Unit : 2 Higher Order Ordinary Linear Differential Equations

Date ______
Page ______

* 2.1 and 2.2 :-

Ex:- (2) Solve $\frac{d^3y}{dx^3} - 4 \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$. —— (1)

→ The operator form of eq^n (1) is given by,

$$(D^3 - 4D^2 + D + 6)y = 0.$$

$R(x) = 0.$
⇓
homogeneous

∴ Auxiliary equation of (1) is,

$$m^3 - 4m^2 + m + 6 = 0$$

∴ $(m+1)(m^2 - 5m + 6) = 0.$

∴ $(m+1)(m-2)(m-3) = 0$

∴ $m = -1, 2, 3$

odd power coefficient = even power coefficient
⇒ -1 is one of the roots.
⇒ $(m+1)(\text{- - - -})$

	1	-4	1	6
-1	0	-1	5	-6
	1	-5	6	0.

→ Since, all roots are distinct and real,
General sol^n of (1) is,

$$y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x},$$

c_i = arbitrary constant
$i = 1$ to 3

Ex:- (4) Solve $y'' + y' - 2y = 0$, $y(0) = 4$ and $y'(0) = -5$

Initial Value Problem

→ Here, we are given $y'' + y' - 2y = 0$ —— (1)
and $y(0) = 4$, $y'(0) = -5$.

→ The operator form of eq^n (1) is given by,

$$(D^2 + D - 2)y = 0$$

$\therefore$ Auxiliary eq^n of (1) is,

$m^2 + m - 2 = 0$

$\therefore (m-1)(m+2) = 0$

$\therefore m = 1, -2.$

$\rightarrow$ Since, all roots are distinct and real,
General sol^n is,

$$y(x) = c_1 \cdot e^{x} + c_2 e^{-2x} \quad \text{——— (2)}$$

c_i = arbitrary constants
$i = 1, 2$

$$\therefore y'(x) = c_1 \cdot e^{x} - 2 \cdot c_2 e^{-2x} \quad \text{——— (3)}$$

$\rightarrow$ Given, $y(0) = 4$ and $y'(0) = -5$.
$\therefore$ substituting these values in eq^n. (2) & (3),
we get,

$4 = y(0) = c_1 + c_2 \Rightarrow c_1 + c_2 = 4$ and
$-5 = y'(0) = c_1 - 2c_2 \Rightarrow c_1 - 2c_2 = -5$ (*)

$\quad - \quad + \quad\quad +$

$3c_2 = 9$

$\therefore \boxed{c_2 = 3}$

from (*) $\boxed{c_1 = 1}$

$\therefore$ Particular sol^n is $\boxed{y(x) = e^{x} + 3 \cdot e^{-2x}}$ Ans

Ex:- (1) Solve $\frac{d^2y}{dx^2} - 5 \cdot \frac{dy}{dx} + 6y = 0$, $y(1) = e^2$, $y'(1) = 3e^2$

$\rightarrow$ Here, we are given,

$$\frac{d^2y}{dx^2} - 5 \cdot \frac{dy}{dx} + 6y = 0. \quad \text{——— (1)}$$

Date
Page

→ The operator form of eq^n ① is given by,

$$(D^2 - 5D + 6)y = 0.$$

∴ Auxiliary eq^n of ① is,

$$m^2 - 5m + 6 = 0$$

$$\therefore (m-3)(m-2) = 0.$$

$$\therefore m = 2, 3.$$

→ Since, all roots are distinct and real,
General sol^n is,

$$y(x) = c_1 e^{2x} + c_2 \cdot e^{3x} \quad \text{—— ②}$$

c_1, c_2 = arbitrary constants

$$\therefore y'(x) = c_1 \cdot 2e^{2x} + c_2 \cdot 3e^{3x} \quad \text{—— ③}$$

→ Given $y(1) = e^2$, $y'(1) = 3e^2$
∴ substituting these values in eq^n ② & ③,
we get,

$$e^2 = c_1 \cdot e^2 + c_2 \cdot e^3 \Rightarrow 2 = 2c_1 + 2c_2 \cdot e \quad \text{—— (*)}$$

$$3e^2 = c_1 \cdot 2e^2 + c_2 \cdot 3e^3 \Rightarrow 3 = c_1 \cdot 2 + c_2 \cdot 3e$$

$$- \quad - \quad -$$

$$-1 = 0 - c_2 e$$

$$\therefore \boxed{c_2 = \frac{1}{e}}$$

∴ from (*) $\quad 2 = 2 \cdot c_1 + 2 \cdot \frac{1}{e} \cdot e$

$$\therefore \boxed{c_1 = 0}$$

∴ Particular sol^n is, $y(x) = 0 \cdot e^{2x} + \frac{1}{e} \cdot e^{3x}$

$$\therefore \boxed{y(x) = e^{3x-1}}$$

Ans.

★ $p \pm q i$, $\pm 2i$ } always consider (2), not (-2)

Date ______ Page ______

Ex:-03 Solve $(D^4 - 16)y = 0$.

→ Here, we are given the operator form,

$$(D^4 - 16)y = 0. \quad \text{——— (1)}$$

∴ A E of (1) is,

$$m^4 - 16 = 0.$$

$$\therefore m^4 = 16$$

$$\therefore m^2 = \pm 4 \Rightarrow +4, -4$$

$$\therefore m = \pm 2, \pm 2i$$

→ $p \pm qi$

→ soln: $e^{px}(c_1 \cos qx + c_2 \sin qx)$

∴ General soln is given by,

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + e^{0x}(c_3 \cos 2x + c_4 \sin 2x)$$

(only positive)

$$\therefore y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cdot \cos 2x + c_4 \sin 2x$$

P: (1). Solve the initial value problem

$$y'' - 5y' + 6y = 0, \quad y(1) = e^2, \quad y'(1) = 3e^2$$

→ Here, we are given,

$$y'' - 5y' + 6y = 0$$

$$\therefore \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0. \quad \text{——— (1)}$$

→ Same as example : (1).

P: (2). Solve the initial value problem,

$$y''' - y'' + 100y' - 100y = 0, \quad y(0) = 4, \; y'(0) = 11$$

$$y''(0) = -299$$

→ Here, we are given,

$$y''' - y'' + 100y' - 100y = 0.$$

$$\therefore \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 100 \cdot \frac{dy}{dx} - 100 \cdot y = 0. \quad \text{--- (1)}$$

→ The operator form of eq^n (1) is,

$$(D^3 - D^2 + 100D - 100)\, y = 0$$

∴ A.E. of (1) is,

$$m^3 - m^2 + 100m - 100 = 0.$$

1	1	-1	100	-100
	0	1	0	100
	1	0	100	0.

$\therefore (m-1)(m^2 + 0m + 100) = 0.$

$\therefore (m-1)(m^2+100) = 0.$

$\therefore m = 1, \quad m^2 = -100$

$\therefore m = 1 \quad \therefore m = 10i$

$\to e^{px}(c_1 \cos qx + c_2 \sin qx)$

∴ General sol^n is given by,

$$y(x) = c_1 e^x + c_2 \cos 10x + c_3 \sin 10x. \quad \text{--- i}$$

$$\therefore y'(x) = c_1 \cdot e^x + c_2 \cdot (-\sin 10x \cdot 10) + c_3 \cos 10x \cdot 10.$$

$$\therefore y'(x) = c_1 e^x - 10c_2 \sin 10x + 10c_3 \cos 10x. \quad \text{--- ii}$$

$$\therefore y''(x) = c_1 \cdot e^x - 100c_2 \cos 10x - 100c_3 \cdot \sin 10x \quad \text{--- iii}$$

→ Given, $y(0) = 4$, $y'(0) = 11$, $y'''(0) = -299$

$\therefore y(0) = c_1 + c_2$

$\therefore c_1 + c_2 = 4$ --- (☆)

$y'(0) = c_1 + 10c_3$

$\therefore c_1 + 10c_3 = 11$ --- (☆☆)

and $y'''(0) = c_1 - 100c_2$

$$\therefore \cancel{c_1} - 100c_2 = -299. \quad \text{--- (☆☆☆)}$$

$$\therefore \underline{-\cancel{c_1} \quad - \quad c_2 \quad = \quad - 4} \quad (\because eq^n \text{ (☆)})$$

$$-101\, c_2 = -303$$

$$\therefore \boxed{c_2 = 3}$$

from eq^n (☆) $\boxed{c_1 = 1}$

from eq^n (☆☆) $c_3 = \frac{11-1}{10} = \boxed{1 = c_3}$

Date
Page

→ Now, substituting these values in eqⁿ (i), we get,

$$y(x) = e^{x} + 3\cos 10x + \sin 10x$$

Ans.

