Unit :- 05 Improper and Multiple Integral

Date: / /
Page No.

★ **Double Integral :-**

$$\iint f(x,y)\cdot dA = \iint f(x,y)\cdot dx\,dy = \iint f(x,y)\cdot dy\,dx$$

★ **Fubini's Theorem :-**

→ Let $f(x,y)$ be a continuous function defined on a rectangular region.

$R: \quad a \le x \le b \;; \quad c \le y \le d.$

$$\iint_R f(x,y)\cdot dA = \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\cdot dx\cdot dy$$

$$= \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\cdot dy\cdot dx.$$

Ex:-01 $\displaystyle\int_1^3 \int_2^3 7xy^2\cdot dx\cdot dy. \Rightarrow \int_{y=1}^{3} \int_{x=2}^{3} 7xy^2\cdot dx\cdot dy.$

$$= \int_1^3 \left[7y^2 \cdot \frac{x^2}{2}\Big|_2^3 \right]\cdot dy$$

$$= \int_1^3 \left[7y^2 \cdot \tfrac{1}{2}(9-4) \right]\cdot dy$$

$$= \frac{35}{2}\int_1^3 y^2\cdot dy \qquad = \frac{35}{2\times 3}(27-1)$$

$$= \frac{35}{2}\left[\frac{y^3}{3}\Big|_1^3 \right] \qquad = \frac{35\times 26}{2\times 3} \qquad = \boxed{\frac{455}{3}} \to \text{Ans}$$

Date: / /
Page No.

→ Now, $\int_{x=2}^{3}\int_{y=1}^{3} 7xy^2 \cdot dy \cdot dx$

$= \int_{2}^{3} \left[7x \cdot \frac{y^3}{3}\Big|_1^3 \right] \cdot dx$

$= \int_{2}^{3} \left[\frac{7}{3} x \cdot (27-1) \right] \cdot dx$

$= \frac{7 \times 26}{3} \int_{2}^{3} x \cdot dx$

$= \frac{7 \times 26}{3} \left[\frac{x^2}{2}\Big|_2^3 \right]$

$= \frac{7 \times 26}{3} \left(\frac{9}{2} - \frac{4}{2} \right) = \frac{7 \times 26 \times 5}{2 \times 3} = \boxed{\frac{455}{3}}$ Ans

∴ it satisfies fubini's theorem.

Ex:-02 Solve : $\int_{y=0}^{1}\int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}} \cdot dx \cdot dy$

$= \int_{0}^{1} \left[\sin^{-1}x\Big|_0^1 \frac{1}{\sqrt{1-y^2}} \right] dy$

$= \int_{0}^{1} \frac{1}{\sqrt{1-y^2}} \cdot \left[\sin^{-1}(1) - \sin^{-1}(0) \right] \cdot dy$

$= \int_{0}^{1} \frac{1}{\sqrt{1-y^2}} \left(\frac{\pi}{2} \right) \cdot dy$

$= \frac{\pi}{2} \left[\sin^{-1}y\Big|_0^1 \right]$

$= \frac{\pi}{2} \cdot \left(\sin^{-1}(1) - \sin^{-1}(0) \right)$

$= \pi/2 \cdot \pi/2$

$= \boxed{\frac{\pi^2}{4}}$

Now, $\int_{x=0}^{1}\int_{y=0}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}} \cdot dy.dx$

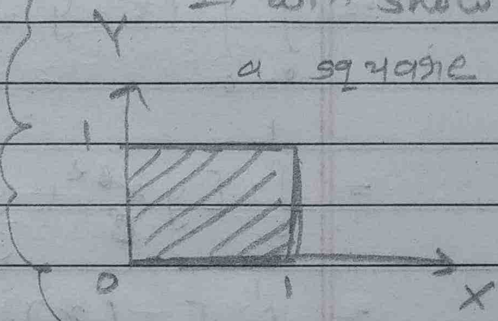
$= \int_0^1 \left[\frac{1}{\sqrt{1-x^2}} \cdot \sin^{-1}y \Big|_0^1 \right] \cdot dx$

$= \int_0^1 \left[\frac{1}{\sqrt{1-x^2}} \cdot \left(\frac{\pi}{2}\right) \right] dx$

$= \frac{\pi}{2} \left[\sin^{-1}x \Big|_0^1 \right]$

$= \pi/2 \cdot \pi/2 = \boxed{\pi^2/4}$ Ans

It will show a square



Ex:- 03 Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy \cdot dA$

Conversion IMP

$= \int_0^a \int_0^{x=\sqrt{ay}} xy \cdot dx \cdot dy$

$= \int_0^a \left[y \cdot \frac{x^2}{2} \Big|_0^{\sqrt{ay}} \right] \cdot dy$

$= \int_0^a \frac{y}{2} \left[(\sqrt{ay})^2 - (0)^2 \right] \cdot dy$

$= \frac{1}{2} \int_0^a \cdot y \cdot ay \cdot dy$

$= \frac{a}{2} \int_0^a \cdot y^2 \cdot dy$

$= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a$

$= \frac{a}{2} \left[\frac{a^3}{3} - 0 \right]$

$= \boxed{\frac{a^4}{6}}$ Ans

Vertical | $\Rightarrow$ limit of x will be constant } → constant से outer integration में change

Horizontal — $\Rightarrow$ limit of y will be constant.

Date: / /

Page No.

Ex:-04 Evaluate $\int_0^1 \int_0^{y=x} (x^2+y^2)\cdot dA$, where dA indicates the small area in X-Y plane.

$y=$

$$= \int_0^1 \int_0^x (x^2+y^2)\, \underline{dy\, dx}$$ why ? $\Rightarrow$ because the inner $\int_0^x$ is the function of x.

$$= \int_0^1 \left[y x^2 + \frac{y^3}{3} \right]_0^x dx$$

$$= \int_0^1 \left[(x)\cdot x^2 + \frac{x^3}{3} \right] \cdot dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{3} \right) \cdot dx$$

$$= \left[\left(\frac{x^4}{4} + \frac{x^4}{12} \right) \Big|_0^1 \right]$$

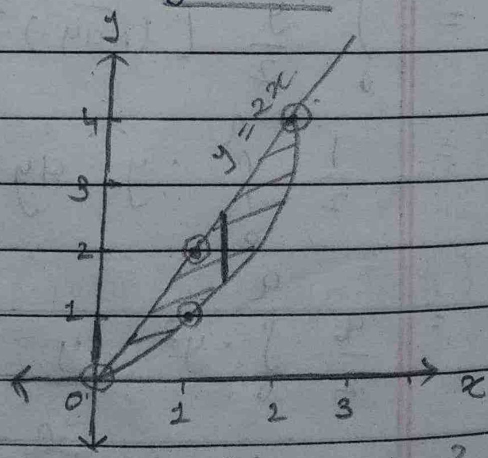
$$= \frac{1}{4} + \frac{1}{12}$$

$$= \frac{3+1}{12} = \frac{4}{12} = \boxed{\frac{1}{3}}$$ Ans.

Ex:-05 Evaluate the $\iint_R (x^2+y^2)\cdot dA$, where R is
IMP
the region bounded by the line $\underline{y=2x}$ and the parabola $\underline{y=x^2}$.

Vertical →

$$\iint_R (x^2+y^2)\cdot dA$$



$$= \int_{x=0}^{2} \int_{y=x^2}^{y=2x} (x^2+y^2)\cdot dy\cdot dx.$$

Write it →
$y = 2x$: $x = 1 \Rightarrow y = 2$
$y = x^2$: $x = 1 \Rightarrow y = 1$
$x = 2 \Rightarrow y = 4$

$y = 2x \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0$
$\Rightarrow x(x-2) = 0 \Rightarrow x = 0$ & $x = 2$
$y = 0$, $y = 4$

$\therefore$ (0,0) and (2,4) are the intersecting points.

Date: / /
Page No.

$$= \int_0^2 \left\{ x^2 y \Big|_{x^2}^{2x} + \frac{y^3}{3}\Big|_{x^2}^{2x} \right\} dx$$

$$= \int_0^2 \left\{ x^2(2x - x^2) + 1/3\left((2x)^3 - (x^2)^3 \right) \right\} dx$$

$$= \int_0^2 \left\{ 2x^3 - x^4 + 1/3(8x^3 - x^6) \right\} dx$$

$$= \int_0^2 \left(14/3\, x^3 - x^4 - 1/3\, x^6 \right) dx$$

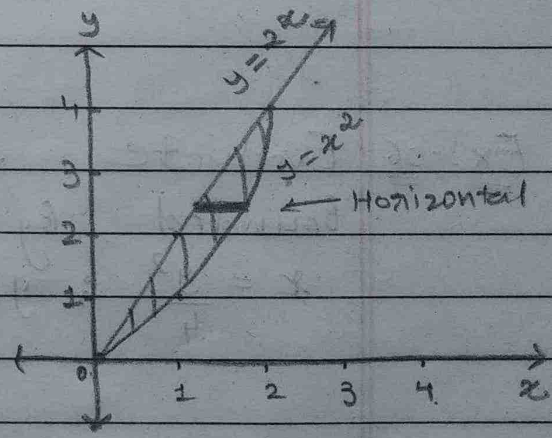
$$= 14/3\, \frac{x^4}{4}\Big|_0^2 - \frac{x^5}{5}\Big|_0^2 - \frac{1}{3}\frac{x^7}{7}\Big|_0^2$$

$$= \frac{14}{3\times 4}[16-0] - \frac{1}{5}[32-0] - \frac{1}{7\times 3}[128-0]$$

$$= \frac{56}{3} - \frac{32}{5} - \frac{128}{21}$$

$$= \boxed{\frac{216}{35}}$$ Ans

Horizontal



$$\rightarrow \iint_R (x^2+y^2) \cdot dA$$

$$= \int_{y=0}^{4} \int_{x=y/2}^{\sqrt{y}} (x^2+y^2) \cdot dx \cdot dy$$

$$= \int_0^4 \left[\frac{x^3}{3}\Big|_{y/2}^{\sqrt{y}} + y^2 \cdot x\Big|_{y/2}^{\sqrt{y}} \right] dy$$

$$= \int_0^4 \left[\frac{1}{3}\left(y^{3/2} - \frac{y^3}{(2)^3} \right) + y^2\left(\sqrt{y} - \frac{y}{2} \right) \right] dy$$

$$= \int_0^4 \left[\frac{y^{3/2}}{3} - \frac{y^3}{24} + y^{5/2} - \frac{y^3}{2} \right] dy$$

$$= \left[\frac{1}{3}\cdot\frac{y^{5/2}}{5/2}\Big|_0^4 - \frac{y^4}{24\times 4}\Big|_0^4 + \frac{y^{7/2}}{7/2}\Big|_0^4 - \frac{y^4}{8}\Big|_0^4 \right].$$

$$= \frac{2}{15}(2)^5 - \frac{1}{96}(4)^4 + \frac{2}{7}(2)^7 - \frac{1}{8}(4)^4$$

$$= \frac{64}{15} - \frac{4\times 4\times 16}{12\times 4\times 2} + \frac{2(128)}{7} - \frac{4\times 2\times 4\times 4\times 2}{8}$$

$$= \frac{64}{15} - \frac{8}{3} + \frac{2(128)}{7} - 32$$

$$= \frac{64-40}{15} + \frac{256-224}{7}$$

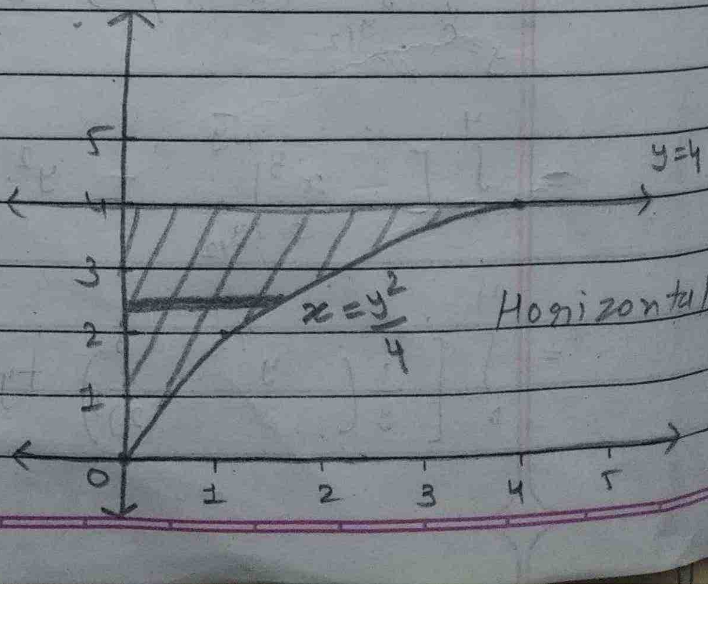
$$= \frac{24}{15} + \frac{32}{7}$$

$$= \frac{8}{5} + \frac{32}{7}$$

$$= \frac{56+160}{35}$$

$$= \boxed{\frac{216}{35}}$$ Ans

Ex:-06 Evaluate $\iint_R x\cdot dx\,dy$; where R is the region bounded by the curves $x=0$, $y=0$, $x=\frac{y^2}{4}$; $y=4$.



→ Here, we are given

$\iint_R x\cdot dx\,dy$ and

$x=0$, $y=0$, $x=\frac{y^2}{4}$, $y=4$.

→ from $x = \frac{y^2}{4}$ $y = 2 \Rightarrow x = 1$
$y = 4 \Rightarrow x = 4$

∴ $(0,0)$, $(4,4)$ are the intensecting points

$$= \int_{y=0}^{4}\int_{x=0}^{x=y^2/4} x \cdot dx \cdot dy$$

$$= \int_0^4 \left.\frac{x^2}{2}\right|_0^{\frac{y^2}{4}} \cdot dy$$

$$= \int_0^4 \frac{1}{2}\left(\left(\frac{y^2}{4}\right)^2 - (0)^2\right) \cdot dy$$

$$= \int_0^4 \frac{1}{32}\, y^4 \cdot dy$$

$$= \frac{1}{32}\left[\left.\frac{y^5}{5}\right|_0^4\right]$$

$$= \frac{1}{32}\left(\frac{(4)^5}{5} - \frac{(0)^5}{5}\right)$$

$$= \frac{1}{32 \times 5}\left(1024\right)$$

$$= \frac{32 \times 32}{32 \times 5}$$

$$= \boxed{\frac{32}{5}}$$ Ans

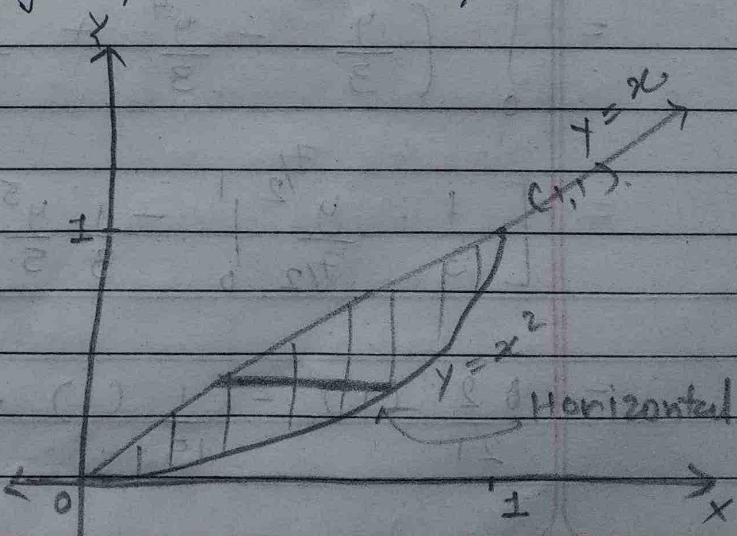
Ex:-07 Evaluate $\iint_R xy(x+y) \cdot dA$, where R is the region bounded by $y = x$ and $y = x^2$.

→ Here, we are given

$$\iint_R xy(x+y) \cdot dA$$

$$= \iint_R (x^2y + xy^2) \cdot dA$$

and $y = x$, $y = x^2$.



Date: / /
Page No.

