Unit : 06 Linear Algebra

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* Vector space :-

$\left(\begin{array}{l} \forall \ u, v, w \in V \\ \forall \ \alpha, \beta \in F \end{array} \right)$

→ Let V be a non-empty set & F be a Field. Then V is called a Vector space over the Field F if, it satisfies the following properties:

1) Closure : $u + v \in \underline{V}$
2) Associ. : $u + (v + w) = (u + v) + w$
3) Identity : $u + 0 = 0 + u = u \ ; \ 0 \in V$
4) Inverse : $u + (-u) = (-u) + u = 0$
5) Commutative : $u + v = v + u$
6) Closure under scalar multi. : $\alpha \cdot u \in \boxed{V}$
7) Associ. under scalar multi : $(\alpha\beta)u = \alpha(\beta u) = \beta(\alpha u)$
8) Distri. under scalar multi : $\alpha \cdot (u + v) = \alpha u + \alpha v$
9) " " " " : $(\alpha + \beta) \cdot u = \alpha u + \beta u$
10) $1 \cdot u = u$ (Identity for '$\cdot$')

Ex :- Show that R is a vector space over R. (i.e., $V = R$, $F = R$) with usual addition and scalar multiplication.

Solⁿ :- We are given that $V = R$ and $F = R$. For all $u, v, w \in R$ and $\alpha, \beta \in R$, we have,

i) $u + v \in R$ (Closure pro.)
ii) $u + (v + w) = (u + v) + w$ (Asso. pro)
iii) Identity :- Here, there exists $0 \in R$ such that $0 + u = u + 0 = u$.
$\therefore$ 0 is an identity element.

4) Inverse : Here, there exists $(-u)$ such that $u + (-u) = (-u) + u = 0$.

$\therefore$ '$-u$' is an inverse.

5) Commutative : $u + v = v + u$.

6) Closure pro. under scalar multiplication :

$\alpha \cdot u \in R$ $(\because \alpha \in R$ & $u \in R)$

7) Associ. under sca. multi. :-

$(\alpha \cdot \beta) \cdot u = \alpha(\beta \cdot u) = \beta(\alpha \cdot u)$

8) Distributive under scalar multiplication :

$\alpha \cdot (u + v) = \alpha u + \alpha v$

9) $(\alpha + \beta) \cdot u = \alpha u + \beta u$

10) $1 \cdot u = u$.

$\rightarrow$ Hence, $V = R$ is a vector space over R.

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Ex:- $\mathbb{C}$ is a vector space over $\mathbb{C}$. (see this method in the print.)

[* Read this solⁿ from print]

Solⁿ:- We are given $V = \mathbb{C}$ and $F = \mathbb{C}$.

For all $u, v, w \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$.

1) Closure : $u + v \in \mathbb{C}$

2) Associ. : $u + (v + w) = (u + v) + w$.

3) Identity : Here, there exists $0 + 0i \in \mathbb{C}$ such that

$(0 + 0i) + (u + wi) = u + wi$

$\therefore$ '$0 + 0i$' is an identity element.

4) Inverse : Here, there exists $(-u) + (-vi)$ such that $(-u) + (-vi) + u + vi = 0$.

$\therefore$ $(-u) + (-vi)$ is an inverse.

5) Commutative : $(u + vi) + (w + xi)$

$= (w + xi) + (u + vi)$; $x \in \mathbb{C}$

6) $\alpha \cdot u \in ¢$ (Closure)
7) $(\alpha \cdot \beta) u = \alpha(\beta u) = \beta(\alpha u)$ (Associ.)
8) $\alpha \cdot (u + v) = \alpha u + \alpha v$ (Distri.)
9) $(\alpha + \beta) \cdot u = \alpha u + \beta u$.
10) $1 \cdot u = u$

→ Hence, ¢ is a vector space over ¢.

Ex:- Prove that ¢(R) is a vector space under the usual addition and scalar multiplication.

→ Here, we are given, $V = ¢$ and $F = R$.
∴ Let $u, v, w \in ¢$
where, $u = a_1 + ib_1$
$v = a_2 + ib_2$
$w = a_3 + ib_3$.
and $\alpha, \beta \in R$.

1) $u + v = a_1 + ib_1 + a_2 + ib_2$ (Closure)
$= a_1 + a_2 + i(b_1 + b_2)$
$= A + iB \in ¢$.

(Asso.) 2) $u + (v+w) = (u+v) + w$. we have to prove.
∴ L.H.S. $= u + (v+w)$
$= u + [a_2 + ib_2 + a_3 + ib_3]$
$= a_1 + ib_1 + [(a_2 + a_3) + i(b_2 + b_3)]$
~~$= a_1 + ib_1 + [A_1 + iB_1]$~~
$= (a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3)$
$= (a_1 + a_2) + a_3 + i(b_1 + b_2) + ib_3$
$= (u + v) + w$.

3) Identity : Here, there exists the identity element which is $0 + 0i \in \mathbb{C}$.

4) Inverse : Here, there exists the element $(-a) + i(-b)$ such that

$a + ib + (-a) + (-b)i = 0 = (-a) + (-b)i + a + ib$

$\therefore (-a) + (-b)i$ is the inverse element of 'u'

5) Commutative : $u + v = v + u$

$\therefore$ L.H.S $= a_1 + b_1 i + a_2 + b_2 i$

$= a_2 + b_2 i + a_1 + b_1 i$

$= v + u$

$=$ R.H.S.

6) For $\alpha \in R$, $\alpha . (u)$ should belongs to $\mathbb{C}$,

$\therefore \alpha . u = \alpha . (a_1 + ib_1)$

$= \alpha a_1 + \alpha b_1 i \in \mathbb{C}$

7) Associ. : $(\alpha\beta) u = \alpha(\beta u) = \beta(\alpha u)$

$\therefore (\alpha\beta) u = \alpha\beta (a_1 + b_1 i)$

$= \alpha\beta a_1 + \alpha\beta b_1 i$

$= \alpha [\beta (a_1 + ib_1)]$

$= \alpha (\beta . u)$ — (1)

and $\beta (\alpha u) = \beta [\alpha (a_1 + b_1 i)]$

$= \beta [\alpha a_1 + \alpha b_1 i]$

$= \alpha\beta a_1 + \alpha\beta b_1 i$

$= \alpha [\beta (a_1 + ib_1)]$

$= \alpha (\beta u)$ — (2)

$\therefore$ From (1) & (2), $(\alpha\beta) u = \alpha(\beta u) = \beta(\alpha u)$

8) Distri : $\alpha(u+v) = \alpha u + \alpha v$

$\therefore$ L.H.S. $= \alpha\,((a_1+ib_1)+(a_2+ib_2))$

~~$= \alpha(a_1+ib_1) + \alpha(a_2+ib_2)$~~

~~$= \alpha u + \alpha v$~~

$= \alpha\,[(a_1+a_2)+i(b_1+b_2)]$

$= \alpha a_1 + \alpha a_2 + i\alpha b_1 + i\alpha b_2$

$= \alpha a_1 + \alpha i b_1 + \alpha a_2 + \alpha i b_2$

$= \alpha(a_1+ib_1) + \alpha(a_2+ib_2)$

$= \alpha u + \alpha v$ $=$ R.H.S

9) $(\alpha+\beta)u = (\alpha u + \beta u)$

$\therefore$ L.H.S. $= (\alpha+\beta)\cdot u$

$= (\alpha+\beta)[a_1+ib_1]$

$= a_1(\alpha+\beta) + ib_1(\alpha+\beta)$

$= \alpha a_1 + \beta a_1 + \alpha i b_1 + \beta\cdot ib_1$

$= \alpha(a_1+ib_1) + \beta(a_1+ib_1)$

$= \alpha u + \alpha v$ $=$ R.H.S.

10) $1\cdot u = u$

$\therefore$ L.H.S. $= 1\cdot u$

$= 1\cdot(a_1+ib_1)$

$= a_1+ib_1$

$= u.$ $=$ R.H.S

$\rightarrow$ Hence, $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Ex:- Show that $\mathbb{R}^3$ (the set of 3-tuples of real number (u_1, u_2, u_3)) is a vector space over $\mathbb{R}$ with usual addition and scalar multiplication

Here, $V = R^3$ and $F = R$.

Let, $u, v, w \in R^3$

where, $u = (u_1, u_2, u_3)$

$v = (v_1, v_2, v_3)$

$w = (w_1, w_2, w_3)$

and $\alpha, \beta \in R$.

1) Closure : $(u + v) \in R^3$.

$\therefore u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in R^3$

$\therefore$ Closure property holds.

2) Commutu: $u + v = v + u$

$\therefore$ L.H.S $= u + v$

$= (u_1, u_2, u_3) + (v_1, v_2, v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$= (v_1 + u_1, v_2 + u_2, v_3 + u_3)$

$= (v_1, v_2, v_3) + (u_1, u_2, u_3)$

$= v + u$

$=$ R.H.S

3) Associ. $u + (v + w) = (u + v) + w$.

$\therefore$ R.H.S $= (u + v) + w$

$= [(u_1, u_2, u_3) + (v_1 + v_2 + v_3)] + (w_1, w_2 + w_3)$

$= [(u_1 + v_1), (u_2 + v_2), (u_3 + v_3)] + (w_1, w_2, w_3)$

$= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3)$ — (1)

and L.H.S $= u + (v + w)$.

$= (u_1, u_2, u_3) + [(v_1 + v_2, v_3) + (w_1, w_2, w_3)]$
$= (u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
$= (v_1 + w_1 + u_1, v_2 + w_2 + u_2, v_3 + w_3 + u_3)$
$= (u_1 + v_1 + w_1, u_2 + w_2 + v_2, u_3 + w_3 + v_3)$ — (2)

$\therefore$ from (1) & (2), R.H.S = L.H.S.

4) Identity: there exists $0 \in R^3$ such that $(u_1, u_2, u_3) + (0, 0, 0)$
$= (0, 0, 0) + (u_1, u_2, u_3)$
$= (u_1, u_2, u_3)$
$\therefore$ '(0,0,0)' is an identity element.

5) Inverse: there exists $-u \in R^3$, $(-u_1, -u_2, -u_3)$
such that $u + (-u)$
$= (u_1, u_2, u_3) + (-u_1, -u_2, -u_3)$
$= (0, 0, 0)$
$= (-u_1, -u_2, -u_3) + (u_1, u_2, u_3)$
$\therefore$ '$(-u_1, -u_2, -u_3)$' is an inverse.

6) Closure for scalar multiplication.
$\alpha \cdot u \in R^3$.
$\therefore \alpha . u = \alpha (u_1, u_2, u_3)$
$= (\alpha u_1, \alpha u_2, \alpha u_3) \in R^3$.