P: (8). Solve the diff. eqⁿ. $\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$

→ Here, we are given,

$$\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0. \quad \text{—(1)}$$

→ The operator form of eqⁿ (1) is,

$$(D^5 - D^3)y = 0.$$

∴ The A.E. of (1) is, $m^5 - m^3 = 0.$

∴ $m^3(m^2-1) = 0$

∴ $m^3 = 0$, $m^2 = 1$

∴ $m = 0.$, ∴ $m = \pm 1$

→ General solⁿ is given by,

$$y(x) = c_1 e^{0x} + c_2 \cdot e^{-x} + c_3 e^{x}$$

∴ $y(x) = c_1 + c_2 e^{-x} + c_3 e^{x}$

P: (10). Solve the differential eqⁿ. $\frac{d^4y}{dx^4} + 32 \cdot \frac{d^2y}{dx^2} + 256y = 0.$

→ Here, we are given, $\frac{d^4y}{dx^4} + 32\frac{d^2y}{dx^2} + 256y = 0$ —(1)

$\therefore$ The operator form of eqⁿ (1) is,

$$(D^4 + 32D^2 + 256)y = 0.$$

$\therefore$ A.E. of (1) is,

$$m^4 + 32m^2 + 256 = 0.$$

$$\therefore m^4 + 16m^2 + 16m^2 + 256 = 0.$$

$$\therefore m^2(m^2+16) + 16(m^2+16) = 0.$$

$$\therefore (m^2+16)(m^2+16) = 0.$$

$\hookrightarrow m^2+16=0.$

$m^2 = -16 \Rightarrow \boxed{m = \pm 4i}$

$$\therefore m^2 + 16 = 0.$$

$$\therefore m^2 = -16. \Rightarrow \boxed{m = \pm 4i}$$

$\hookrightarrow e^{px}(c_1 \cos qi + c_2 \sin qi)$

$\therefore$ General solⁿ is given by,

~~y(x) =~~

$$y(x) = e^{0x}[(c_1 + c_2 x)\cdot \cos 4x + (c_3 + c_4 x)\cdot \sin 4x]$$

$$\therefore y(x) = (c_1 + c_2 x)\cos 4x + (c_3 + c_4 x)\cdot \sin 4x$$

Ans

* Linear Dependent And Linear Independent :- functions

Ex:- Check whether the following functions are linearly dependent or linearly independent.

(1) $\cosh x, e^x, e^{-x}$

$$W = \begin{vmatrix} \cosh x & e^x & e^{-x} \\ \sinh x & e^x & -e^{-x} \\ \cosh x & e^x & e^{-x} \end{vmatrix}$$

$$= 0 \qquad (\because R_1 = R_3).$$

(2). $e^x \cdot \cos x,\ e^x \sin x,\ e^x.$

$\rightarrow$ $$W = \begin{vmatrix} e^x \cdot \cos x & e^x \sin x & e^x \\ e^x(-\sin x) + \cos x \cdot e^x & e^x \cdot \cos x + \sin x \cdot e^x & e^x \\ -(e^x \cos x + e^x \sin x) + e^x(\cos x - \sin x) & e^x(\cos x - \sin x) + e^x(\cos x + \sin x). & e^x \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cdot \cos x & e^x \cdot \sin x & e^x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) & e^x \\ e^x(2\cos x) & e^x(2\cos x). & e^x. \end{vmatrix}$$

$$= e^{3x} \begin{vmatrix} \cos x & \sin x & 1 \\ \cos x - \sin x & \cos x + \sin x & 1 \\ 2\cos x & 2\cos x & 1 \end{vmatrix}$$

$$= e^{3x} \begin{vmatrix} -\cos x & \sin x - 2\cos x & 0 \\ -\cos x - \sin x & \sin x - \cos x & 0 \\ 2\cos x & 2\cos x & 1 \end{vmatrix} \quad (\because R_1 \rightarrow R_1 - R_3,\ R_2 \rightarrow R_2 - R_3)$$

$$= e^{3x}\left[1\left((\sin x - \cos x)(-\cos x)\right) - \left((-\cos x - \sin x)(\sin x - 2\cos x)\right)\right].$$

$$= e^{3x}\left[\cancel{-\sin x \cos x} + \cos^2 x + \cancel{\sin x \cos x} - 2\cos^2 x + \sin^2 x - 2\sin x \cdot \cos x\right].$$

$$= e^{3x}\,(1)$$

$$= e^{3x} \neq 0.\ \forall x$$

$\rightarrow$ Hence, $\{e^x \cdot \sin x,\ e^x \cos x,\ e^x\}$ are linearly independent.

Date
Page

* 2.3 : Non-homogeneous linear diff. equation :-

$$f(D)\,y = R(x)$$

$R(x) \neq 0$.

$y = y_c + y_p$ → Variation of parameters / → Undetermined coefficient

y_c: Complementary function (homogeneous)

y_p: particular Integral

Ex:- (2nd order) Solve $(D^2+1)y = \operatorname{cosec} x$ using the method of variation of parameters.

→ Given, $(D^2+1)y = \operatorname{cosec} x$ —— (1).

$\therefore R(x) = \operatorname{cosec} x$.

∴ The auxiliary equation of eqⁿ (1) is,

$m^2+1 = 0$.

$\Rightarrow m^2 = -1$. $p \pm qi$ { imaginary & distinct: $e^{px}[c_1 \cos qx + c_2 \sin qx]$ }

$\Rightarrow m^2 = \pm i$

∴ C.F. $y_c = c_1 \cos x + c_2 \sin x$ —— (2)

c_1, c_2 = arbitrary constants

→ Let $y_1(x) = \cos x$ and $y_2(x) = \sin x$

$\therefore W(y_1, y_2)$ OR $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= 1 \neq 0$.

∴ linearly independent

→ C.F. involves arbitrary constants.

→ P.I. never involves arbitrary constants.
(Particular integral)

→ $W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x$

→ $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$

∴ By method of variation of parameters,

$$y_p(x) = y_1(x) \cdot u_1 + y_2(x) \cdot u_2$$

$$= \cos x \cdot \int \frac{R(x) \cdot W_1}{W} \cdot dx + \sin x \cdot \int \frac{R(x) \cdot W_2}{W} dx$$

$$= \cos x \cdot \int \frac{\operatorname{cosec} x \cdot (-\sin x)}{1} \cdot dx + \sin x \int \frac{\operatorname{cosec} x \cdot \cos x}{1} \cdot dx$$

$$= \cos x (-1) \int \frac{\sin x}{\sin x} \cdot dx + \sin x \int \frac{\cos x}{\sin x} \cdot dx$$ છેદનું વિકલન અંશમાં

$$= -\cos x \cdot (x) + \sin x \cdot \ln(\sin x) \quad \text{———} \quad (3)$$

→ From, (2) & (3) the general solⁿ of (1) is,

$$y(x) = y_c + y_p$$

$$\therefore y(x) = c_1 \cos x + c_2 \sin x - x \cdot \cos x + \sin x \cdot \ln(\sin x)$$

Ans

Ex:- Solve $\frac{d^3y}{dx^3} + \frac{dy}{dx} = \operatorname{cosec} x$ using the method of variation of parameters.

→ Given, $\frac{d^3y}{dx^3} + \frac{dy}{dx} = \operatorname{cosec} x$ ——— (1)

$R(x) = \operatorname{cosec} x$

The operator form of eq^n (1) is,

$(D^3 + D)y = \operatorname{cosec} x$

∴ Auxiliary eq^u of (1) is,

$m^3 + m = 0.$

∴ $m(m^2+1) = 0.$

∴ $m = 0$, $m = \pm i$

∴ C.F. $y_c = c_1 e^{0x} + e^{0x}[c_2 \cos x + c_3 \sin x]$.

∴ $y_c = c_1 + c_2 \cos x + c_3 \sin x$ ——— (2)

where c_1, c_2, c_3 = arbitrary constants

∴ Let $y_1(x) = 1$, $y_2(x) = \cos x$, $y_3(x) = \sin x$.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$= 1[\sin^2 x + \cos^2 x] = 1 \neq 0.$

∴ linearly independent.