$\therefore$ from $y = x$ and $y = x^2$

$$\therefore \quad x = x^2$$

$$\therefore \quad x^2 - x = 0$$

$$\therefore \quad x(x-1) = 0$$

$$\therefore \quad x = 0, \quad x = 1$$

$$\Downarrow \qquad \Downarrow$$

$$y = 0 \qquad y = 1$$

$$\rightarrow \iint_R (x^2 y + x y^2) \cdot dA$$

$$= \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} (x^2 y + x y^2) \cdot dx\,dy$$

$$= \int_0^1 \left[y \frac{x^3}{3} \Big|_y^{\sqrt{y}} + \frac{x^2}{2} y^2 \Big|_y^{\sqrt{y}} \right] dy$$

$$= \int_0^1 \left[\frac{y}{3} \left(y^{3/2} - y^3 \right) + \frac{y^2}{2} \left(y - y^2 \right) \right] dy$$

$$= \int_0^1 \left(\frac{y^{5/2}}{3} - \frac{y^4}{3} + \frac{y^3}{2} - \frac{y^4}{2} \right) dy$$

$$= \left[\frac{1}{3} \cdot \frac{y^{7/2}}{7/2} \Big|_0^1 - \frac{1}{3} \frac{y^5}{5} \Big|_0^1 + \frac{1}{2} \frac{y^4}{4} \Big|_0^1 - \frac{1}{2} \frac{y^5}{5} \Big|_0^1 \right]$$

$$= \frac{2}{21}(1) - \frac{1}{15}(1) + \frac{1}{8}(1) - \frac{1}{10}(1)$$

Date: / /
Page No.

$$= \frac{2}{21} - \frac{1}{15} + \frac{5-4}{40} = \frac{2}{21} - \frac{1}{15} + \frac{1}{40}$$

$$= \frac{10-7}{105} + \frac{1}{40}$$

$$= \frac{3}{105} + \frac{1}{40}$$

$$= \frac{1}{35} + \frac{1}{40}$$

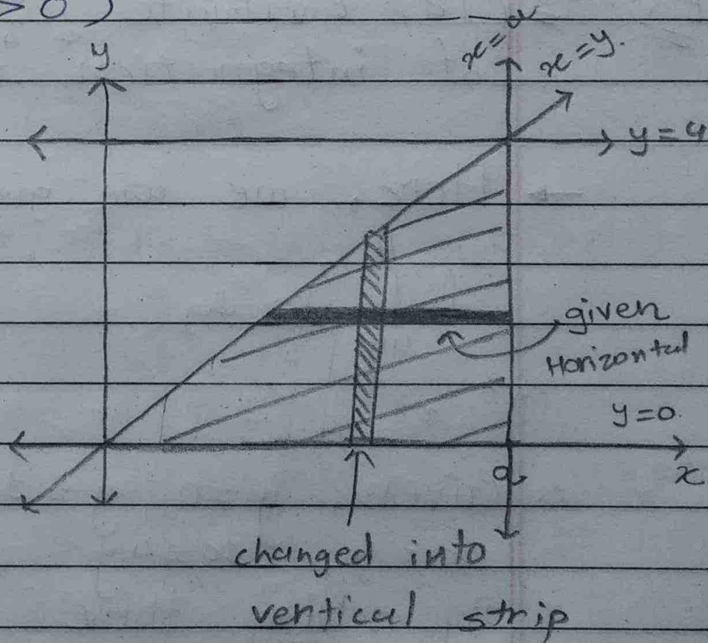
$$= \frac{40+35}{1400}$$

$$= \frac{75}{1400} = \boxed{\frac{3}{56}}$$ Ans

Change of order of double integration :-

Ex:-08 Change the order of integration in

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\,dy,$$ and evaluate the same. $(a>0)$



→ Here, we are given

$$\int_{y=0}^{y=a} \int_{x=y}^{x=a} \frac{x}{x^2+y^2} \cdot dx \cdot dy.$$

→ Now, changing the order of the integration,

$$\int_{x=0}^{x=a} \int_{y=0}^{y=x} \frac{x}{x^2+y^2} \cdot dy \cdot dx$$

Date: / /
Page No.

$$= \int_0^a \left[\cancel{x} \cdot \left(\frac{1}{\cancel{x}} \cdot \tan^{-1} y/x \right) \Big|_0^x \right] \cdot dx \qquad \int \frac{1}{a^2+y^2}\, dy = \frac{1}{a} \cdot \tan^{-1} \frac{y}{a}$$

$$= \int_0^a \left(\tan^{-1} \frac{x}{x} - \tan^{-1} \frac{0}{x} \right) \cdot dx$$

$$= \int_0^a \left(\frac{\pi}{4} - 0 \right) dx$$

$$= \boxed{\frac{\pi}{4} a}$$ Ans.

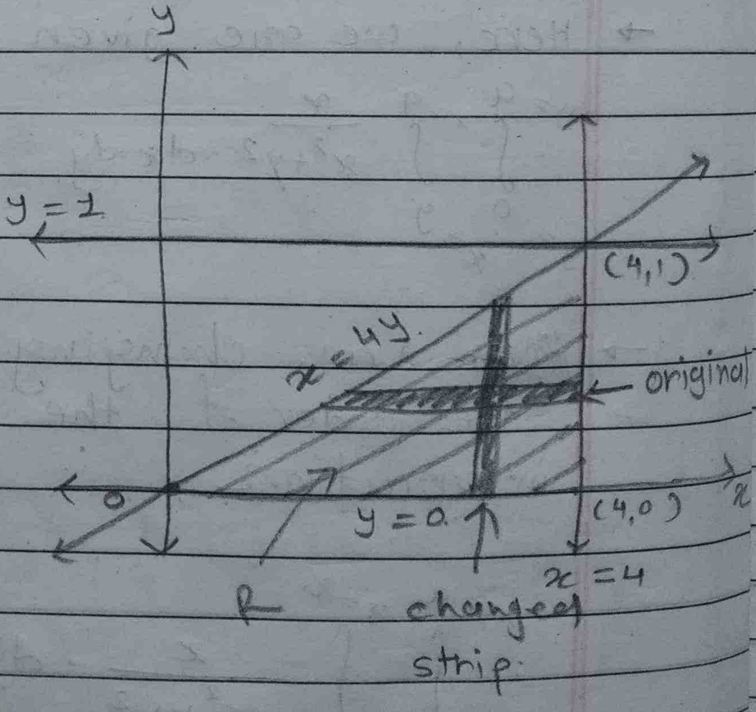
Ex:- 09 (18) Evaluate $\int_0^1 \int_{4y}^4 dx dy$ by changing the order of integration.

→ Here, we are given

$$\int_0^1 \int_{4y}^4 dx\, dy$$

$y=$ $x=$

∴ Given, $y=0$, $y=1$
$x=4$, $x=4y$.
By solving these equations, we come to know that the region R is bounded by $x=0$, $x=4$, $x=4y$
and it is a triangle with vertices



Date: / /

Page No.

(0,0), (4,0), and (4,1) as shown in the figure. By changing the order of integration we have,

$$\int_{y=0}^{1}\int_{x=4y}^{4} dx\cdot dy = \int_{x=0}^{4}\int_{y=0}^{x/4} dy\cdot dx$$

$$= \int_{0}^{4} y\Big|_{0}^{x/4} dx$$

$$= \int_{0}^{4} x/4 \cdot dx$$

$$= \frac{x^2}{8}\Big|_{0}^{4}$$

$$= \frac{16}{8} - 0$$

$$= \boxed{2}$$ Ans

Ex:- 10 (17) Evaluate $\int_{0}^{a}\int_{\sqrt{ax}}^{a} \frac{y^2}{\sqrt{y^4-a^2x^2}} dy\cdot dx$ by changing the order of integration. $(a>0)$

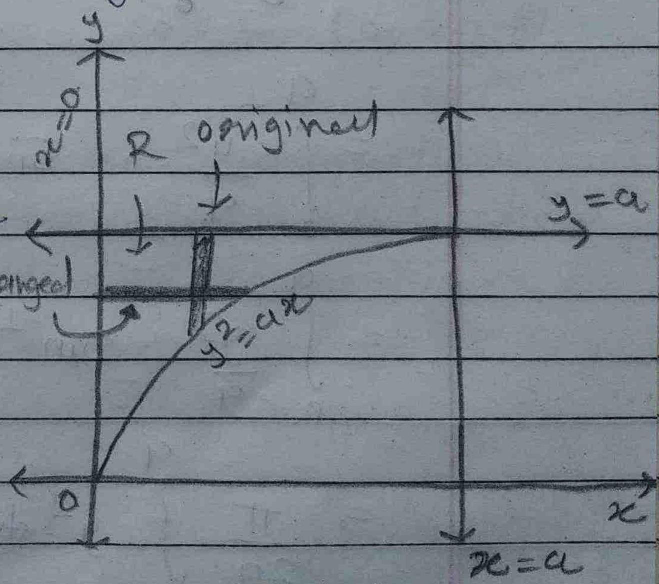
→ Here, we are given

$$\int_{x=0}^{a}\int_{y=\sqrt{ax}}^{a} \frac{y^2}{\sqrt{y^4-a^2x^2}} dy\cdot dx$$

and $x=0$, $x=a$

$y=\sqrt{ax}$, $y=a$

$y^2=ax$



$\rightarrow$ From the figure, it can be seen that the region (R) is bounded by the lines $x=0$, $y^2=ax$ and $y=a$ having intersecting points $(0,0)$, $(0,a)$ & (a,a), $(a>0.)$

$\therefore$ By changing the order of the integration,

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4-a^2x^2}}\, dy \cdot dx$$

$$= \int_{y=0}^a \int_{x=0}^{y^2/a} \frac{y^2}{\sqrt{y^4-a^2x^2}}\, dx\, dy$$

$$= \int_0^a \int_0^{y^2/a} \frac{y^2}{a} \cdot \frac{1}{\sqrt{\left(\frac{y^2}{a}\right)^2-(ax)^2}}\, dx \cdot dy$$

$$= \int_0^a \frac{y^2}{a} \cdot \sin^{-1}\left(\frac{x}{y^2/a}\right)\Big|_0^{y^2/a} \cdot dy$$

$$= \int_0^a \frac{y^2}{a} \cdot \sin^{-1}\left(\frac{y^2/a}{y^2/a}\right) dy$$

$$= \frac{\pi}{2a} \int_0^a y^2 \cdot dy$$

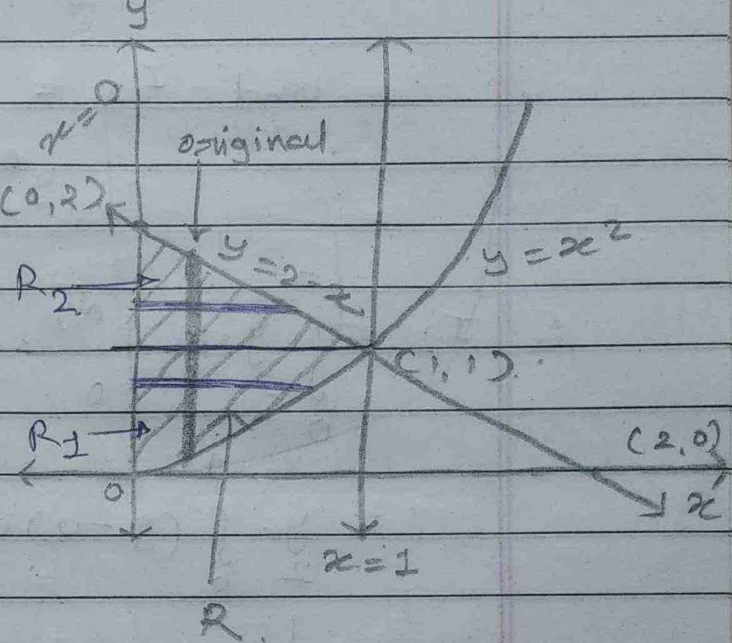
$$= \frac{\pi}{2a} \left.\frac{y^3}{3}\right|_0^a$$

$$= \frac{\pi}{2a}\left[\frac{a^3}{3}-0\right]$$

$$= \boxed{\frac{\pi a^2}{6}}$$

Date: / /
Page No.

Ex:- 11 Change the order of integration $\int_0^1 \int_{x^2}^{2-x} xy\, dy \cdot dx$ and Evaluate it.



→ Here, we are given

$$\int_{x=0}^{1} \int_{y=x^2}^{2-x} xy\, dy \cdot dx$$

and $x=0$, $x=1$

$y=x^2$, $y=2-x$

$\therefore$ from $y=2-x$

if $x=1 \Rightarrow y=1$

$x=0 \Rightarrow y=2$

$x=2 \Rightarrow y=0$.

→ As shown in the figure, the region R is bounded by $x=0$, $x=1$, $y=x^2$ and $y=2-x$ having boundary points as $(0,0)$, $(0,2)$ & $(1,1)$.

→ By changing the order of the integration we have to ~~divide~~ consider horizontal strip and for that we have to divide the region R into two ~~parts~~ parts R_1 and R_2 as shown in the figure.

$$\therefore \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy\, dy\, dx = I_1 + I_2 \quad \text{—— } (\star)$$

Date: / /

Page No.

where, $I_1 = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \cdot dx\,dy.$

and $I_2 = \int_{y=1}^{2} \int_{x=0}^{2-y} xy \cdot dx\,dy.$

$$\therefore \quad I_1 = \int_0^1 y \cdot \frac{x^2}{2}\Big|_0^{\sqrt{y}} \cdot dy$$

$$= \frac{y}{2} \cdot \int_0^1 (y-0)\,dy.$$

$$= \int_0^1 \frac{y^2}{2}\,dy.$$

$$= \frac{y^3}{6}\Big|_0^1 = \boxed{\frac{1}{6}} \text{ --- } (1).$$

$$\text{and} \quad I_2 = \int_1^2 y\,\frac{x^2}{2}\Big|_0^{2-y} \cdot dy$$

$$= \int_1^2 \frac{y}{2}\,(2-y)^2 \cdot dy$$

$$= \int_1^2 \left(\frac{y}{2}\right)(4 - 4y + y^2) \cdot dy$$

$$= \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3) \cdot dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2}\Big|_1^2 - \frac{4y^3}{3}\Big|_1^2 + \frac{y^4}{4}\Big|_1^2\right].$$

Date: / /
Page No.

$\therefore I_2 = \frac{1}{2}\left(\left(\frac{16}{2} - \frac{4}{2}\right) - \left(\frac{32}{3} - \frac{4}{3}\right) + \frac{16}{4} - \frac{1}{4}\right)$

$= \frac{1}{2}\left\{ 6 - \frac{28}{3} + \frac{15}{4} \right\}$

$= \left\{ 3 - \frac{14}{3} + \frac{15}{8} \right\}$

$= -\frac{5}{3} + \frac{15}{8}$

$= \frac{-40 + 45}{24}$

$= \boxed{\frac{5}{24}}$ —— (2)

$\therefore$ Substituting the values of (1) & (2) in eq^n (A), we get,

$$\int_0^1 \int_{x^2}^{2-x} xy \cdot dy \cdot dx = \frac{1}{6} + \frac{5}{24}$$

$$= \frac{4+5}{24} = \boxed{\frac{9}{24}}$$

Extra Now, considering vertical strip,

$$\int_0^1 \int_{x^2}^{2-x} xy \cdot dy\, dx = \int_0^1 x \cdot \frac{y^2}{2}\Big|_{x^2}^{2-x} \cdot dx$$

$x =$ $y =$

$$= \int_0^1 \frac{x}{2} \cdot \left\{ (2-x)^2 - (x^2)^2 \right\} dx$$

$$= \int_0^1 \frac{x}{2}\left\{ 4 - 4x + x^2 - x^4 \right\} dx$$

$$= \int_0^1 \left(2x - 2x^2 + \frac{x^3}{2} - \frac{x^5}{2} \right) dx$$

$$= 2\cdot\frac{x^2}{2}\Big|_0^1 - 2\,\frac{x^3}{3}\Big|_0^1 + \frac{x^4}{8}\Big|_0^1 - \frac{x^6}{12}\Big|_0^1$$

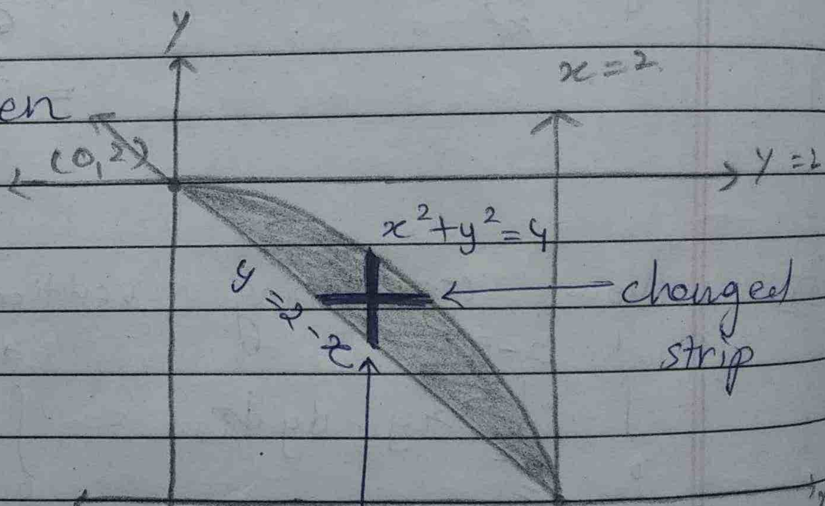
$$= 1 - \frac{2}{3} + \frac{1}{8} - \frac{1}{12}$$

$$= \frac{24 - 16 + 3 - 2}{24}$$

$$= \boxed{\frac{9}{24}}$$

Ex:- 12 (19) Evaluate $\int_0^2 \int_{2-x}^{\sqrt{4-x^2}} x \cdot dA$ by changing the order of integration.

