CHAROTAR UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF MATHEMATICAL SCIENCES

SEMESTER 3 B.Tech. (CE/ IT/ CSE)

DISCRETE MATHEMATICS AND ALGEBRA

MA253

Unit-2 Relations and Lattice

CARTESIAN PRODUCT: Let A and B be sets. Cartesian product of A and B , denoted by A×B, is defined as

$$A \times B = \{(a,b) : a \in A \ and \ b \in B\}$$

RELATION ON SETS: Let A and B be two sets. A relation from A to B is a subset of the Cartesian product AXB. Suppose R is a relation from A to B. Then R is a set of ordered pairs (a, b) or aRb where a is in A and b is in B. Every such ordered pair is written as (a, b) or aRb and read as " a is related to b by R".

DOMAIN AND RANGE OF RELATION R:

Domain: The set $\{a \in A : (a,b) \in R \ for \ some \ b \in B\}$ is called the domain of R and denoted by Dom(R).

Range: The set $\{b \in B : (a,b) \in R \ for \ some \ a \in A\}$ is called the range of R and denoted by Ran(R).

Thus, the domain of relation R is the set of all first elements of the ordered pairs which belong to R and the range is the set of second element.

Example: Let A={ 2, 3, 5} , B={ 2, 4, 6, 10} . A relation from A to B is given as follows: {2R2, 2R4,2R6, 2R10,3R6,5R10}.Write R as a set of ordered pair.

Solution: R={ (2,2),(2,4), (2,6),(2,10), (3,6),(5,10)}.

Total number of distinct relation from a set A to a set B .

Let the number of element of A and B be m and n respectively. Then the number of elements in AXB is mn. Therefore, the number of elements of the power set of AXB is 2^{mn}. Thus AXB has 2^{mn} subsets. Now every subset of AXB is a relation from A to B. Hence ,the number of different relation from A to B is 2^{mn}.

TYPES OF RELATION

Inverse Relation: Let R be the relation from set A to set B. The inverse of R is denoted by R^{-1} is the relation from B to A which consist of those ordered pairs which , when reversed, belong to R that is

$R^{-1}=\{(b, a): (a, b)\in R\}$.

Example: Let A={1,2,3} and B={a, b} and R={ (1,a), (1,b),(3,a), (2,b)} be a relation defined from A to B. Then find R^{-1}.

Solution: R^{-1}={ (a,1), (b,1), (a,3), (b,2)}.

Identity Relation: A relation R on a set A to A is said to be identity relation $I_{A.}$

i.e., $I_A=\{(x, x): x\in A\}$.

For example, Let A={2,4,6}. $I_A=\{(2,2),(4,4),(6,6)\}$ is an identity relation on A.

Complement of Relation: Let R be a relation defined on set $A\times B$ .Then complement of the relation is denoted by R^c or $\bar{R}$. Then $\bar{R}$ **is defined as** ={ $(a,b):(a,b)\in A\times B \, and \, (a,b)\notin R$}

i.e., =(A×B)-R.

Combining Relation: Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$ and $R_2 - R_1$.

For example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations

$R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

can be combined using basic set operations to form new relations:

$\boldsymbol{R_1 \cap R_2 = \{(1,1)\}}$

$\boldsymbol{R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}}$

$\boldsymbol{R_2 - R_1 = \{(1,2),(1,3),(1,4)\}}$

PROPERTIES (TYPES) OF RELATION

REFLEXIVE RELATION: Let R be a relation on set A to A. Then R is said to be reflexive relation if (a, a) $\in R\ \forall a \in A$. (a, a) are known as diagonal element.

Let A={ 1,2,3,4}

RELATION	REFLEXIVE	REASON
R_1={(1,1),(2,4),(3,3),(4,1),(4,4)}	Not Reflexive	(2,2) $\notin R$
R_2={(1,1),(2,2),(2,3),(3,3),(4,4)}	Reflexive	(1,1),(2,2),(3,3),(4,4) $\in R$
R_3={(1,1),(2,2),(3,3),(4,4)}	Reflexive	It contain all diagonal element.
R_4={(1,2),(2,3),(2,4),(3,4),(4,2)}	Not Reflexive	(1,1),(2,2),(3,3),(4,4) $\notin R$
$R_5=\phi$	Not Reflexive	(1,1) $\notin R$,(2,2) $\notin R$,(3,3) $\notin R$, (4,4) $\notin R_5$. No diagonal element belongs to relation R_3.
R_6=AXA	Reflexive	(1,1),(2,2),(3,3),(4,4) $\in R_6$ i.e. all diagonal element belong to relation R_3.

Remark: Total number of element in A =n. Therefore no. of element in Cartesian product AXA =nXn.

<u>IRREFLEXIVE RELATION:</u> A relation R on a set A to A is said to be irreflexive if a∈ A, (a,a) does not belong to R. It is also called as antireflexive.

Let A={ 1,2,3,4}

RELATION	REFLEXIVE	REASON
R_1={(1,1),(2,4),(3,3),(4,1),(4,4)}	Not Irreflexive	((1,1) belong to the set R_1
R_2={(1,1),(2,2),(2,3),(3,3),(4,4)}	Not Irreflexive	(1,1),(2,2),(3,3),(4,4) ∈ R
R_3={(1,1),(2,2),(3,3),(4,4)}	Not Irreflexive	It contains all diagonal elements.
R_4={(1,2),(2,3),(2,4),(3,4),(4,2)}	Irreflexive	Diagonal element does not belong to set R_4.
$R_5=\phi$	Irreflexive	Diagonal element does not belong to set R_5.
R_6=AXA	Not Irreflexive	all diagonal element belong to relation R_3.

Remark: If the relation is not reflexive then it is not compulsory that it is irreflexive.

For example relation R_1 is neither reflexive nor irreflexive.

SYMMETRIC RELATION: A relation R on a set A to A is said to symmetric if for $\forall$ a, b $\in$ A (a, b) $\in$ R , then (b, a) $\in$ R.

Let A= {a, b, c}

RELATION	SYMMERTIC	REASON
$R_1=\phi$	Symmetric	No pair exist so no need to check to symmetric pair.
R_2= AXA	Symmetric	All symmetric pair exist in relation
R_3={(a, b),(b, a)}	Symmetric	$(a,b)\in R$ & $(b,a)\in R$
R_4={(b, c),(c, b),(b, b),(c, c)}	Symmetric	Symmetric pairs exist in R_4
R_5={(a, a),(b, b),(c, c)}	Symmetric	Diagonal element are symmetric to itself
R_6={(a, b),(b, c),(a, c)}	Not symmetric	$(a,b)\in R$ *but* $(b,a)\notin R$
R_7={(a,b),(b,a),(a,c)}	Not symmetric	$(a,c)\in R$ *but* $(c,a)\notin R$

Antisymmetric Relation:

A relation R on set *X* is said to be **antisymmetric** if aRb and bRa, then $a = b$.

Remark: Antisymmetric is not the same as symmetric. A relation may be symmetric as well as antisymmetric at the same time.

For example, R_5={(a,a),(b,b),(c,c)} is both symmetric and antisymmetric on A ={a,b,c}.

ASYMMETRIC RELATION: A relation R on set A to A. is asymmetric if (a, b) $\in$ R, then (b, a) $\notin$ R). This means that the presence of (a, b) in R excludes the possibility of presence of (b, a) in R.

RELATION	ASYMMETRIC	REASON
$R_1=\phi$	Asymmetric	No symmetric pair exists.
R_2= AXA	Not Asymmetric	Symmetric pairs exist for example (a, b) and (b, a)
R_3={(a, a), (b, b), (c, c)}	Not Asymmetric	Diagonal elements are symmetric to its self.
R_4={(a, b), (b, c), (a, c)}	Asymmetric	No symmetric pair exits.
R_5={(a, b), (b, a), (b, c)}	Not Asymmetric	Symmetric pair (a, b) and (b, a) exist
R_6={(a, a),(b, b),(a, b),(b, a)}	Not Asymmetric	Diagonal element as well as symmetric pair exists.

Remarks:

1. A relation can be symmetric, antisymmetric and asymmetric for example: $R_1=\phi$
2. It is also possible that relation can be neither symmetric, antisymmetric nor asymmetric. R={(a, a), (a, b), (b, a),(b, c)}

Transitive:

A relation R over a set X is said to be **transitive** if for all

elements a, b, c in X, whenever R relates aRb and bRc, then aRc.

