rightarrow 4A + 10B = 0$$

$$\Rightarrow \boxed{A = \frac{5}{2}} \qquad \Rightarrow 4\left(\frac{5}{2}\right) + 10B = 0$$

$$\Rightarrow \boxed{B = -1}$$

and $\Rightarrow 2A + 2B + 10C = 3$

$$\Rightarrow 2\left(\frac{5}{2}\right) + 2(-1) + 10C = 3$$

$$\Rightarrow 5 - 2 + 10C = 3$$

$$\Rightarrow \boxed{C = 0}$$

→ From, (3), $y_p = \frac{5}{2}x^2 + (-1)x + 0$

$$= \frac{5}{2}x^2 - x \quad \text{--- (4)}$$

→ From (2) and (3),
the g. sol^n of (1) is,

$$y = y_c + y_p$$

$$= e^{-x}\left[c_1 \cdot \cos 3x + c_2 \cdot \sin 2x \right] + \frac{5x^2}{2} - x.$$

Ans.

Case : 2 : If a term of $R(x)$ occurs in the C.F. [Modification Rule]

Ex :- $y'' - y' - 2y = 3 \cdot e^{2x}$.

→ Given, $y'' - y' - 2y = 3 \cdot e^{2x}$ —— (1)

∴ The operator form of (1) is given by,

$$(D^2 - D - 2)y = 3 \cdot e^{2x}$$

∴ The A. E. is,

$$m^2 - m - 2 = 0.$$

$$\therefore (m-2)(m+1) = 0$$

$$\therefore m = 2, -1.$$

∴ C.F. $y_c = c_1 \cdot e^{2x} + c_2 \cdot e^{-x}$ —— (1)

c_1, c_2 = arbitrary constants

→ Let $y_1(x) = e^{2x}$, $y_2(x) = e^{-x}$.

Here, $y_1(x) \in R(x)$, therefore we have to modify the choice of y_p as follows :

$$y_p = A \cdot \boxed{x} \cdot e^{2x}$$

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$\therefore y_p' = A \cdot e^{2x} + 2Ax \cdot e^{2x}$
$= A \cdot e^{2x} [1 + 2x]$

$\therefore y_p'' = 2 \cdot A \cdot e^{2x} (1+2x) + A \cdot e^{2x} (2)$
$= 4 \cdot A \cdot e^{2x} + 4Ax \cdot e^{2x}$

---- Same as case: 1.

* Legendre's linear differential equation :-

→ $(ax+b) \cdot \frac{dy}{dx} = a \cdot D \cdot y$

→ $(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 \cdot D(D-1) \cdot y$

→ $(ax+b)^3 \cdot \frac{d^3y}{dx^3} = a^3 \cdot D(D-1)(D-2) \cdot y$

where, $D = \frac{d}{dt}$

and let $ax + b = e^t$

$\therefore t = \log(ax+b)$

Ex:- Solve: $(2x+3)^2 \cdot \frac{d^2y}{dx^2} - (2x+3) \cdot \frac{dy}{dx} - 12y = 6x$

→ Given, $(2x+3)^2 \cdot \frac{d^2y}{dx^2} - (2x+3) \cdot \frac{dy}{dx} - 12y = 6x$ —— (1)

→ Let, $(2x+3) = e^t$ $\Rightarrow x = \frac{e^t - 3}{2}$

$\therefore t = \log(2x+3)$

→ Now, from $(ax+b)\cdot\frac{dy}{dx} = a\cdot D\cdot y$ and

$$(ax+b)^2\cdot\frac{d^2y}{dx^2} = a^2\cdot D(D-1)\cdot y$$

where, $D = \frac{d}{dt}$

we have,

$$(2x+3)\cdot\frac{dy}{dx} = 2\cdot D\cdot y \quad \text{and}$$

$$(2x+3)^2\cdot\frac{d^2y}{dx^2} = 4\cdot D(D-1)\cdot y$$

where, $D = \frac{d}{dt}$

→ Now, substituting these values in eq.ⁿ (1), we have,

$$[4(D^2-D) - 2D - 12]y = 6\left(\frac{e^t-3}{2}\right)$$

$$\Rightarrow (2D^2 - 3D - 6)y = \frac{3\cdot e^t - 9}{2}$$

∴ The A.E. is $2m^2 - 3m - 6 = 0$

--- same as case : 2 [∵ Non homogeneous]

* Special Case : Cauchy - Euler L.D.E :-

→ $a = 1$ and $b = 0$ in L.D.E.

$$x\cdot\frac{dy}{dx} = D\cdot y$$

$$x^2\cdot\frac{d^2y}{dx^2} = D(D-1)\cdot y$$

$$x^n\cdot\frac{d^ny}{dx^n} = D(D-1)\ldots(D-n+1)\cdot y$$

where, $D = \frac{d}{dt}$

and $x = e^t$

∴ $t = \log x$

∴ $\frac{dt}{dx} = \frac{1}{x}$

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* System of simultaneous first order linear differential equations :-

[Refer it from the book]

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Probability and Statistics

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* Mean :-

i) For ungrouped data :-

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

ii) For grouped data :- (Discrete data and continuous series)

Method : 1 : Direct Method

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad \text{where, } \sum_{i=1}^{n} f_i = n.$$

Method : 2 : Assumed Mean Method.

$$\bar{x} = a + \frac{\sum f_i d_i}{\sum f_i}$$

where, a = Assumed mean
$d_i = x_i - a$.

* Median :-

Case :- 4 : For continuous frequency distribution

$$\text{Median} = L + \frac{N/2 - c.f}{f} \times h.$$

* Mode :-

Case: 1 : For ungrouped data
use the definition of Mode.

Case: 2 : For grouped data (Continuous frequency distribution.)

$$\text{Mode} = L + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

* Standard deviation :-

Case: 1 : For ungrouped data :-

$$\sigma = \sqrt{\frac{\Sigma (x - \bar{x})^2}{N}}$$

Case: 2 : For grouped data :-

$$\sigma = \sqrt{\frac{\Sigma f x^2}{N} - \left(\frac{\Sigma f x}{N}\right)^2}$$

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* Bionomial Distribution :-

$$P(X = x) = \binom{n}{x} p^x \cdot q^{n-x}$$

Mean : np

Variance : npq

Standard deviation : $\sqrt{npq}$

* Poisson Distribution :-

$$\begin{cases} n \to \infty \\ p \to 0 \end{cases} \quad \text{and} \quad \lambda = np = \text{finite.}$$

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad ; \quad x = 0, 1, 2, \ldots$$

* Probability Density Function :-

Mean $\bar{x} = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$ | $f(x)$ be a continuous function

Variance $= \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f(x) \cdot dx$

($\bar{x}$ = mean)

⇝ $f(x) \geq 0$ for every values of x.

⇝ $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

⇝ $\int_{a}^{b} f(x) \cdot dx = P(a < x < b)$

* Exponential Distribution :-

$$f_x(x) = \begin{cases} \frac{1}{\beta} \cdot e^{-\frac{x}{\beta}} & ; \ x > 0 \ , \ \beta > 0 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

* Normal Distribution :-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$-\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0$$