→ Here, we are given



$$\int_{x=0}^{2} \int_{y=2-x}^{\sqrt{4-x^2}} \cdot x \cdot dA.$$

$$= \int_{x=0}^{2} \int_{y=2-x}^{\sqrt{4-x^2}} x \cdot \cancel{dx} \cdot dy \cdot dx$$

and $x=0$, $x=2$

$y = 2-x$, $y = \sqrt{4-x^2}$

$\therefore y^2 = 4 - x^2$

$\therefore x^2 + y^2 = 4$

Date: / /
Page No.

→ from, $y = 2 - x$
if $x = 0 \Rightarrow y = 2.$
$x = 2 \Rightarrow y = 0.$

and from $x^2 + y^2 = 4$
$x = 0 \Rightarrow y = 2$
$x = 2 \Rightarrow y = 0.$

→ As shown in the figure, Region R is bounded by $y = 2-x$, $x^2+y^2=4$ having boundary point $(0,2)$, $(2,0)$.

→ Now, changing the order of the integration, we get,

$$\int_{y=0}^{2} \int_{x=2-y}^{\sqrt{4-y^2}} x \cdot dx \cdot dy$$

$$= \int_0^2 \frac{x^2}{2} \Big|_{2-y}^{\sqrt{4-y^2}} \cdot dy$$

$$= \int_0^2 \frac{1}{2}\left\{ 4 - y^2 - (2-y)^2 \right\} dy$$

$$= \int_0^2 \frac{1}{2}\left\{ 4 - y^2 - 4 + 4y - y^2 \right\} dy$$

$$= \int_0^2 (-y^2 + 2y) \cdot dy$$

$$= -\frac{y^3}{3}\Big|_0^2 + y^2\Big|_0^2$$

$$= -\left(\frac{8}{3} - 0\right) + 4$$

$$= -\frac{8}{3} + 4$$

$$= \boxed{\frac{4}{3}}$$ Ans

Page No.

Ex:- 12 P:⑥ $\iint_R y^2 \cdot dy \cdot dx$ $\boxed{\text{Evaluate}}$

where R is the region bounded by the curves $y^2 = x$ and $y = x^3$.

→ $\iint_R \cdot y^2 \cdot dy\, dx$.

→ $y^2 = x$ and $y = x^3$

$\therefore y^2 = x^6$

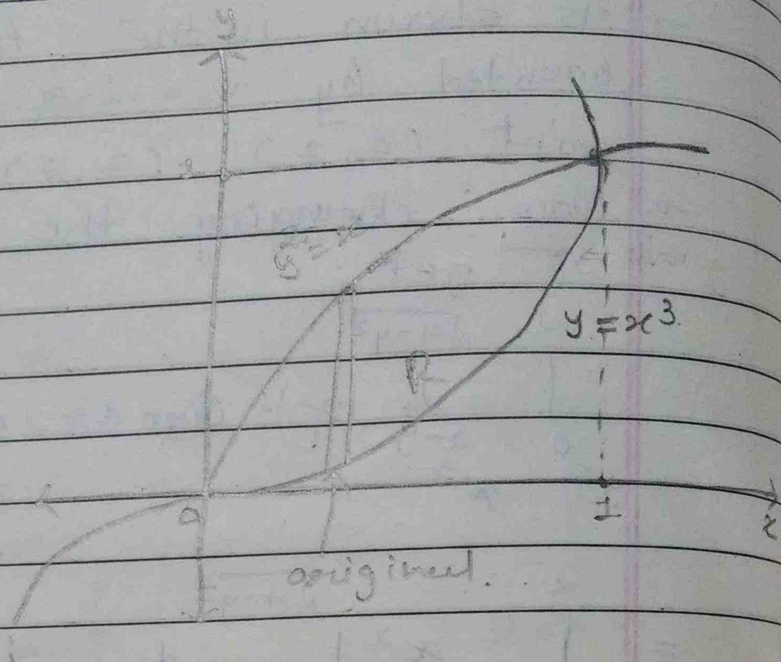
$\therefore x = x^6$

$\therefore x(x^5 - 1) = 0$

$\therefore x = 0, 1$.

→ $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$.



→ As shown in the figure, Region R is bounded by $y = x^2$, $y = x^3$ and boundary points are $(1,1)$, $(0,0)$.

→ ~~Now, changing the order of the integration, we get,~~

$\boxed{\text{Only Evaluation}}$ is asked

$$\int_{y=0}^{1} \int y^2 \cdot dx\, dy$$

→ $$\int_{x=0}^{1} \int_{y=x^3}^{\sqrt{x}} y^2 \cdot dy \cdot dx$$

$$= \int_0^1 \frac{y^3}{3}\Big|_{x^3}^{\sqrt{x}} \cdot dx$$

$$= \int_0^1 \frac{1}{3}\left(x^{3/2} - x^9\right) dx$$

$$= \frac{1}{3}\left[\frac{x^{5/2}}{5/2}\Big|_0^1 - \frac{x^{10}}{10}\Big|_0^1\right]$$

$$= \frac{1}{3}\left[\frac{2}{5} - \frac{1}{10}\right]$$

$$= \frac{1}{3}\left[\frac{4-1}{10}\right]$$

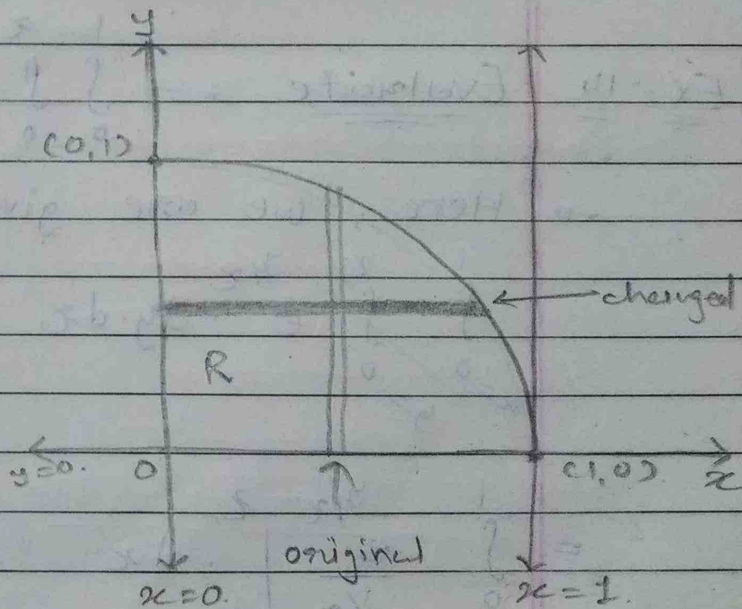
$$= \frac{1}{3}\left[\frac{3}{10}\right]$$

$$= \boxed{\frac{1}{10}}$$ Ans.

Ex: 13 P: 7 Change the order of the integration in $\int_0^1 \int_0^{\sqrt{1-x^2}} x \cdot dy\, dx$ and hence evaluate it.

→ Here, we are given,

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} x \cdot dy \cdot dx$$



$\therefore$ $x=0$, $x=1$, $y=0$, $y^2+x^2=1$

$\therefore$ from $x^2+y^2=1$

if $x=0 \Rightarrow y=1$

$x=1 \Rightarrow y=0$.

→ As show in the figure, region R is bounded by $x=0$, $y=0$, $x=1$ and $y^2+x^2=1$ and boundary points are $(0,0)$, $(0,1)$, $(1,0)$.

→ Now, changing the order of the integration,

$$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} x \cdot dx\, dy$$

$$= \int_0^1 \frac{x^2}{2}\Big|_0^{\sqrt{1-y^2}} \cdot dy$$

$$= \int_0^1 \frac{1}{2}(1-y^2) \cdot dy$$

$$= \frac{1}{2}\left[y\Big|_0^1 - \frac{y^3}{3}\Big|_0^1 \right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)$$

$$= \boxed{\frac{1}{3}}$$ Ans.

Ex :- 14 Evaluate :- $\int_0^1\int_0^x e^{\frac{y}{x}} \cdot dA$

→ Here, we are given,

$$\int_{x=0}^{1}\int_{y=0}^{x} e^{y/x} \cdot dy \cdot dx$$

$$= \int_0^1 \frac{e^{y/x}}{1/x}\Big|_0^x \cdot dx$$

$$= \int_0^1 (x \cdot e^1 - x)\, dx$$

$$= e \cdot \frac{x^2}{2}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1$$

$$= e\,\frac{1}{2} - \frac{1}{2}$$

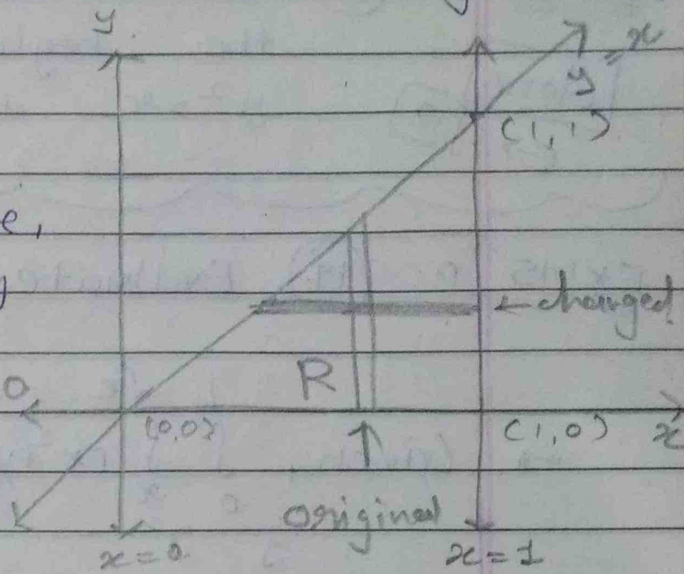
$$= \boxed{\frac{e-1}{2}}$$ Ans.

Extra :- Now, changing the order of the integration,

$x = 0$, $y = 0$

$x = 1$, $y = x$

→ As shown in the figure, region R is bounded by $y = x$, $y = 0$, $x = 0$ and $x = 1$ and boundary $y = 0$ points are $(0,0)$, $(1,1)$, $(1,0)$.



→ Now, changing the order of the integration,

$$\int_{y=0}^{1}\int_{x=y}^{1} e^{y/x} \cdot dx\,dy$$

$$= \int_0^1 \left[\frac{e^{y/x}}{(-y/x^2)} \Big|_y^1 \right] dy$$

$$= \int_0^1 \left(-\frac{x^2}{y} \cdot e^{y/x} \Big|_y^1 \right) dy$$

$$= \int_0^1 \left[-\frac{1}{y} e^{y} + y \cdot e \right] dy$$

$$=$$

Date: / /
Page No.

~~Ex: 15~~ P: 20 Evaluate $\iint_R y^2 \cdot dx\,dy$ where R is the region bounded by the curves $y^2 = x$ and $y = x^3$

$\boxed{\text{done}}$ Ex: $\boxed{12}$

Ex: 15 P: 19 Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) \cdot dy \cdot dx$.

→ Given, $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) \cdot dy \cdot dx$

$y = \quad y =$

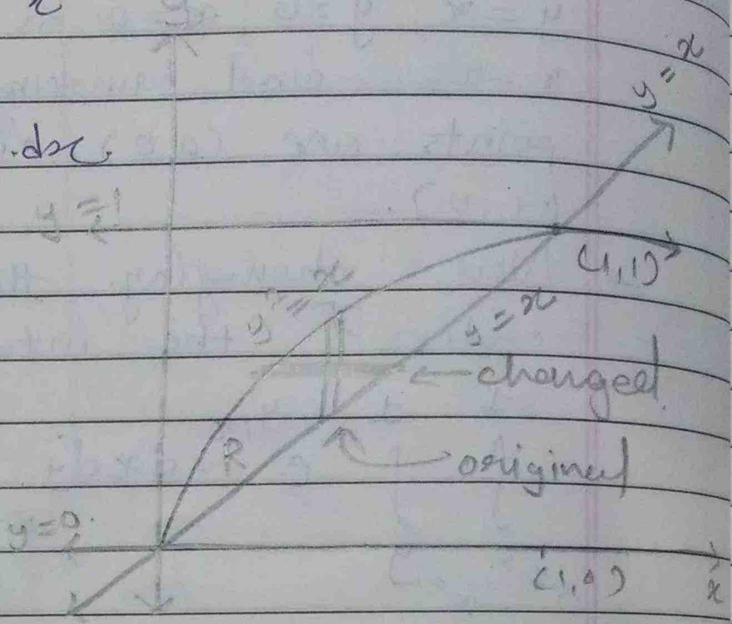
∴ $y = 0,\ y = 1,\ y = x$ and $y^2 = x$

~~$\therefore y^2 - y = 0$~~

~~$\therefore y(y-1) = 0$~~

~~$\therefore y = 0$~~

→ ~~$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) \cdot dy \cdot dx$~~



$$= \int_0^1 \left. \frac{y^3}{3} \right|_x^{\sqrt{x}} \cdot dx$$

$$= \int_0^1 \left(\frac{x^{3/2}}{3} - \frac{x^3}{3} \right) dx$$

$$= \left(\left. \frac{x^{5/2}}{15/2} \right|_0^1 - \left. \frac{x^4}{12} \right|_0^1 \right)$$

$$= \frac{2}{15}(1) - \frac{1}{12}$$

$$= \frac{8}{60} - \frac{5}{60}$$

$$= \frac{3}{60}$$

$$= \boxed{\frac{1}{20}}$$

Date: / /
Page No.

$$\rightarrow \int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\cdot dy \cdot dx$$

$$= \int_0^1 \left(x^2 \cdot y \Big|_x^{\sqrt{x}} + \frac{y^3}{3} \Big|_x^{\sqrt{x}} \right) \cdot dx$$

$$= \int_0^1 \left\{ \left(x^{5/2} - x^3\right) + \left(\frac{x^{3/2}}{3} - \frac{x^3}{3}\right) \right\} dx$$

$$= \int_0^1 \left(x^{5/2} - x^3 + \frac{x^{3/2}}{3} - \frac{x^3}{3} \right) dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3} \right) dx$$

$$= \frac{x^{7/2}}{7/2} \Big|_0^1 + \frac{x^{5/2}}{15/2} \Big|_0^1 - \frac{4}{3} \frac{x^4}{4} \Big|_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{2}{7} + \frac{2-5}{15}$$

$$= \frac{2}{7} - \frac{3}{15}$$

$$= \frac{2}{7} - \frac{1}{5}$$

$$= \frac{10-7}{35}$$

$$= \boxed{\frac{3}{35}}$$

Date: / /
Page No.