Following are the example of

R={(a, b),(b, a),(a, a),(b, b)} and R=AXA transitive relation.

EQUIVALENCE RELATION: A relation R on set S to S is called an **EQUIVALENCE RELATION** if it is reflexive, symmetric and transitive (RST). That is, R is an equivalence relation on A if it satisfy the following properties:

1. $(a,a) \in R$ *for all* $a \in A$
2. $(a,b) \in R$ *implies* $(b,a) \in R$
3. (a,b) and $(b,c) \in R$ *implies* $(a,c) \in R$

Remark: If R is an equivalence relation then R^{-1} is also an equivalence relation.

Example: Consider the following relation on {1, 2, 3, 4, 5, 6}. R={ (i, j): $|i-j|=2$ }.Is R reflexive , symmetric and transitive?

Solution : Let A={ 1,2,3,4,5,6}.Then R= { (1,3), (2,4), (3,1),(4,2),(3,5) ,(5,3),(4,6),(6,4)}

R is not reflexive as $(1,1) \notin R$.

R is symmetric since $(i,j) \in R \Rightarrow (j,i) \in R, \forall i,j \in R$

R is not transitive as $(2,4)$ *and* $(4,2) \in R$ *but* $(2,2) \notin R$

R is not a reflexive and transitive. R is a symmetric. Therefore R is not an Equivalence Relation.

Example: Let R be a binary relation on the set of all integers of 0's and 1's such that R={ (a, b): a and b are strings, that they have the same numbers of 0's } .Is R reflexive? Symmetric? Antisymmetric? Transitive ? An equivalence relation?

Solution: R is reflexive since $(a,a) \in R \;\; \forall a \in R$.

R is symmetric since when a and b have same numbers of 0's then b and a will also have same number of 0's .Hence

$(a,b) \in R \Rightarrow (b,a) \in R$

R is transitive since when a and b have the same number of 0's and b and c have the same number of 0's then a and c will also have same number of 0's .Hence $(a,b) \in R, (b,c) \Rightarrow (a,c) \in R$

Thus, R is reflexive, symmetric, transitive and hence an equivalence relation.

R is not antisymmetric since (a, b) and (b, a) belongs to R does not imply a=b.

EQUIVALENCE CLASSES: If R is an equivalence relation on a set S and (x, y) or xRy, then x and y are called respect to R. The set of all elements of S that are equivalent to a given element x constitute the equivalence class of x and equivalence class is denoted by [x].

Thus [x] = {y$\in$ S:yRx}.

PARTITIONS OF SET: A partition of a set A is a set of non –empty subsets of A denoted by { A_1,A_2,.....A_n} such that the union of A_i ‘s is equal to A and intersection of A_i and A_j is empty for any distinct A_i and A_j.

Example : Let A={ 1,2,3,4,5} and R={(1,2), (1,1),(2,1),(2,2),(3,3),(4,4),(4,5),(5,4),(5,5)} Show that R is an equivalence relation on A. Determine the partition corresponding to R.

Solution: <u>show that R is an equivalence relation on (Homework)</u>

Then Equivalence class of

[1]={1,2}; [2]={1,2}; [3]={3}; [4]={4,5} ; [5]={4,5}

Thus partition set corresponding to R

P={ {1,2},{3}, {4,5}}

Note: Using Cross product between element to it self in a partition set, One can find equivalence relation from partition.

Example: Let R is an equivalence relation on set A = {a, b, c, d} defined by partitions of P{{a, d},{b, c}}. Determine the element of equivalence relation and also find the equivalence classes of R.

Solution:

The elements of equivalence relation defined by partition P is

R={(a, a), (d, d), (a, d),(d, a), (b, b),(c, c), (b, c),(c, b)}.

The equivalence classes of R are [a]=[d]={a, d} and [b]=[c]={b, c}.

Example: Show that the relation (x, y)R(a, b), $x^2+y^2= a^2+b^2$ is an equivalence relation on plane.

Solution:

Reflexive :

(x, y)R(x, y) $\Leftrightarrow x^2+y^2= x^2+y^2$

Therefore R is reflexive .

Symmetric : Let $(x, y)R(a, b) \Longrightarrow x^2+y^2= a^2+b^2$

$\Longrightarrow a^2+b^2 =x^2+y^2$

$\Longrightarrow (a, b)R\,(x, y)$

R is symmetric.

Transitive:

Let $(x,y)R(a,b)$ and $(a,b)R(c,d)$, $x^2+y^2= a^2+b^2$ and $a^2+b^2=c^2+d^2$

$\Longrightarrow x^2+y^2= c^2+d^2$

$\Longrightarrow (x,y)R\,(c,d)$

Therefore R is transitive.

Since R is reflexive , symmetric and transitive therefore R is an equivalence relation.

Example: If R be a relation in the set of integers Z defined by

R={ (x, y): $x \in Z, y \in Z$,x-y is divisible by 3}

Then prove that R is an equivalence relation. Describe the distinct equivalence classes of R.

Solution: Reflexive: Let $x \in Z$ .Then x-x=0 and 0 is divisible by 3.

Therefore $(x,x) \in R$ for all $x \in Z$.

Hence, R is reflexive.

Symmetric: Let $(x,y) \in R \Rightarrow$ (x-y) is divisible by 3.

$\Rightarrow$ -(y-x) is divisible by 3.

$\Rightarrow$ (y-x) is divisible by 3.

$\Rightarrow (y,x) \in R$

Hence, R is symmetric

Transitive: Let $(x,y) \in R$ *and* $(y,z) \in R \Rightarrow$ (x-y) is divisible by 3 and (y-z) is divisible by 3 .

$\Rightarrow$ (x-y) + (y-z) is divisible by 3.

$\Rightarrow$ (x-z) is divisible by 3.

$\Rightarrow (x,z) \in R$

Hence R is transitive

Thus R is an equivalence relation.

EQUIVALENCE CLASSES

For each integer a

$$[a] = \{x \in Z : xRa\}$$

$$= \{x \in Z : \text{x} - \text{a is divisible by } 3\}$$

$$= \{x \in Z : \text{x} - \text{a} = 3k \ \textit{for some integer } k\}$$

$$= \{x \in Z : \text{x} = 3k + a \ \textit{for some integer } k\}$$

In particular (put a=0 ,1,2,.....in above)

$$[0] = \{x \in Z : xR0\}$$

$$= \{x \in Z : \text{x} - 0 \text{ is divisible by } 3\}$$

$$= \{x \in Z : \text{x} - 0 = 3k \ \textit{for some integer } k\}$$

$$= \{x \in Z : \text{x} = 3k + 0 \ \textit{for some integer } k\} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$

$$[1] = \{x \in Z : xR1\}$$

$$= \{x \in Z : \text{x} - 1 \text{ is divisible by } 3\}$$

$$= \{x \in Z : \text{x} - 1 = 3k \ \textit{for some integer } k\}$$

$$= \{x \in Z : \text{x} = 3k + 1 \ \textit{for some integer } k\} = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$

$$[2] = \{x \in Z : xR2\}$$

$$= \{x \in Z : x - 2 \text{ is divisible by } 3\}$$

$$= \{x \in Z : x - 2 = 3k \text{ for some integer } k\}$$

$$= \{x \in Z : x = 3k + 2 \text{ for some integer } k\} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$

Remark: There are no other equivalence classes because every integer is already accounted for , in one of [0],[1],[2].

<u>PARTIAL ORDERED RELATION</u>: A relation R on set S is called a partial order if it is reflexive, antisymmetric and transitive (RAT). That is

1. $(a,a)\in R$ *for all* $a\in A$
2. $(a,b)\in R$ *and* $(b,a)\in R \Rightarrow a=b$
3. (a,b) and $(b,c)\in R$ *implies* $(a,c)\in R$

<u>POSET:</u> A set S together with a partial order relation R or $\preccurlyeq$ is called **partial order set or POSET** and is denoted by (S, R) or (S, $\preccurlyeq$).

Example: The greater or equal to '$\geq$' relation is a partial ordering on Z- The set of integers.