7) Associ: $(\alpha\beta)u = \alpha(\beta u) = \beta(\alpha u)$
$\therefore (\alpha\beta).u = (\alpha.\beta)(u_1, u_2, u_3)$
$= (\alpha\beta u_1, \alpha\beta u_2, \alpha\beta u_3)$
$= \alpha(\beta u_1, \beta u_2, \beta u_3)$
$= \alpha(\beta u)$ — (1).

and $\beta(\alpha u) = \beta(\alpha.(u_1, u_2, u_3))$

$= \beta(\alpha u_1, \alpha u_2, \alpha u_3)$

$= (\alpha\beta u_1, \alpha\beta u_2, \alpha\beta u_3)$

$= \alpha(\beta u_1, \beta u_2, \beta u_3)$

$= \alpha[\beta(u_1, u_2, u_3)]$

$= \alpha(\beta u)$ —— 2

$\therefore$ from eq. 1 & 2,

$(\alpha\beta)u = \alpha(\beta u) = \beta(\alpha u)$

8) Distri : $\alpha.(u+v) = \alpha u + \alpha v$

$\therefore$ L.H.S $= \alpha.[(u_1, u_2, u_3)+(v_1, v_2, v_3)]$

$= \alpha(u_1+v_1, u_2+v_2, u_3+v_3)$

$= (\alpha u_1+\alpha v_1, \alpha u_2+\alpha v_2, \alpha u_3+\alpha v_3)$

$= [\alpha u_1, \alpha u_2, \alpha u_3) + (\alpha v_1, \alpha v_2, \alpha v_3)$

$= \alpha u + \alpha v$ = R.H.S

9) $(\alpha+\beta)u = \alpha u + \beta u$

$\therefore$ L.H.S $= (\alpha+\beta).u$

$= (\alpha+\beta)[u_1, u_2, u_3)$

$= [(\alpha+\beta)u_1, (\alpha+\beta)u_2, (\alpha+\beta)u_3]$

$= [\alpha u_1+\beta u_1, \alpha u_2+\beta u_2, \alpha u_3+\beta u_3]$

$= (\alpha u_1, \alpha u_2, \alpha u_3)+(\beta u_1, \beta u_2, \beta u_3)$

$= \alpha u + \beta u$

= RHS

10) $1.u = u$ $\therefore$ L.H.S. $= 1.(u_1, u_2, u_3)$

$= (1.u_1, 1.u_2, 1.u_3)$

$= (u_1, u_2, u_3)$

$= u$ = R.H.S.

$\rightarrow$ Hence, R^3 is a vector space over R.

★ **Note :** $\boxed{\mathbb{R}^n}$, set of n-tuples of real numbers $\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, i \in \{1, 2, \ldots, n\}\}$ with usual vector addition & scalar multiplication forms a vector space over $\mathbb{R}$.

Ex :- Prove that M_{33}, the set of all 3×3 matrices of real numbers forms a vector space over $\mathbb{R}$ with the matrix addition and scalar multiplication.

→ Here, $V = M_{33}$ and $F = \mathbb{R}$.

(1) For $A, B \in M_{33}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & b_{23} + a_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix} \in M_{33}$$

∴ Closure property is satisfied.

(2) Associative :- For $A, B, C \in M_{33}$,

$A + (B + C) = (A + B) + C$

∴ L.H.S. $= A + (B + C)$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} & b_{13} + c_{13} \\ b_{21} + c_{21} & b_{22} + c_{22} & b_{23} + c_{23} \\ b_{31} + c_{31} & b_{32} + c_{32} & b_{33} + c_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} & a_{13}+b_{13}+c_{13} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} & a_{23}+b_{23}+c_{23} \\ a_{31}+b_{31}+c_{31} & a_{32}+b_{32}+c_{32} & a_{33}+b_{33}+c_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$= (A+B)+C$

$=$ R.H.S

(3) Identity :

→ There exists $0 \in M_{33}$, i.e. $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

$$\therefore A+0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and $0+A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\therefore A+0 = 0+A = A$.

$\therefore$ $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is an identity element ~~off~~ M_{33}.

(4) Inverse :

→ There exists $-A \in M_{33}$,

$$-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}$$

Such that $A+(-A) = 0 = (-A)+A$.

$$\therefore A+(-A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

and $(-A)+A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$\therefore$ $-A$ is an inverse element of M_{33}.

(5) Commutative :

$\rightarrow$ For $A, B \in M_{33}$, we have,

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \\ b_{31}+a_{31} & b_{32}+a_{32} & b_{33}+a_{33} \end{bmatrix}$$

$= B+A$

(6) Closure for s. multi. :

For $\alpha \in R$ and $A \in M_{33}$.

$\therefore \alpha \cdot A \in M_{33}$.

$$\therefore \alpha \cdot A = \alpha \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \alpha \cdot a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha \cdot a_{22} & \alpha \cdot a_{23} \\ \alpha \cdot a_{31} & \alpha \cdot a_{32} & \alpha \cdot a_{33} \end{bmatrix} \in M_{33}$$

(7) Associ : $(\alpha.\beta)A = \alpha(\beta A) = \beta(\alpha.A)$

$\Rightarrow \beta.(\alpha A) = \beta \begin{bmatrix} \alpha.a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha.a_{21} & \alpha.a_{22} & \alpha.a_{23} \\ \alpha.a_{31} & \alpha.a_{32} & \alpha.a_{33} \end{bmatrix}$ ($\because$ from ⓐ)

$= \alpha.\beta \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (\alpha\beta).A$

$= \alpha. \begin{bmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{bmatrix} = \alpha(\beta.A)$

$\therefore \beta(\alpha A) = \alpha(\beta A) = (\alpha\beta)A$

(8) Distri : $\alpha.(u+v) = \alpha u + \alpha v$.

$\therefore \alpha.(A+B) = \alpha A + \alpha B$; $\alpha \in R$ & $A, B \in M_{33}$

$\therefore \alpha.(A+B) = \alpha. \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$

$= \begin{bmatrix} \alpha a_{11}+\alpha b_{11} & \alpha a_{12}+\alpha b_{12} & \alpha a_{13}+\alpha b_{13} \\ \alpha.a_{21}+\alpha a_{21} & \alpha.a_{22}+\alpha b_{22} & \alpha.a_{23}+\alpha.b_{23} \\ \alpha.a_{31}+\alpha.a_{31} & \alpha.a_{32}+\alpha.b_{32} & \alpha.a_{33}+\alpha.b_{33} \end{bmatrix}$

$= \begin{bmatrix} \alpha.a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha.a_{21} & \alpha.a_{22} & \alpha a_{23} \\ \alpha.a_{31} & \alpha.a_{32} & \alpha.a_{33} \end{bmatrix} + \begin{bmatrix} \alpha.b_{11} & \alpha.b_{12} & \alpha b_{13} \\ \alpha.b_{21} & \alpha.b_{22} & \alpha b_{23} \\ \alpha.b_{31} & \alpha b_{32} & \alpha.b_{33} \end{bmatrix}$

$= \alpha.A + \alpha.B$

(9) $(\alpha+\beta).A = \alpha.A + \beta.A$

$\alpha, \beta \in R$ and $A \in M_{33}$.

$\therefore$ L.H.S. $= (\alpha+\beta).A$

$= (\alpha+\beta). \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} \alpha \cdot a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix} + \begin{bmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{bmatrix}$$

$= \alpha \cdot A + \beta \cdot A$

$=$ R.H.S.

★ (10) $1 \cdot u = u$. Identity.

→ Let $1 \in R$ and $A \in M_{33}$,

$$\therefore 1 \cdot A = 1 \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$= A$.

→ Hence, M_{33} is a vector space over R.

★ **Note** :- The M_{mn}, the set of all $m \times n$ matrices of real numbers forms a vector space over R with the matrix addition and scalar multiplications.

Ex :- Show that P_2, the set of all polynomials of degree ≤ 2 with addition and scalar multiplication of polynomials forms a vector space over R.

→ Here, $V = P_2$ and $F = R$.

(1) For $p_1, p_2 \in P_2$; $p_1 = a_0 + a_1 x + a_2 x^2$

$p_2 = b_0 + b_1 x + b_2 x^2$

~~and~~ $\therefore p_1 + p_2 = (a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)$

$\therefore p_1 + p_2 = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$

$\in P_2$

$\therefore$ Closure property holds.

(2) Associ : $p_1, p_2, p_3 \in P_2$.

$\therefore p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$

$\therefore$ R.H.S. $= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + c_0 + c_1 x + c_2 x^2$

$= a_0 + b_0 + a_1 x + b_1 x + a_2 x^2 + b_2 x^2 + c_0 + c_1 x + c_2 x^2$

$= a_0 + a_1 x + a_2 x^2 + (b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2$

$= p_1 + (p_2 + p_3)$

$=$ L.H.S.

(3) Identity :

→ There exists $p_0 \in P_2$ such that

$p_0 + p_1 = p_1 + p_0 = 0$.

where $p_0 = 0 + 0x + 0x^2$

$p_1 = a_0 + a_1 x + a_2 x^2$.

$\therefore p_0 + p_1 = 0 + 0x + 0x^2 + (a_0 + a_1 x + a_2 x^2)$

$= (a_0 + a_1 x + a_2 x^2) + 0 + 0x + 0x^2$

$= p_1 + p_0$

$= 0$.

(4) Inverse :

→ There exists $-p_1 \in P_2$ such that

$p_1 + (-p_1) = (-p_1) + p_1 = 0.$

where, $p_1 = a_0 + a_1 x + a_2 x^2$

$-p_1 = (-a_0) + (-a_1) x + (-a_2) x^2$

$\therefore p_1 + (-p_1) = a_0 + a_1 x + a_2 x^2 + [-(a_0 + a_1 x + a_2 x^2)]$

$= -(a_0 + a_1 x + a_2 x^2) + a_0 + a_1 x + a_2 x^2$

$= (-p_1) + p_1$

$= 0.$

(5) Commute :

→ For, $p_1, p_2 \in P_2$

$p_1 + p_2 = p_2 + p_1$

$\therefore$ L.H.S $= p_1 + p_2$

$= a_0 + a_1 x + a_2 x^2 + b_0 + b_1 x + b_2 x^2$

$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2$

$= (b_0 + a_0) + (b_1 + a_1) x + (b_2 + a_2) x^2$

$= b_0 + b_1 x + b_2 x^2 + a_0 + a_1 x + a_2 x^2$

$= p_2 + p_1$

$=$ R.H.S.

(6). Closure under scalar multiplication.

→ $\alpha . p_1 \in P_2$; where, $p_1 \in P_2$ and $\alpha \in R$.

$\therefore \alpha . p_1 = \alpha [a_0 + a_1 x + a_2 x^2)$

$= \alpha a_0 + \alpha . a_1 x + \alpha . a_2 x^2 \in P_2$

7) Associ : $(\alpha.\beta)p_1 = \alpha(\beta p_1) = \beta(\alpha.p_1)$
where, $\alpha, \beta \in R$ and $p_1 \in P_2$

$\therefore (\alpha.\beta).p_1 = (\alpha.\beta)[a_0 + a_1 x + a_2 x^2]$
$= \alpha.[\beta a_0 + \beta a_1 x + \beta a_2 x^2]$
$= \alpha.(\beta p_1)$ — (1)

and $\alpha.(\beta p_1) = \alpha[\beta a_0 + \beta a_1 x + \beta a_2 x^2]$
$= \alpha\beta[a_0 + a_1 x + a_2 x^2]$
$= \beta(\alpha.a_0 + \alpha.a_1 x + \alpha a_2 x^2)$
$= \beta.(\alpha p_1)$ — (2)

$\therefore$ from (1) & (2),
$(\alpha\beta)p_1 = \alpha(\beta p_1) = \beta(\alpha p_1)$.

8) Distri : $\alpha.(p_1 + p_2) = \alpha p_1 + \alpha p_2$
where, $\alpha \in R$ and
$p_1 + p_2 \in P_2$

$\therefore$ L.H.S. $= \alpha.(p_1 + p_2)$
$= \alpha[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2]$
$= \alpha a_0 + \alpha b_0 + \alpha a_1 x + \alpha b_1 x + \alpha a_2 x^2 + \alpha b_2 x^2$
$= \alpha.(a_0 + a_1 x + a_2 x^2) + \alpha(b_0 + b_1 x + b_2 x^2)$
$= \alpha p_1 + \alpha p_2$
= R.H.S.

9) $(\alpha + \beta).p_1 = \alpha p_1 + \beta p_1$. where, $\alpha, \beta \in R$
$p_1 \in P_2$.