→ Now, changing the order of the integration

$$\int_{y=0}^{1}\int_{x=y^2}^{y} (x^2+y^2)\, dx\, dy$$

$$= \int_0^1 \left(\frac{x^3}{3}\Big|_{y^2}^{y} + x\cdot y^2 \Big|_{y^2}^{y} \right) dy$$

$$= \int_0^1 \left\{ \left(\frac{y^3}{3} - \frac{y^6}{3} \right) + \left(y^3 - y^4 \right) \right\} dy$$

$$= \int_0^1 \left\{ \frac{y^3}{3} - \frac{y^6}{3} + y^3 - y^4 \right\} dy$$

$$= \int_0^1 \left(\frac{4y^3}{3} - \frac{y^6}{3} - y^4 \right) dy$$

$$= \frac{4}{3}\cdot\frac{y^4}{4}\Big|_0^1 - \frac{y^7}{21}\Big|_0^1 - \frac{y^5}{5}\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{21} - \frac{1}{5}$$

$$= \frac{5-3}{15} - \frac{1}{21}$$

$$= \frac{2}{15} - \frac{1}{21}$$

$$= \frac{42-15}{15\times 21}$$

$$= \frac{27}{5\times 3\times 7\times 3} = \frac{3\times \cancel{3}\times \cancel{3}}{5\times \cancel{3}\times 7\times \cancel{3}}$$

$$= \boxed{\frac{3}{35}}$$ — Ans

* $x = r\cos\theta$ $\qquad$ $\theta = \tan^{-1}(y/x)$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\therefore r = \sqrt{x^2+y^2}$

→ Ex: 1 $\quad r = 2\sin\theta$

$\therefore \sqrt{x^2+y^2} = 2\left(\dfrac{y}{\sqrt{x^2+y^2}}\right)$

$\therefore x^2 + y^2 = 2y$

$\therefore x^2 + y^2 - 2y = 0$

$\therefore \cancel{(x-0)^2 + (y-0)^2 - 2y = 0}$

→ Standard eqn of a circle centered at (a,b).

$$(x-a)^2 + (y-b)^2 = r^2$$

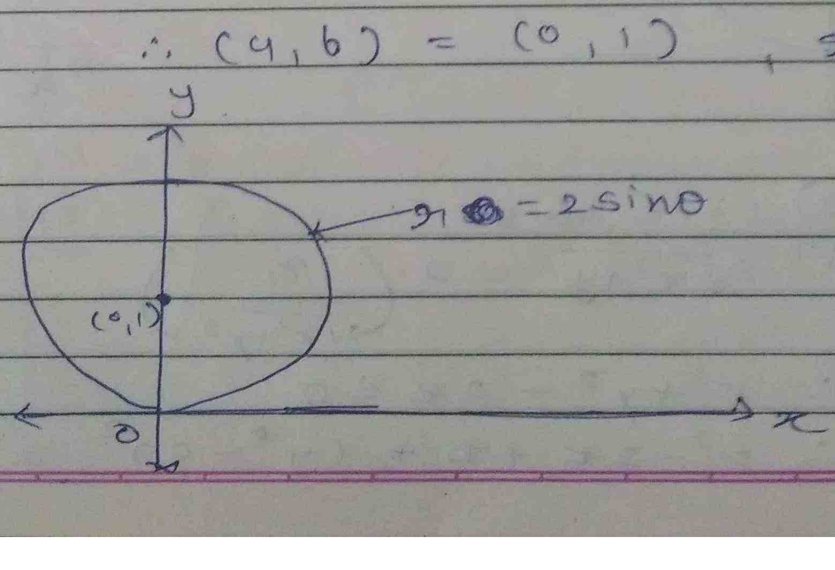
(a,b), r.

$\cancel{\therefore (x-0)^2 + (y-0)^2}$

$\therefore (x-0)^2 + y^2 - 2y + 1 = 1$

$\therefore (x-0)^2 + (y-1)^2 = 1^2$

$\therefore (a,b) = (0,1)$, $r = 1$.



Date: / /

Page No.

→ **Ex :- ②** $r = 4\sin\theta$

$\therefore \sqrt{x^2+y^2} = 4\left(\dfrac{y}{\sqrt{x^2+y^2}}\right)$

$\therefore x^2+y^2-4y = 0.$

$\therefore$ Converting it into the standard eqⁿ of a circle,

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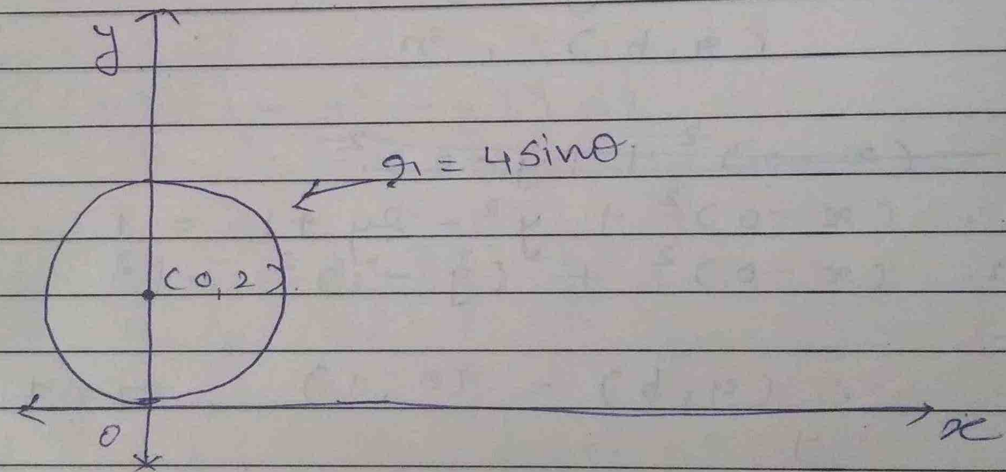
$(x-0)^2 + (y^2-4y+2^2) = 2^2.$

$\therefore (x-0)^2 + (y-2)^2 = (2)^2.$

$\therefore (a,b) = (0,2)$, $r = 2.$

→ because standard eqⁿ is

$(x-a)^2 + (y-b)^2 = r^2$



→ **Ex :- ③** $r = 2\cos\theta$

$\therefore \sqrt{x^2+y^2} = 2\cdot\left(\dfrac{x}{\sqrt{x^2+y^2}}\right)$

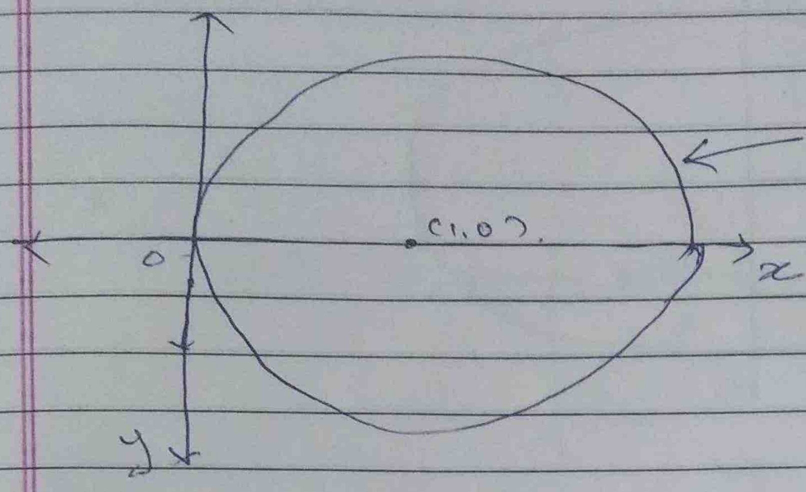
$\therefore x^2+y^2-2x = 0.$

$\therefore x^2-2x+1+(y^2-0)^2 = 1$

Date: / /
Page No.

$\therefore (x-1)^2 + (y-0)^2 = 1$

$\therefore (a,b) = (1,0) \quad , \quad r = 1$



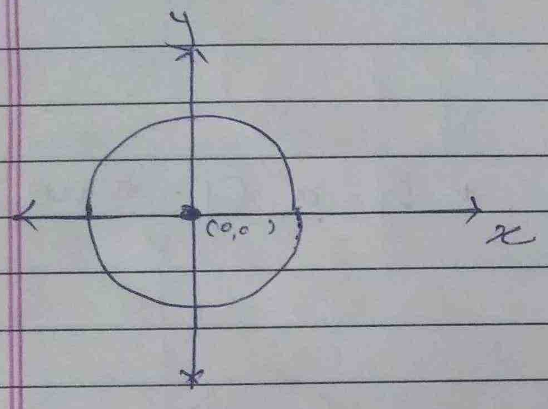
Symmetric about x-axis

→ Ex:- (4) $\quad r = c \quad , \quad c > 0$

$\therefore \sqrt{x^2+y^2} = c$

$\therefore x^2 + y^2 = c^2$

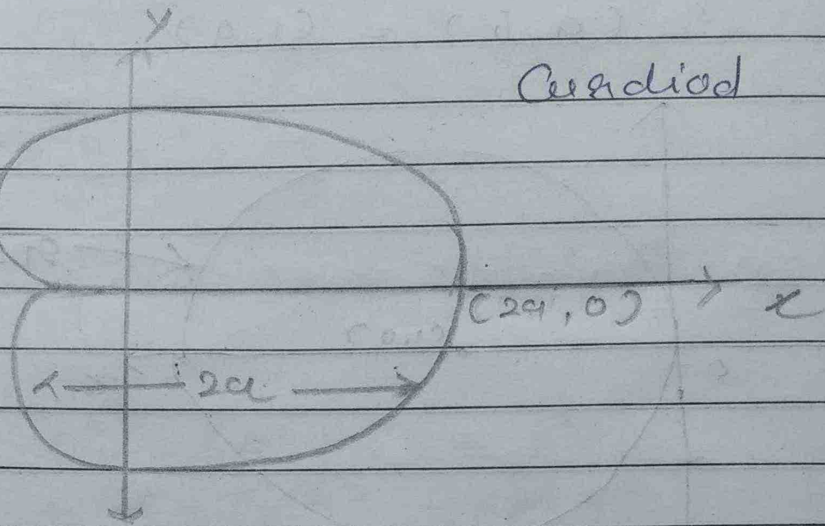
$(a,b) = (0,0) \quad , \quad r = c$ radius.



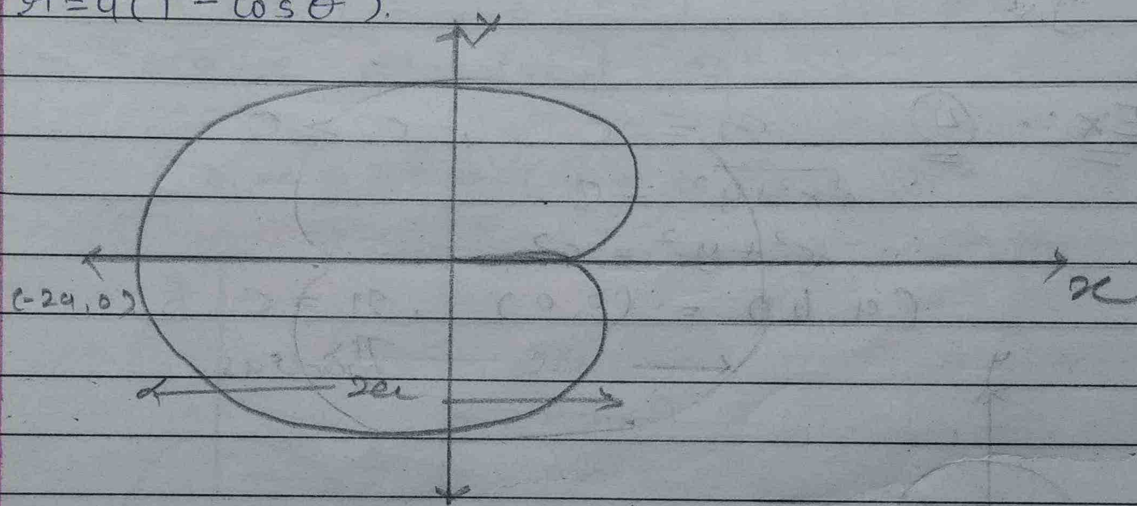
→ Ex:- (5) $\quad \theta = c \quad$ it will represt the ray.

* $r = a(1 + \cos\theta)$.

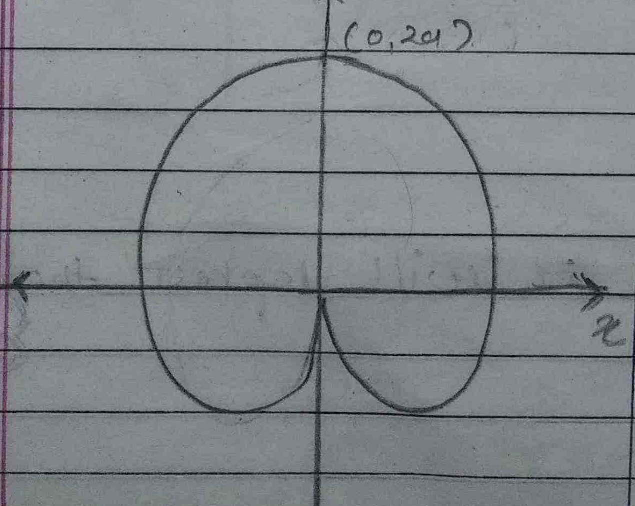
Cardiod



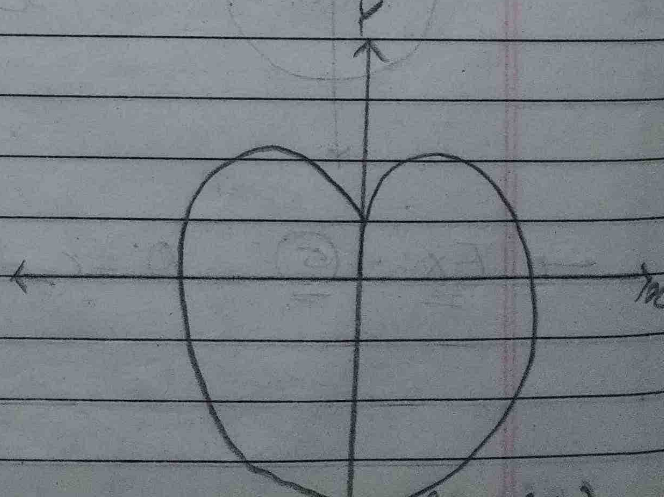
* $r = a(1 - \cos\theta)$.



* $r = a(1 + \sin\theta)$



* $r = a(1 - \sin\theta)$.



Date: / /

Page No.

* $$\int_{-a}^{a} f(x)\cdot dx = \begin{cases} 2\int_0^a f(x)\cdot dx & , f \text{ is even} \\ 0 & , f \text{ is odd} \end{cases}$$

* $$\int_0^{2a} f(x)\cdot dx = \begin{cases} 2\int_0^a f(x)\cdot dx & , f(2a-x) = f(x) \\ 0 & , f(2a-x) = -f(x) \end{cases}$$

* $\underbrace{\int \sin^4\theta\cdot d\theta \quad \& \quad \int \sin^{16}\theta\cdot d\theta}_{\text{Reduction formula}}$ } Very Helpful.

$n \geq 2 \rightarrow$ $$\int_{\theta=0}^{\pi/2} \sin^n\theta\cdot d\theta = \int_{\theta=0}^{\pi/2} \cos^n\theta\cdot d\theta$$

$$= \frac{(n-1)(n-3)\ \ldots\ldots\ 2 \text{ OR } 1}{n(n-2)(n-4)\ \ldots\ldots\ 2 \text{ OR } 1}\ K$$

limit must be $\boxed{0 \text{ to } \pi/2}$

$$K = \begin{cases} \pi/2 & , \text{if } n \text{ is even} \\ 1 & , \text{otherwise} \end{cases}$$

→ Ex: ① $\int_0^{\pi/2} \sin^4\theta \cdot d\theta$ $n = 4$.