Solution:

Reflexive: Since $a\geq a$ for every integer a, $\geq$ is reflexive.

Antisymmetric: Since $a\geq b$ and $b\geq a$ imply a=b, $\geq$ is antisymmetric.

Transitive: Since $a\geq b$ and $b\geq c$ imply a$\geq$c, $\geq$ is transitive.

Hence, $\geq$ is partial ordering on Z and (Z, $\geq$) is a POSET.

Example: Consider P(S) as the power set i.e., the set of all subsets of a given set S. Show that the inclusion relation $\subseteq$ is a partial ordering on power set P(S).

Solution: Reflexive: $A \subseteq A$ for all $A \subseteq S$, $\subseteq$ is reflexive

Antisymmetric: $A \subseteq B$ and $B \subseteq A$ imply $A=B$, $\subseteq$ is antisymmetric.

Transitive: $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$, $\subseteq$ is transitive.

It follows that $\subseteq$ is a partial ordering on P(S) and $(P(S), \subseteq)$ is a Poset.

Practice Example

Example: Show that the set $(Z^+, /)$ is a POSET where the relation a/b means a is divisible by b.

Example: Check that the set $(R, \geq)$ is a POSET or not, where the relation $a \geq b$ means a is greater or equal to b.

Comparability:

The elements a and b of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. or we can say that a and b are comparable if either "a" is related to" b "i.e., $(a, b) \in R$ or "b" is related to "a" i.e., $(b, a) \in R$.

Incomparable: When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

Example: In the poset (Z+, /), are the integers 9 and 3 comparable? Are 5 and 7 comparable?

Solution: The integers 9 and 3 are comparable, because 9 / 3. The integers 5 and 7 are incomparable, because 5 / 7 and 7 / 5.

REPRESENTATION OF RELATION

There are many ways of representing relations on finite sets. Graphical methods are particularly useful for visualising the information about relations. It is more convenient to represent Relations as matrices to do mathematical calculations.

MATRIX REPRESENTATION:

Let $A=\{a_1, a_2, a_3,\ldots,a_n\}$ and $B=\{b_1,b_2,b_3,\ldots,b_m\}$ are finite sets containing n and m elements respectively and let R be a relation from A to B. The relation R can be represented by $n\times m$ matrix, $M_R=[m_{ij}]$, where

$$M_R = m_{ij} = \begin{cases} 1, & if\ (a_i, b_j) \in R \\ 0 & if\ (a_i, b_j) \notin R \end{cases}.$$

Then matrix M_R is called the matrix of R.

Example: Let R be the relation from set A={1,3,4} on itself and defined by R={(1,1), (1,3), (3,3), (4,4)}. Then find the matrix of R.

Solution: Let M_R denotes the matrix of R.

The number of rows in M_R=number of elements in A=3. Since the relation from the set A on itself, the number of column in M_R is also 3. So M_R is 3x3 matrix.

$M_R = \begin{array}{cc} & \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$	(1,1) $\in$ R so entry is 1 (1,3) $\in$R so entry is 1 (1,4) $\notin$R so entry is 0 (3,1) $\notin$ so entry is 0 (3,3) $\in$ so entry is 1 (3,4) $\notin$ so entry is 0 (4,,1) $\notin$ so entry is 0 (4,3) $\notin$ so entry is 0 (4,4) $\in$ so entry is 1

Example : Let A={ a_1, a_2, a_3} and B={ b_1,b_2,b_3, b_4} which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Find the relation R from given Matrix.

Solution:

Since R contains of those ordered pairs (a_i, b_j) with m_{ij} = 1.

It follows that R={ (a_1,b_2),(a_2,b_1),(a_2,b_3),(a_2,b_4),(a_3,b_1), (a_3,b_3) }

Remarks:

The matrix representing a relation can be used to determine whether the relation has various properties i.e. reflexive, irreflexive, symmetric, asymmetric, antisymmetric and transitive.

1) <u>Reflexive:</u> If all the elements in the main diagonal of the matrix representation of relation are 1 i.e., $m_{ii}=1$, then the relation is reflexive. For example,

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2) <u>Irreflexive:</u> If all the elements in the main diagonal of the matrix representation of relation are 0, then the relation is irreflexive. For example,

$$M_R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

3) <u>Symmetric:</u> If the representative matrix of a relation is symmetric with respect to the main diagonal, i.e., $m_{ij}=m_{ji}$ for all value of i and j then the relation is symmetric. (i.e., $M_R=M_R^T$).

4) <u>Antisymmetric:</u> A relation is antisymmetric if and only if $m_{ij}=1$ **at a same time** $m_{ji}=0$.The following matrices illustrate the notions of symmetry and antisymmetric.

$$M=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \qquad M=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$(Symmetric)$ $\qquad$ $(Anisymmetric)$

5) **<u>Asymmetric:</u>** A relation is asymmetric iff $m_{ii}=0$, $m_{ij}=1$ and $m_{ji}=0$.

For example, $M=\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

6) **<u>Transitive:</u>** A relation R is transitive if and only if $M_R^2+M_R=M_R$.

<u>Note:</u>

When we take product or sum of M_R (Matrix of relation R) at that time entries of new matrix of product or sum must be 1 or 0. If it is more than one, then take that entry as 1.

Example: Let A={1,2,3,4} and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

. Show that R is transitive.

Solution:

$$M^2_R = M_R . M_R$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^2_R + M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_R$$

Therefore, relation R is transitive.

GRAPHICAL REPRESENTATION

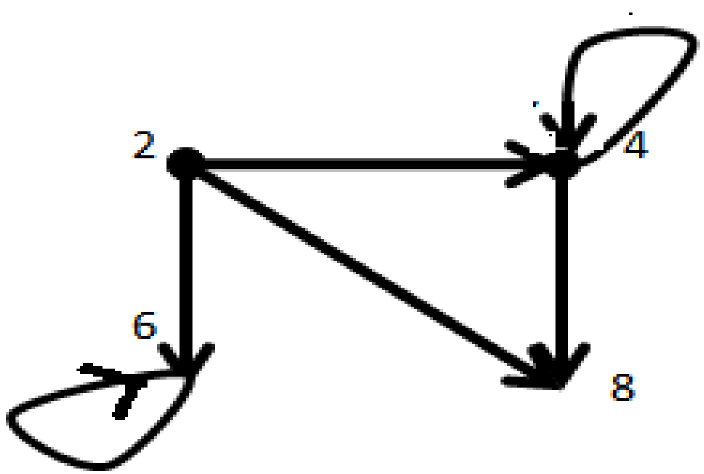
Let A and B are two finite sets and R is a relation from A to B.

For graphical representation of a relation on a set, each element of the set is represented by a point. These points are called nodes or vertices.

An arc is drawn from each point to its related point. The arcs start at the first element of the pair, and they go to the second element of the pair. The direction is indicated by an arrow. All arcs with an arrow are called directed arcs. The resulting image is called a directed graph or diagraph of R. An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called loop.

For example, Let A={ 2,4,6} and B={ 4,6,8} and R be the relation from the set A to the set B given by xRy means x is a factor of y, then **R={(2,4),(2,6), (2,8),(4,4),(6,6),(4,8)}.**

This relation R from A to B is represented by the arrow diagram as shown in figure:



Remarks:

The directed graph representing a relation can be used to determine various properties a relation:

(1) A relation is **reflexive** if and only if there is a loop at every vertex of the directed graph, so that ordered pair of the form (a,a) occurs in the relation. If no vertex has a loop then the relation is **irreflexive.**

(2) A relation is **symmetric** if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, that is if (b, a) is in the relation whenever (a, b) is in relation. A relation is **antisymmetric** if no two distinct points in the diagraph have an edge going between them in both directions.

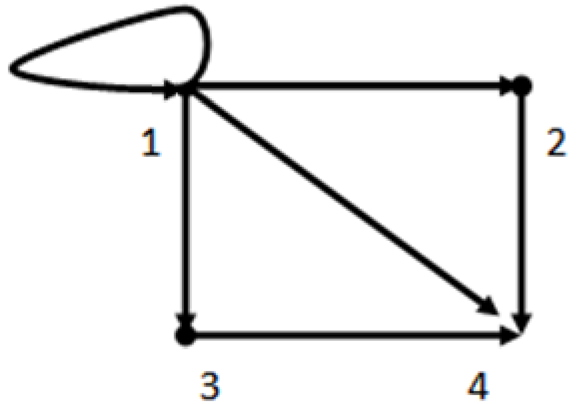
(3) A relation is **transitive** if and only if whenever there is a directed edge from to vertex **a** to a vertex **b** and from a vertex **b** to vertex **c** , then there is also a directed edge from **a** to **c**.