$\therefore$ L.H.S. $= (\alpha + \beta).p_1$
$= (\alpha + \beta)[a_0 + a_1 x + a_2 x^2]$
$= \alpha.a_0 + \alpha.a_1 x + \alpha.a_2 x^2 + \beta.a_0 + \beta.a_1 x + \beta.a_2 x^2$
$= \alpha.[a_0 + a_1 x + a_2 x^2] + \beta[a_0 + a_1 x + a_2 x^2]$

R^+ does NOT include $\underline{\underline{0}}$.

$= \alpha \cdot p_1 + \beta \cdot p_2$

$= R.H.S$

(10) $1 \cdot p_1 = p_1$ where, $1 \in R$

$p_1 \in P_1$

$\therefore 1 \cdot p_1 = 1 \cdot (a_0 + a_1 x + a_2 x^2)$

$= a_0 + a_1 x + a_2 x^2$

$= p_1$

→ Hence, P_2 is a vector space.

→ HomeWork :- P_3 and P_4 is a vector space over R

★ Note :- The set $\boxed{P_n}$ of all polynomials of degree $\leq n$ with addition and scalar multiplication of polynomials forms a vector space.

Ex :- Check whether R^+ with usual vector addition and scalar multiplication forms a vector space over R or not.

→ Here we are given $V = R^+$ and $F = R$ and we know that,

R^+ is the set of all positive real numbers.

$\therefore \; 0 \notin R^+$

$\therefore$ There does not exists identity for R^+

Hence, R^+ is not a vector space over R.

Note :-

→ The sets Q, R forms a vector space over R with usual addition and scalar multiplica. of numbers.

→ N is NOT a vector space over R. with usual addition and scalar multiplication.

New Binary Operations

Ex :- Check whether the set R^+ with the binary operations $x + y = xy$ and $k x = x^k$, where k is any scalar, forms a vector space over R or not.

→ Here, $V = R^+$ and $F = R$.

(1) Closure : for any $x, y \in R^+$, we have $x + y = xy \in R^+$ as x, y are positive real numbers.

(2) Associ. : for $x, y \in R^+$.

$$x + (y + z) = x + (y \cdot z)$$
$$= xyz.$$

and $$(x + y) + z = (x \cdot y) + z$$
$$= xyz.$$

$$\therefore \quad x + (y + z) = (x + y) + z.$$

(3) Identity :

→ Let $e \in R^+$. then we want to prove for any $x \in R^+$,

$$x + e = x = e + x.$$

$\therefore \quad x + e = x$ (Given)

$\therefore \quad xe = x$ (Binary operation)

$\therefore \quad e = 1$

similarly $\quad e + x = x$

$\therefore \quad ex = x$

$\therefore \quad e = 1$

$\therefore \quad e = 1$ is an identity element.

(4) Inverse :

→ Let $a \in R^+$ where $a = a^{-1}$.

Then we need to prove that $a + x = e = x + a$ where $x \in R^+$.

→ Consider, $x + a = e$

$\therefore \quad xa = 1$

$\therefore \quad a = \frac{1}{x}$

and $a + x = e$

$\therefore \quad ax = 1$

$\therefore \quad a = \frac{1}{x}$

$\therefore \quad a^{-1} = \frac{1}{x}$ is an inverse element.

(5) Commu.

→ for any $x, y \in R^+$,

$x + y = xy$

$= yx = y + x$.

(6) Closure :

→ for any $x \in R^+$ and $\alpha \in R$.

[($\alpha . x \in R^+$) we want to prove.]

$\therefore \quad \alpha \cdot x = x^{\alpha} \in R^+$.

(7) Associ. : for any $\beta, \alpha \in R^+$, we need to prove

$(\alpha\beta) x = \alpha(\beta x) = \beta(\alpha . x)$

→ $(\alpha\beta)x = x^{\alpha\beta} = (x^{\beta})^{\alpha} = \alpha . x^{\beta} = \alpha(\beta x)$ — (1)

and

$\beta.(\alpha x) = \beta.(x^{\alpha}) = (x^{\alpha})^{\beta} = x^{\alpha\beta} = (x^{\beta})^{\alpha}$

$= \alpha . x^{\beta}$

$= \alpha(\beta x)$ — (2)

(8) Distri. : for $\alpha, \beta \in R$ and $x, y \in R^+$.
we need to prove $\alpha . (x+y) = \alpha x . + \alpha . y$.

$$\therefore \quad L.H.S. = \alpha . (x+y)$$
$$= (x+y)^{\alpha}$$
$$= (x \cdot y)^{\alpha} \qquad * \quad (\because \text{Binary operation})$$
$$= x^{\alpha} \cdot y^{\alpha}$$
$$= x^{\alpha} + y^{\alpha} \qquad * \quad (\because \text{Binary operation})$$
$$= \alpha . x + \alpha . y \quad = R.H.S$$

(9) we need to prove $(\alpha + \beta) . x = \alpha . x + \beta x$.

$$\therefore \quad L.H.S. = (\alpha + \beta) . x$$
$$= x^{(\alpha + \beta)} \qquad * \quad (\text{Bin})$$
$$= x^{\alpha . \beta}$$
$$= x^{\alpha} . x^{\beta}$$
$$= x^{\alpha} + x^{\beta} \qquad * \quad (\text{Bin})$$
$$= \alpha x + \beta x. \quad = R.H.S.$$

(10) Let $1 \in R$ and for $x \in R^+$,
we need to prove $1 . x = x$.

$$\therefore \quad L.H.S. = 1 . x$$
$$= x^1$$
$$= x \quad = R.H.S.$$

→ Hence, R^+ is a vector space over R with the given operations.

Ex:- Determine whether the set V of all pairs of real numbers (x, y) with the operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and $k(x, y) = (kx, ky)$, where $k \in R$ is a vector space over R or not.

→ Here, $V = R^2$ and $F = R$.

(1) Closure :

→ For $u, v \in V$, $u = (x_1, y_1)$, $v = (x_2, y_2)$

$\therefore u + v = (x_1, y_1) + (x_2, y_2)$
$= (x_1 + x_2 + 1, y_1 + y_2 + 1) \in R^2$.

(2) Associ. :

→ We need to prove, $u + (v + w) = (u + v) + w$
where, $u, v, w \in R^2$
and $u = (x_1, y_1)$
$v = (x_2, y_2)$, $w = (x_3, y_3)$

$\therefore u + (v + w) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$
$= (x_1, y_1) + [(x_2 + x_3 + 1, y_2 + y_3 + 1)]$
$= (x_1 + x_2 + x_3 + 2, y_1 + y_2 + y_3 + 2)$ ①

and $(u + v) + w = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$
$= [(x_1 + x_2 + 1, y_1 + y_2 + 1)] + (x_3, y_3)$
$= (x_1 + x_2 + x_3 + 2, y_1 + y_2 + y_3 + 2)$ ②

$\therefore$ from ① & ②, Associ. pro. is satisfied

~~$\therefore$ Consider $x + e = (x_1, y_1) + (a + b)$~~
~~$= (x_1 + a + 1, y_1 + b + 1)$~~
~~$=$~~

(3) Identity :

→ Let $e = (a, b) \in \mathbb{R}^2$ and $x \in \mathbb{R}^2$.
$\therefore x + e = x = e + x$

Consider $x + e = x$
$\therefore (x_1, y_1) + (a, b) = (x_1, y_1)$
$\therefore (x_1 + a + 1, y_1 + b + 1) = (x_1, y_1)$
$\therefore \Rightarrow x_1 + a + 1 = x_1$ and $\Rightarrow y_1 + b + 1 = y_1$
$\Rightarrow a + 1 = 0$ and $\Rightarrow b + 1 = 0$
$\Rightarrow \boxed{a = -1}$ $\Rightarrow \boxed{b = -1}$

~~$\therefore e = (a, b) = (-1, -1) \in \mathbb{R}^2$ is an identity element.~~

~~(4) Inverse :~~
~~→ Let $(a, b) = (a^{-1}, b^{-1})$ be an inverse~~
~~$\therefore x + e = 0 = e + x$~~
~~$\therefore (x_1, y_1) + (a$~~

$\therefore$ Now, consider $e + x = x$
$\therefore (a, b) + (x_1, y_1) = (x_1, y_1)$
$\therefore (a + x_1 + 1, b + y_1 + 1) = (x_1, y_1)$
$\Rightarrow a + x_1 + 1 = x_1$ $\Rightarrow b + y_1 + 1 = y_1$
$\Rightarrow a + 1 = 0$ $\Rightarrow b + 1 = 0$
$\Rightarrow \boxed{a = -1}$ $\Rightarrow \boxed{b = -1}$

$\therefore e = (a, b) = (-1, -1) \in \mathbb{R}^2$ is an identity element

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(4) Inverse :

→ Let $(a^{-1}, b^{-1}) = (a, b) \in R^2$, we need to prove for $u \in R^2$,

$$(x_1, y_1) + (a, b) = (-1, -1) = (a, b) + (x_1, y_1)$$

$$(u + \text{Inverse} = e)$$

→ Consider $(x_1, y_1) + (a, b) = (-1, -1)$

$\therefore (x_1 + a + 1, \quad y_1 + b + 1) = (-1, -1)$

$\Rightarrow x_1 + a + 1 = -1 \qquad \Rightarrow y_1 + b + 1 = -1$

$\Rightarrow x_1 + a = -2 \qquad \Rightarrow \boxed{b = -2 - y_1}$

$\Rightarrow \boxed{a = -2 - x_1}$

~~$\therefore (a, b) = (-2 - x_1, -2 - y_1)$ is an inverse element.~~

→ Now consider, $(a, b) + (x_1, y_1) = (-1, -1)$

$\therefore (a + x_1 + 1, \quad b + y_1 + 1) = (-1, -1)$

$\Rightarrow a + x_1 + 1 = -1 \qquad \Rightarrow b + y_1 + 1 = -1$

$\Rightarrow a = -2 - x_1 \qquad \Rightarrow b = -2 - y_1$

$\therefore (a, b) = (-2 - x_1, -2 - y_1)$ is an inverse element.

(5) Commuta. :

→ For $u, v \in R^2$ we need to prove,

$u + v = v + u$ where, $u = (x_1, y_1)$

$v = (x_2, y_2)$

$\therefore$ L.H.S. $= u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2 + 1, \quad y_1 + y_2 + 1)$

$= (x_2 + x_1 + 1, \quad y_2 + y_1 + 1)$

$= (x_2, y_2) + (x_1, y_1)$

$= v + u$

$=$ R.H.S.

(6) Closure for scalar multipli. :

$\rightarrow$ For, $\alpha \in R$, $u \in R^2$

$$\alpha \cdot u = \alpha \cdot (x_1, y_1)$$
$$= (\alpha x_1, \alpha y_1) \in R^2$$

(7) Associ.

$\rightarrow$ for $\alpha, \beta \in R$ and $u \in R^2$

$$(\alpha\beta) u = \alpha(\beta u) = \beta(\alpha u)$$

$\therefore$
$$(\alpha\beta) u = (\alpha\beta)(x_1, y_1)$$
$$= (\alpha\beta x_1, \alpha\beta y_1)$$
$$= \alpha(\beta x_1, \beta y_1)$$
$$= \alpha[\beta(x_1, y_1)]$$
$$= \alpha(\beta u) \quad \text{--- (1)}$$

$\therefore$
$$(\alpha\beta) u = (\alpha\beta)[(x_1, y_1)$$
$$= (\alpha\beta x_1, \alpha\beta y_1)$$
$$= \beta[\alpha(x_1, y_1)]$$
$$= \beta(\alpha u) \quad \text{--- (2)}$$

$\therefore$ from (1) and (2), Associ. pro. is satisfied

(8). Distri : for $\alpha \in R$ and $u, v \in R^2$.

$$\alpha \cdot (u + v) = \alpha u + \alpha v.$$

$\therefore$
$$\text{L.H.S.} = \alpha((x_1, y_1) + (x_2, y_2))$$
$$= \alpha(x_1 + x_2 + 1, y_1 + y_2 + 1)$$
$$= (\alpha x_1 + \alpha x_2 + \alpha, \alpha y_1 + \alpha y_2 + \alpha)$$
$$\neq \alpha u + \alpha v$$

$\rightarrow$ Hence, the given set R^2 with the given binary operation R is <u>not</u> a vector space

* Subspace :-

→ A non empty subset S of a vector space V is said to be a subspace if S itself a vector space under the operations defined on V.