$= \frac{3 \times 1}{4\,(2)} \cdot \frac{\pi}{2}$ ($\because$ n is even)

$= \boxed{\frac{3\pi}{8}}$

* $\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \cdot d\theta$

$= \frac{(m-1)(m-3)\ldots(2 \text{ OR } 1)\;(n-1)(n-3)\ldots(2 \text{ OR } 1)}{(m+n)(m+n-2)(m+n-4)\ldots(2 \text{ OR } 1)}\,k$

where, $k = \begin{cases} \frac{\pi}{2}, & \text{both } m, n \text{ are even} \\ 1, & \text{otherwise} \end{cases}$

→ Ex: ① $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\,dx$

let $x = \sin\theta$ $x = 0 \Rightarrow \theta = 0$

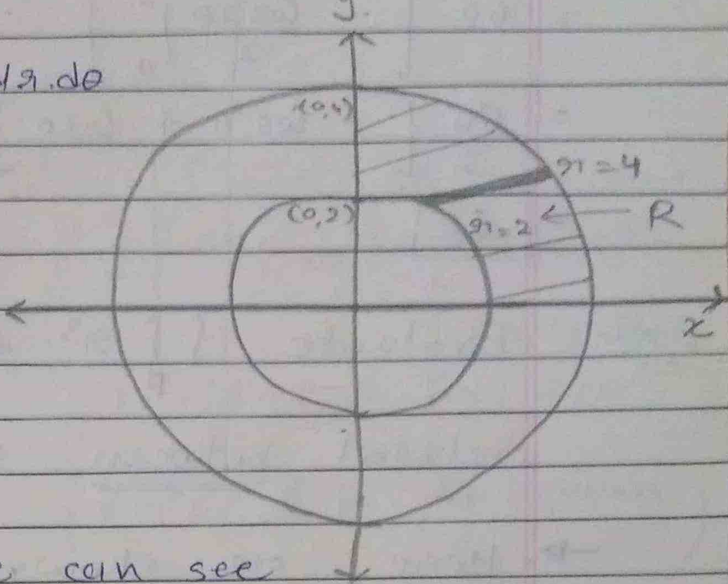
$\therefore dx = \cos\theta \cdot d\theta$ $x = 1 \Rightarrow \theta = \pi/2$

$= \int_0^{\pi/2} \frac{\sin^6\theta}{\cancel{\cos\theta}}\,\cancel{\cos\theta} \cdot d\theta$

$= \int_0^{\pi/2} \sin^6\theta \cdot d\theta = \frac{5 \times 3 \times 1}{\cancel{6}_{2} \times 4 \times 2} \cdot \frac{\pi}{2} = \boxed{\frac{5\pi}{32}}$

Ex:- Evaluate $\iint_R r^3 \cdot \sin 2\theta \cdot dr \cdot d\theta$, where R is the region bounded in the first quadrant between $r = 2$ and $r = 4$.

→ Given, $\iint_R r^3 \cdot \sin 2\theta \cdot dr \cdot d\theta$



and $r = 2$

$\therefore \sqrt{x^2+y^2} = 2$

$\therefore x^2+y^2 = 4$

& $r = 4$

$\therefore x^2+y^2 = 16$

→ From the figure we can see that the limits of θ is 0 to $\pi/2$ and it is varying from 2 to 4.

$\therefore \iint_R r^3 \cdot \sin 2\theta \cdot dr \cdot d\theta$

$$= \int_{\theta=0}^{\pi/2} \int_{r=2}^{4} r^3 \cdot \sin 2\theta \cdot dr \cdot d\theta$$

$$= \int_0^{\pi/2} \sin 2\theta \cdot \left. \frac{r^4}{4} \right|_2^4 \cdot d\theta$$

$$= \int_0^{\pi/2} \sin 2\theta \, (64 - 4) \cdot d\theta$$

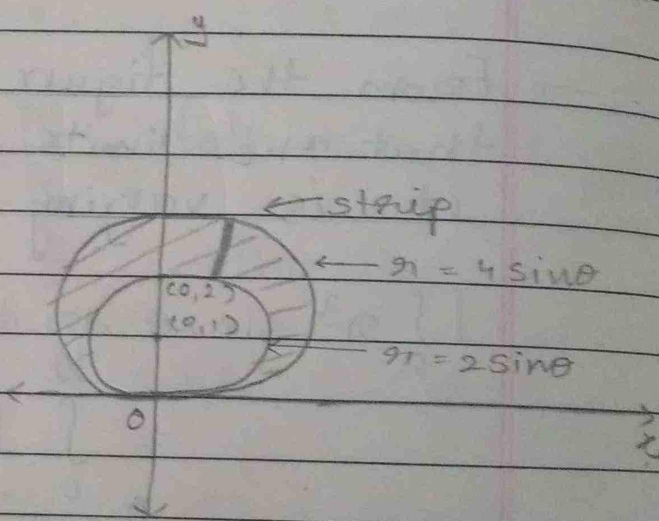
$$= 60 \int_0^{\pi/2} \sin 2\theta \cdot d\theta$$

$$= 60 \cdot \left[-\frac{\cos 2\theta}{2} \Big|_0^{\frac{\pi}{2}} \right]$$

$$= \frac{60}{2} \left[-\cos\pi + \cos 0 \right] = \frac{60}{2}\left[-(-1) + 1 \right] = \frac{60}{2} \times 2$$

$$= \boxed{60}$$

Ex:- Evalute $\iint_R r^3 \cdot dr \cdot d\theta$, over the area included between $r = 2\sin\theta$ & $r = 4\sin\theta$.



→ Here, as shown in the figure, the region R is bounded between $r = 2\sin\theta$ & $r = 4\sin\theta$ and limits of r and θ is... ~~0 to π and~~

$\theta = 0$ to π

$r = 2\sin\theta$ to $r = 4\sin\theta$

$$\therefore \iint_R r^3 \cdot dr \cdot d\theta = \int_{\theta=0}^{\pi} \int_{r=2\sin\theta}^{4\sin\theta} r^3 \cdot dr \cdot d\theta$$

$$= \int_0^{\pi} \frac{r^4}{4} \Big|_{2\sin\theta}^{4\sin\theta} \cdot d\theta$$

$$= \frac{1}{4}\int_0^{\pi} (256 \cdot \sin^4\theta - 16 \cdot \sin^4\theta) \cdot d\theta$$

$$= \int_0^{\pi} \sin^4\theta \cdot \left(\frac{240}{4}\right) \cdot d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta \cdot d\theta$$

$$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta \cdot d\theta$$

$\left(\because \sin(\pi - x) = \sin x \text{ and } \int_0^{2a} f(x) \cdot dx = 2 \cdot \int_0^{a} f(x) \cdot dx \text{ if } f(2a - x) = f(x) \right)$

$$= 120 \cdot \frac{3 \times 1}{4 \times 2} \cdot \frac{\pi}{2}$$

$$= \frac{15 \times 3\pi}{2}$$

$\left(k = \pi/2 \;\; \because n \text{ is even} \right)$

$$= \boxed{\frac{45\pi}{2}}$$

Ex:- Evaluate the integral $\int_0^{\frac{\pi}{2}} \int_0^{1-\sin\theta} r^2 \cdot \cos\theta \cdot dr\, d\theta$.

→ Given, $\int_0^{\frac{\pi}{2}} \int_0^{1-\sin\theta} r^2 \cdot \cos\theta \cdot dr\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \cdot \left.\frac{r^3}{3}\right|_0^{1-\sin\theta} \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{3} \cdot (1 - \sin\theta)^3 \cdot d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \cos\theta \left(1 - \sin^3\theta - 3\sin\theta\,(1 - \sin\theta) \right) \cdot d\theta$$

$$\int f(x) \cdot f'(x) \cdot dx = \frac{f(x)^2}{2}$$

$$= \frac{1}{3} \int_0^{\pi/2} \cos\theta \left[1 - \sin^3\theta - 3\sin\theta + 3\sin^2\theta \right] . d\theta .$$

[illegible]

$$= \frac{1}{3} \int_0^{\pi/2} (\cos\theta - \sin^3\theta \cdot \cos\theta - 3 \sin\theta \cdot \cos\theta + 3\sin^2\theta . \cos\theta) \, d\theta$$

$$= \frac{1}{3} \left[\sin\theta \Big|_0^{\pi/2} - \frac{\sin^4\theta}{4} \Big|_0^{\pi/2} - 3 \cdot \frac{\sin^2\theta}{2} \Big|_0^{\pi/2} + 3 \cdot \frac{\sin^3\theta}{3} \Big|_0^{\pi/2} \right]$$

$$= \frac{1}{3} \left[(1-0) - \left(\frac{1}{4} - 0 \right) - \frac{3}{2} (1-0) + (1-0) \right]$$

$$= \frac{1}{3} \left(1 - \frac{1}{4} - \frac{3}{2} + 1 \right)$$

$$= \frac{1}{3} \left(\frac{4 - 1 - 6 + 4}{4} \right) = \boxed{\frac{1}{12}}$$

$$* \quad \iint_R f(x,y) \cdot dx\,dy = \iint_S f(u,v) \cdot \frac{\partial(x,y)}{\partial(u,v)} du . dv$$

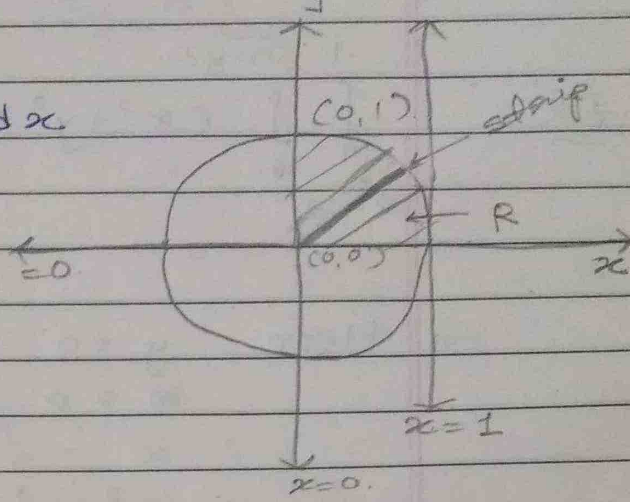
→ let $y = r \sin\theta$, $x = r \cos\theta$.

$$\iint_R f(x,y) . \, dx\,dy = \iint_S f(r\cos\theta, r\sin\theta) . \frac{\partial(x,y)}{\partial(u,v)} dr\,d\theta$$

$$= \iint_S f(r . \cos\theta , r \sin\theta) . r . \, dr . d\theta$$

Date: / /
Page No.

Ex: Change the Cartesian integral $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \cdot dy\,dx$ into an equivalent polar integral and evaluate the same.



→ Given, $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \cdot dy\,dx$

→ Here, $x = 0$ to 1

$y = 0$ to $\sqrt{1-x^2}$

$\therefore y = \sqrt{1-x^2}$

$\therefore x^2 + y^2 = 1.$

→ let $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore dx\,dy = r \cdot dr\,d\theta$

$\therefore \iint_R e^{-(x^2+y^2)} \cdot dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} e^{-r^2} r \cdot dr\,d\theta$

let $r^2 = t \Rightarrow 2r \cdot dr = dt.$

$r = 0 \Rightarrow t = 0.$

$r = 1 \Rightarrow t = 1$

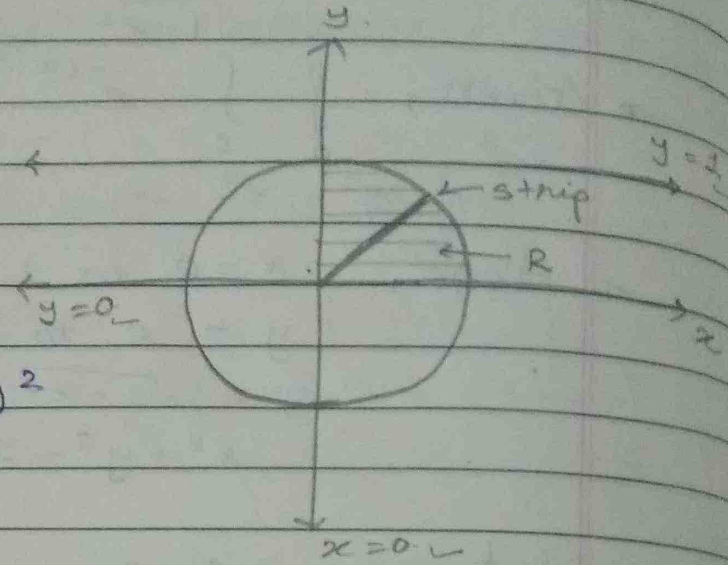
$\therefore I = \frac{1}{2} \int_0^{\pi/2} \int_0^1 e^{-t}\,dt\,d\theta$

$= \frac{1}{2} \int_0^{\pi/2} \left(\frac{e^{-t}}{-1} \Big|_0^1 \right) \cdot d\theta$

$$\therefore I = \frac{-1}{2}\int_0^{\pi/2} (e^{-1}-1)\cdot d\theta$$

$$= \frac{(1-e^{-1})}{2} \cdot \frac{\pi}{2} = \boxed{(1-e^{-1})\cdot\frac{\pi}{4}}$$

Ex: $\displaystyle\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy.$



→ Here, $y = 0$ to 1

$x = 0$ to $\sqrt{1-y^2}$

$\therefore x = \sqrt{1-y^2}$

$\therefore x^2+y^2 = 1$

→ From the figure, we can see that if $x = r\cos\theta$ & $y = r\sin\theta$, then r is varying from 0 to 1 and θ is varying from 0 to $\pi/2$.

$$\therefore \iint_R (x^2+y^2)\cdot dx\, dy.$$

$$= \int_0^{\pi/2}\int_0^1 r^2\cdot r\cdot dr\cdot d\theta$$

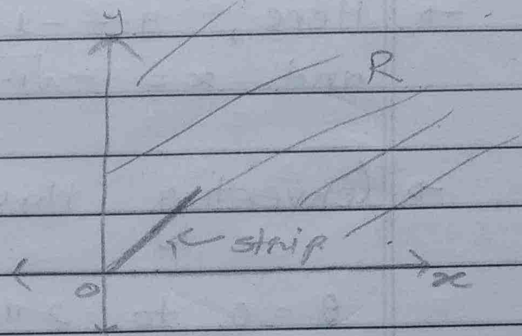
$$= \int_0^{\pi/2}\int_0^1 r^3\cdot dr\, d\theta.$$

Date: / /
Page No.

$$= \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^1 \cdot d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 \cdot d\theta = \frac{1}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{8}}$$ Ans

Ex:-24(b) $\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)} \cdot dx\,dy$



→ Let $x = r\cos\theta$, $y = r\sin\theta$
then $dx\,dy = r \cdot dr\,d\theta$

→ From the figure we can see that $\theta = 0$ to $\pi/2$
and $r = 0$ to ∞.

$$\therefore \int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)} \cdot dx \cdot dy = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r \cdot dr \cdot d\theta$$

$\Rightarrow$ let $r^2 = t \Rightarrow 2r \cdot dr = dt$

$\therefore$ $r = 0 \Rightarrow t = 0$

$r = \infty \Rightarrow t = \infty$

$$\therefore I = \int_0^{\pi/2}\int_0^{\infty} e^{-t} \frac{dt}{2} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^{\infty} e^{-t} \cdot dt \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{e^{-t}}{-1}\Big|_0^{\infty}$$

$$= \frac{1}{2}\int_0^{\pi/2} (-(0-1)) \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 \cdot d\theta$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \boxed{\frac{\pi}{4}}$$ Ans

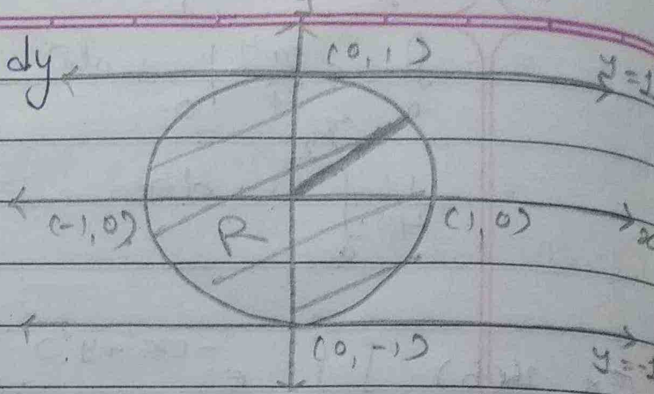
$\rightarrow \int \ln t \cdot dt = t(\ln t - 1)$

Date: / /

Page No.

Ex:- 24(c) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1) \cdot dx\,dy$

y" x"



$\rightarrow$ Here, $y = -1$ to 1
and $x = -\sqrt{1-y^2}$ to $\sqrt{1-y^2}$

$\rightarrow$ Converting this in terms of polar co-ordinates,

$\theta = 0$ to 2π and $r = 0$ to 1.
let $x = r\cos\theta$ & $y = r\sin\theta$.
$\therefore dx \cdot dy = r \cdot dr \cdot d\theta$.