Every diagraph determines a relation, so that we may recover R from the graph. Domain and Range can be found easily if the relation is represented by a graph. Every node with an outgoing arc belongs to the domain and every node with an incoming arc belongs to the range.

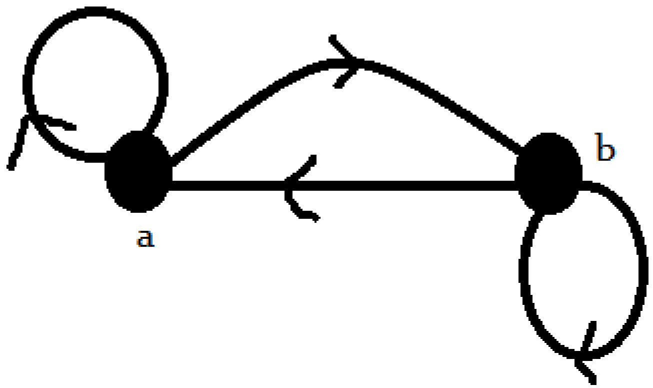
<u>EXAMPLE:</u> Draw the directed graph that represents the relation R={ (1,1),(2,2),(1,2),(2,3),(3,2),(3,1),(3,3)} on X.

Solution:

<u>EXAMPLE:</u> Determine whether the relation for the directed graph shown in figures are reflexive, symmetric , antisymmetric and transitive.



R1



R2

REPRESENTATION AND HASSE DIAGRAM

HASSE DIAGRAM: A partial order relation on X can be represented by means of a diagram known as Hasse diagram of X.

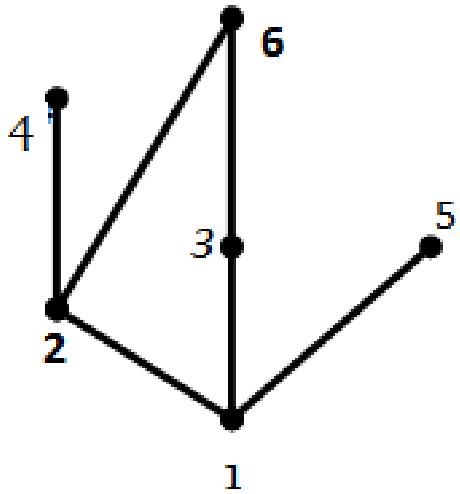
This gives a method of representing finite poset. We represent the element of x by points and if y is an immediate successor of x we take y at a higher level than x and join x and y by a straight line. A diagram formed as above is known as Hasse diagram.

Thus, there will not be any horizontal lines in the diagram of a poset.

Example: Let A={ 1,3,9,27,81} , draw the Hasse diagram of the Poset (A,/).

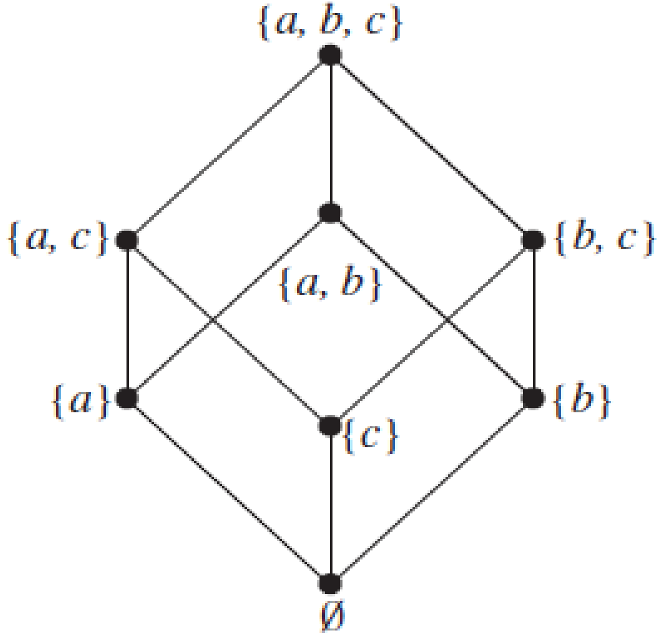
Example: Let X={1,2,3,4,5,6}, then / is partial order relation on X. Draw the Hasse diagram of (X,/)

Solution



Example: Draw the Hasse diagram for the partial ordering { (A,B): A ⊆ **B**} on the power set P(S) where S= { a,b,c}

Solution: Here P(S)={ Ø,{a}, {b},{c}, {a,b},{a,c},{b,c}, {a,b,c}}. The Hasse diagram of the poset (P(S), ⊆) is shown below

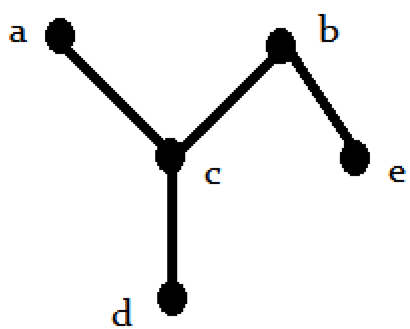


SPECIAL ELEMENT IN POSETS

MAXIMAL ELEMENT: An element '*a*' in the poset is called maximal element of P if $x \prec a$ for every x in P, i.e. No element of P strictly succeeds '*a*'.

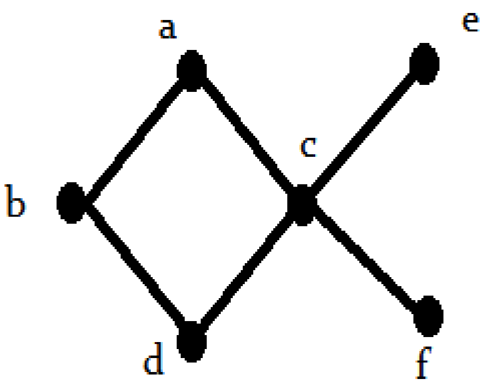
MINIMAL ELEMENT: An element '*b*' in P is called minimal element of P if $b \prec x$ for every x in P.

Remark: Maximal and minimal element are easy to spot in the Hasse diagram, They are the top and bottom elements in the diagram. That is a maximal element has no connections leading up and minimal element has no connection leading down. **Also, it is not necessary that Maximal and minimal element are unique.**

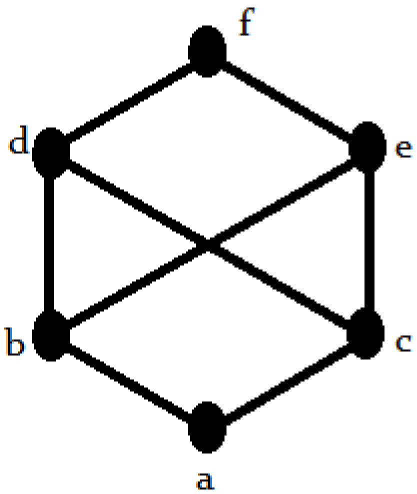


Maximal elements : a, b

Minimal elements : d, e



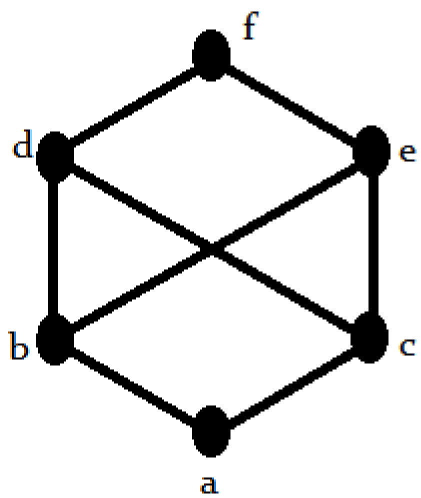
Maximal elements : a, e
Minimal elements : d, f