* Note :-

→ Every vector space has at least two subspaces, vector space itself V and $\{0\}$.
→ The subspace $\{0\}$ is called the zero subspace containing the zero element or vector.

* Condition to check whether a non-empty subset of a vector space is subspace or not :-

→ Theorem :-

→ A non-empty subset S of a vector space V is a subspace if and only if

(i) $u + v \in S, \quad \forall u, v \in S$

(ii) $\alpha \cdot u \in S, \quad \forall u \in S$ and $\alpha \in R$.

→ Closure property for scalar multiplication.

→ Closure property for '+'.

Ex :- Check whether the following sets are subspace over the respective vector space or not.

(1) $S = \{ (x, y) / x = 3y \}$, $V = R^2$

→ We are given that $S = \{ (x, y) / x = 3y \}$ and $V = R^2$.

→ let $u, v \in S$.
Then $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are in S such that, $x_1 = 3y_1$, $x_2 = 3y_2$.

→ Now, $u + v = (x_1, y_1) + (x_2, y_2)$
$= (x_1 + x_2, y_1 + y_2)$

But here, $x_1 = 3y_1$ and $x_2 = 3y_2$.
Then $x_1 + x_2 = 3y_1 + 3y_2$
$\therefore (x_1 + x_2) = 3(y_1 + y_2)$
Therefore $u + v \in S$.

→ Also let $\alpha \in R$ and $u \in S$.
Then $\alpha . u = \alpha (x_1, y_1)$
$= (\alpha x_1, \alpha y_1)$

But here, $x_1 = 3y_1$
$\therefore \alpha . x_1 = \alpha . (3y_1)$
$\therefore \alpha x_1 = 3(\alpha y_1)$

→ Hence S is a subspace of V.

(2) $S = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 / a_0 = 0 \}$, $V = P_3$

→ We are given that $S = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 / a_0 = 0 \}$ and $V = P_3$.

→ Let $p_1, p_2 \in S$. Then
$p_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and
$p_2 = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ in S
such that $a_0 = 0$ and $b_0 = 0$.

→ Now, $p_1 + p_2 = (a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3)$

$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3.$

But here, $a_0 = 0$ and $b_0 = 0$

$\therefore \; a_0 + b_0 = 0 + 0$

$= 0.$

$\therefore \; p_1 + p_2 = 0 + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3.$

$\therefore \; p_1 + p_2 \in S.$

→ Also, $\alpha \in R$ and $p_1 \in P_3$.

Then $\alpha \cdot p_1 = \alpha \cdot [a_0 + a_1 x + a_2 x^2 + a_3 x^3]$

$= \alpha \cdot a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3$

But here $a_0 = 0.$

$\therefore \; \alpha p_1 = 0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3$

$\therefore \; \alpha p_1 \in S.$

→ Hence, S is a subspace of V.

(3) $S = \{ (x, y) \mid y = x^2 \}$, $V = R^2$.

→ We are given that $S = \{ (x, y) \mid y = x^2 \}$ and $V = R^2$.

→ Let $u, v \in S$.

Then $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are in S such that,

$y_1 = x_1^2$ and $y_2 = x_2^2$

→ Now, $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, \; y_1 + y_2)$

But here, $y_1 = x_1^2$ and $y_2 = x_2^2$

Then $y_1 + y_2 = x_1^2 + x_2^2$

$\therefore u + v \notin S$ /* ($\because$ we can NOT make $(x_1+x_2)^2$) */

→ Hence, S is not a subspace of $V = R^2$.

(4) $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ a+b+c+d=0 \right\}$, $V = M_{22}$

→ ~~we are given....~~

→ Let $A, B \in S$. then $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and

$B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ in S such that

$a_1 + b_1 + c_1 + d_1 = 0$ and $a_2 + b_2 + c_2 + d_2 = 0$.

→ Now, $A + B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

But here, $a_1 + b_1 + c_1 + d_1 = 0$ and

$a_2 + b_2 + c_2 + d_2 = 0$.

$\therefore (a_1 + b_1 + c_1 + d_1) + (a_2 + b_2 + c_2 + d_2) = 0$.

$\therefore a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + d_1 + d_2 = 0$.

$\therefore A + B \in S$.

→ Also, let $\alpha \in R$ and $A \in S$.

Then $\alpha . A = \alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$

But here, we need

$\alpha a_1 + \alpha b_1 + \alpha c_1 + \alpha d_1 = 0$.

and we are given, $a_1 + b_1 + c_1 + d_1 = 0$

$\therefore \alpha a_1 + \alpha b_1 + \alpha c_1 + \alpha d_1 = 0$

$\therefore \alpha A \in S$.

→ Hence, S is a subspace of V.

H.W.(5) $S = \{ (x, y) / x^2 = y^2 \}, \quad V = \mathbb{R}^2$.

→ We are given $S = \{ (x, y) / x^2 = y^2 \}$ and $V = \mathbb{R}^2$.

→ Let $u, v \in S$.

Then $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are in S such that $x_1^2 = y_1^2$ and $x_2^2 = y_2^2$.

→ Now, $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$.

But $x_1^2 = y_1^2$ and $x_2^2 = y_2^2$.

$\therefore x_1^2 + x_2^2 = y_1^2 + y_2^2$.

$\therefore u + v \notin S$.

we need $(x_1 + x_2)^2 = (y_1 + y_2)^2$

→ Hence, S is not a subspace.

→ Let $u, v \in S$.

Then $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ are in S such that

$$y_1 = x_1 + z_1 + 1 \quad \text{and} \quad y_2 = x_2 + z_2 + 1$$

→ Now, $u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

But here, $y_1 = x_1 + z_1 + 1$ and
$y_2 = x_2 + z_2 + 1$

and $y_1 + y_2 = (x_1 + z_1 + 1) + (x_2 + z_2 + 1)$
$= (x_1 + x_2) + (z_1 + z_2) + 2$

$\therefore y_1 + y_2 \notin S$.

→ Hence, S is not a subspace of $V = \mathbb{R}^3$.

$V = M_{nn}$

(6) $S = \{ A_{nn} \mid AB = BA$ for any fixed $B_{nn} \}$,

→ Given : $S = \{ A_{nn} \mid AB = BA$ for any fixed $B_{nn}\}$ and $V = M_{nn}$.

→ Let $A_1, A_2 \in S$. Then $[A_1]_n$ and $[A_2]_n$ are in S such that for any fixed B_{nn} $A_1 B = BA_1$ and $A_2 B = BA_2$.

→ Here, first we want to prove $A_1 + A_2 \in S$
i.e. we want to prove for any fixed B_{nn}
$(A_1 + A_2) \cdot B = B \cdot (A_1 + A_2)$.

$$\begin{aligned} \therefore L.H.S &= (A_1 + A_2) \cdot B \\ &= A_1 \cdot B + A_2 \cdot B \\ &= B \cdot A_1 + B \cdot A_2 \\ &= B \cdot (A_1 + A_2) = R.H.S \end{aligned}$$

$\therefore A_1 + A_2 \in S$.

→ Also, for $\alpha \in R$ and $A_1 \in S$.
we want to prove $\alpha \cdot A_1 \in S$.
i.e we want to prove : for any fixed B_{nn}, $(\alpha A_1) \cdot B = B \cdot (\alpha A_1)$.

$$\begin{aligned} \therefore L.H.S &= (\alpha A_1) \cdot B \\ &= \alpha \cdot (A_1 \cdot B) \\ &= \alpha (B A_1) \\ &= B (\alpha A_1) = R.H.S \end{aligned}$$

$\therefore \alpha A_1 \in S$.

→ Hence, S is a subspace for $V = M_{nn}$.

H.W (7) $S = \{(x, kx) / k, x \in R\}$, $V = R^2$

→ Given : $S = \{(x, kx) / k, x \in R\}$, $V = R^2$

→ Let $u, v \in S$. Then $u = (x_1, kx_1)$ and $v = (x_2, kx_2)$ are in S such that $k, x_1 \in R$ and $k, x_2 \in R$.

→ Now,
$$u + v = (x_1, kx_1) + (x_2, kx_2)$$
$$= (x_1 + x_2, kx_1 + kx_2)$$
$$= (x_1 + x_2, k(x_1 + x_2))$$

$\therefore$ $u + v \in S$.

→ Also for any $\alpha \in R$ and $u = (x_1, kx_1)$

$$\alpha \cdot u = \alpha \cdot (x_1, kx_1)$$
$$= (\alpha x_1, k \cdot \alpha x_1)$$

$\therefore$ $\alpha \cdot u \in S$

→ Hence, S is a subspace with $V = R^2$.

* Linear Combinations :-

→ A vector $v \in V$ is said to be a linear combination of vectors $v_1, v_2, v_3, \ldots, v_n$ if it can be expressed as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n.$$

where, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

* Method :-

(1) Express v as a linear combination of $v_1, v_2, \dots, v_n$ and form a system of linear equations.
i.e. $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

(2) If the system of equations is consistent [has soln] then v is a linear combination of $v_1, v_2, \dots, v_n$ and if it is inconsistent [has NO soln] then v is not a linear combination of $v_1, v_2, \dots, v_n$.

Ex:- Which of the following vectors is/are linear combinations of $v_1 = (0, -2, 2)$ and $v_2 = (1, 3, -1)$?
(i) $(3, 1, 5)$, (ii) $(0, 4, 5)$.

→ (i) Let $v = (3, 1, 5)$ and $v_1 = (0, -2, 2)$,
$v_2 = (1, 3, -1)$

∴ for α_1 and α_2 scalar, we have

$$v = \alpha_1 v_1 + \alpha_2 v_2$$

$$\therefore (3, 1, 5) = \alpha_1 (0, -2, 2) + \alpha_2 (1, 3, -1) \quad (1)$$

$$= (0, -2\alpha_1, 2\alpha_1) + (\alpha_2, 3\alpha_2, -\alpha_2)$$

$$\therefore (3, 1, 5) = (\alpha_2, -2\alpha_1 + 3\alpha_2, 2\alpha_1 - \alpha_2)$$

$$\Rightarrow \boxed{\alpha_2 = 3}$$

$$\Rightarrow -2\alpha_1 + 3\alpha_2 = 1$$
$$\Rightarrow -2\alpha_1 + 3(3) = 1$$
$$\Rightarrow -2\alpha_1 = -8$$
$$\Rightarrow \boxed{\alpha_1 = 4}$$

$$\cancel{\Rightarrow 2\alpha_1 - \alpha_2 = 5}$$
$$\cancel{\Rightarrow 8 - \alpha_2}$$

→ Substituting these values in (i),

$(3,1,5) = 4(0,-2,2) + 3(1,3,-1)$
$= (0,-8,8) + (3,9,-3)$
$= (3,1,5)$

$\therefore$ v is a linear combination of v_1 and v_2.