→ $W_1 = \begin{vmatrix} \boxed{0} & \cos x & \sin x \\ \boxed{0} & -\sin x & \cos x \\ \boxed{1} & -\cos x & -\sin x \end{vmatrix} = 1[\cos^2 x + \sin^2 x]$

$= 1.$

$\rightarrow W_2 = \begin{vmatrix} 1 & \boxed{0} & \sin x \\ 0 & 0 & \cos x \\ 0 & 1 & -\sin x \end{vmatrix} = 1[-\cos x]$

$= -\cos x$

$\rightarrow W_3 = \begin{vmatrix} 1 & \cos x & \cancel{\sin x}\ \boxed{0} \\ 0 & -\sin x & \cancel{\cos x}\ 0 \\ 0 & -\cos x & \cancel{\sin x}\ 1 \end{vmatrix}$ $\cancel{= 1[\sin^2 x + \cos^2 x]}$

$= 1.(-\sin x)$

$= -\sin x$

$\therefore$ By method of variation of parameter,

$$y_p = y_1(x) \cdot u_1 + y_2(x) \cdot u_2 + y_3(x) \cdot u_3.$$

$$\therefore y_p = y_1(x) \cdot \int \frac{R(x) \cdot W_1}{W} \cdot dx + y_2(x) \int \frac{R(x) \cdot W_2}{W} \cdot dx + y_3(x) \int \frac{R(x) \cdot W_3}{W} \cdot dx$$

$$\therefore y_p = 1 \int \frac{\operatorname{cosec} x \cdot (1)}{1} \cdot dx + \cos x \int \frac{\operatorname{cosec} x \cdot (-\cos x)}{1} \cdot dx + \sin x \int \frac{\operatorname{cosec} x \ (-\sin x)}{1} \cdot dx$$

$$\therefore y_p = \int \operatorname{cosec} x \cdot dx - \cos x \int \frac{\cos x}{\sin x} \cdot dx - \sin x \int 1 \, dx$$

$$\therefore y_p = \ln(\operatorname{cosec} x - \cot x) - \cos x \cdot \ln(\sin x) + \sin x \cdot x \quad \text{--- (3)}.$$

$\therefore$ From eqn (2) & (3), g.s. of (1) is,

$$y(x) = y_c + y_p.$$

$\therefore$ $y(x) = c_1 + c_2 \cdot \cos x + c_3 \cdot \sin x + \ln|\operatorname{cosec} x - \cot x|$
$- \cos x \cdot \ln(\sin x) + x \cdot \sin x$

Ex:-5 Apply the method of variation of parameters to solve $(D^2 - 2D)y = e^x \cdot \sin x$

$\rightarrow$ Given, $(D^2 - 2D)y = e^x \cdot \sin x$ —— (1)

where, $R(x) = e^x \cdot \sin x$

$\rightarrow$ The operator form is eq^n (1).

$\therefore$ the A.E. is,

$$m^2 - 2m = 0.$$

$$\therefore m(m-2) = 0.$$

$$\therefore m = 0, \quad m = 2$$

$\therefore$ C.F. $= y_c = c_1 \cdot e^{0x} + c_2 \cdot e^{2x}$
$= c_1 + c_2 \cdot e^{2x}$ —— (2).

c_1, c_2 = arbitrary constants

$\therefore$ let, $y_1(x) = 1$, $y_2(x) = e^{2x}$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2 \cdot e^{2x} \end{vmatrix} = 2 \cdot e^{2x} \neq 0.$$

$\therefore$ linearly independent

$$\therefore W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{2x} \\ 1 & 2 \cdot e^{2x} \end{vmatrix} = -e^{2x}$$

$$\therefore W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\rightarrow$ $\int e^{ax} \cdot \sin(bx+k) \cdot dx$

$= \dfrac{e^{ax}}{a^2+b^2} \left[\underline{a} \cdot \sin(bx+k) - \underline{b} \cos(bx+k) \right]$

$\uparrow \quad (a, b \neq 0)$

$\therefore$ By method of variation of parameters,

$$y_p = y_1(x) \cdot u_1 + y_2(x)\, u_2$$

$$\therefore y_p = y_1(x) \cdot \int \frac{R(x) \cdot W_1}{W} \, dx + y_2(x) \int \frac{R(x) \cdot W_2}{W} \, dx$$

$\rightarrow$ We can $\boxed{\text{NOT}}$ cancel this

$$\therefore y_p = 1 \int \frac{e^{x} \sin x \cdot (-e^{2x})}{2 \cdot e^{2x}} \, dx + e^{2x} \int \frac{e^{x} \cdot \sin x \cdot (1)}{2 \cdot e^{2x}} \, dx$$

$$\therefore y_p = -\frac{1}{2} \int e^{x} \cdot \sin x \cdot dx + \frac{e^{2x}}{2} \int e^{-x} \cdot \sin x \cdot dx$$

$a = 1, \; b = 1 \qquad\qquad a = -1, \; b = 1.$

$$\therefore y_p = -\frac{1}{2} \left(\frac{e^{x}}{2} [\sin x - \cos x] \right) + \frac{e^{2x}}{2} \left(\frac{e^{-x}}{2} [-\sin x - \cos x] \right)$$

$$\therefore y_p = -\frac{e^{x}}{4} \sin x + \cancel{\cos x \cdot \frac{e^{x}}{4}} - \frac{e^{x}}{4} \cdot \sin x - \cancel{\frac{e^{x}}{4} \cdot \cos x}$$

$$\therefore y_p = -\frac{e^{x}}{2} \cdot \sin x \qquad \text{---} \; (3)$$

$\rightarrow$ From (2) & (3), the g. sol^n of (1) is

$$y(x) = y_c + y_p$$

$\rightarrow \int e^{ax} \cdot \cos(bx+k) \cdot dx$

$= \frac{e^{ax}}{a^2+b^2} [a \cdot \cos(bx+k) + b \sin(bx+k)]$

$(a, b \neq 0)$

$\therefore y(x) = c_1 + c_2 \cdot e^{2x} - \frac{e^{x}}{2} \cdot \sin x$

c_1, c_2 = arbitrary constants

Ex:-06 Apply the method of variation of parameters to solve $(D^2+1)y = \sec x$.

$\rightarrow$ Given, $(D^2+1)y = \sec x$ ——— (1).

$R(x) = \sec x.$

$\therefore$ the A.E. of (1) is,

$m^2+1 = 0$

$\therefore m^2 = \pm i$

$\therefore$ C.F. $= y_c = e^{0x}[c_1 \cos x + c_2 \sin x]$

$= c_1 \cos x + c_2 \sin x$ ——— (2).

c_1, c_2 = arbitrary constants

$\rightarrow$ Let $y_1(x) = \cos x$ & $y_2(x) = \sin x$

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1 \neq 0$

$\therefore$ linearly independent

$\therefore W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x.$

$\therefore W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x.$

Date
Page

$\therefore$ By method of variation of parameters,

$$y_p = y_1(x) \cdot v_1 + y_2(x) \cdot v_2$$

$$= y_1(x) \int \frac{R(x) \cdot W_1}{W} \cdot dx + y_2(x) \int \frac{R(x) \cdot W_2}{W} \cdot dx$$

$$= \cos x \int \frac{\sec x \cdot (-\sin x)}{1} \cdot dx + \sin x \int \frac{\sec x \cdot \cos x}{1} \cdot dx$$

$$= \cos x \int \frac{(-\sin x)}{\cos x} \cdot dx + \sin x \int 1 \cdot dx$$

$$= \cos x \cdot \ln|\cos x| + \sin x \cdot x \quad \text{——— (3)}$$

$\rightarrow$ From, (2) & (3), the g. solⁿ of (1) is,

$$y(x) = y_c + y_p$$

$$\therefore y(x) = C_1 \cos x + C_2 \sin x + \cos x \cdot \ln|\cos x| + x \cdot \sin x$$

Ex:- 07 Apply the method of variation of parameters to solve $(D^2 + 4D + 4)y = \frac{e^{-2x}}{x^2}$.

$\rightarrow$ Given, $(D^2 + 4D + 4)y = \frac{e^{-2x}}{x^2}$ ——— (1).