$\therefore \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1) \cdot dx\,dy$

$$= \int_{0}^{2\pi}\int_{0}^{1} \ln(r^2+1) \cdot r \cdot dr \cdot d\theta$$

Let $r^2+1 = t$
$\therefore 2r \cdot dr = dt$.
$\therefore$ if $r = 0 \Rightarrow t = 1$
$r = 1 \Rightarrow t = 2$.

$$\therefore I = \int_{0}^{2\pi}\int_{1}^{2} \ln t \cdot \frac{dt}{2} \cdot d\theta$$

$$= \int_{0}^{2\pi} t(\ln t - 1)\Big|_{1}^{2} \cdot \frac{dt}{2} \cdot d\theta$$

Date: / /
Page No.

$$\therefore I = \frac{1}{2}\int_0^{2\pi} \left[2(\ln 2 - 1) - 1(\ln 1 - 1) \right] \cdot dt \cdot d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (2\ln 2) \cdot d\theta$$

$$= \frac{2 \cdot \ln 2}{2} \cdot (2\pi - 0)$$

$$= \boxed{2\pi \ln 2} \text{ Ans.}$$

Date: / /
Page No.

* Triple Integrals :-

Ex:- Evaluate $\int_1^2 \int_x^{2x} \int_0^{y-x} \cdot dv.$

$$= \int_{x=1}^{2} \int_{y=x}^{2x} \int_{z=0}^{y-x} \cdot dz \cdot dy \cdot dx$$

$$= \int_1^2 \int_x^{2x} (z)\Big|_0^{y-x} \cdot dy \cdot dx$$

$$= \int_1^2 \int_x^{2x} (y-x) \cdot dy \cdot dx$$

$$= \int_1^2 \left(\frac{y^2}{2}\Big|_x^{2x} - xy\Big|_x^{2x} \right) \cdot dx$$

$$= \int_1^2 \left\{ \tfrac{1}{2}(4x^2 - x^2) - x(2x^2 - x^2) \right\} \cdot dx$$

$$= \int_1^2 \left\{ \frac{3}{2}x^2 - x^2 \right\} \cdot dx$$

$$= \frac{1}{2} \int_1^2 x^2 \cdot dx$$

$$= \frac{1}{2} \frac{x^3}{3}\Big|_1^2$$

$$= \frac{1}{2} \cdot \frac{1}{3} (8 - 1)$$

$$= \boxed{\frac{7}{6}} \text{ Ans}$$

Date: / /
Page No.

EX: 25

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \cdot dz \cdot dy \cdot dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \cdot \frac{z^2}{2}\Big|_0^{\sqrt{1-x^2-y^2}} \cdot dy \cdot dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{2} \left(1-x^2-y^2-0\right) \cdot dy \cdot dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} \left(xy - x^3y - xy^3\right) \cdot dy \cdot dx$$

$$= \frac{1}{2}\int_0^1 \left(x \cdot \frac{y^2}{2}\Big|_0^{\sqrt{1-x^2}} - x^3 \cdot \frac{y^2}{2}\Big|_0^{\sqrt{1-x^2}} - x \cdot \frac{y^4}{4}\Big|_0^{\sqrt{1-x^2}} \right) \cdot dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x}{2}\left\{(1-x^2)-0\right\} - \frac{x^3}{2}\left(1-x^2-0\right) - \frac{x}{4}\left(1-x^2\right)^2 \right) dx$$

$$= \frac{1}{2}\int_0^1 \left\{ \frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4}\left(1-2x^2+x^4\right) \right\} dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x}{2} - x^3 + \frac{x^5}{2} - \frac{x}{4} + \frac{2x^3}{2} + \frac{x^5}{4} \right) \cdot dx$$

$$= \frac{1}{4}\int_0^1 \left(x - 2x^3 + x^5 - \frac{x}{2} + x^3 + \frac{x^5}{2} \right) \cdot dx$$

$$= \frac{1}{4}\left\{ \frac{x^2}{2}\Big|_0^1 - \frac{2x^3}{3}\Big|_0^1 + \frac{x^6}{6}\Big|_0^1 - \frac{x^2}{4}\Big|_0^1 + \frac{x^4}{4}\Big|_0^1 + \frac{x^6}{12}\Big|_0^1 \right\}$$

Date: / /
Page No.

$$= \frac{1}{4}\left\{ \frac{1}{2} - \frac{2}{3} + \frac{1}{6} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} + \frac{1}{12} \right\}$$

$$= \frac{1}{4}\left\{ \frac{6-8+2+1}{12} \right\}$$

$$= \boxed{\frac{1}{48}} \text{ Ans.}$$

Ex:- 27 (a) $\int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)\, dz\, dy \cdot dx$

$$= \int_0^1 \int_0^1 \left(x^2 z\Big|_0^1 + y^2 z\Big|_0^1 + \frac{z^3}{3}\Big|_0^1 \right) \cdot dy \cdot dx$$

$$= \int_0^1 \int_0^1 \left(x^2 + y^2 + \frac{1}{3} \right) dy \cdot dx$$

$$= \int_0^1 \left(x^2 y\Big|_0^1 + \frac{y^3}{3}\Big|_0^1 + \frac{y}{3}\Big|_0^1 \right) \cdot dx$$

$$= \int_0^1 \left(x^2 + \frac{1}{3} + \frac{1}{3} \right) \cdot dx$$

$$= \frac{x^3}{3}\Big|_0^1 + \frac{x}{3}\Big|_0^1 + \frac{x}{3}\Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

$$= \boxed{1} \text{ Ans.}$$

Date: / /

Page No.

Ex:- 27 (c) Evaluate : $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^{x} \cdot dx\, dt\, dv$.

$$= \int_0^{\pi/4} \int_0^{\ln \sec v} e^{x} \Big|_{-\infty}^{2t} \cdot dt\, dv.$$

$$= \int_0^{\pi/4} \int_0^{\ln \sec v} (e^{2t} - e^{-\infty}) \cdot dt \cdot dv$$

$$= \int_0^{\pi/4} \frac{e^{2t}}{2} \Big|_0^{\ln \sec v} \cdot dv$$

$$= \frac{1}{2} \int_0^{\pi/4} (e^{2 \ln \sec v} - e^{2(0)}) \cdot dv.$$

$$= \frac{1}{2} \int_0^{\pi/4} (\sec^2 v - 1) \cdot dv.$$

$$= \frac{1}{2} \left(\tan v \Big|_0^{\pi/4} - v \Big|_0^{\pi/4} \right).$$

$$= \frac{1}{2} (1 - \pi/4)$$

$$= \boxed{\frac{4 - \pi}{8}}$$ Ans.

Ex:- 26 Evaluate $\int_0^{1} \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2\theta + z^2) r \cdot d\theta \cdot dr \cdot dz$

$$= \int_0^{1} \int_0^{\sqrt{z}} \int_0^{2\pi} (r^3 \cdot \cos^2\theta + z^2 r) \cdot d\theta \cdot dr \cdot dz.$$

Date: / /
Page No.

$$= \int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left(r^3 \cdot \left(\frac{1 + \cos 2\theta}{2} \right) + z^2 r \right) \cdot d\theta \cdot dr \cdot dz$$

$$= \int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left\{ \left(\frac{r^3}{2} + \frac{r^3}{2} \cdot \cos 2\theta \right) + z^2 r \right\} \cdot d\theta \, dr \, dz$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left\{ \frac{r^3 \theta}{2} \Big|_0^{2\pi} + \frac{r^3}{2} \frac{\sin 2\theta}{2} \Big|_0^{2\pi} + z^2 r \theta \Big|_0^{2\pi} \right\} dr \cdot dz$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left\{ \frac{r^3}{2} \cdot 2\pi + \frac{r^3}{4} (\sin 4\pi - 0) + z^2 r \cdot 2\pi \right\} \cdot dr \cdot dz$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left(\pi \cdot r^3 + 0 + 2\pi z^2 \cdot r \right) dr \cdot dz$$

$$= \int_0^1 \left(\pi \frac{r^4}{4} \Big|_0^{\sqrt{z}} + 2\pi z^2 \cdot \frac{r^2}{2} \Big|_0^{\sqrt{z}} \right) \cdot dz$$

$$= \int_0^1 \left(\frac{\pi}{4} (z^2 - 0) + \pi z^2 (z - 0) \right) \cdot dz$$

$$= \int_0^1 \left(\frac{\pi}{4} z^2 + \pi z^3 \right) \cdot dz$$

$$= \frac{\pi z^3}{12} \Big|_0^1 + \pi \frac{z^4}{4} \Big|_0^1$$

$$= \frac{\pi}{12}(1 - 0) + \frac{\pi}{4}(1 - 0)$$

$$= \frac{\pi}{12} + \frac{\pi}{4}$$

$$= \frac{4\pi}{12}$$

$$= \boxed{\frac{\pi}{3}}$$ Ans

* $A = \iint dA = \iint dx\,dy = \iint dy\,dx$

→ $A = \int_a^b f(x)\cdot dx$

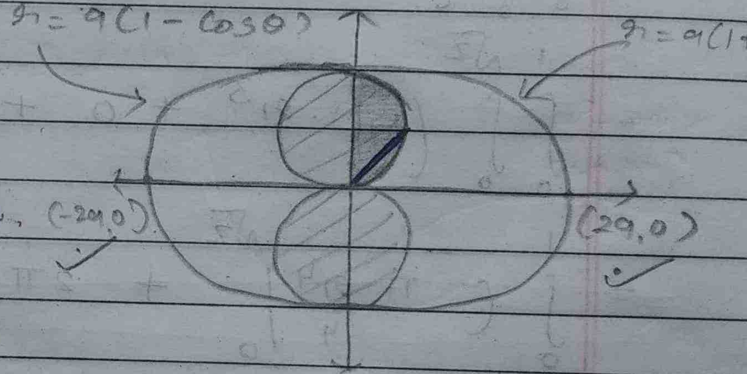
$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\cdot dr\,d\theta$

$A = \iint r\cdot dr\cdot d\theta$

Ex: 29 Find the area common to the cardioids $r = (1+\cos\theta)$ and $r = a(1-\cos\theta)$.

→ Area between two cardioids,



$$A = 4\int_0^{\pi/2}\int_0^{a(1-\cos\theta)} r\cdot dr\,d\theta$$

$$\therefore A = 4\int_0^{\pi/2} \left.\frac{r^2}{2}\right|_0^{a(1-\cos\theta)} \cdot d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2} a^2(1-2\cos\theta+\cos^2\theta)\cdot d\theta$$

$$= 2a^2\int_0^{\pi/2}\left[1-2\cos\theta+\frac{1+\cos2\theta}{2}\right]\cdot d\theta$$

$$= 2a^2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin2\theta}{2}\cdot\frac{1}{2}\right]$$

$$= 2a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\cdot\sin2\theta\right]$$

Date: / /
Page No.

$$\therefore A = 2a^2 \left[\theta \Big|_0^{\pi/2} - 2\sin\theta \Big|_0^{\pi/2} + \frac{\theta}{2} \Big|_0^{\pi/2} + \frac{\sin 2\theta}{2} \cdot \frac{1}{2} \Big|_0^{\pi/2} \right]$$

$$= 2a^2 \left[\frac{\pi}{2} - 2\cdot(1) + \frac{\pi}{4} + 0 \right]$$

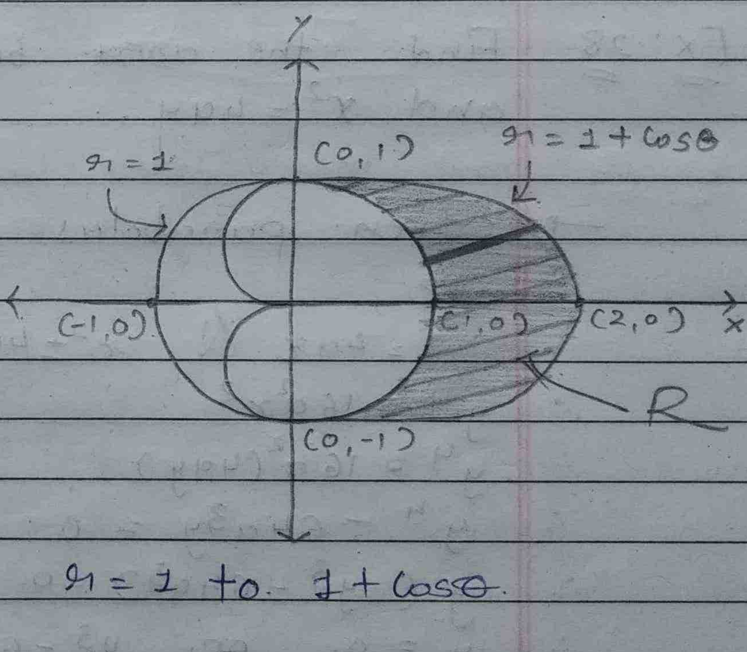
$$= 2a^2 \left[\frac{3\pi}{4} - 2 \right]$$

$$= \frac{2a^2}{4} [3\pi - 8] = \boxed{\frac{a^2}{2}(3\pi - 8)} \text{ Ans.}$$

Ex:- Find the area of the region lies ~~between~~ inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

→ Here, $r = 1 + \cos\theta$

$\therefore a = 1.$



→ From the figure, the region $R/2$ having the limits,

$\theta = 0$ to $\pi/2$ and $r = 1$ to $1 + \cos\theta$.

$\therefore$ Area of the region,

$$A = 2 \int_0^{\pi/2} \int_1^{1+\cos\theta} r \, dr \, d\theta$$

$$= 2 \int_0^{\pi/2} \frac{r^2}{2} \Big|_1^{1+\cos\theta} d\theta$$

$$\therefore A = \int_0^{\pi/2} \left[(1 + 2\cos\theta + \cos^2\theta) - 1 \right] d\theta$$

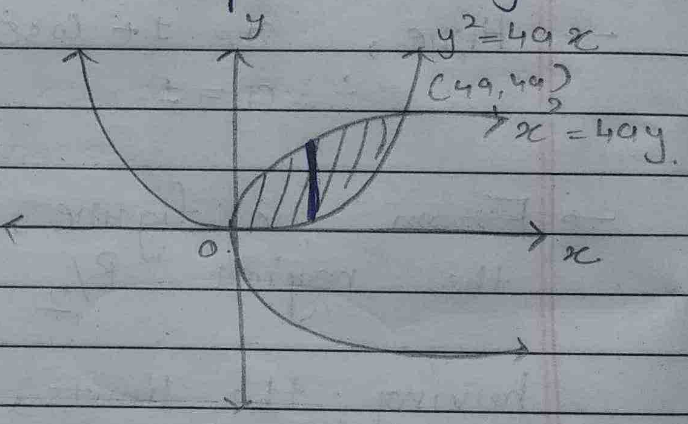
$$= \int_0^{\pi/2} \left(1 + 2\cos\theta + \tfrac{1}{2} + \frac{\cos 2\theta}{2} - 1\right) . d\theta$$

$$= \left[2 \cdot \sin\theta \Big|_0^{\pi/2} + \frac{\theta}{2}\Big|_0^{\pi/2} + \frac{1}{2} \cdot \frac{\sin 2\theta}{2}\Big|_0^{\pi/2} \right]$$

$$= 2(1) + \frac{\pi}{4} + 0$$

$$= \boxed{\frac{\pi}{4} + 2} \text{ Ans}$$

Ex:-28 Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.



→ Given parabolas are,

$y^2 = 4ax$ & $x^2 = 4ay$

$\therefore y^4 = 16a^2x^2$

$\therefore y^4 = 16a^2(4ay)$

$\therefore y^4 - 64a^3y = 0$

$\therefore y(y^3 - 64a^3) = 0$

$\therefore y = 0$ or $y^3 - 64a^3 = 0$

$\Rightarrow x = 0$ $\quad \therefore y^3 = 64a^3$

$\therefore y = 4a \Rightarrow x^2 = 4a(4a)$

$\Rightarrow x = 4a$

$\therefore$ Intersecting points are $(0, 0)$, $(4a, 4a)$.

$\therefore$ Area of the region R,

$$A = \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy.dx$$

$$= \int_0^{4a} y \Big|_{\frac{x^2}{4a}}^{2\sqrt{ax}} .dx$$

$$= \int_0^{4a} \left(-\frac{x^2}{4a} + 2\sqrt{ax}\right).dx$$

$$= 2\sqrt{a} \; \frac{x^{3/2}}{3/2} \Big|_0^{4a} - \frac{1}{4a} \; \frac{x^3}{3} \Big|_0^{4a}$$

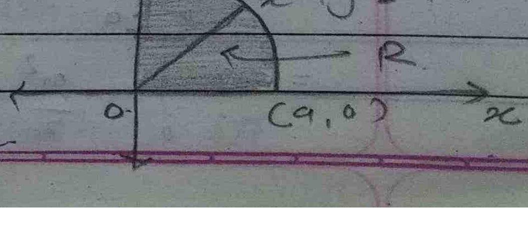
$$= \frac{4}{3} \sqrt{a} \; (4a)^{3/2} - \frac{1}{12a} \; (4a)^3$$

$$= \frac{4}{3} (8) a^2 - \frac{1}{12a} \cdot 64 \cdot a^3$$

$$= \frac{32a^2}{3} - \frac{16}{3} \cdot a^2$$

$$= \boxed{\frac{16a^2}{3}}$$ Ans.