→ (ii) Let $v = (0,4,5)$ and $v_1 = (0,-2,2)$
$v_2 = (1,3,-1)$

$\therefore$ for α_1 and α_2 scalar, we have

$v = \alpha_1 v_1 + \alpha_2 v_2$

$\therefore (0,4,5) = \alpha_1(0,-2,2) + \alpha_2(1,3,-1)$ — (1)

$\therefore (0,4,5) = (0, -2\alpha_1, 2\alpha_1) + (\alpha_2, 3\alpha_2, -\alpha_2)$

$\therefore (0,4,5) = (\alpha_2, -2\alpha_1 + 3\alpha_2, 2\alpha_1 - \alpha_2)$

$\Rightarrow \boxed{\alpha_2 = 0}$ $\Rightarrow -2\alpha_1 + 3\alpha_2 = 4$ — (2) $\Rightarrow 2\alpha_1 - \alpha_2 = 5$ — (3)

Now, (2) + (3),

$3\alpha_2 - \alpha_2 = 9$

$\therefore 2\alpha_2 = 9$

$\therefore \boxed{\alpha_2 = 9/2}$

Here, there is Not a unique value obtained for α_2, and we can not get the solⁿ.

→ Hence, v is NOT a linear combination of v_1 and v_2.

Ex:- Express the vector $(2,-2,3)$ as a linear combination of the set of vectors $\{(0,1,-1), (2,0,1), (-3,2,5)\}$ of R^3.

→ Let $v = (2,-2,3)$ and
$v_1 = (0,1,-1)$, $v_2 = (2,0,1)$, $v_3 = (-3,2,5)$

→ For $\alpha_1, \alpha_2, \alpha_3 \in R$, we have,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\therefore (2,-2,3) = \alpha_1 (0,1,-1) + \alpha_2 (2,0,1) + \alpha_3 (-3,2,5) \quad (1)$$

$$\therefore (2,-2,3) = (0, \alpha_1, -\alpha_1) + (2\alpha_2, 0, \alpha_2) + (-3\alpha_3, 2\alpha_3, 5\alpha_3)$$

$$\therefore (2,-2,3) = (2\alpha_2 - 3\alpha_3, \alpha_1 + 2\alpha_3, -\alpha_1 + \alpha_2 + 5\alpha_3)$$

By comparing both the sides,

* $2\alpha_2 - 3\alpha_3 = 2$; $\alpha_1 + 2\alpha_3 = -2$; $-\alpha_1 + \alpha_2 + 5\alpha_3 = 3$.

* $\therefore$ The matrix form is :

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 2 & -3 & 2 \\ 1 & 0 & 2 & -2 \\ -1 & 1 & 5 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 2 & -3 & 2 \\ -1 & 1 & 5 & 3 \end{array}\right] \quad (\because R_{12})$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 2 & -3 & 2 \\ 0 & 1 & 7 & 1 \end{array}\right] \quad (\because R_3 \to R_3 + R_1)$$

→ rank : No of non-zero rows

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 7 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right] \quad (\because R_{23})$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & -17 & 0 \end{array}\right] \quad (\because R_3 \to R_3 + (-2)R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \left(\because R_3 \to \left(\frac{-1}{17}\right) R_3\right) \quad \text{—— (2)}$$

→ ∴ (2) is in row-echelon form.

∴ $r([A|B]) = 3$ and $r(A) = 3$.

∴ $r([A/B]) = r(A) = 3$ = number of unknowns

∴ System is consistent.

∴ v can be expressed as linear combination of v_1, v_2, v_3.

→ Now, from (2),

$\alpha_1 + 0\alpha_2 + 2\alpha_3 = -2$

∴ $\alpha_1 + 2\alpha_3 = -2$

∴ $\boxed{\alpha_1 = -2}$

$\boxed{\alpha_3 = 0}$

and $0 + \alpha_2 + \alpha_3 = 1$

∴ $\alpha_2 = 1 - 0$

∴ $\boxed{\alpha_2 = 1}$

→ ∴ from (1),

$(2,-2,3) = (-2)(0,1,-1) + (1)(2,0,1) + 0(-3,2,5)$

$= (2,-2,3)$

$\therefore$ v is a linear combination of v_1 and v_2.

* Span of a set :-

→ The span of a non-empty subset S of a vector space V is the set of all linear combinations of finite number of elements of S.

→ It is denoted by 'span(S)' or $L(S)$ or $[S]$.

i.e. $\text{span}(S) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \,/\, v_i \in S, \alpha_i \in R, 1 \le i \le n \}$.

Ex :- Determine the span of $(4,2)$ in R^2.

→ Let $v_1 = (4,2)$ and $v \in R^2$.
Then for $\alpha_1 \in R$, we have

$v = \alpha_1 v_1$

$\therefore v = \alpha_1 (4,2)$

$\therefore v = (4\alpha_1, 2\alpha_1)$

→ Hence, $\text{span}(v_1) = \{ v \in R^2 / v = (4\alpha_1, 2\alpha_1), \alpha_1 \in R \}$.

Ex :- Determine the span of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ in R^3

→ Let $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$

Then for $\alpha_1, \alpha_2, \alpha_3 \in R$ and $v \in R^3$, we have

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$\therefore v = \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1)$

$\therefore v = (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3)$

$\therefore v = (\alpha_1, \alpha_2, \alpha_3)$

→ Hence, $\text{span}(v_1, v_2, v_3) = \{ v \in R^3 / (\alpha_1, \alpha_2, \alpha_3), \; \alpha_1, \alpha_2, \alpha_3 \in R \}$

Ex:- Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in M_{22}. Then determine the $\text{span}(A_1, A_2, A_3)$

Sol^n → Given : $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then for $\alpha_1, \alpha_2, \alpha_3 \in R$ and $A \in M_{22}$, we have,

$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$

$$\therefore A = \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_2 \\ \alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha_3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{bmatrix}$$

Hence, $\text{span}(A_1, A_2, A_3) = \{ A \in M_{22} / A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{bmatrix} ; \alpha_1, \alpha_2, \alpha_3 \in R \}$.

Ex:- Let $p_1(x) = 1 + 3x$; $p_2(x) = x + x^2$
Then find the span(p_1, p_2) in P_2.

Sol. → Given, $p_1 = 1 + 3x$; $p_2 = x + x^2$
Then $\alpha_1, \alpha_2 \in R$ and $p \in P_2$, we have

$$p = \alpha_1 p_1 + \alpha_2 p_2$$
$$= \alpha_1 (1 + 3x) + \alpha_2 (x + x^2)$$
$$\therefore p = \alpha_1 + 3\alpha_1 x + \alpha_2 x + \alpha_2 x^2$$
$$\therefore p = \alpha_1 + (3\alpha_1 + \alpha_2) x + \alpha_2 \cdot x^2.$$

Hence, span$(p_1, p_2) = \{ p \in P_2 : p = \alpha_1 + (3\alpha_1 + \alpha_2)x + \alpha_2 x^2 ; \alpha_1, \alpha_2 \in R \}$

Ex :- Determine whether $v_1 = (2, 2, 2)$, $v_2 = (0, 0, 3)$, $v_3 = (0, 1, 1)$ span the vectors of R^3 or not.

→ Given : $v_1 = (2, 2, 2)$, $v_2 = (0, 0, 3)$, $v_3 = (0, 1, 1)$
Then for $\alpha_1, \alpha_2, \alpha_3 \in R$ and $v = (x_1, x_2, x_3) \in R^3$, we have,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$
$$\therefore (x_1, x_2, x_3) = \alpha_1 (2, 2, 2) + \alpha_2 (0, 0, 3) + \alpha_3 (0, 1, 1).$$
$$\therefore (x_1, x_2, x_3) = (2\alpha_1, 2\alpha_1, 2\alpha_1) + (0, 0, 3\alpha_2) + (0, \alpha_3, \alpha_3)$$
$$\therefore (x_1, x_2, x_3) = (2\alpha_1, 2\alpha_1 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3)$$

$\therefore$ By comparing on both the sides,
$x_1 = 2\alpha_1$; $x_2 = 2\alpha_1 + \alpha_3$; $x_3 = 2\alpha_1 + 3\alpha_2 + \alpha_3$.

$\therefore$ The matrix form of a given system is,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2(0-3) = -6 \neq 0.$$

$\therefore$ The system has solution.
$\therefore$ Given vector v_1, v_2, v_3 spans the vectors v of R^3.

* Linear Dependence and Independence of a set :-

$\ast\to$ Linearly dependent set :-

$\to$ A finite set $\{v_1, v_2, \ldots, v_n\}$ of a vector space V over a field F is said to be a linearly dependent (LD) set if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\boxed{\text{not}}$ all zero (i.e., at least one of them is non-zero) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0.$$

$\ast\to$ Linearly independent set :-

$\to$ A finite set $\{v_1, v_2, \ldots, v_n\}$ of a vector space V over a field F is said to be a linearly independent (LI) set if there exists scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0. \quad \text{(Homogeneous)}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

(all are zero)

LI - trivial if $\alpha^{\circ} = 0$. $\boxed{\text{for } \forall \alpha}$
LD - non-trivial if $\alpha \neq 0$.

* **Method :-**

(1) Method to check a set to be LI or LD :

→ If the system of equations $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ has trivial solution then the set of vectors is LI otherwise LD. (i.e. for non-trivial solutions)

(2) In a vector space V, any set of vectors containing zero vector is LD set.

(3) An infinite set of vectors in V(F) is said to be LI if every finite subset of it is L.I. (*Result to be remembered)

Ex :- Check whether the following set of vectors are LI or LD :

(1) $\{(1,2,3), (0,2,1), (0,1,3)\}$, $V = R^3$.

→ Let, $v_1 = (1,2,3)$, $v_2 = (0,2,1)$
$v_3 = (0,1,3)$

$r(A) =$ no. of unknowns
$\Rightarrow$ trivial $\Rightarrow$ LI
$r(A) <$ no. of unknown
$\Rightarrow$ non-trivial $\Rightarrow$ LD

and for $\alpha_1, \alpha_2, \alpha_3 \in R$,
we consider,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\therefore \alpha_1(1,2,3) + \alpha_2(0,2,1) + \alpha_3(0,1,3) = (0,0,0)$$

$$\therefore (\alpha_1, 2\alpha_1, 3\alpha_1) + (0, 2\alpha_2, \alpha_2) + (0, \alpha_3, 3\alpha_3) = (0,0,0)$$

$$\therefore (\alpha_1, 2\alpha_1 + 2\alpha_2 + \alpha_3, 3\alpha_1 + \alpha_2 + 3\alpha_3) = (0,0,0)$$

By comparing both the sides,

$\Rightarrow \alpha_1 = 0$ | $\Rightarrow 2\alpha_2 + 2\alpha_1 + \alpha_3 = 0$ | $\Rightarrow 3\alpha_1 + \alpha_2 + 3\alpha_3 = 0$.

$\cancel{\Rightarrow 2\alpha_2 + \alpha_3 = 0}$

Then the matrix form is,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 01 & 3. \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-3)R_1. \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \quad R_{23}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3. \\ 0 & 0 & -5 \end{bmatrix} \quad R_3 \to R_3 + (-2)R_2.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to \left(\frac{-1}{5}\right) R_3.$$

which is in Row-echelon form.

$\therefore r(A) =$ no. of non-zero rows
$= 3.$

But here no of unknowns is 3.

$\therefore r(A) =$ no. of unknowns $= 3$

$\therefore$ System has trivial solution.

$\therefore \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0.$

→ Hence, given set is LI.

* → If we get $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \ r(A) = 2$ (row echelon form)

$<$ no. of unknowns

∴ given ~~system has~~ set is $\boxed{LD}$ ~~there is a possibility of any scalars to be non-zero~~

(∵ non trivial) ~~⇒ given set is L.D.~~ ⇒ ~~The system has infinitely many sol.~~

Hence, given set is

(2) $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} \right\}, V = M_{22}$.