$\therefore$ the A.E. of eqⁿ (1) is, $\quad R(x) = \frac{e^{-2x}}{x^2}$

$$m^2 + 4m + 4 = 0.$$

$$\therefore (m+2)^2 = 0.$$

$$\therefore m = -2, -2 \longrightarrow \text{Real and Equal}$$

Real & Equal → $e^{-2x} \cdot c_1 + e^{-2x} \cdot c_2 \cdot x + e^{-2x} \cdot c_3 \cdot x^2 + \ldots$

$\therefore$ C.F. $= y_c = c_1 \cdot e^{-2x} + c_2 \cdot e^{-2x} \cdot x$ —— (2)

c_1, c_2 = arbitrary constants

→ Let, $y_1(x) = e^{-2x}$ and $y_2(x) = e^{-2x} \cdot x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-2x} \cdot x \\ -2e^{-2x} & -2e^{-2x} \cdot x + e^{-2x}(1) \end{vmatrix}$$

$$= e^{-2x} (-2 \cdot e^{-2x} \cdot x) + e^{-2x} \cdot e^{-2x} + 2e^{-2x} \cdot e^{-2x} \cdot x$$

$$= -2x \cdot e^{-4x} + e^{-4x} + 2x e^{-4x}$$

$$= e^{-4x} \neq 0$$

$\therefore \{y_1, y_2\}$ is linearly independent

$$\rightarrow W_1 = \begin{vmatrix} 0 & e^{-2x} \cdot x \\ 1 & -2e^{-2x} \cdot x + e^{-2x} \end{vmatrix}$$

$$= -x \cdot e^{-2x}$$

$$\rightarrow W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 1 \end{vmatrix} = e^{-2x}$$

$\therefore$ By method of variation of parameters,

$$y_p = y_1(x) \cdot \int \frac{R(x) \cdot W_1}{W} \cdot dx + y_2(x) = \int \frac{R(x) \cdot W_2}{W} \cdot dx$$

$$\therefore y_p = e^{-2x} \int \frac{e^{-2x}}{x^2} \frac{(-x \cdot e^{-2x})}{e^{-4x}} \cdot dx + x \cdot e^{-2x} \int \frac{e^{-2x}}{x^2} \cdot \frac{e^{-2x}}{e^{-4x}} \cdot dx$$

$$\therefore y_p = -e^{-2x} \int \frac{1}{x} \cdot dx + x \cdot e^{-2x} \int \frac{1}{x^2} \cdot dx$$

Date
Page

$\therefore$ $y_p = -e^{-2x}\ln|x| + x\cdot e^{-2x}\left(\frac{-1}{x}\right)$

$= -e^{-2x}\cdot\ln x - e^{-2x}$ —— (3)

$\rightarrow$ From, (2) & (3), the g. solⁿ of (1) is,

$y(x) = y_c + y_p$

$\therefore$ $y(x) = c_1\cdot e^{-2x} + c_2\cdot e^{-2x} - e^{-2x}(\ln x + 1)$

Ex:- 08 Apply the method of variation of parameters to solve $y'' - 2y' + y = e^x \log x$

$\rightarrow$ Given, $y'' - 2y' + y = e^x \log x$ —— (A)

$R(x) = e^x \cdot \log x$

$\therefore$ The operator form of eqⁿ (A) is,

$(D^2 - 2D + 1)y = e^x \cdot \log x$ —— (1)

$\therefore$ The A.E. of (1) is, $m^2 - 2m + 1 = 0$

$\therefore$ $m^2 - m - m + 1 = 0$

$\therefore$ $(m-1)(m-1) = 0$

$\therefore$ $m = 1, 1$ Real and Equal

$\therefore$ C.F. $= y_c = c_1\cdot e^x + c_2\cdot e^x\cdot x$. (2)

c_1, c_2 = arbitrary constants

$\rightarrow$ Let, $y_1(x) = e^x$, $y_2(x) = e^x\cdot x$

$\therefore$ $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x\cdot e^x \\ e^x & x\cdot e^x + e^x(1) \end{vmatrix}$

$= e^{2x}\cdot x + e^{2x} - x\cdot e^{2x} = e^{2x} \neq 0$

$\rightarrow \int u \cdot v \cdot dx = u \int v \cdot dx - \int \left(\frac{d}{dx} \cdot u \int v \cdot dx \right) dx$

$\therefore \{ y_1, y_2 \}$ is linearly independent.

$$\therefore W_1 = \begin{vmatrix} 0 & x \cdot e^{x} \\ 1 & x \cdot e^{x} + e^{x} \end{vmatrix} = -x \cdot e^{x}$$

$$\therefore W_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & 1 \end{vmatrix} = e^{x}.$$

$\therefore$ By the method of variation of parameters,

$$y_p = y_1(x) \cdot \int \frac{R(x) \cdot W_1}{W} \cdot dx + y_2(x) \cdot \int \frac{R(x) \cdot W_2}{W} \cdot dx$$

$$\therefore y_p = e^{x} \int \frac{\cancel{e^{x}} \log x \cdot (-x \cdot \cancel{e^{x}})}{\cancel{e^{2x}}} \cdot dx + e^{x} \cdot x \int \frac{\cancel{e^{x}} \cdot \log x \cdot \cancel{e^{x}}}{\cancel{e^{2x}}} \cdot dx$$

$$\therefore y_p = -e^{x} \int \underset{v}{x} \cdot \underset{u}{\log x} \cdot dx + e^{x} \cdot x \int \underset{u}{\log x} \cdot dx \quad v = 1$$

$$\therefore y_p = -e^{x} \left\{ \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \cdot dx \right\} + x \cdot e^{x} \left\{ \log x \cdot x - \int \frac{1}{x} \cdot x \cdot dx \right\}$$

$$\therefore y_p = -e^{x} \left\{ \log x \cdot \frac{x^2}{2} - \frac{1}{2} \frac{x^2}{2} \right\} + x \cdot e^{x} \left\{ x \cdot \log x - x \right\}$$

$$\therefore y_p = \underline{- e^{x} \cdot \frac{x^2}{2} \cdot \log x} + \frac{e^{x}}{2} \cdot \frac{x^2}{2} + \cancel{x^2 \cdot e^{x} \cdot \log x} - x^2 \cdot e^{x}$$

$\therefore\ y_p = e^x \cdot x^2 \cdot \log x \left[-\frac{1}{2}+1\right] + e^x \cdot x^2 \left[\frac{1}{4}-1\right]$

$\therefore\ y_p = e^x \cdot \frac{x^2}{2} \cdot \log x - e^x \cdot \frac{3x^2}{4}$ —— (3)

→ From (2) & (3), the g. solⁿ of (1) is,

$y(x) = y_c + y_p$

$\therefore\ y(x) = c_1 \cdot e^x + c_2 \cdot e^x \cdot x + e^x \cdot \frac{x^2}{2} \cdot \log x - e^x \cdot \frac{3x^2}{4}$

P.Ex:- 17. Solve $y'' - 2y' = e^x \sin x$ using the method of variation of parameters.

Ans → Ex:- 5.

* Undetermined Co-efficient :-

Ex :- Find the general solution of the differential eq^n $y'' + 2y' + 10y = 25x^2 + 3$ by the method of undetermined co-efficient.

→ Given, $y'' + 2y' + 10y = 25x^2 + 3$ —— (1)

The operator form of (1) is given by,

$$(D^2 + 2D + 10)y = 25x^2 + 3$$

∴ A.E. is $m^2 + 2m + 10 = 0$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2}$$

$$= \frac{-2 \pm \sqrt{4 - 40}}{2}$$

$$= \frac{-2 \pm 6i}{2}$$

$$= -1 \pm 3i$$

∴ C.F. $= y_c = e^{-x}[c_1 \cos 3x + c_2 \sin 3x]$ —— (2)

→ Here, we have $R(x) = 25x^2 + 3$. which is not in C.F.

∴ we can choose $y_p = Ax^2 + Bx + C$ —— (3)

∴ $y_p' = 2Ax + B$

∴ $y_p'' = 2A$

→ Since y_p is trial solution, substituting these values in ①, we get,

$$y_p'' + 2y_p' + 10y_p = 25x^2 + 3$$

$$\therefore 2A + 2[2Ax + B] + 10[Ax^2 + Bx + C] = 25x^2 + 3$$

$$\therefore 2A + 4Ax + 2B + 10Ax^2 + 10Bx + 10C = 25x^2 + 3$$

$$\therefore 10Ax^2 + (4A + 10B)x + 2A + 2B + 10C = 25x^2 + 0x + 3$$

$$\therefore 10A = 25 \Rightarrow \boxed{A = 5/2}$$

$$\therefore 4A + 10B = 0 \Rightarrow \cancel{4}^{2}(5/\cancel{2}) + 10B = 0$$

$$\Rightarrow \boxed{B = -1}$$

$$\therefore 2A + 2B + 10C = 3 \Rightarrow 2\left(\frac{5}{2}\right) + 2(-1) + 10C = 3$$

$$\Rightarrow 5 - 2 + 10C = 3$$

$$\Rightarrow 10C = 0 \Rightarrow \boxed{C = 0}$$

→ From ③, $y_p = \frac{5}{2}x^2 + (-1)x + 0$

$$= 5/2\, x^2 - x \quad \text{——— ④}$$

Date
Page

→ From (2) & (3), we can write the g. solⁿ of (1),
as

$$y = y_c + y_p$$

$$\therefore y = e^{-x}[c_1 \cos 3x + c_2 \sin 2x] + \frac{5}{2}x^2 - x$$

Ex: Find the general solution of the differential equation $y'' + 6y' + 9y = 50e^{-x}.\cos x$ by the method of undetermined co-efficients.