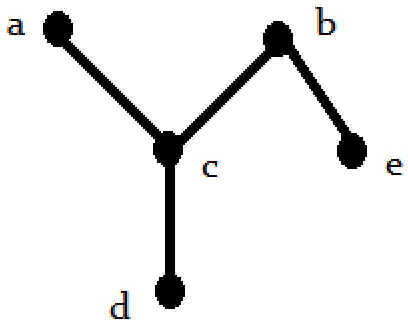
Maximal element : f
Minimal element : a

MAXIMUM ELEMENT (GREATEST ELEMENT): Let (P, $\leq$) be a poset. An element a in P is the greatest element of P if $x \leq a$ for all $x \in P$ i.e., every element in P precede a. OR An element of poset is said to be maximum if it is maximal and every element is related to it. **Also, maximum element is a unique point.**

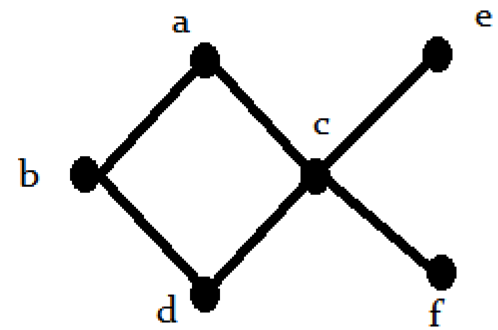
MINIMUM ELEMENT (LEAST ELEMENT) : Let (P, $\leq$) be a poset. An element a in P is the least element of P if $a \leq x$ for all $x \in P$ i.e., every element in P succeeds a. OR An element of poset is said to be minimum if it is minimal and every element is related to it. **Also, minimum element is a unique point.**

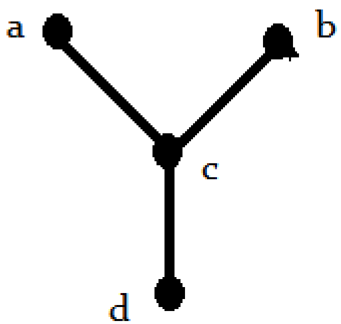


Maximal and maximum element : f
Minimal and minimum element : a



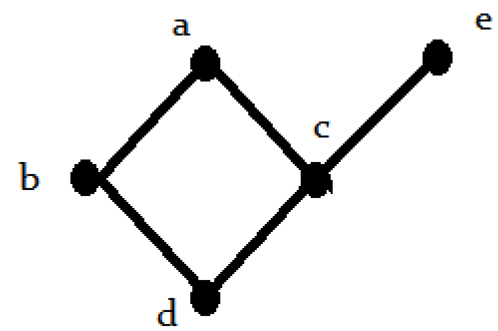
Maximum and minimum element does not exist

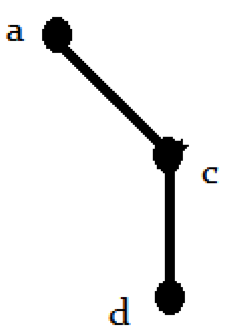




Maximum element does not exist

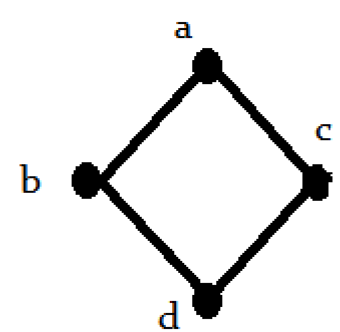
Minimum element is d.





Maximum element is a

Minimum element is d.



Remarks: The following points are to be noted

1) A poset may not have maximal element. For instance, the set of natural numbers N under the operation $\leq$ have no maximal element. Minimal element exists and it is 1.

2)A poset may have a maximal element but no minimal elements, or minimal element but no maximal elements.

For example, the poset $(Z^-, \leq)$ has a maximal element as -1 but no minimal element, whereas the poset $(Z^+, \leq)$ has minimal element as 1 but no maximal element.

3) A poset may not have a maximal element and minimal elements. For example, the poset $(R, \leq)$ has no maximal element and minimal element.

UPPER BOUND: Let B be a subset of a poset (A, ≤). An element '*u*' in A is called an upper bound of B if '*u*' succeeds every element of B i.e., $x \leq u$ for all x in B.

LOWER BOUND: Let B be a subset of a poset (A, ≤). An element '*l*' in A is called a lower bound of B if '*l*' precedes every element of B i.e., $l \leq x$ for all x in B.

LEAST UPPER BOUND : Let B be a subset of a poset (A, ≤). An element '*a*' in A is called least upper bound (lub) of B. If '*a*' is an upper bound of B and $a \leq a'$ whenever *a'* is the upper bound of B. **That means it is one of the upper bound element which is less than all the other upper bounds.(It is a unique.)**

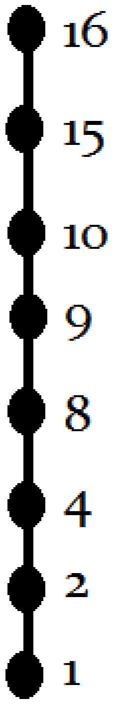
GREATEST LOWER BOUND: Let B be a subset of a poset (A,≤). An element '*a*' in A is called the greatest lower bound (glb) of B if '*a*' is lower bound of B and $a' \leq a$, whenever *a'* is a lower bound of B. i.e. it is one of the lower bound element which is greater than all the other lower bounds. **(It is a unique.)**

There are the elements which are greater than or **equal to** all the element of subset B of a poset.

Find the Maximal element, Minimal element, Maximum element, Minimum element, Upper bound, Lower bound, Least upper bound and Greatest lower bound for the sub set A={8, 9} of poset (S, ≤), where S={1, 2, 4, 8, 9, 10, 15, 16}.

Solution : Here we have a poset (S,≤), where S={1, 2, 4, 8, 9, 10, 15, 16}

. A ={8, 9} is a subset of S={1, 2, 4, 8, 9, 10, 15, 16}



Maximal elements are 9,10, 15 and 16.

Minimal elements are 1,2, 4 and 8.

Maximum element is 16.

Minimum element is 1.

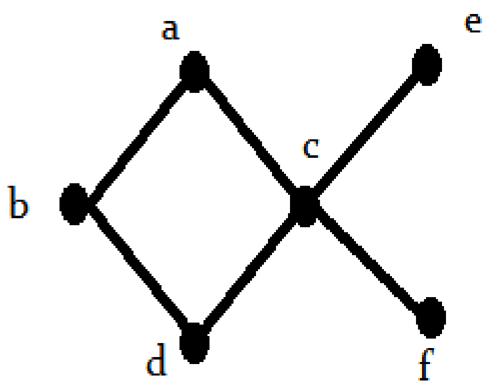
Upper bounds are 9, 10, 15 and 16.

Lower bounds are 1, 2, 4 and 8.

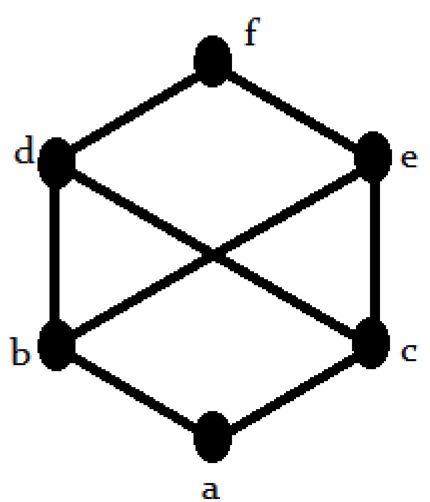
Least upper bound is 9.

Greatest lower bound is 8.

Find the Upper bound, Lower bound, Least upper bound and Greatest lower bound for the following Hasse diagram for given set if exists.

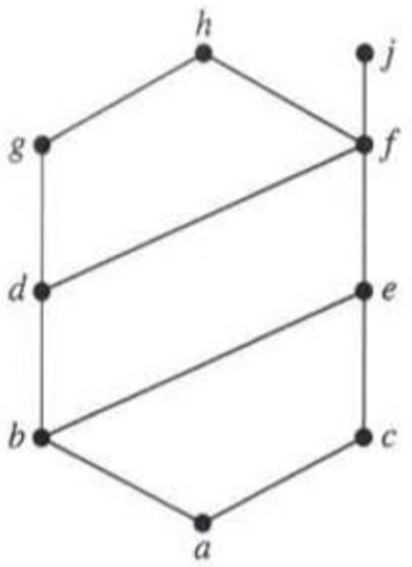


B={c, d}	B={a, b }
UB is a, c and e.	UB is a.
LB is d.	LB is b and d.
LUB is c.	LUB is a.
GLB is d.	GLB is b.