→ Let, $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix},$

$A_4 = \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}$.

∴ for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$, we consider,

$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = 0.$

$\therefore \ \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} +$

$\alpha_4 \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \ \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 2\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 \\ 0 & 2\alpha_2 \end{bmatrix} + \begin{bmatrix} 0 & 3\alpha_3 \\ \alpha_3 & 2\alpha_3 \end{bmatrix} + \begin{bmatrix} 2\alpha_4 & 6\alpha_4 \\ 4\alpha_4 & 6\alpha_4 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \ \begin{bmatrix} \alpha_1 + \alpha_2 + 2\alpha_4 & \alpha_1 + 3\alpha_3 + 6\alpha_4 \\ \alpha_1 + \alpha_3 + 4\alpha_4 & 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 6\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

Now, by comparing both the sides,

$\alpha_1 + \alpha_2 + 2\alpha_4 = 0$ and

$\alpha_1 + 3\alpha_3 + 6\alpha_4 = 0$ and

$\alpha_1 + \alpha_3 + 4\alpha_4 = 0$ and

$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 6\alpha_4 = 0.$

$\therefore \alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 = 0.$

$\therefore$ The matrix of is given by,

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 3 & 6 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + (-R_1) \\ R_3 \to R_3 + (-R_1) \\ R_4 \to R_4 + (-R_1). \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 \to (-1) R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2.$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \quad R_3 \leftrightarrow R_4 \mid R_{34}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + (2)R_3.$$

which is in row-echelon form.

$\therefore r(A) = 3$

But here, no of unknown $= 4$

$\therefore r(A) \neq$ no of unknowns

$\Rightarrow$ system has non-trivial sol^n.

$\therefore$ Given set is LD.

HW (3) $\left\{ \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \right\}, V = M_{22}$

* Basis :-

→ A subset S of vectors of a vector space V is said to be a basis for V if,

(i) S is LI.

(ii) S spans V or S generates V.

* Notes :-

(1) Basis for a vector space is NOT unique.

→ For example,

Take $S = \{2\}$,

Then for any $\alpha_1 \in \mathbb{R}$, we have

$\alpha_1 v_1 = 0$

$\therefore \alpha_1 (2) = 0$

$\therefore \boxed{\alpha_1 = 0}$

$\therefore \{2\}$ is LI as $\alpha_1 = 0$ / system has trivial solution.

Now, for any $\alpha_1 \in R$ and $v \in R$,
we have $v = \alpha_1 v_1$
$\therefore v = \alpha_1 (2)$
$\therefore v = 2\alpha_1$
$\therefore 2\alpha_1 = v$
Therefore, $\{2\}$ spans R.
Hence, $\{2\}$ is a basis of R.

→ It is also true for $\{1\}$ or $\{4\}$, $4 \in R$.

MCQ (2) Some standard Basis for various vector space.

→ $\{(1,0), (0,1)\}$ is basis for R^2.
→ $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for R^3.
→ $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for M_{22}.

→ $\{1, x, x^2\}$ is a basis for $P_2(x)$.
In general, $\{1, x, \dots, x^n\}$ is a basis for $P_n(x)$.

Ex:- Show that $S = \{(1,0,0), (2,2,0), (3,3,3)\}$ is a basis for R^3.

→ Given: $S = \{(1,0,0), (2,2,0), (3,3,3)\}$.
Let $v_1 = (1,0,0)$, $v_2 = (2,2,0)$, $v_3 = (3,3,3)$
Then for any $\alpha_1, \alpha_2, \alpha_3 \in R$. and for $v \in R^3$, i.e. $v = (x_1, x_2, x_3)$,

we have, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

$\therefore (x_1, x_2, x_3) = \alpha_1 (1,0,0) + \alpha_2 (2,2,0) + \alpha_3 (3,3,3)$

$\therefore (x_1, x_2, x_3) = (\alpha_1, 0, 0) + (2\alpha_2, 2\alpha_2, 0) + (3\alpha_3, 3\alpha_3, 3\alpha_3)$

$\therefore (x_1, x_2, x_3) = (\alpha_1 + 2\alpha_2 + 3\alpha_3, 2\alpha_2 + 3\alpha_3, 3\alpha_3)$.

By comparing both the sides,

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = x_1$$
$$2\alpha_2 + 3\alpha_3 = x_2$$
$$3\alpha_3 = x_3.$$

$\therefore$ The matrix form is :

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 2 & 3 & x_2 \\ 0 & 0 & 3 & x_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 3/2 & x_2/2 \\ 0 & 0 & 1 & x_3/3 \end{array}\right] \quad (1) \qquad \begin{array}{l} R_2 \to R_2\left(\frac{1}{2}\right) \\ R_3 \to R_3\left(\frac{1}{3}\right). \end{array}$$

which is in Row-echelon form.

→ Here, for any x_3, i.e. $x_3 \in R$ we have $r([A|B]) = 3$ and $r(A) = 3$.

$\therefore r([A|B]) = r(A) = 3$.

$\therefore$ The system is consistent

$\therefore$ S spans R^3. $\therefore$ $span(S) = R^3$.

→ Now, to prove given S to be LI / LD, we consider, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 = (0,0,0)$.

Then from matrix (1),

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ $r(A) = 3$ and no of unknowns $= 3$

$\therefore$ $r(A) =$ no of unknowns $= 3$.

$\therefore$ The system has trivial solⁿ.

$\therefore$ $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$.

$\therefore$ S is LI

→ Hence, S is the basis for R^3.

EX:- Determine whether the set of vectors $\{1-3x+2x^2,\ 1+x+4x^2,\ 1-7x\}$ forms a basis for P_2 or not.

→ Given: $S = \{1, -3x+2x^2,\ 1+x+4x^2,\ 1-7x\}$.

Let $p_1 = 1-3x+2x^2$

$p_2 = 1+x+4x^2$, $p_3 = 1-7x$.

Then for $\alpha_1, \alpha_2, \alpha_3 \in R$ and for $p \in P_2$ i.e. $p = a_0 + a_1 x + a_2 x^2$.

we have,

$$p = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3.$$

$\therefore$ $a_0 + a_1 x + a_2 x^2 = \alpha_1(1-3x+2x^2) + \alpha_2(1+x+4x^2) + \alpha_3(1-7x)$

$\therefore$ $a_0 + a_1 x + a_2 x^2 = (\alpha_1 - 3\alpha_1 x + 2\alpha_1 x^2) + (\alpha_2 + \alpha_2 x + 4\alpha_2 x^2) + (\alpha_3 - 7\alpha_3 x)$

$\therefore a_0 + a_1 x + a_2 x^2 = (\alpha_1 + \alpha_2 + \alpha_3) +$
$(-3\alpha_1 x + \alpha_2 x - 7\alpha_3 x) + (2\alpha_1 x^2 +$
$4\alpha_2 x^2)$

$\therefore a_0 + a_1 x + a_2 x^2 = (\alpha_1 + \alpha_2 + \alpha_3) +$
$(-3\alpha_1 + \alpha_2 - 7\alpha_3) x + (2\alpha_1 + 4\alpha_2) x^2$

By comparing both the sides,

$\alpha_1 + \alpha_2 + \alpha_3 = a_0$;
$-3\alpha_1 + \alpha_2 - 7\alpha_3 = a_1$;
$2\alpha_1 + 4\alpha_2 = a_2$

The matrix form is,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ -3 & 1 & -7 & a_1 \\ 2 & 4 & 0 & a_2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 4 & -4 & a_1 + 3a_0 \\ 0 & 2 & -2 & a_2 - 2a_0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + (-2)R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 1 & -1 & \frac{a_1 + 3a_0}{4} \\ 0 & 1 & -1 & \frac{a_2 - 2a_0}{2} \end{array}\right] \quad \begin{array}{l} R_2 \to (1/4) R_2 \\ R_3 \to (1/2) R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 1 & -1 & \frac{a_1 + 3a_0}{4} \\ 0 & 0 & 0 & \underbrace{\frac{2a_2 - a_1 - 7a_0}{4}}_{\text{zero and non zero}} \end{array}\right] \quad R_3 \to R_3 + (-R_2)$$

which is in the Ref.

બંને માટે rank same આવે તો ~ consitant.

→ for any $a_0, a_1, a_2 \in R$, we have

$r([A|B]) = 3 \neq r(A) = 2$

∴ The system is inconsistent.

∴ S not span P_2.

→ Hence, S is not a basis for P_2.

Ex :- Determine whether the set of vectors $\{(1,0), (0,1), (2,-1)\}$ forms a basis for R^2 or not.

→ Let $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (2,-1)$

Then for $\alpha_1, \alpha_2, \alpha_3 \in R$ and $v \in R^2$.

i.e. $v = (x_1, x_2)$ we have,

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

∴ $(x_1, x_2) = \alpha_1(1,0) + \alpha_2(0,1) + \alpha_3(2,-1)$

$= (\alpha_1, 0) + (0, \alpha_2) + (2\alpha_3, -\alpha_3)$

∴ $(x_1, x_2) = (\alpha_1 + 2\alpha_3, \alpha_2 - \alpha_3)$

∴ Comparing both the sides,

$\alpha_1 + 2\alpha_3 = x_1$ and $\alpha_2 - \alpha_3 = x_2$

∴ The matrix form is,

$$[A|B] = \left[\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}\right] \quad \text{---} (1).$$

which is in Ref.

for any $x \in R$, we have
$r([A/B]) = 2 = r(A)$.
but $r([A|B]) = r(A) < 3 =$ no of unknowns.
∴ The system has infinitely many solⁿ.
∴ The system is consistent.
∴ S spans R^2.

→ To check S to be LI / LD,
we consider,
$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$.

from (1), we get,

∴ $[A/B] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

∴ $r(A) = 2$ and no of unknowns $= 3$.
∴ $r(A) \neq$ no of unknowns.
∴ System has non-trivial solⁿ.
∴ $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
∴ S is LD.

→ Hence, S is not a basis for R^2.

Ex :- Determine whether the set of vectors
HW.
$\left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$.
forms a basis for M_{22} or not.

→ Given: $S = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$.

→ Let, $A_1 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$,

$A_3 = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$.

∴ Then for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and $A \in M_{22}$, $A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

we have,

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$$

$$\therefore \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}.$$

$$\therefore = \begin{bmatrix} \alpha_1 & 2\alpha_1 \\ \alpha_1 & -2\alpha_1 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_2 \\ -\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_3 \\ 3\alpha_3 & \alpha_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\alpha_4 & 2\alpha_4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 2\alpha_1 - \alpha_2 + 2\alpha_3 \\ \alpha_1 - \alpha_2 + 3\alpha_3 - \alpha_4 & -2\alpha_1 + \alpha_3 + 2\alpha_4 \end{bmatrix}$$

∴ By comparing both the sides,

$\alpha_1 = x_1$;

$2\alpha_1 - \alpha_2 + 2\alpha_3 = x_2$;

$\alpha_1 - \alpha_2 + 3\alpha_3 - \alpha_4 = x_3$;

$-2\alpha_1 + \alpha_3 + 2\alpha_4 = x_4$.