→ Given, $y'' + 6y' + 9y = 50e^{-x}.\cos x$ —— (1)

→ The operator form of (1) is,

$$(D^2 + 6D + 9)y = 50e^{-x}.\cos x$$

$\therefore$ A.E. is $m^2 + 6m + 9 = 0$

$\therefore (m+3)^2 = 0$

$\therefore m = -3, -3.$

$\therefore$ C.F. $= y_c = e^{-3x}[c_1 + x \cdot c_2]$ —— (2)

c_1, c_2 = arbitrary constants

→ Since, C.F. does not involve any functions of $R(x)$,
we can assume,

$$y_p = e^{-x}[A\cos x + B\sin x] \quad \text{—— (3)}$$

Date
Page

$\therefore y_p' = -e^{-x}[A\cos x + B\sin x] + e^{-x}[-A\sin x + B\cos x]$

$\therefore y_p' = -e^{-x} A\cos x - e^{-x} B\sin x - e^{-x} A\sin x + e^{-x} B\cos x.$

$\therefore y_p' = e^{-x}[B-A]\cos x - e^{-x}[B+A]\sin x$

$\therefore y_p'' = (B-A)(-1)e^{-x}\cos x + (B-A)e^{-x}(-\sin x)$

$- (B+A)[e^{-x}\cdot\cos x + (-1)e^{-x}\cdot\sin x]$

$= (B-A)(-1)e^{-x}[\cos x + \sin x]$
$- (A+B)e^{-x}[\cos x - \sin x]$

$= (A-B)e^{-x}[\cos x + \sin x]$
$- (A+B)e^{-x}[\cos x - \sin x]$

$= e^{-x}\{A\cos x + A\sin x - B\cos x - B\sin x$
$- A\cos x + A\sin x - B\cos x + B\sin x\}$

$= e^{-x}\{2A\sin x - 2B\cos x\}$

$\therefore y_p'' = 2e^{-x}(A\sin x - B\cos x).$

→ Substituting these values in eq^n (1),

$y_p'' + 6y_p' + 9y_p = 50e^{-x}\cdot\cos x.$

$\therefore 2\cdot e^{-x}[A\sin x - B\cos x] + 6[e^{-x}]\{B\cos x - A\cos x - B\sin x - A\sin x\} + 9\{e^{-x}(A\cos x + B\sin x)\} = 50e^{-x}\cos x$

$\therefore e^{-x}[2A\sin x - 2B\cos x + 6B\cos x - 6A\cos x - 6B\sin x - 6A\sin x + 9A\cos x + 9B\sin x]$
$= 50e^{-x}\cos x.$

$\therefore \quad e^{-x}\{ -4A\sin x + 4B\cos x + 3A\cos x + 3B\sin x \} = 50\,e^{-x}\cdot\cos x$

$\therefore \quad e^{-x}\{ \sin x\,(-4A + 3B) + \cos x\,(3A + 4B)\} = e^{-x}[\,50\cos x + 50\,(0)\cdot\sin x\,]$

Comparing,

$-4A + 3B = 0$ and $3A + 4B = 50.$

$\times 3 \quad -12A + 9B = 0. \quad \leftarrow (A)$ $\quad \times 4 \quad \therefore\ 12A + 16B = 200 \quad \leftarrow (AA)$

$\therefore$ Substituting these eq^n.

$$-\cancel{12A} + 9B + \cancel{12A} + 16B = 200.$$

$\therefore \quad 25B = 200$

$\therefore \quad \boxed{B = 8}$

$\therefore$ From eq^n (A). $-12A = -72 \quad \therefore \boxed{A = 6}$

→ From, ② and ③, we can write the g. solⁿ of ①,

as, $y = y_c + y_p$

$$\therefore \; y = e^{-3x}[c_1 + x c_2] + e^{-x}[6\cos x + 8\sin x]$$

Ex: Find general solution of the differential eqⁿ $y'' - y' - 2y = 3e^{2x}$ by the method of undetermined co-efficients.

→ Given, $y'' - y' - 2y = 3e^{2x}$ —— ①

→ The operator form of ① is,

$$(D^2 - D - 2)y = 3e^{2x} \qquad \text{where, } R(x) = 3e^{2x}.$$

∴ A.E. is $m^2 - m - 2 = 0$.

$\therefore (m-2)(m+1) = 0$.

$\therefore m = 2, -1$.

∴ C.F. $= y_c = c_1 e^{2x} + c_2 e^{-x}$ —— ②

c_1, c_2 = arbitrary constants.

→ Let, $y_1(x) = e^{2x}$, $y_2(x) = e^{-x}$.

$y_1 \in R(x)$.

∴ We have to modify the choice of y_p.

$y_p = A x \cdot e^{2x}$

Date
Page

$\therefore y_p' = Ae^{2x} + 2Ax \cdot e^{2x}$
$= Ae^{2x} [1 + 2x]$

$\therefore y_p'' = 2Ae^{2x}(1+2x) + Ae^{2x}(2)$
$= 4Ae^{2x} + 4Ax \cdot e^{2x}$

From (1), we can write,

$y'' - y' - 2y = 3e^{2x}$

$\Rightarrow y_p'' - y_p' - 2y_p = 3e^{2x}$

$\Rightarrow 4Ae^{2x} + 4Ax \cdot e^{2x} - Ae^{2x} - 2A \cdot x \cdot e^{2x}$
$- 2Ax \cdot e^{2x} = 3e^{2x}$

$\Rightarrow 3A \cdot e^{2x} + (0) \cdot x \cdot e^{2x} = 3 \cdot e^{2x}$

$\Rightarrow 3Ae^{2x} = 3 \cdot e^{2x}$

$\Rightarrow \boxed{A = 1}$

$\therefore y_p = x \cdot e^{2x}$ ——— (3).

→ From (2) & (3), the general solⁿ is given by,

$y = y_c + y_p$

$\therefore y = c_1 \cdot e^{2x} + c_2 \cdot e^{-x} + x \cdot e^{2x}$

$\therefore y(x) = (c_1 + x) \cdot e^{2x} + c_2 \cdot e^{-x}$

c_1, c_2 = constants.

Ex:: Apply the method of undetermined co-efficient to solve $y'' + 9y = 6 \cdot \cos 3x$, $y(0) = 1$, $y'(0) = 0$.

→ Given, $y'' + 9y = 6 \cdot \cos 3x$ ——— (1)

→ The operator form of (1) is,

$(D^2 + 9)y = 6 \cos 3x$

$\therefore$ A.E. is $m^2 + 9 = 0$.

$\therefore m = \pm 3i$

$\therefore y_c = c_1 \cdot \cos 3x + c_2 \sin 3x$, c_1, c_2 = constants.

→ let $y_1 = \cos 3x$, $y_2 = \sin 3x$

→ Since, $R(x) = 6 \cdot \cos 3x$, choice of

$$y_p = A \cdot \cos 3x + B \cdot \sin 3x$$

But $y_1 \in \{\cos 3x, \sin 3x\}$ OR $y_1 \in R(x)$.

we have to modify the choice of y_p as

$x (A \cos 3x + B \cdot \sin 3x)$

$$y_p' = (A \cos 3x + B \sin x) + x[-3A \sin 3x + 3B \cos 3x]$$

$$y_p'' = -3A \sin 3x + 3B \cos 3x + 3B \cos 3x - 3A \sin 3x + x[-9A \cos 3x - 9B \sin 3x]$$

$$\therefore y_p'' = -6A \sin 3x + 6B \cdot \cos 3x - 9Ax \cdot \cos 3x - 9Bx \cdot \sin 3x$$

→ Substituting these in (1), we get,

$$y_p'' + 9y_p = 6 \cos 3x$$

$$\Rightarrow -6A \sin 3x + 6B \cos 3x - 9Ax \cos 3x - 9Bx \sin 3x + 9Ax \cos 3x + 9Bx \cdot \sin 3x = 6 \cos 3x.$$