B={d, e}	B={b, c}
UB is f.	UB is d e and f.
LB is a b and c.	LB is a.
LUB is f.	LUB does not exist as it is unique.
GLB does not exist as it is unique.	GLB is a.

Find the Upper bound, Lower bound, Least upper bound and Greatest lower bound for the following Hasse diagram for given set if exists.



B={b, d, g}

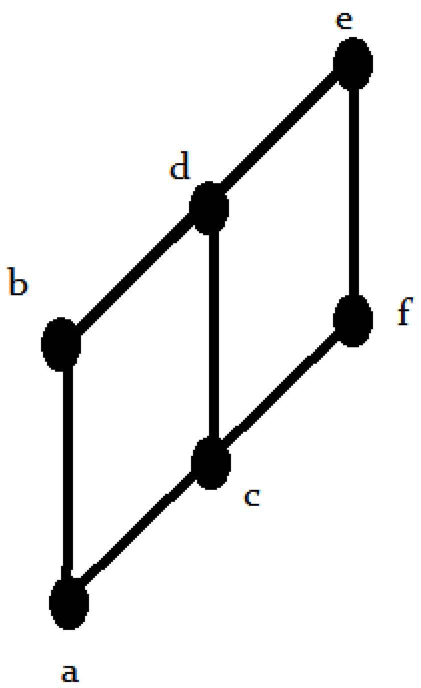
UB is g and h.

LB is a and b.

LUB is g.

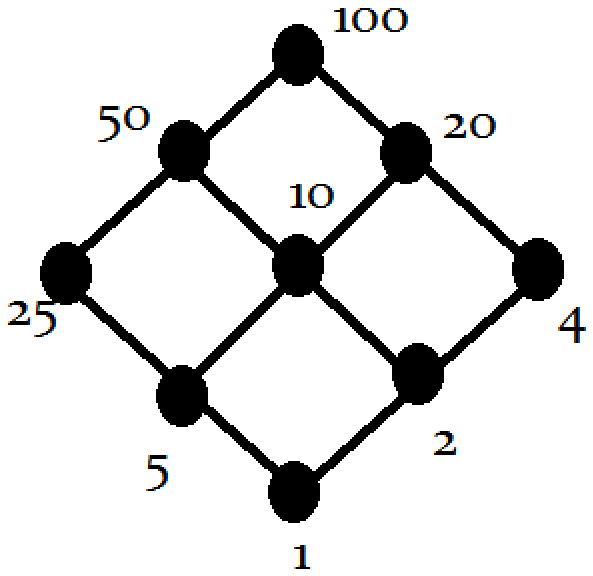
GLB is b.

Find the Upper bound, Lower bound, Least upper bound and Greatest lower bound for the following Hasse diagram for given sets if exists.



1) B={a, c, f}

2) B={c, d}



1)B={5, 10}

2) B={5, 10, 25}

JOIN SEMI LATTICE : In a poset if **LUB/JOIN/ SUPREMUM/ ∨ (LUB)** exist **for every pair of elements then poset is called join semi lattice.**

MEET SEMI LATTICE: In a poset of **GLB/MEET/INFIMUM/ ∧ (GLB)** exist **for every pair of elements then poset is called meet semi lattice or MEET SEMI LATTICE.**

LUB: In set of upper bound elements, Lub is the smallest element as per relation R or lies below all upper bound element as per Hasse diagram. **It is unique element out of upper bound elements.**

GLB: In set of lower bound elements, GLB is the largest element as per relation R or lies above all upper bound element as per Hasse diagram. **It is unique element out of lower bound elements.**

LATTICE: A poset is called lattice if it is both MEET and JOIN SEMI LATTICE. Lattice is denoted by (L, ∨, ∧).

Find the Upper bound, Lower bound, Least upper bound and Greatest lower bound for the sub set A={8, 9} of poset (S, ≤), where S={1, 2, 4, 8, 9, 10, 15, 16}.

Solution : Here we have a poset (S,≤), where S={1, 2, 4, 8, 9, 10, 15, 16}

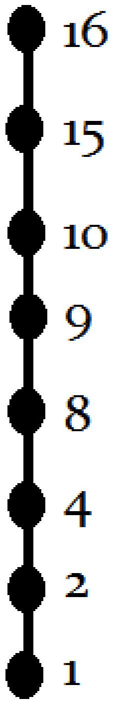
. A ={8, 9} is a subset of S={1, 2, 4, 8, 9, 10, 15, 16}

Upper bounds are 9, 10, 15 and 16.

Lower bounds are 1, 2, 4 and 8.

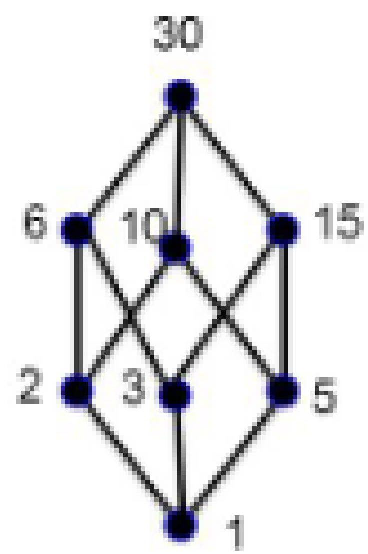
Least upper bound is 9.

Greatest lower bound is 8.



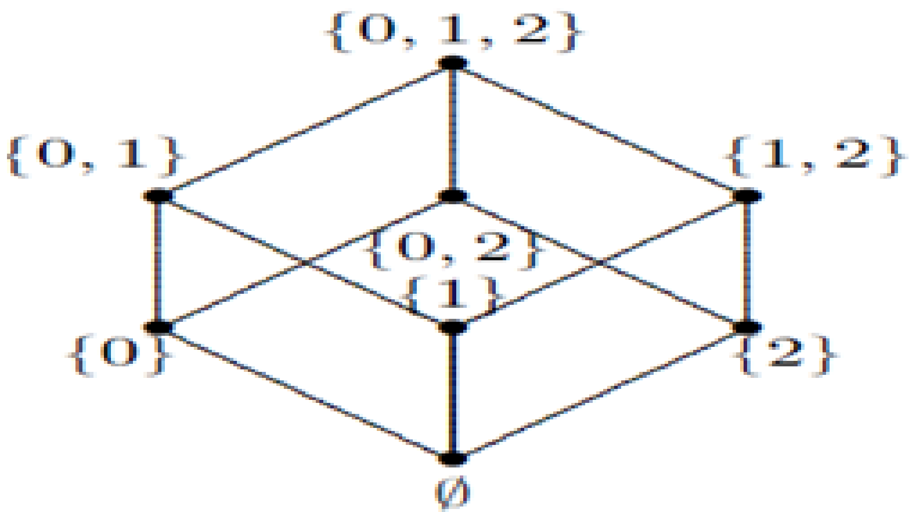
Example : The poset of the divisors of 30 ordered by divisibility forms lattice.

Solution : The Hasse diagram of the divisors of 30 is given below.



The poset consisting of all the divisors of 30 is a lattice as every pair of elements has both a meet and a join.

Example : Show that the poset of the subsets of {0, 1, 2},ordered by the relation ⊆ is a lattice.



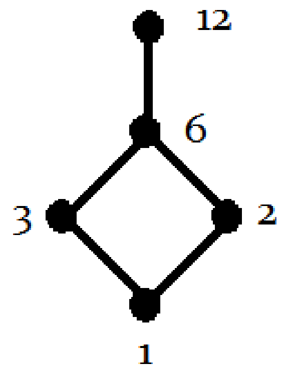
Solution: The poset of the subsets of {0, 1, 2}, ordered by the subset relation as every pair of elements has both a meet and a join.