$\therefore$ The matrix form is :

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 2 & -1 & 2 & 0 & x_2 \\ 1 & -1 & 3 & -1 & x_3 \\ -2 & 0 & 1 & 2 & x_4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & -1 & 2 & 0 & x_2 - 2x_1 \\ 0 & -1 & 3 & -1 & x_3 - x_1 \\ 0 & 0 & 1 & 2 & x_4 + 2x_1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-1)R_1 \\ R_4 \to R_4 + (2)R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & 1 & -2 & 0 & 2x_1 - x_2 \\ 0 & 0 & 1 & 2 & 2x_1 + x_4 \\ 0 & -1 & 3 & -1 & x_3 - x_1 \end{array}\right] \quad \begin{array}{l} R_2 \to (-1)R_2 \\ R_3 \leftrightarrow R_4 \,/\, R_{34} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & 1 & -2 & 0 & 2x_1 - x_2 \\ 0 & 0 & 1 & 2 & 2x_1 + x_4 \\ 0 & 0 & 1 & -1 & x_1 - x_2 + x_3 \end{array}\right] \quad R_4 \to R_4 + R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & 1 & -2 & 0 & 2x_1 - x_2 \\ 0 & 0 & 1 & 2 & 2x_1 + x_4 \\ 0 & 0 & 0 & -3 & -x_1 - x_2 + x_3 - x_4 \end{array}\right] \quad R_4 \to R_4 + (-1)R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & 1 & -2 & 0 & 2x_1 - x_2 \\ 0 & 0 & 1 & 2 & 2x_1 + x_4 \\ 0 & 0 & 0 & 1 & \frac{-x_1 - x_2 + x_3 - x_4}{(-3)} \end{array}\right] \quad R_4 \to \left(-\frac{1}{3}\right)R_4 \quad \text{—— (1)}$$

which is in the Ref.

→ for any $x_1, x_2, x_3, x_4 \in R$, we have
$r([A|B]) = 4 = r(A)$
∴ The system is consistent.
∴ S spans M_{22}.
∴ $\text{span}(S) = M_{22}$

→ Now, to prove given S to be LI / LD, we consider,
$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = 0$.
∴ from (1),

$$\therefore [A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

∴ $r(A) = 4$ = no of unknowns
∴ The system has trivial solⁿ.
∴ $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$.
∴ S is LI.

→ Hence, S is the basis for M_{22}.

* Dimension :-

→ The number of vectors in a basis of a non-zero vector space V is called the dimension of V and it is denoted by $\dim(V)$.

→ Dimensions of some standard vector spaces can be obtained directly from their basis :

(i) $\dim(R^n) = n \quad ; \quad n \geq 1$

(ii) $\dim(M_{mn}) = m \cdot n \quad$ i.e. $M_{22} = 2 \cdot 2 = 4$

(iii) $\dim(P_n) = n+1 \quad$ i.e. $P_2 = \{1, x, x^2\} = 3$.

(iv) $\dim(\{0\}) = 0$. ($\because$ $\{0\}$ is L.D. vector space $\therefore$ $\{0\}$ has no basis)

(If we can not generate the basis of a vector space then the dimension will be '0'.)

* Linear Transformation :-

→ Let V and W be two vector spaces. Then a linear transformation is a function $T : V \to W$ such that,

(i) $T(u+v) = T(u) + T(v)$, $\forall\, u, v \in V$.

(ii) $T(\alpha u) = \alpha \cdot T(u)$, $\forall\, \alpha \in R$

→ Note :-

(1) Above condition can also be written as for all $u, v \in V$ and $\alpha, \beta \in R$.

$$T(\alpha u + \beta v) = \alpha \cdot T(u) + \beta \cdot T(v).$$

MCQ

(2) If $V = W$ then the linear transformation $T : V \to V$ is called a linear operator.

Ex:- Determine whether the following functions are linear transformations or not :-

1. $T : R^2 \to R^2$, $T(x,y) = (x+2y, 3x-y)$

→ Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ be in R^2.

$\therefore T(u) = T(x_1, y_1) = (x_1 + 2y_1, 3x_1 - y_1)$
and $T(v) = T(x_2, y_2) = (x_2 + 2y_2, 3x_2 - y_2)$

Now, $u + v = (x_1, y_1) + (x_2, y_2)$
$\therefore u + v = (x_1 + x_2, y_1 + y_2)$
$\therefore T(u+v) = ((x_1+x_2) + 2(y_1+y_2), 3(x_1+x_2) - (y_1+y_2))$
$= (x_1 + x_2 + 2y_1 + 2y_2, 3x_1 + 3x_2 - y_1 - y_2)$
$= (x_1 + 2y_1 + x_2 + 2y_2, 3x_1 - y_1 + 3x_2 - y_2)$
$= (x_1 + 2y_1, 3x_1 - y_1) + (x_2 + 2y_2, 3x_2 - y_2)$
$\therefore T(u+v) = T(u) + T(v)$

→ For any $\alpha \in R$, we have
$\alpha u = \alpha(x_1, y_1)$
$\therefore \alpha u = (\alpha x_1, \alpha y_1)$
$\therefore T(\alpha u) = T(\alpha x_1, \alpha y_1)$
$\therefore T(\alpha u) = (\alpha x_1 + 2\alpha y_1, 3\alpha x_1 - y_1)$
$= \alpha(x_1 + 2y_1, 3x_1 - y_1)$
$\therefore T(\alpha u) = \alpha \cdot T(u)$.

→ Hence, given T is a linear transformation.

2. $T : R^3 \to R^2$, $T(x, y, z) = (2x - y + z, y - 4z)$.

$\to$ Let $u = (x_1, y_1, z_1)$ and
$v = (x_2, y_2, z_2)$ be in $\underline{\underline{R^3}}$.

$\therefore T(u) = T(x_1, y_1, z_1)$
$= (2x_1 - y_1 + z_1, y_1 - 4z_1)$
and $T(v) = T(x_2, y_2, z_2)$
$= (2x_2 - y_2 + z_2, y_2 - 4z_2)$

Now, $u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$\therefore u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$\therefore T(u+v) = (2(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2),$
$(y_1 + y_2) - 4(z_1 + z_2))$
$= (2x_1 + 2x_2 - y_1 - y_2 + z_1 + z_2,$
$y_1 + y_2 - 4z_1 - 4z_2)$
$= (2x_1 - y_1 + z_1 + 2x_2 - y_2 + z_2,$
$y_1 - 4z_1 + y_2 - 4z_2)$
$= (2x_1 - y_1, y_1 - 4z_1) +$
$(2x_2 - y_2, y_2 - 4z_2)$.
$\therefore T(u+v) = T(u) + T(v)$.

$\to$ For any $\alpha \in R$, we have,
$\alpha \cdot u = \alpha(x_1, y_1, z_1)$
$\therefore \alpha u = (\alpha x_1, \alpha y_1, \alpha z_1)$
$\therefore T(\alpha u) = T(\alpha x_1, \alpha y_1, \alpha z_1)$
$\therefore T(\alpha u) = (2\alpha x_1 - \alpha y_1 + z_1, \alpha y_1 - 4z_1\alpha)$
$= \alpha(2x_1 - y_1 + z_1, y_1 - 4z_1)$
$= \alpha \cdot (T(x_1, y_1, z_1))$
$\therefore T(\alpha u) = \alpha \cdot T(u)$

$\to$ Hence, given T is a linear transformation.

3. $T : P_2 \to P_2$, $T(a_0 + a_1 x + a_2 x^2) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$.

→ Let $p_1 = a_0 + a_1 x + a_2 x^2$ and $p_2 = b_0 + b_1 x + b_2 x^2$ be in P_2.

Then $T(p_1) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$
and $T(p_2) = (b_0+1) + (b_1+1)x + (b_2+1)x^2$

Now, $p_1 + p_2 = (a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)$

$\therefore p_1 + p_2 = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$

$\therefore T(p_1 + p_2) = T[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2]$

$= ((a_0+b_0)+1) + ((a_1+b_1)+1)x + ((a_2+b_2)+1)x^2$

$= ((a_0+1) + (a_1+1)x + (a_2+1)x) + (b_0 + b_1 x + b_2 x^2)$

$\therefore T(p_1 + p_2) \neq T(p_1) + T(p_2)$

→ Hence, given T is not a linear transformation.

4. $T : M_{nn} \to R$, where $T(A) = \det(A)$.

→ Let, A_1 and A_2 be in M_{nn}.

Then $T(A_1) = \det(A_1)$ and $T(A_2) = \det(A_2)$.

/* $\det(A_1 \cdot A_2) = \det(A_1) + \det(A_2)$ */

$$\left.\begin{aligned} \text{Now, } T(A_1+A_2) &= \det(A_1+A_2) \\ &\neq \det(A_1) + \det(A_2) \\ \therefore T(A_1+A_2) &\neq T(A_1) + T(A_2) \end{aligned}\right\} \text{L*}$$

→ Hence, T is not a linear transformation.

5. $T : M_{nn} \to \mathbb{R}$, $T(A) = tr(A)$,
where $tr(A)$ = sum of diagonal elements of A.

→ Let, $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

Then
$$\begin{aligned} T(A_1) &= tr(A_1) \\ &= (a_1 + d_1) \quad \text{and} \\ T(A_2) &= tr(A_2) \\ &= (a_2 + d_2) \end{aligned}$$

Now,
$$A_1 + A_2 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$
$$\therefore A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$\begin{aligned} \therefore T(A_1 + A_2) &= tr(A_1 + A_2) \\ &= (a_1 + a_2) + (d_1 + d_2) \\ &= (a_1 + d_1) + (a_2 + d_2) \\ &= tr(A_1) + tr(A_2) \\ \therefore T(A_1 + A_2) &= T(A_1) + T(A_2) \end{aligned}$$

→ Now, for any $\alpha \in \mathbb{R}$, we have
$$\alpha \cdot A_1 = \alpha \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

$\therefore \quad \alpha \cdot A_1 = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$

$\therefore \quad T(\alpha A_1) = tr(\alpha A_1)$
$= \alpha a_1 + \alpha d_1$
$= \alpha (a_1 + d_1)$

$\therefore \quad T(\alpha A_1) = \alpha \cdot tr(A_1)$

$\therefore \quad T(\alpha A_1) = \alpha \cdot T(A_1)$.

→ Hence, given T is linear transformation.

MCQ

* Matrix Representation of a linear transformation :-

→ Let $T : R^n \to R^m$ be a linear transformation. Then there exists a matrix A of order $m \times n$ such that $T(X) = AX$.

For example,

Let $T : R^3 \to R^3$ be a linear transformation, where

$$T(x, y, z) = (x - y + 2z,\ 2x + y - z,\ -x - 2y)$$

Then its matrix representation is,

$$T(x, y, z) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= AX$.

where, $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$.

→ **Ex:-** $T(x,y) = (2x, -y)$.

∴ The matrix representation is

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

→ **Ex:-** $T(x,y) = (x+y, x-y)$

∴ The matrix representation is.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

→ **Ex:-** $T(x,y,z) = (x+y+z, x-y, -z)$.

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

* Range and Kernel [Null space] of a linear transformation :-

/* → Let V and W be two vector spaces and $T: V \to W$ be a linear transformation. Then the range of T, denoted by R(T), is the set of all vectors in W which are images of at least one vector in V under T. */

i.e. $R(T) = \{ v \in V : T(v) = w, w \in W \}$.

→ The kernel or null space of T, denoted by N(T), is the set of all vectors in V that maps into the zero vector.

i.e. $N(T) = \{ v \in V : T(v) = 0 \}$

→ The dimension of range of T is called rank of T and all the dimension of kernel of T is called nullity of T.

* Theorem :-

→ If $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a transformation multiplication by $[A]_{m \times n}$, then the kernel of T_A is the null-space of the matrix A and range of T_A is the column space of A.

[Column space : The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called a column space]

i.e. Basis for ker(T) = Basis for the null space of A.

Basis for R(T) = Basis for the column space of A.