$\Rightarrow -6A = 0$ & $6B = 6$

$\Rightarrow \boxed{A = 0}$ $\Rightarrow \boxed{B = 1}$

$\therefore y_p = x[0 + \sin 3x] = x \cdot \sin 3x$

$\therefore$ the general soln of (1), is given by,

Date
Page

$y = y_c + y_p$

$\therefore y = c_1 \cos 3x + c_2 \sin 3x + x \cdot \sin 3x$

$\therefore y'(x) = -3 \cdot c_1 \sin 3x + 3c_2 \cdot \cos 3x + 3x \cdot \cos 3x + \sin 3x \cdot (1)$

→ Now, $y(0) = 1 \Rightarrow c_1 (1) + 0 + 0 \Rightarrow 1$

$\therefore \boxed{c_1 = 1}$

$\therefore y'(x) = -3 \sin 3x + 3c_2 \cos 3x + 3x \cdot \cos 3x + \sin 3x$

and $y'(0) = 0$

$\therefore -3(0) + 3c_2 \cdot (1) + 3(0)(1) + 0 = 0$

$\therefore 3c_2 = 0$

$\therefore \boxed{c_2 = 0}$

$\therefore$ P. Solⁿ. $\boxed{y(x) = \cos 3x + x \cdot \sin 3x}$

Ex: Apply the method of undetermined co-efficient to solve $y''' - 3y'' + 3y' - y = 4e^t$

→ Given, $y''' - 3y'' + 3y' - y = 4e^t$ — (1)

→ The homogeneous part of (1) is,

$(D^3 - 3D^2 + 3D - 1)y = 0$

$\therefore$ A.E. is $m^3 - 3m^2 + 3m - 1 = 0$

$\therefore (m-1)^3 = 0$

$\therefore m = 1, 1, 1.$

$\therefore y_c = e^t c_1 + c_2 \cdot t \cdot e^t + c_3 \cdot t^2 \cdot e^t$; (2)

c_1, c_2, c_3 = constants

→ let $y_1 = e^t$, $y_2 = t \cdot e^t$, $y_3 = t^2 \cdot e^t$

→ Since all these functions involved in $R(x)$, we have to consider modified particular integral y_p as,

$$y_p = A t^3 \cdot e^t$$

$$\therefore y_p' = A \cdot e^t \cdot (3t^2) + A t^3 \cdot e^t$$
$$\therefore y_p' = A \cdot e^t [3t^2 + t^3]$$

$$\therefore y_p'' = A \cdot e^t [6t + 3t^2] + (3t^2 + t^3) \cdot A \cdot e^t$$
$$\therefore y_p'' = A \cdot e^t [t^3 + 6t^2 + 6t]$$

$$\therefore y_p''' = A \cdot e^t [3t^2 + 12t + 6] + [t^3 + 6t^2 + 6t] \cdot A e^t$$
$$\therefore y_p''' = A \cdot e^t [t^3 + 9t^2 + 18t + 6]$$

→ Substituting all these in (1), we get,

$$y_p''' - 3y_p'' + 3y_p' - y_p = 4 \cdot e^t$$

$$\Rightarrow A \cdot e^t [t^3 + 9t^2 + 18t + 6]$$
$$- 3 \cdot A \cdot e^t [t^3 + 6t^2 + 6t]$$
$$+ 3 \cdot A \cdot e^t [3t^2 + t^3] - At^3 e^t = 4 \cdot e^t$$

$$+ 3t^3 - t^3 \qquad +6$$

$$\Rightarrow A \cdot e^t [t^3 - 3t^3 + 9t^2 - 18t^2 + 9t^2 + \cancel{18t} - \cancel{18t}] = 4 \cdot e^t$$

$$\Rightarrow A \cdot e^t \cdot 6 = 4 \cdot e^t$$

$$\Rightarrow 6A = 4$$

$$\Rightarrow \boxed{A = \frac{2}{3}}$$

$\therefore y_p = \frac{2}{3} t^3 \cdot e^t$ —— (3)

→ From (2) & (3), the general solⁿ of (1) is given by,

$y = y_c + y_p$

$\therefore y = c_1 \cdot e^t + c_2 \cdot t \cdot e^t + c_3 \cdot t^2 \cdot e^t + \frac{2}{3} \cdot t^3 \cdot e^t$

where, c_1, c_2, c_3 = constants.

Ex:- Apply the method of undetermined co-efficient to solve $\frac{d^2y}{dx^2} + 2 \cdot \frac{dy}{dx} - 35y = 12 \cdot e^{5x} + 37 \cdot \sin 5x$.

→ Given, $\frac{d^2y}{dx^2} + 2 \cdot \frac{dy}{dx} - 35y = 12 \cdot e^{5x} + 37 \cdot \sin 5x$

$\therefore y'' + 2y' - 35y = 12 e^{5x} + 37 \cdot \sin 5x$ —— (1)

→ The homogeneous part of (1) is,

$(D^2 + 2D - 35)y = 0$

$\therefore$ AE is $m^2 + 2m - 35 = 0$

$\therefore (m+7)(m-5) = 0$

$\therefore m = 5, -7$

$\therefore y_c = c_1 e^{5x} + c_2 \cdot e^{-7x}$, c_1, c_2 = constants —— (2)

→ let, $y_1(x) = e^{5x}$, $y_2(x) = e^{-7x}$

Here, $R(x) = 12 e^{5x} + 37 \cdot \sin 5x$

Since, $e^{5x} \in R(x)$, then we have to modify the choice of y_p as,

$$y_p = Ax \cdot e^{5x} + B \cdot \cos 5x + C \cdot \sin 5x$$

$$\therefore y_p' = Ax \cdot 5 \cdot e^{5x} + A \cdot e^{5x}(1) + B \cdot [-\sin 5x \cdot 5]$$
$$+ C \cdot \cos 5x \cdot 5$$

$$\therefore y_p' = 5Ax \cdot e^{5x} + A \cdot e^{5x} - 5B \cdot \sin 5x + 5C \cdot \cos 5x$$

$$\therefore y_p'' = 5Ax \cdot e^{5x}(5) + e^{5x} \cdot (5A(1)) - 5B \cdot \cos 5x \cdot (5)$$
$$+ 5C \cdot (-\sin 5x\,(5)) + A \cdot e^{5x} \cdot (5)$$

$$\therefore y_p'' = 25Ax \cdot e^{5x} + 10A \cdot e^{5x} - 25 \cdot B \cos 5x$$
$$- 25 \cdot C \cdot \sin 5x$$

→ Substituting these values in (1),

$$y_p'' + 2y_p' - 35y_p = 12e^{5x} + 37 \cdot \sin 5x$$

$$\Rightarrow 25Ax\,e^{5x} + 10A \cdot e^{5x} - 25B \cdot \cos 5x - 25C \cdot \sin 5x$$
$$+ 10Ax \cdot e^{5x} + 2 \cdot A \cdot e^{5x} - 10B \cdot \sin 5x$$
$$+ 10C \cdot \cos 5x - 35Ax \cdot e^{5x} - 35B \cdot \cos 5x$$
$$- 35C \cdot \sin 5x = 12e^{5x} + 37 \cdot \sin 5x$$

$$\Rightarrow 12A \cdot e^{5x} + \cos 5x[-25B + 10C - 35B]$$
$$+ \sin 5x[-25C - 10B - 35C]$$
$$= 12 \cdot e^{5x} + 37 \cdot \sin 5x$$

$$\Rightarrow 12A \cdot e^{5x} = 12e^{5x} \quad \text{and} \quad -60B + 10C = 0$$

$$\Rightarrow \boxed{A = 1} \qquad \therefore 10C = 60B$$

$$\therefore C = 6B$$

and $-60C - 10B = 37$

$$\therefore -60(6B) - 10B = 37 \qquad \therefore C = 6\left(\frac{-1}{10}\right)$$

$$\therefore -360B - 10B = 37 \qquad \therefore \boxed{C = \frac{-3}{5}}$$

$$\therefore \boxed{B = -1/10}$$

$\therefore\ y_p = x \cdot e^{5x} + \left(\frac{-1}{10}\right)\cos 5x + \left(\frac{-3}{5}\right)\sin 5x$

→ From, ② & ③, the g. solⁿ of ① is given by,

$y = y_c + y_p$

$\therefore\ y(x) = c_1 \cdot e^{5x} + c_2 \cdot e^{-7x} + x \cdot e^{5x} - \frac{\cos 5x}{10}$

$- \frac{3}{5} \cdot \sin 5x$

Ex: Apply the method of undetermined co-efficient to solve $y'' - 4y = e^{-2x} - 2x$. $y(0) = y'(0) = 0$.