Ex: 30 Find the area of a plate in the form of a quadrant of a circle $x^2+y^2=a^2$.



→ From the figure,
θ is varying from 0 to $\pi/2$
and r is varying from 0 to a.

Date: / /

Page No.

$\therefore$ Area of the region R,

$$\therefore A = \int_0^{\pi/2} \int_0^{a} r \cdot dr \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{r^2}{2} \Big|_0^{a} \cdot d\theta$$

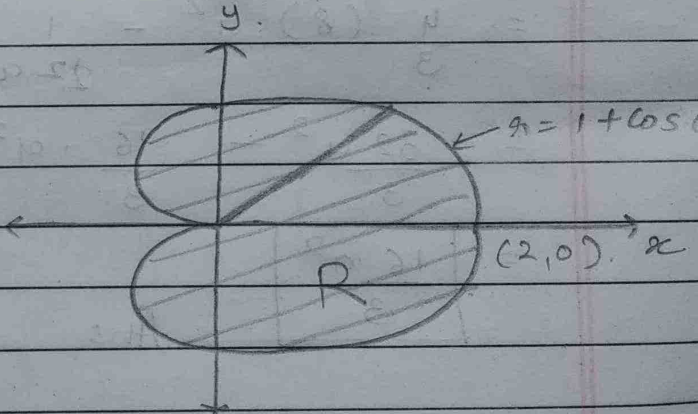
$$= \frac{1}{2} \int_0^{\pi/2} a^2 \cdot d\theta$$

$$= \frac{a^2}{2} \theta \Big|_0^{\pi/2}$$

$$= \boxed{\frac{\pi a^2}{4}}$$ Ans

Ex:-31 Find the area of a region enclosed by the cardioid $r = 1 + \cos\theta$.

→ From the figure,
θ is varying from 0 to π and
r is varying from 0 to $1 + \cos\theta$



$\therefore$ the Area of the region R is,

$$\therefore A = 2 \int_0^{\pi} \int_0^{1+\cos\theta} r \cdot dr \cdot d\theta$$

$$= 2 \int_0^{\pi} \frac{r^2}{2} \Big|_0^{1+\cos\theta} \cdot d\theta$$

$$\therefore A = \int_0^{\pi} (1+2\cos\theta + \cos^2\theta) \cdot d\theta$$

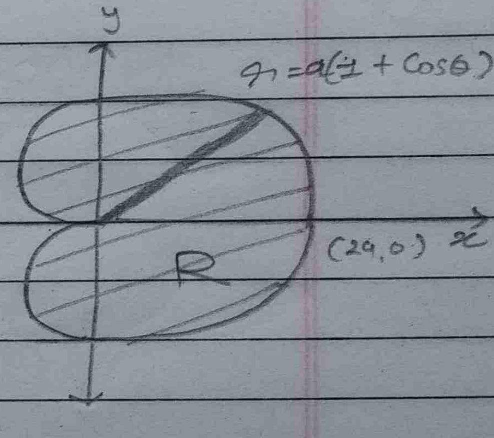
$$= \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) \cdot d\theta$$

$$= \left[\theta \Big|_0^{\pi} + 2\sin\theta \Big|_0^{\pi} + \frac{\theta}{2}\Big|_0^{\pi} + \frac{1}{2} \cdot \frac{\sin 2\theta}{2}\Big|_0^{\pi} \right]$$

$$= \pi + 0 + \frac{\pi}{2} + 0$$

$$= \boxed{\frac{3\pi}{2}} \text{ Ans.}$$

Extra H.W. Find the area of a region enclosed by the cardioid $r = a(1+\cos\theta)$



→ From the figure, θ is varying from 0 to π and r is varying from 0 to $a(1+\cos\theta)$

∴ the area of the region R is,

$$\therefore A = 2\int_0^{\pi} \int_0^{a(1+\cos\theta)} r \cdot dr \cdot d\theta$$

$$= 2\int_0^{\pi} \frac{r^2}{2}\Big|_0^{a(1+\cos\theta)} \cdot d\theta$$

$$= \int_0^{\pi} a^2 (1 + 2\cos\theta + \cos^2\theta) \cdot d\theta$$

$$= a^2 \int_0^{\pi} \left[1 + 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right] \cdot d\theta$$

Date: / /
Page No.

$\therefore A = a^2 \left[\theta \Big|_0^{\pi} + 2\sin\theta \Big|_0^{\pi} + \frac{\theta}{2} \Big|_0^{\pi} + \frac{1}{2} \frac{\sin 2\theta}{2} \Big|_0^{\pi} \right]$

$\therefore A = a^2 \left[\pi + 0 + \frac{\pi}{2} + 0 \right]$

$\therefore A = \boxed{\frac{3\pi a^2}{2}}$ Ans.

Ex:- Find the volume of the sphere $x^2+y^2+z^2=a^2$.

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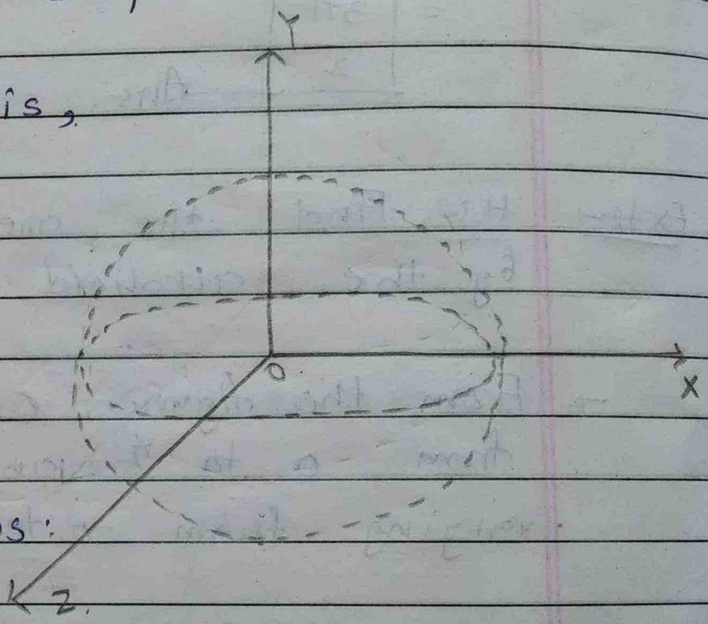
→ The required volume is,

$V = \iiint dv.$

→ From the figure we can see that the limits of variables are varying as follows:



$z = -\sqrt{a^2-x^2-y^2}$ to $z = \sqrt{a^2-x^2-y^2}$

$y = -\sqrt{a^2-x^2}$ to $y = \sqrt{a^2-x^2}$

$x = -a$ to $x = a$

→ $v = \iiint dv.$

$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1 \, dz \, dy \, dz.$

$f(-z) = 1 = f(z)$ $\therefore 1$ is even
$f(z) = 1$

Date: / /
Page No.

$$\therefore V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2 \int_{0}^{\sqrt{a^2-x^2-y^2}} 1 \cdot dz \cdot dy \cdot dx \quad (\because 1 \text{ is even})$$

$(\because f(x) = f(-x))$

$$\int_{-a}^{a} f(x) \cdot dx = 2\int_{0}^{a} f(x) \cdot dx.$$

$$\therefore V = 2 \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (\sqrt{a^2-x^2-y^2}) \cdot dy \cdot dx$$

$$\therefore V = 2\int_{-a}^{a} 2\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} \cdot dy \cdot dx$$

$\rightarrow$ We know that,

$$\int \sqrt{a^2-x^2} \cdot dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Let, $a^2 - x^2 = b^2$ then,

$$V = 4\int_{-a}^{a}\int_{0}^{b} \sqrt{b^2-y^2} \cdot dy \cdot dx$$

$$= 4\int_{-a}^{a} \left[\frac{y}{2}\sqrt{b^2-y^2} + \frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b \cdot dx.$$

$$= 4\int_{-a}^{a} \frac{b^2}{2}\frac{\pi}{2} \cdot dx$$

$$= \frac{4\pi}{4}\int_{-a}^{a} b^2 \cdot dx$$

$$= \pi \int_{-a}^{a} (a^2-x^2) \cdot dx$$

$$= \pi \, 2\int_{0}^{a} (a^2-x^2) \, dx$$

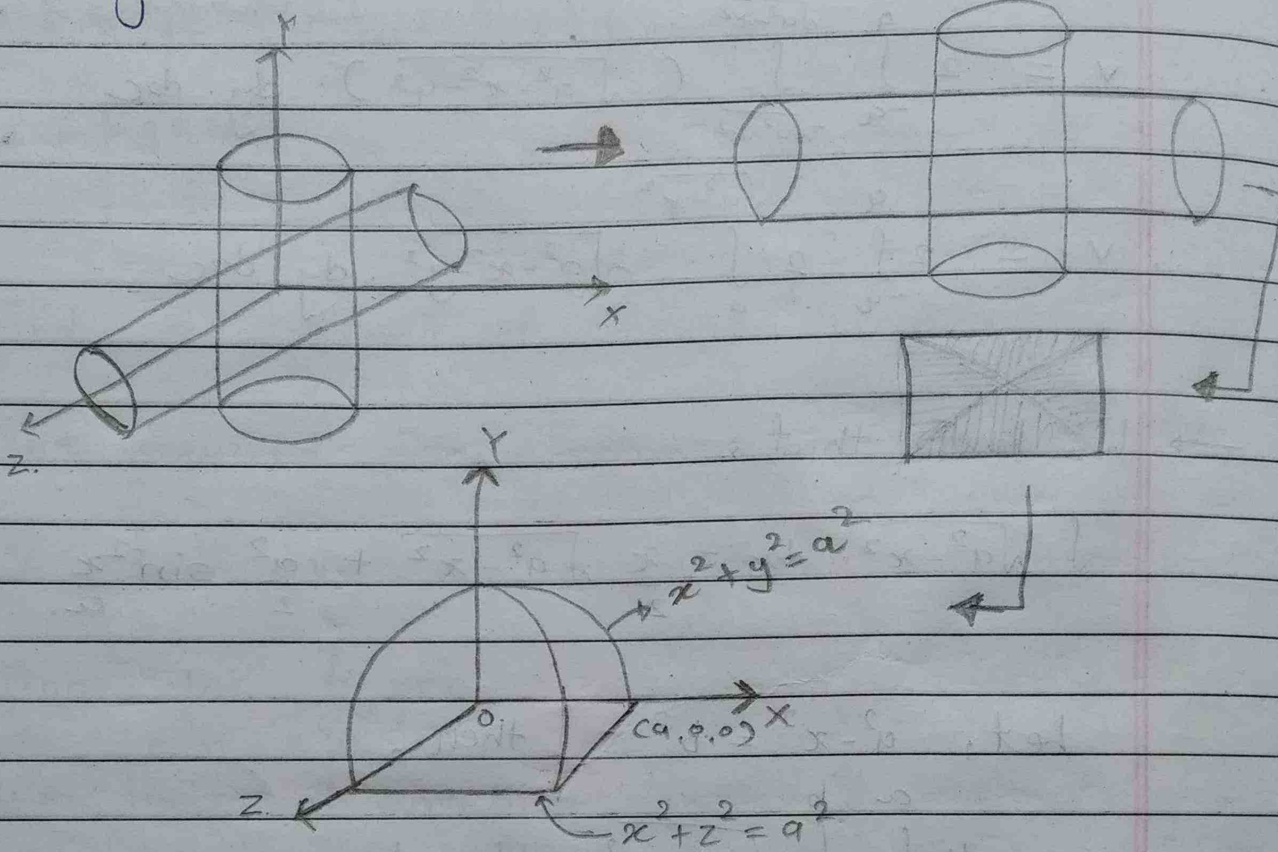
$$= 2\pi \left[a^2 \cdot x \Big|_0^a - \frac{x^3}{3}\Big|_0^a \right]$$

$$= 2\pi \left[a^3 - \frac{a^3}{3}\right]$$

$$= \boxed{\frac{4\pi a^3}{3}}$$ Ans.

Ex:- 32
L.15
27:08

Find the volume common to the cylinder $x^2+y^2=a^2$ and $x^2+z^2=a^2$.



$$\rightarrow \quad V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$\therefore \quad V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2 \int_{0}^{\sqrt{a^2-x^2}} dz\, dy\, .dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\, z \Big|_{0}^{\sqrt{a^2-x^2}} .dy\, dx$$

$$= 2\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, .dy\, dx$$

Date: / /
Page No.

$$\therefore \quad V = 4\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\cdot dy\,dx$$

$$= 4\int_{-a}^{a} \left(\sqrt{a^2-x^2}\right) y\Big|_{0}^{\sqrt{a^2-x^2}} \cdot dx$$

$$= 4\int_{-a}^{a} (a^2-x^2)\cdot dx$$

$$= 8\int_{0}^{a} (a^2-x^2)\cdot dx$$

$$= 8\left[a^2 x\Big|_{0}^{a} - \frac{x^3}{3}\Big|_{0}^{a}\right]$$

$$= 8\left[a^3 - \frac{a^3}{3}\right]$$

$$\therefore \quad \boxed{V = \frac{16a^3}{3}}$$ Ans.

Improper Integrals :-

→ The integrals of the form $\int_a^b f(x) \cdot dx$ is said to be an improper integral if,

i) One or both limits of integrations are infinite.

Ex:- $\int_a^{\infty} f(x) \cdot dx$, $\int_{-\infty}^{b} f(x) \cdot dx$, $\int_{-\infty} f(x) \cdot dx$

ii) function $f(x)$ becomes infinite at the end points of the interval or at a point within the interval of integrations.

Ex:- $\int_0^4 \frac{1}{3-x} \cdot dx$ Not defined at 3.

→ Though the interval length is finite, but integrand is unbounded and we will get infinite value.

→ Type : 1 :

function has to be continuous at $[a, \infty)$

1) $\int_a^{\infty} f(x) \cdot dx = \lim_{b \to \infty} \int_a^b f(x) \cdot dx$

2) $\int_{-\infty}^{b} f(x) \cdot dx = \lim_{a \to -\infty} \int_a^b f(x) \cdot dx$ $(-\infty, b]$

3) $\int_{-\infty}^{\infty} f(x) \cdot dx = \int_{-\infty}^{c} f(x) \cdot dx + \int_c^{\infty} f(x) \cdot dx$ $(-\infty, \infty)$

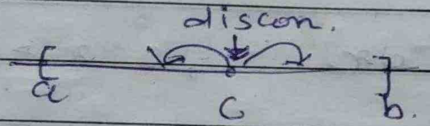
Type : 2 :

[discontinuous at a ($\because f(a) = \infty$)]

1) $\int_a^b f(x) \cdot dx = \lim_{\delta \to 0} \int_{a+\delta}^{b} f(x) \cdot dx$; $f(x) \in C(a, b]$, $\delta > 0$.

2) $\int_a^b f(x) \cdot dx = \lim_{\delta \to 0} \int_a^{b-\delta} f(x) \cdot dx$; $f(x) \in [a, b)$, $\delta > 0$.

3) $\int_a^b f(x) \cdot dx = \lim_{\delta \to 0} \int_a^{c-\delta} f(x) \cdot dx + \lim_{\delta \to 0} \int_{c+\delta}^{b} f(x) \cdot dx$



* Direct Comparison Test :-

→ If $f(x)$ and $g(x)$ are continuous on $[a, \infty]$ and $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then ...

if $\int_a^\infty g(x) \cdot dx$ converges $\Rightarrow \int_a^\infty f(x) \cdot dx$ converges.

if $\int_a^\infty g(x) \cdot dx$ diverges $\Rightarrow \int_a^\infty f(x) \cdot dx$ diverges.

* Limit Comparison Test :-

→ If $f(x)$ and $g(x)$ are positive and continuous on $[a, \infty)$ and

Date: / /
Page No.

if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$, $0 < l < \infty$
finite number

Then, $\int_a^\infty f(x) \cdot dx$ and $\int_a^\infty g(x) \cdot dx$ both

converges or diverges together.