~~* Rank - Nullity~~ theorem ~~[Dimension theorem]~~

~~→ If T~~

* Rank - Nullity theorem [Dimension theorem] :-

→ If $T : V \to W$ is a linear transformation from a finite dimensional vector space V to a vector space W, then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

i.e. $r(T) + n(T) = \dim($ Domain of a linear transformation $)$.

Ex :- Let $T : R^2 \to R^2$ be the linear operator defined by $T(x, y) = (2x - y, -8x + 4y)$. Then find basis for $\ker(T)$ and $R(T)$.

Sol^n :- We are given that,

$T : R^2 \to R^2$ defined by

$$T(x, y) = (2x - y, -8x + 4y)$$

→ The basis for $\ker(T)$ is the basis for the solution space of the homogeneous system,

$\therefore T(x, y) = (0, 0)$

$\therefore (2x - y, -8x + 4y) = (0, 0)$

$\Rightarrow 2x - y = 0 \qquad \Rightarrow -8x + 4y = 0$

$\therefore 2x - y = 0 \quad \text{--- (1)} \qquad \Rightarrow 2x - y = 0. \quad \text{--- (2)}$

→ Let $y = K$, where $K \in R$

Then $x = \frac{K}{2}$.

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} K/2 \\ K \end{bmatrix} = K \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

→ Hence, basis for $\ker(T) = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

So, $N(T) = 1$.

→ The basis for $R(T)$ is the basis for the column space of T.

Here, $A = \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$.

Now, we need to reduce it into Ref.

$\sim \begin{bmatrix} 1 & -1/2 \\ -2 & 1 \end{bmatrix}$ $\quad R_1 \to (1/2) R_1$
$\quad R_2 \to (1/4) R_2$

$\sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$. $\quad R_2 \to R_2 + (2) R_1$

which is in Ref.

Here the leading element '1' appears in the first column only.

Hence, basis for $R(T)$ = basis for column space of $A = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$.

So, $\text{rank}(T) = 1$.

$\boxed{\text{Note}}$ → Here, $\dim(R^2) = 2$ and $R(T) = 1$ and $N(T) = 1$.

So, dimension theorem is verified.

can include in the sol^n.

Ex:- Verify the dimension theorem or rank-nullity theorem for a linear transformation $T: R^3 \to R^3$ defined by $T(x, y, z) = (x + 2y + 5z, 3x + 5y + 13z, -2x - y - 4z)$

$\rightarrow$ Given: $T: R^5 \to R^3$ defined by,

$T(x, y, z) = (x + 2y + 5z, 3x + 5y + 13z, -2x - y - 4z)$.

The basis for $\ker(T)$ is the basis for the solution space of the homogeneous system.

$x + 2y + 5z = 0$; $3x + 5y + 13z = 0$; $-2x - y - 4z = 0$.

The matrix form of the system is,

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 13 \\ -2 & -1 & -4 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & 3 & 6 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + (-3)R_1 \\ R_3 \to R_2 + (2)R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \quad R_2 \to (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + (-3)R_2$$

which is in Ref.

$\therefore r(A) = 2 <$ no. of unknowns $= 3$.

→ From above the system is,

$$x + 2y + 5z = 0. \quad \text{—} ①$$
$$y + 2z = 0 \quad \text{—} ②$$

let $z = k$, where $k \in R$.
Then $y = -2k$ and $x = -k$.

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Hence, basis for $\ker(T) = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

So, $N(T) = 1$.

The basis for $R(T)$ is the basis for the column space of T.

Here, $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 13 \\ -2 & -1 & -4 \end{bmatrix}$

$\therefore$ from ①, the Ref is,

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, the leading element '1' appears in the first and second columns.

Hence, the basis for $R(T)$ = basis for column space of $A = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \right\}$.

So, rank $(T) = 2$

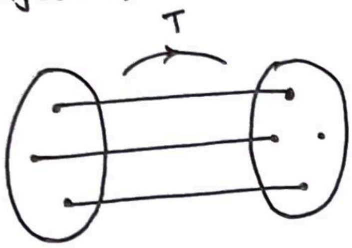
Now, by the dimension theorem,

$$\text{rank}(T) + \text{nullity}(T) = 2+1 = 3 = \dim(R^3).$$

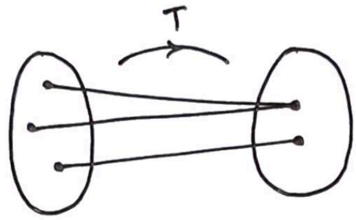
Hence, the theorem is verified.

One to One Transformation:

Let V & W be two vector spaces. A linear transformation $T: V \to W$ is one-one if T maps distinct vectors in V to distinct vectors of W.
(injective) i.e. $x = y \Rightarrow T(x) = T(y)$, $\forall\ x, y \in V$.



T is one-one.

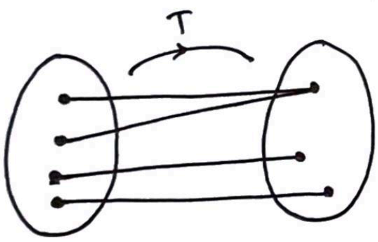


T is not one-one

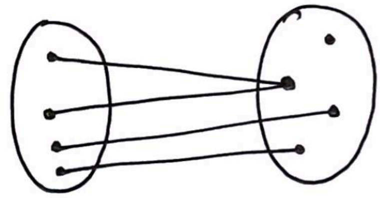
Theorem: 1. A linear transformation $T: V \to W$ is one to one iff $\ker(T) = \{0\}$.
2. " " " " " iff $\dim(\ker(T)) = 0$ i.e. $\text{nullity}(T) = 0$.
3. " " " " " iff $\text{rank}(T) = \dim V$.

Onto Transformation:

Let V & W be two vector spaces. A linear transformation $T: V \to W$ is onto if the range of T is W. i.e. T is onto (surjective). iff for every $w \in W$, there is a $v \in V \ni w = T(v)$.



T is onto



T is not onto.

Theorem: A linear transformation $T: V \to W$ is onto iff $\text{rank}(T) = \dim(W)$ $(R(T) = W.)$

Bijective Transformation:

If a transformation $T: V \to W$ is both one to one & onto then it is called bijective transformation.

Isomorphism:

A bijective transformation from V to W is known as an isomorphism between V & W.

Theorem 1. Let V be a finite dimensional real vector space. If $\dim(V) = n$, then there is an isomorphism from V to $\mathbb{R}^n$.

2. Let V & W be a finite dimensional vector spaces. If $\dim(V) = \dim(W)$ then V & W are isomorphic.

Examples:

(I) In each case, determine whether the linear transformation is 1-1, onto or both or neither.

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+y, x-y)$

H.W. (ii) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (x-y, y-x, 2x-2y)$

(iii) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x+y+z, x-y-z)$

H.W. (iv) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (x+3y, y, z+2x)$.

Sol^n:- (i) We know that linear transformation is 1-1 iff $\ker(T) = \{\bar{0}\}$.

Let $T(x,y) = \bar{0}$ $\quad$ $(\because \ker(T) = \{v \in V : T(v) = \bar{0}\}$

$\Rightarrow (x+y, x-y) = (0,0)$ $\quad$ $= \{(x,y) \in \mathbb{R}^2 : T(x,y) = \bar{0}\})$

$\Rightarrow$ $x+y = 0$,
$x-y = 0$

Solving these equations we get $x=0$ & $y=0$.

$\therefore (x,y) = (0,0)$.

$\therefore \ker(T) = \{\bar{0}\}$

Hence T is one-one.

(ii) A linear transformation is onto if $R(T) = W$.

Let $v \in V = \mathbb{R}^2$ as $v = (x,y)$ & $\omega = (a,b) \in W = \mathbb{R}^2$, where a & b are real no. $\ni T(v) = \omega$.

$\therefore T(x,y) = (a,b)$

$\therefore (x+y, x-y) = (a,b)$

$\therefore x+y = a$
$x-y = b$

Solving these equations,

$$x = \frac{a+b}{2}, \quad y = \frac{a-b}{2}$$

Thus $\forall\, \omega \in W = \mathbb{R}^2$, $\exists\, v = \left(\frac{a+b}{2}, \frac{a-b}{2}\right) \in \mathbb{R}^2 \ni T(v) = \omega$.

Hence T is onto.

(iii) Let $v = (x,y,z)$ be in $\mathbb{R}^3$ & $\omega = (a,b) \in \mathbb{R}^2$, where $a, b \in \mathbb{R} \ni T(v) = \omega$.

$\therefore T(x,y,z) = (a,b)$

$\Rightarrow (x+y+z, x-y-z) = (a,b)$

$\Rightarrow$ $x+y+z = a$
$x-y-z = b$

Solving these equations, $x = \frac{a+b}{2}$.

Let $z = t$, $t \in \mathbb{R}$.

$$y = a - \left(\frac{b+a}{2}\right) - t = \frac{a-b-2t}{2}$$

Thus, for every $\omega = (a, b) \in \mathbb{R}^2$, $\exists\, v = \left(\frac{a+b}{2}, \frac{a-b-2t}{2}, t \right)$ in $\mathbb{R}^3 \ni$

$T(v) = \omega$.

$\therefore$ T is onto.

To check: T is 1-1 or not.

Let $T(x, y, z) = \bar{0}$

$(x+y+z, x-y-z) = (0, 0)$

$x+y+z = 0$

$x-y-z = 0$

Solving these equations, $x = 0$

Let $z = t$

$y = -t$, $t \in \mathbb{R}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

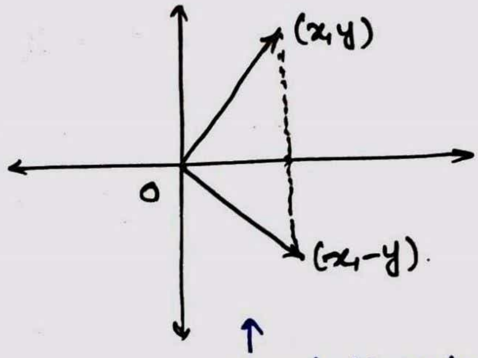
$\therefore \ker(T) \neq \{0\}$

$\therefore$ Hence T is not one-one.

Types of linear operator:

Reflection: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection operator defined by

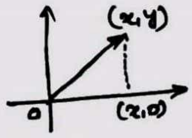
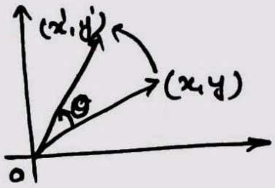
$$T(x, y) = (x, -y).$$

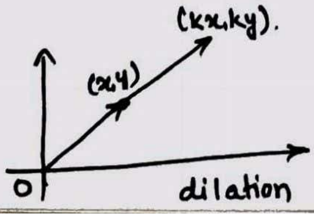


In a matrix form,

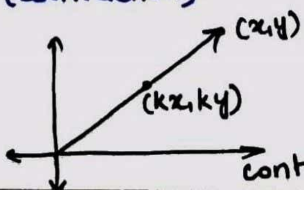
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore$ The standard matrix of T is $[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Operator	Equation	Standard Matrix
1. Reflection about the X-axis on $\mathbb{R}^2$	$T(x, y) = (x, -y)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
2. Reflection about the Y-axis on $\mathbb{R}^2$	$T(x, y) = (-x, y)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
3. Reflection about the line $y = x$ on $\mathbb{R}^2$	$T(x, y) = (y, x)$.	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
4. Projection on the X-axis of $\mathbb{R}^2$	$T(x, y) = (x, 0)$ 	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
5. Rotation operator	$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$	
6. dilation & contraction operator.	$T(x, y) = (kx, ky)$, $k \geq 1$ (dilation) $0 \leq k < 1$ (contraction)	



dilation



contraction.