→ Given, $y'' - 4y = e^{-2x} - 2x$ — ①

→ The homogeneous form of ① is,

$D^2 - 4 = 0$

$\therefore$ A.E. is $m^2 - 4 = 0$

$\therefore\ m^2 = 4$

$\therefore\ m = 2, -2$

$\therefore\ y_c = c_1 e^{2x} + c_2 \cdot e^{-2x}$ — ②

→ Here, $R(x) = e^{-2x} - 2x$.

Since, the $e^{-2x} \in R(x)$, we have to modify the choice of y_p as,

$y_p = A x \cdot e^{-2x} + Bx + C$

$\therefore\ y_p' = Ax \cdot e^{-2x}(-2) + e^{-2x} \cdot A(1) + B + 0$

$\therefore y_p' = -2Ax \cdot e^{-2x} + A \cdot e^{-2x} + B.$

$\therefore y_p'' = 4Ax \cdot e^{-2x} - 2A \cdot e^{-2x} - 2A \cdot e^{-2x}$

→ Substituting these values in (1),

$$y_p'' - 4y_p = e^{-2x} - 2x$$

$$\therefore \cancel{4Ax \cdot e^{-2x}} - 2A \cdot e^{-2x} - 2A \cdot e^{-2x} - \cancel{4Ax \cdot e^{-2x}} - 4Bx - 4C = e^{-2x} - 2x$$

$$\therefore -4A \cdot e^{-2x} - x[4B + 4C] = e^{-2x} - 2x$$

$\Rightarrow -4A \cdot e^{-2x} = e^{-2x}$ and $-4Bx = -2x$

$\Rightarrow \boxed{A = -\frac{1}{4}}$ $\therefore \boxed{B = \frac{1}{2}}$

and $-4C = 0$

$\therefore \boxed{C = 0}$

$$\therefore y_p = -\frac{1}{4} \cdot e^{-2x} x + \frac{1}{2} \cdot x + 0.$$

→ From (2) & (3), the g. solⁿ of (1) is,

$y(x) = y_c + y_p$

$$\therefore y(x) = c_1 \cdot e^{2x} + c_2 \cdot e^{-2x} - \frac{1}{4} x e^{-2x} + \frac{1}{2} \cdot x$$

$$\therefore y'(x) = 2 \cdot c_1 \cdot e^{2x} - 2 \cdot c_2 \cdot e^{-2x} \cancel{+ \frac{1}{2} \cdot e^{-2x} + \frac{1}{2}}$$

$$+ \frac{1}{2} \cdot e^{-2x} \cdot x - \frac{1}{4} \cdot e^{-2x}(1) + \frac{1}{2}$$

→ We are given, $y(0) = y'(0) = 0$

$\therefore$ $y(0) = 0$

$\therefore$ $c_1 \cdot (1) + c_2 (1) - \frac{1}{4}(0) + 0 = 0.$

$\therefore$ $c_1 + c_2 = 0 \Rightarrow \boxed{c_1 = -c_2}$

and $y'(0) = 0$

$\therefore$ $2c_1 (1) - 2c_2 \cdot (1) + 0 - \frac{1}{4} + \frac{1}{2} = 0$

$\therefore$ $-2 \cdot c_2 - 2c_2 + \frac{1}{4} = 0.$

$\therefore$ $4c_2 = \frac{1}{4} \Rightarrow \boxed{c_2 = \frac{1}{16}}$

$\Rightarrow \boxed{c_1 = -c_2 = -\frac{1}{16}}$

$\therefore$ The p. solⁿ of (1) is given by,

$$y(x) = -\frac{1}{16} \cdot e^{2x} + \frac{1}{16} e^{-2x} - \frac{1}{4} \cdot x \cdot e^{-2x} + \frac{1}{2} x.$$

* Legendre's Linear Differential Equation :-

→ $(ax+b) \cdot \frac{dy}{dx} = a \cdot D \cdot y$ where, $D = \frac{d}{dt}$

→ $(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 D (D-1) y$ | $t = \log(ax+b)$, $ax+b = e^t$

→ $(ax+b)^3 \cdot \frac{d^3y}{dx^3} = a^3 D \cdot (D-1)(D-2) \cdot y$

Date
Page

* Special Case :- Cauchy - Euler linear diff. eqⁿ :-

→ $a = 1$ and $b = 0$ in Legendre's dff. eqⁿ.

→ $x \cdot \frac{dy}{dx} = D \cdot y$

→ $x^2 \cdot \frac{d^2y}{dx^2} = D(D-1) \cdot y$

→ $x^3 \cdot \frac{d^3y}{dx^3} = D(D-1)(D-2)\, y$

/* → $x^n \cdot \frac{d^ny}{dx^n} = D(D-1) \ldots (D-n+1) \cdot y$ */

Here, $x = e^t$

$\therefore t = \log x$

$\frac{dt}{dx} = \frac{1}{x}$

Ex :- Solve $(x^2 \cdot D^2 - 3x \cdot D + 3)\, y = 3 \cdot \log x - 4$

(Don't read it first, go for the another and come back)

→ Given, $(x^2 D^2 - 3x D + 3)\, y = 3 \cdot \log x - 4$

/* → Here, let $D = D'$ */ (સરળતા માટે)

$\therefore (x^2 D'^2 - 3x D' + 3)\, y = 3 \log x - 4$ ——— (1).

where, $D' = \frac{d}{dx}$

→ Let, $x = e^t$

$\therefore t = \log x$ ——— (☆)

$\therefore \frac{dt}{dx} = \frac{1}{x}$

→ $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$

$\therefore x \cdot \frac{dy}{dx} = \frac{dy}{dt}$

$\therefore x \cdot \frac{dy}{dx} = D \cdot y$

where, $D = \frac{d}{dt}$

$$\cancel{\frac{d^2y}{dx^2} = D(D-1)y}$$

$$\rightarrow \quad x^2 \cdot \frac{d^2y}{dx^2} = D(D-1)y$$

→ Substituting these values in ①, it reduces to,

$$[D(D-1) - 3D + 3]y = 3t - 4$$

where, $D = \frac{d}{dt}$

$$\therefore [D^2 - D - 3D + 3]y = 3t - 4 \quad \text{——— ②}$$

→ The A.E. is $m^2 - 4m + 3 = 0$

$$\therefore (m-3)(m-1) = 0$$

$$\therefore m = 1, 3$$

x → t

$$\therefore y_c = c_1 e^{t} + c_2 \cdot e^{3t} \quad ; \; c_1, c_2 = \text{constants}$$

→ Since, $R(t) = 3t - 4$, the choice of y_p is $At + B$, and A, B are to be determined.

$$\therefore y_p = At + B$$

$$\therefore y_p' = A$$

$$\therefore y_p'' = 0$$

→ Substituting these values in ②,

$$y_p'' - 4y_p' + 3y_p = 0$$

$\therefore\ 0 - 4A + 3At + 3B = 3t - 4$

$\therefore\ 3At + 3B - 4A = 3t - 4.$

$\Rightarrow 3At = 3t \qquad\qquad 3B - 4A = -4$

$\Rightarrow \boxed{A = 1} \qquad\qquad \therefore\ 3B - 4 = -4$

$\therefore\ \boxed{B = 0.}$

$\rightarrow \therefore\ y_p = At + B$

$\therefore\ y_p = t$

$\therefore$ The g. solⁿ of (1) is, $y = y_c + y_p$

$\therefore\ y = c_1 \cdot e^{t} + c_2 \cdot e^{3t} + t$

$\therefore\ y = c_1 \cdot e^{\log x} + c_2 e^{3\log x} + \log x$

$(\because *)$

$$\therefore\ \boxed{y = c_1 \cdot x + c_2 \cdot x^3 + \log x.}$$

Ex :- Solve $(2x+3)^2 \cdot \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$

$\rightarrow$ Given, $(2x+3)^2 \cdot \frac{d^2y}{dx^2} - (2x+3) \cdot \frac{dy}{dx} - 12y = 6x.$ ——— (1)

$\rightarrow$ Let, $2x+3 = e^t \Rightarrow t = \log(2x+3)$

and $x = \frac{e^t - 3}{2.}$

$\rightarrow \therefore$ From, $(ax+b)\frac{dy}{dx} = a \cdot Dy$ where, $D = \frac{d}{dt}$

Date
Page

$$(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 \cdot D(D-1)\,y.$$
and

we have, $(2x+3) \cdot \frac{dy}{dx} = 2 \cdot D \cdot y.$

and $(2x+3)^2 \cdot \frac{d^2y}{dx^2} = 4 \cdot D(D-1) \cdot y$

where, $D = \frac{d}{dt}.$

→ Now, substituting these values in (1),

$$[4(D^2 - D) - 2D - 12]\,y = 6\left(\frac{e^t - 3}{2}\right)$$

$\Rightarrow (4D^2 - 6D - 12)\,y = 3e^t - 9.$

$\Rightarrow (2D^2 - 3D - 6)\,y = \frac{3e^t - 9}{2} \rightarrow R(t)$ — (2).