* Gamma function :-

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \cdot dx \quad , \quad x > 0.$$

* Beta function :-

→ m, n any two positive real numbers.

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$$

* Properties :-

1) $\Gamma(1) = 1$

2) $n \in \mathbb{N}$, $\Gamma(n+1) = n! = n\Gamma(n)$

Ex:- $\Gamma(2) = 1$

$\Gamma(3) = 2$

$\Gamma(4) = 6$

Date: / /
Page No.

3) $\Gamma(1/2) = \sqrt{\pi}$

$2m-1 = p \Rightarrow m = \frac{p+1}{2}$

4) $B(m,n) = B(n,m)$

$2n-1 = q \Rightarrow n = \frac{q+1}{2}$

5) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$

6) $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

7) $2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$= \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

Ex:- 01 Prove that $\Gamma(1/2) = \sqrt{\pi}$.

→ We know that, $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$; $m,n>0$ — (1)

& $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$ — (2)

Now, take $m = n = 1/2$ in eq^n (2),

$\Rightarrow B(1/2, 1/2) = 2\int_0^{\pi/2} \sin^{2(1/2)-1}\theta \cdot \cos^{2(1/2)-1}\theta \cdot d\theta$

$\Rightarrow \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = 2\int_0^{\pi/2} d\theta$

Date: / /

Page No.

$$\Rightarrow \frac{[\Gamma(1/2)]^2}{\Gamma(1)} = 2\left[\pi/2 - 0\right].$$

$$\Rightarrow [\Gamma(1/2)]^2 = \pi \qquad (\because \Gamma(1) = 1)$$

$$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}.}$$

Extra :- IMP

$$\Gamma(n+1) = n\,\Gamma(n).$$

1) $\Gamma(3/2) = \Gamma(1/2 + 1) = 1/2\,\Gamma(1/2)$
$= \frac{1}{2}\sqrt{\pi}$

2) $\Gamma(5/2) = \Gamma(3/2 + 1) = \frac{3}{2}\Gamma(3/2)$
$= \frac{3}{2}\Gamma(1/2 + 1)$
$= \frac{3}{2}\frac{1}{2}\Gamma(1/2) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

3) $\Gamma(7/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

4) $\Gamma(9/2) = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$

5) $\Gamma(n+1) = n!$
$\Gamma(2021) = \Gamma(2020+1) = 2020!$

Date: / /

Page No.

Ex: Evaluate : $\int_0^{\pi/2} \sqrt{\tan x} \cdot dx$.

→ We know that, for $p, q > 0$.

$$2\int_0^{\pi/2} \sin^p x \cdot \cos^q x \cdot dx = B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\Rightarrow \int_0^{\pi/2} \sin^p x \cdot \cos^q x \cdot dx = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad (1)$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{(2)\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

→ Let, $I = \int_0^{\pi/2} \sqrt{\tan x} \cdot dx = \int_0^{\pi/2} \sin^{1/2} x \cdot \cos^{-1/2} x \cdot dx$.

Now, with the use of eq^n (1), by taking

$p = \frac{1}{2}$ & $q = -\frac{1}{2}$,

we get,

$$I = \frac{1}{2} B\left(\frac{1/2+1}{2}, \frac{-1/2+1}{2}\right)$$

$$= \frac{1}{2} B\left(3/4, 1/4\right)$$

Ex. 2 Evaluate :- $\int_0^{\infty} x^7 \, 7^{-x} \cdot dx$.

→ Let, $I = \int_0^{\infty} x^7 \cdot 7^{-x} \cdot dx$

Date: / /

Page No.

$\therefore \quad I = \int_0^\infty x^7 \cdot e^{-x \cdot \log 7} \cdot dx$

now, let $x \cdot \log 7 = t$

$\therefore \; x = \frac{t}{\log 7} \qquad \therefore \; dx = \frac{1}{\log 7} \cdot dt$

and $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\therefore \quad I = \int_0^\infty \left(\frac{t}{\log 7}\right)^7 \cdot e^{-t} \cdot \frac{1}{\log 7} \cdot dt$

$= \frac{1}{(\log 7)^8} \int_0^\infty t^7 \cdot e^{-t} \cdot dt$

$= \frac{1}{(\log 7)^8} \int_0^\infty t^{7+1-1} \, e^{-t} \cdot dt$

$= \frac{1}{(\log 7)^8} \, \Gamma(8)$

$\therefore \quad I = \frac{7!}{(\log 7)^8}$

Ex. 3 Evaluate : $\int_0^1 \sqrt{x} \cdot (1-x^2)^{1/3} \cdot dx$

$\rightarrow$ Let, $I = \int_0^1 x^{1/2} \cdot (1-x^2)^{1/3} \cdot dx$

Date: / /

Page No.

Now, let $x^2 = t$

$\therefore \quad 2x \cdot dx = dt$

$\therefore \quad dx = \frac{1}{2x} \cdot dt = \frac{1}{2\sqrt{t}} \cdot dt$

and $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$$\therefore \quad I = \int_0^1 t^{1/4} \cdot (1-t)^{1/3} \cdot \frac{1}{2t^{1/2}} \cdot dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{1}{2}} (1-t)^{\frac{1}{3}} \cdot dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{1}{2}+1-1} (1-t)^{\frac{1}{3}+1-1} \cdot dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{3}{2}-1} \cdot (1-t)^{\frac{4}{3}-1} \cdot dt$$

$$= \frac{1}{2} B\left(3/2, 4/3\right)$$

Evaluate : $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$

$\rightarrow$ Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$

$= \int_0^1 (1-x^n)^{-1/2} \cdot dx$

Now, let $x^n = t \Rightarrow x = t^{\frac{1}{n}}$

$\therefore n \cdot x^{n-1} \cdot dx = dt$

$\therefore dx = \frac{1}{n} \cdot \frac{1}{x^{n-1}} \cdot dt$

$= \frac{1}{n} x^{1-n} \cdot dt$

$= \frac{1}{n} (t^{\frac{1}{n}})^{1-n} \cdot dt$

$\therefore dx = \frac{1}{n} \cdot t^{(\frac{1}{n}-1)} \cdot dt$

and $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$\rightarrow$ $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$

Date: / /

Page No.

$$\therefore I = \int_0^1 (1-t)^{-\frac{1}{2}} \cdot \frac{1}{n} t^{\frac{1}{n}-1} \cdot dt.$$

$$= \frac{1}{n} \int_0^1 t^{1/n - 1} \cdot (1-t)^{-\frac{1}{2}+1-1}$$

$$= \frac{1}{n} \cdot B(1/n, 1/2).$$

$$= \frac{1}{n} \frac{\Gamma(1/n) \cdot \Gamma(1/2)}{\Gamma(1/n + 1/2)}$$

$$\therefore I = \frac{1}{n} \frac{\Gamma(1/n) \sqrt{\pi}}{\Gamma(1/n + 1/2)}.$$

Ex: 8 Evaluate : $\int_0^1 x^4 \left(\log\left(\frac{1}{x}\right)\right)^3 \cdot dx$

$\rightarrow$ Let $I = \int_0^1 x^4 \left[\log \frac{1}{x}\right]^3 \cdot dx.$

let $\log \frac{1}{x} = t \Rightarrow \frac{1}{x} = e^t$

$\Rightarrow x = \frac{1}{e^t}$

$\Rightarrow dx = -e^{-t} \cdot dt.$

As, $x \to 0 \Rightarrow t = \infty.$

$x \to 1 \Rightarrow t = 0.$

$$\therefore I = \int_\infty^0 (e^{-t})^4 \cdot t^3 \cdot (-e^{-t}) \cdot dt.$$

$$\Gamma(n) = \int_0^\infty x^{n-1} \cdot e^{-x} \cdot dx$$

Date: / /

Page No.

$$\therefore \quad I = \int_0^\infty e^{-5t} \cdot t^3 \cdot dt.$$

let, $5t = y$

$\therefore 5 \cdot dt = dy.$

& $t \to 0 \Rightarrow y \to 0.$

$t \to \infty \Rightarrow y \to \infty.$

$$\therefore \quad I = \int_0^\infty e^{-y} \cdot \left(\frac{y}{5}\right)^3 \cdot \frac{dy}{5.}$$

$$= \int_0^\infty \frac{1}{(5)^4} \cdot e^{-y} \cdot y^{3+1-1} dy.$$

$$= \frac{1}{(5)^4} \int_0^\infty y^{4-1} \cdot e^{-y} \cdot dy$$

$$= \frac{1}{(5)^4} \Gamma(4)$$

$$= \frac{1}{625} \, 3!$$

$$= \boxed{\frac{6}{625}} \text{ Ans.}$$

Ex:- 5 Prove that for $n > 0$, $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} \cdot dx$

→ Given, $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} \cdot dx.$

Date: / /
Page No.

→ R.H.S. $= \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} . dx$

let, $\log \frac{1}{x} = t \Rightarrow \frac{1}{x} = e^t \Rightarrow x = e^{-t}$

$\Rightarrow dx = -e^{-t} . dt.$

and $x = 0 \Rightarrow t = \infty$
$x = 1 \Rightarrow t = 0.$

∴ R.H.S. $= \int_\infty^0 t^{n-1} . (-e^{-t}) . dt.$

$= \int_0^\infty t^{n-1} . e^{-t} . dt.$

$= \Gamma(n)$

$=$ L.H.S.

Ex: 7 Evaluate : $\int_0^\infty a^{-bx^2} . dx$

→ Let $I = \int_0^\infty a^{-bx^2} . dx$

let $bx^2 = t$

$\Rightarrow x^2 = \frac{t}{b} \Rightarrow x = \sqrt{\frac{t}{b}}$

$\Rightarrow dx = \frac{1}{\sqrt{b}} \frac{1}{2\sqrt{t}} . dt.$

and $x \to 0 \Rightarrow t \to 0$
$x \to \infty \Rightarrow t \to \infty.$

Date: / /
Page No.

$$\therefore I = \int_0^\infty a^{-t} \cdot \frac{1}{\sqrt{b}} \frac{1}{2\sqrt{t}} \cdot dt$$

$$\therefore I = \frac{1}{2\sqrt{b}} \int_0^\infty e^{-t \cdot \log a} \cdot t^{-\frac{1}{2}} \cdot dt$$

let, $t \cdot \log a = y$

$$\Rightarrow t = \frac{y}{\log a}$$

$$\Rightarrow dt = \frac{1}{\log a} \cdot dy$$

and $t \to 0 \Rightarrow y \to 0$

$t \to \infty \Rightarrow y \to \infty$.

$$\therefore I = \frac{1}{2\sqrt{b}} \int_0^\infty e^{-y} \cdot \left(\frac{y}{\log a}\right)^{-\frac{1}{2}} \cdot \frac{dy}{\log a}$$

$$= \frac{1}{2\sqrt{b} \cdot} \frac{1}{(\log a)^{1/2}} \int_0^\infty e^{-y} \cdot y^{-\frac{1}{2}} \cdot dy$$

$$= \frac{1}{2\sqrt{b \cdot \log a}} \int_0^\infty y^{-\frac{1}{2}+1-1} \cdot e^{-y} \cdot dy$$

$$= \frac{1}{2\sqrt{b \cdot \log a}} \Gamma(1/2)$$

$$\therefore I = \frac{\sqrt{\pi}}{2\sqrt{b \cdot \log a}}$$

Date: / /
Page No.

Ex :- 6 Evaluate $\int_0^\infty \frac{x^c}{c^x} \cdot dx$

→ Let $I = \int_0^\infty \frac{x^c}{c^x} \cdot dx$

$= \int_0^\infty \frac{x^c}{e^{x \cdot \log c}} \cdot dx$

$\therefore I = \int_0^\infty x^c \cdot e^{-x \cdot \log c} \cdot dx$

Now, let, $x \cdot \log c = t$

$\Rightarrow x = \frac{t}{\log c}$

$\Rightarrow dx = \frac{dt}{\log c}$

and $x \to 0 \Rightarrow t \to 0$

$x \to \infty \Rightarrow t \to \infty$.

$\therefore I = \int_0^\infty \left(\frac{t}{\log c}\right)^c \cdot e^{-t} \cdot \frac{dt}{\log c}$

$= \frac{1}{(\log c)^{c+1}} \int_0^\infty t^c \cdot e^{-t} \cdot dt$

$= \frac{1}{(\log c)^{c+1}} \int_0^\infty t^{c+1-1} \cdot e^{-t} \cdot dt$

$= \frac{1}{(\log c)^{c+1}} \Gamma(c+1)$

___ Ans.

Date: / /
Page No.

* Error Function :-

→ $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \cdot dt$$

→ Complimentary error function

$$\operatorname{erf}_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \cdot dt$$

→ Properties :-

1) $\operatorname{erf}(0) = 0$
2) $\operatorname{erf}(\infty) = 1$
3) $\operatorname{erf}(x) + \operatorname{erf}_c(x) = 1$
4) $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ → error function is an odd function.

Ex:- 10 Show that $\int_a^\infty e^{-(2x-a)^2} \cdot dx = \frac{\sqrt{\pi}}{4}(1 - \operatorname{erf}(a))$

→ $$\text{L.H.S.} = \int_a^\infty e^{-(2x-a)^2} \cdot dx$$

let, $2x - a = t$

$\therefore 2 \cdot dx = dt$

and $x \to a \Rightarrow t \to a$

$x \to \infty \Rightarrow t \to \infty$.

Date: / /

Page No.

$\therefore$ L.H.S. $= \int_a^\infty e^{-t^2} \cdot \frac{dt}{2}$

$= \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-t^2} \cdot \frac{dt}{2}$

$= \frac{\sqrt{\pi}}{4} \cdot \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-t^2} \cdot dt$

$= \frac{\sqrt{\pi}}{4} \cdot erf_c(a)$

$= \frac{\sqrt{\pi}}{4} \left[1 - erf(a) \right]$

$=$ R.H.S.

Date: / /
Page No.

Ex: 11 Prove that, $\operatorname{erf}(x) = \alpha(x\sqrt{2})$,

where $\alpha(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} \cdot dt$.

→ Here, we are given that,

$$\alpha(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} \cdot dt \quad \text{—— (1)}.$$

Now, To prove $\operatorname{erf}(x) = \alpha(x\sqrt{2})$

$$\Rightarrow \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \cdot dt = \alpha(x\sqrt{2})$$

→ From (1),

$$\alpha(x\sqrt{2}) = \sqrt{\frac{2}{\pi}} \int_0^{x\sqrt{2}} e^{-t^2/2} \cdot dt$$

let, $\frac{t^2}{2} = y^2$

$\Rightarrow t^2 = 2y^2 \quad t = \sqrt{2}\,y$

$\Rightarrow \cancel{2}t \cdot dt = \cancel{2} \cdot 2y \cdot dy$

$\Rightarrow dt = \frac{2y}{\sqrt{2}\,y} \cdot dy$

$\Rightarrow dt = \sqrt{2} \cdot dy$.

and $t \to 0 \Rightarrow y \to 0$

$t \to x\sqrt{2} \Rightarrow y \to x$

Date: / /
Page No.

$$\therefore \alpha(x\sqrt{2}) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-u^2} \cdot \sqrt{2} \cdot du.$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \cdot du.$$

$$= \mathrm{erf}(x).$$

Ex: 9 Show that $\int_{-a}^{a} e^{-(x+a)^2} \cdot dx = \frac{\sqrt{\pi}}{2} \mathrm{erf}(2a).$

→ Here, we are given,

$$\int_{-a}^{a} e^{-(x+a)^2} \cdot dx = \frac{\sqrt{\pi}}{2} \mathrm{erf}(2a).$$

→ $\text{L.H.S.} = \int_{-a}^{a} e^{-(x+a)^2} \cdot dx$

let, $x + a = t$

$\therefore dx = dt.$

and $x \to -a \Rightarrow t \to 0.$

$x \to a \Rightarrow t \to 2a.$

$$\therefore \text{L.H.S.} = \int_0^{2a} e^{-t^2} \cdot dt.$$

$$= \frac{\sqrt{\pi}}{2} \frac{2}{\sqrt{\pi}} \int_0^{2a} e^{-t^2} \cdot dt$$

$$\therefore \text{L.H.S.} = \frac{\sqrt{\pi}}{2} \mathrm{erf}(2a)$$