SUBLATTICE : A nonempty subset L' of a lattice L is called a sub lattice of L if $a \vee b \in L'$ and $a \wedge b \in L'$ for all $a, b \in L'$ or the algebra $(L', \vee, \wedge)$ is a sublattice of $(L, \vee, \wedge)$ iff L' is closed under both operations $\vee$ and $\wedge$.

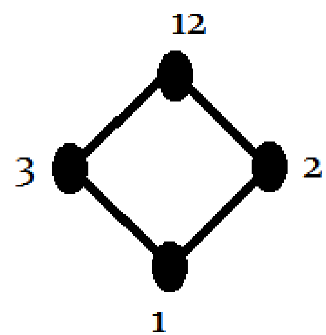
Remark:

1) From the definition it follows that a sub lattice itself is a lattice .

2) Every singleton of a lattice L is a sublattice. However, any subset of L, which is a lattice need not be a sub lattice.



(L, /) is a poset
, where L={1,2,3,6,12}
$(L, \vee, \wedge)$ is a lattice.



(L', /) is a poset
, where L'={1,2,3,12}
$(L', \vee, \wedge)$ is a lattice.

L' is a subset of L.
But L' is not a sublattice of L

Practice Example : The poset of the divisors of 60 ordered by divisibility forms lattice.

Note: '∨' stands for LUB, '∧' stands for GLB and '≤' stands for relation.

PROPERTIES OF LATTICE

Let L be a lattice. Then, for every a, b and c in L

1. **Commutative :** $a \vee b = b \vee a$; $a \wedge b = b \wedge a$
2. **Associativity:** $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
3. **Impotence:** $a \vee a = a$; $a \wedge a = a$
4. **Absorption:** $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$
5. If $a \leq b$, then (a) $a \vee c \leq b \vee c$; (b) $a \wedge c \leq b \wedge c$
6. $a \leq c$ and $b \leq c$ if and only if $a \vee b \leq c$
7. $c \leq a$ and $c \leq b$ if and only if $c \leq a \wedge b$
8. If $a \leq b$ and $c \leq d$, then (a) $a \vee c \leq b \vee d$ (b) $a \wedge c \leq b \wedge d$

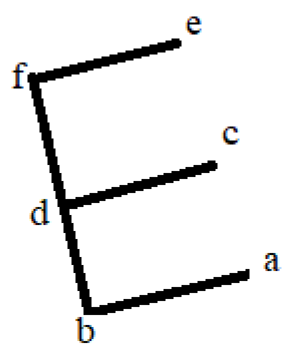
Example: Let T={ ∅, {a}, {c}, {a,c}, {a,b,c}} be a subset of P(S) for S={a, b, c} and L=(P(s), ∩, ∪) be a lattice. Show that T is a sub lattice of L.

Solution: Here P(S)={∅, {a}, {b}, {c}, {a,b}, {a, c}, {b, c}, {a,b,c}}.
$T \subseteq P(S)$is a closed under $\cap$ *and* $\cup$, **i. e.** for any set A and B in T, the A $\cap B$ and A $\cup B$ is in T. Therefor, T is a sub lattice.

Example: Let T={ ∅, {a}, {b}, {a, c}, {a, b, c}} be a subset of P(S) for S={a, b, c} and L=(P(s), ∩, ∪) be a lattice. Show that T is not a sub lattice of L.

Solution: Here P(S)={∅, {a}, {b},{c}, {a,c}, {b,c}, {a,b,c}}.
$T \subseteq P(S)$ is not closed under $\cup$, beause A={a} and B={b} in T, For A={a} and B={b}, the A $\cup B$ is not in a T. Therefor, T is not a sub lattice.

Example : The Hasse diagram of a poset (S, R) is given below. Check that it is a lattice or not.



Solution: we take pair {e, c},

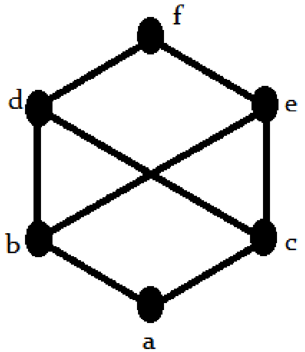
Upper bound of e is e itself and Upper bound of c is c it self. Common upper bound of {e, c} does not exist. Therefore, LUB of {e,c} does not exist.

Thus, The poset (S, R) is not a Joint semi lattice.

Therefore The poset (S, R) is not a lattice.

Note : Similarly you can take pair {c, a}.

Example : The Hasse diagram of a poset (S, R) is given below. Check that it is a lattice or not.



Solution: we take pair {d, e},

Lower bounds of d are a, b, c and d and Lower bounds of e are a, b, c and e. Common lower bounds of {d, e} are a, b and c. GLB of {d, e} does not exist.

Thus, The poset (S, R) is not a Meet semi lattice.

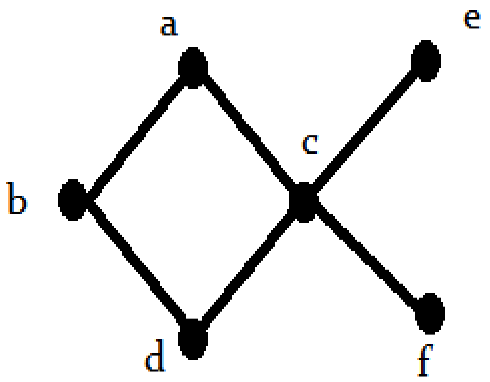
Therefore The poset (S, R) is not a lattice.

OR

Similarly take {b, c}, this pair does not have LUB.

Thus, The poset (S, R) is not Joint semi lattice and it is not a lattice.

Practice Example : The Hasse diagram of a poset (S, R) is given below. Check that it is a lattice or not.



SOME SPECIAL LATTICES

COMPLETE LATTICE: A lattice is called complete lattice if each of its non empty subsets has a least upper bound and greatest lower bound.

DISTRIBITIVE LATTICE:

A lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any a,b, c $\in$ L

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Remark: Note that both the equalities are equivalent to one another, hence to check whether the lattice is distributive or not , it is sufficient to verify one of them.

If L is not distributive, we say that L is non distributive.

Note: The distributive property holds when

1. any two of the elements a, b and c are equal.
2. when any one of the elements is 0 or 1.

BOUNDED LATTICE: A lattice L is said to be bounded if it has a greatest (maximum) element and a least (minimum) element which are denoted by 1 (maximum) or I and 0 or O (minimum) respectively.

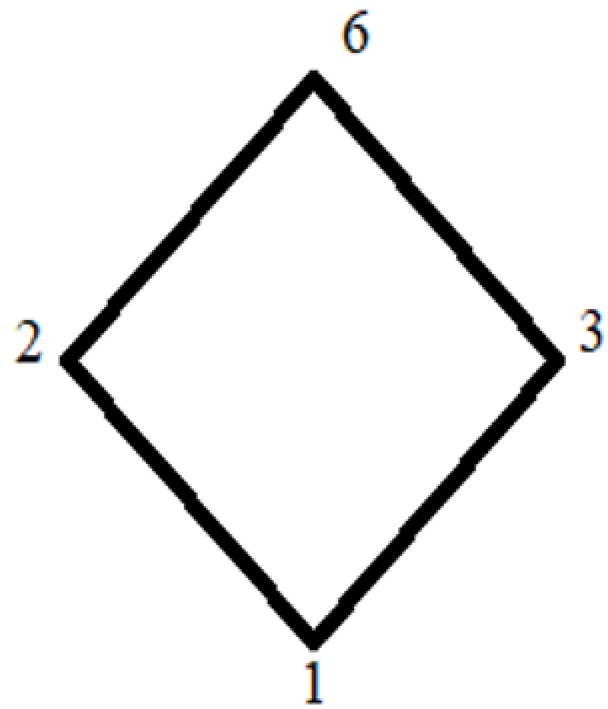
COMPLEMENT OF ELEMENT: Let L be bounded lattice with greatest element 1 and least element 0, and let *a* in L. An element b in L is called a complement of *a* if $a \vee b = 1$(maximum) and $a \wedge b = 0$(minimum).

Note: $\mathbf{0'/0^c = 1}$ **and** $\mathbf{1'/1^c = 0}$

COMPLEMENTED LATTICE: A lattice L is said to be complemented if it is bounded and every element in it has a complement.

Example Show that (D_6, /) is a complete lattice.

Solution: We have D_6={1, 2, 3, 6}. Hasse Diagram of D_6 is given by



There are 4 elements. So the number of subsets D_6 of are 2^4=16.

$P(D_6)$={Ø, {1}, {2}, {3}, {6}, {1 2}, {1 3}, {1 6}, {2 3}, {2 6}, {3 6}, {1 2 3}, {1 2 6}, {1 3 6}, {2 3 6},{1 2 3 6}}.

The set of non-empty subset of $P(D_6)$ is ={{1}, {2}, {3}, {6}, {1 2}, {1 3}, {1 6}, {2 3}, {2 6}, {3 6}, {1 2 3}, {1 2 6}, {1 3 6}, {2 3 6},{1 2 3 6}}.

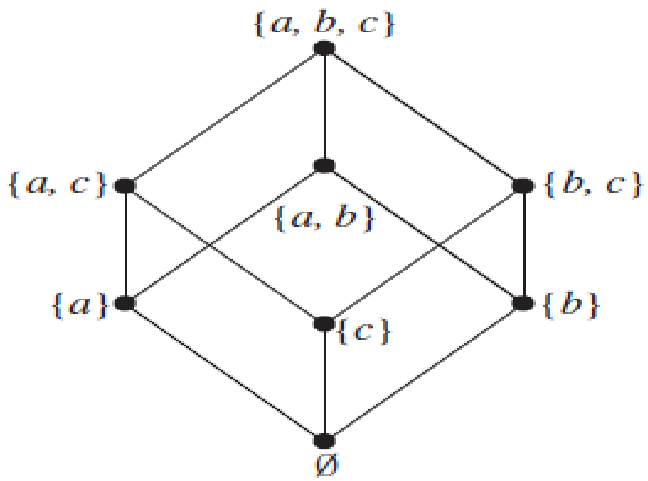
Sr no	Set	Lower bound	Greatest Lower bound	Upper bound	Least upper bound
1	{1}	1	1	1 2 3 6	1
2	{2}	2 1	2	2 6	2
3	{3}	1 3	3	3 6	3
4	{6}	6 1 2 3	1	6	6
5	{1 2}	L b of 1→1 Lb of 2-> 1 2 So Lb of {1 2 } is 1	1	Ub of 1-> 1 2 3 6 Ub of 2->2 6 Ub of {1 2} is 2 6	2
6	{1 3}	Lb of 1→1 Lb of 3→1 3 So lb of {1 3} is 1	1	Ub of 1→1 2 3 6 Ub of 3→ 3 6 So ub of {1 3} is 3 6	3
7	{1 6}	Lb of 1 →1 Lb of → 1 2 3 6 So lb of {1 6 } is 1	1	Ub of 1 →1 2 3 6 Ub of 6→ 6 So Ub of {1 6 } is 6	6
8	{2 3}	Lb of 2 →2 1 Lb of 3 →3 1 So lb of {2 3 } is 1	1	Ub of 2 →2 6 Ub of 3 →3 6 So Ub of {2 3 } is 6	6
9	{2 6}	Lb of 2→1 2 Lb of 6→1 2 3 6 So lb of {2 6} are 1 2	2	Ub of 2→2 6 Ub of 6→6 So Ub of {2 6} is 6	6
10	{3 6}	Lb of 3 →1 3 Lb of 6→1 2 3 6 So lb of {3 6 } are 1 3	3	Ub of 3 →3 6 Ub of 6→6 So Ub of {3 6 } is 6	6
11	{1 2 3}	Lb of 1→1 Lb of 2→ 2 1 Lb of 3→ 1 3 So lb of {1 2 3 } is 1	1	Ub of 1→1 2 3 6 Ub of 2→ 2 6 Ub of 3→ 3 6So Ub of {1 2 3 } is 6	6
12	{1 2 6}	Lb of 1→1 Lb of 2→2 1 Lb of 6→ 1 2 3 6 So lb of {1 2 6 } is 1	1	Ub of 1→1 2 3 6 Ub of 2→2 6 Ub of 6→ 6 So Ub of {1 2 6 } is 6	6
13	{1 3 6}	Lb of 1→1 Lb of 3→ 1 3 Lb of 6→ 1 2 3 6 So lb of {1 3 6 } is 1	1	Ub of 1→1236 Ub of 3→ 3 6 Ub of 6→6 So Ub of {1 3 6 } is 6	6
14	{2 3 6}	Lb of 2→1 2 Lb of 3→ 1 3 Lb of 6→1 2 3 6 So lb of {2 3 6 } is 1	1	Ub of 2→2 6 Ub of 3→ 3 6 Ub of 6→6 So Ub of {2 3 6 } is 6	6
15	{1 2 3 6}	Lb of 1→1 Lb of 2→ 1 2 Lb of 3→1 3 Lb of 6→1 2 3 6 So lb of {1 2 3 6 } is 1	1	Ub of 1→1 2 3 6 Ub of 2→ 2 6 Ub of 3→3 6 Ub of 6→6 So Ub of {1 2 3 6 } is 6	6

Example: Prove that P(S) **is a bounded and distributive lattice under** $\cap$ $\boldsymbol{and}$ $\cup$**, where S= {a, b, c}.**

Solution: We know that (P(S), $\cap$, $\cup$) is a lattice. Here P(S)={ $\emptyset$,{a}, {b},{c}, {a, b},{a, c},{b, c}, {a, b, c}}. Maximum element is {a, ,b, c} and minimum element is $\emptyset$. Therefore P(S) is a bounded under $\cap$ and $\cup$.

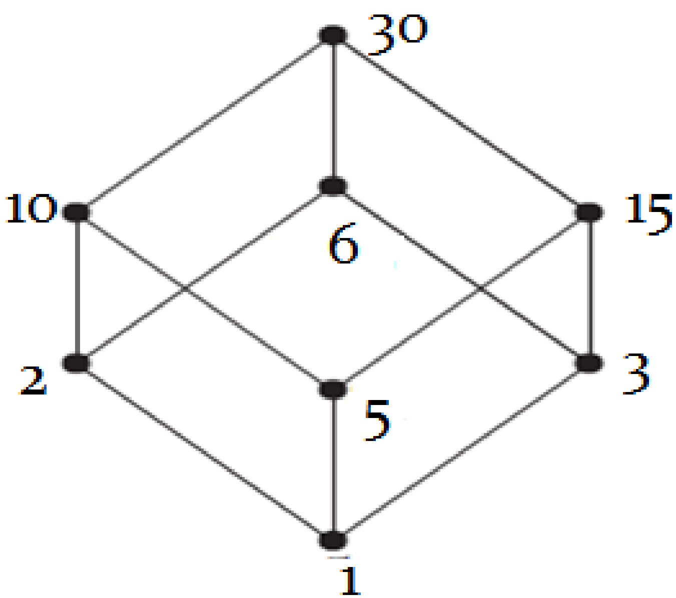
Also we know that P(S) satisfier the distributive relation

a $\cup$ ($b \cap c) =$ (a $\cup$ $b) \cap$ (a $\cup$ c), therefore it is a distributive lattice



Example: Prove that D_{30} with relation '/' is a complemented lattice, where a/b means a is divisible by b.

Solution: D_{30} (divisors of 30) ={ 1,2,3,5,6,10,15,30}



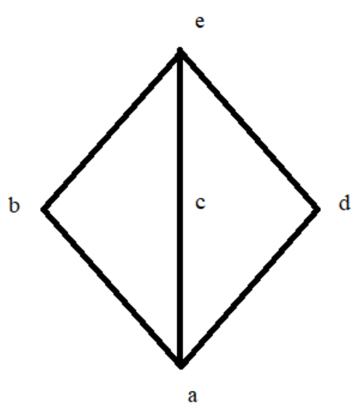
Element	Its Complement
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

It is a complemented lattice.

Note : Complement of an element is not unique.

Example: The Hasse diagram of a lattice (S, R) is given below. Check that it is a complemented lattice or not.

Maximum element is e and minimum element is a. Therefore it is a bounded lattice. It is a complemented lattice because each element has /have its complement as shown in table.



Element	Complement
a	e
b	c and d(not unique)
c	b and d(not unique)
d	b and c(not unique)
Complement of an element need not to be unique.	

It is a complemented lattice.