→ The A.E. is $2m^2 - 3m - 6 = 0.$

$a = 2, \quad b = -3, \quad c = -6.$

$/* \; x = \frac{-b \pm \sqrt{D}}{2a}, \quad D = b^2 - 4ac \; */$

$\therefore \; x = \frac{3 \pm \sqrt{57}}{4}.$

$$\therefore \; y_c = c_1 \cdot e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 \cdot e^{\left(\frac{3-\sqrt{57}}{4}\right)t}$$

→ Since, $R(t) = \frac{3}{2} \cdot e^t - \frac{9}{2}$, the choice of

y_p is, $y_p = A \cdot e^t + B.$

$\therefore y_p' = A \cdot e^t$

$\therefore y_p'' = A \cdot e^t$

→ Substituting these in eqⁿ ②, we get,

$$2y_p'' - 3y_p' - 6y_p = \frac{3}{2} \cdot e^t - \frac{9}{2}$$

/*

$$\cancel{\therefore 2Ae^t - 3A \cdot e^t - 6 \cdot A \cdot e^t + 6B = \frac{3}{2} e^t + \frac{9}{2}}$$

$$\cancel{\therefore e^t[-9A] + 2A + 6B = \frac{3}{2} e^t + \frac{9}{2}}$$

$$\cancel{\Rightarrow -9A = \frac{3}{2}} \quad \text{and} \quad \cancel{\Rightarrow 2A + 6B = \frac{9}{2}}$$

$$\cancel{\Rightarrow \boxed{A = -\frac{1}{6}}} \qquad \cancel{\Rightarrow -\frac{1}{3} + 6B = \frac{9}{2}}$$

$$\cancel{\Rightarrow 6B = \frac{27}{6} + \frac{2}{6}}$$

$$\cancel{\Rightarrow B = \frac{29}{36}}$$

*/

$$\Rightarrow 2A \cdot e^t - 3A \cdot e^t - 6A \cdot e^t - 6B = \frac{3}{2} e^t - \frac{9}{2}$$

$$\Rightarrow -7 \cdot A e^t - 6B = \frac{3}{2} \cdot e^t - \frac{9}{2}$$

$$\Rightarrow -7 \cdot A \cdot e^t = \frac{3}{2} \cdot e^t \quad \text{and} \quad -6B = -\frac{9}{2}$$

$$\Rightarrow -7A = \frac{3}{2} \qquad \Rightarrow \boxed{B = \frac{3}{4}}$$

$$\Rightarrow \boxed{A = -\frac{3}{14}}$$

$\therefore\ y_p = \frac{-3}{14} \cdot e^t + \frac{3}{4}.$

$\therefore$ The g. solⁿ of (2) is,

$y = y_c + y_p$

$\therefore\ y = c_1 \cdot e^{\left(\frac{3+\sqrt{57}}{4}\right) \cdot t} + c_2 \cdot e^{\left(\frac{3-\sqrt{57}}{4}\right) \cdot t}$

$- \frac{3}{14} \cdot e^t + \frac{3}{4}$

the g. solⁿ of (1)

$\therefore\ y = c_1 \cdot e^{\left(\frac{3+\sqrt{57}}{4}\right) \cdot \log(2x+3)} +$

$c_2 \cdot e^{\left(\frac{3-\sqrt{57}}{4}\right) \log(2x+3)} - \frac{3}{14} e^{\log(2x+3)} + \frac{3}{4}$

$\therefore\ y = c_1 \cdot (2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)} + c_2 \cdot (2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)}$

$- \frac{3}{14} \cdot (2x+3) + \frac{3}{4}$; c_1, c_2 = constants

Ex:- Solve $(x+2)^2 \cdot \frac{d^2y}{dx^2} - (x+2) \cdot \frac{dy}{dx} + y = 3x+4$

→ Given, $(x+2)^2 \cdot \frac{d^2y}{dx^2} - (x+2) \cdot \frac{dy}{dx} + y = 3x+4$ —— (1)

→ Let, $x+2 = e^t \Rightarrow t = \log(x+2)$

and $x = e^t - 2.$

$\therefore$ From, $(ax+b) \cdot \frac{dy}{dx} = a \cdot D \cdot y$ and

$$(ax+B)^2 \cdot \frac{d^2y}{dx^2} = a^2 \cdot D(D-1) \cdot y$$

where, $D = \frac{d}{dt}$

we can get,

$$(x+2) \cdot \frac{dy}{dx} = D \cdot y \quad \text{and}$$

$$(x+2)^2 \cdot \frac{d^2y}{dx^2} = D(D-1) \cdot y$$

→ Substituting these values in eq.^{n} (1),

$$[D^2 - D - D + 1] \cdot y = 3(e^t - 2) + 4$$

$$\Rightarrow (D^2 - 2D + 1)y = \underbrace{3 \cdot e^t - 2}_{R(t)} \quad \text{—— (2)}$$

→ The A.E is $m^2 - 2m + 1 = 0$

$\therefore (m-1)(m-1) = 0$

$\therefore m = 1, 1$

$\therefore y_c = e^t [c_1 + c_2 \cdot t]$

$\therefore y_c = c_1 \cdot e^t + c_2 \cdot e^t \cdot t$; c_1, c_2 = constants

→ Since, $R(t)$ of eq^n (2) is $3 \cdot e^t - 2$, the choice of $y_p = A \cdot e^t + B$ but $e^t, t \cdot e^t$ are in y_c, we have to modify the choice of y_p as follows :

$$y_p = At^2 e^t + B$$

$$\therefore y_p' = At^2 \cdot e^t + e^t \cdot 2At + 0$$

$$\therefore y_p' = A \cdot e^t [t^2 + 2t]$$

$\therefore y_p'' = At^2 \cdot e^t + e^t \cdot 2A \cdot t + e^t\, 2A + 2At \cdot e^t$

$\therefore y_p'' = At^2 \cdot e^t + 4At \cdot e^t + 2A \cdot e^t$

$\therefore y_p'' = A \cdot e^t \, [t^2 + 4t + 2]$

$\rightarrow$ Replacing y_p by y_p in eq^n (2), we get,

$y_p'' - 2y_p' + y_p = 3e^t - 2$

$\therefore A \cdot e^t \, [t^2 + 4t + 2] - 2A \cdot e^t \, [t^2 + 2t] + At^2 \cdot e^t + B = 3e^t - 2$

$\therefore A \cdot e^t \, [\cancel{t^2} + \cancel{4t} + 2 - 2t^2 - \cancel{4t} + t^2] + B = 3 \cdot e^t - 2$

$\therefore A \cdot e^t \, [2] + B = 3 \cdot e^t - 2$

$\Rightarrow 2A = 3$ and $\boxed{B = -2}$

$\Rightarrow \boxed{A = \frac{3}{2}}$

$\Rightarrow \therefore y_p = \frac{3}{2} \cdot t^2 \cdot e^t - 2$

$\therefore$ The g. sol^n of (2) is,

$y = y_c + y_p$

$\therefore y = c_1 \cdot e^t + c_2 t \cdot e^t + \frac{3}{2} t^2 e^t - 2$

$\therefore$ The g. solⁿ of (1) is,

$$\therefore y = C_1 \cdot (x+2) + C_2 \cdot \log(x+2) \cdot (x+2) + \frac{3}{2}[\log(x+2)]^2 \cdot (x+2) - 2.$$

$$\therefore y(x) = C_1(x+2) + C_2(x+2)\cdot\log(x+2) + \frac{3}{2}(x+2)\cdot[\log(x+2)]^2 - 2.$$

C_1, C_2 = constants

Ex:- Solve $(3x+2)^2 \cdot \frac{d^2y}{dx^2} + 3(3x+2)\cdot\frac{dy}{dx} - 36y = 3x^2+4x+1.$

$\rightarrow$ Given, $(3x+2)^2 \cdot \frac{d^2y}{dx^2} + 3(3x+2)\cdot\frac{dy}{dx} - 36y = 3x^2+4x+1.$ —— (1)

$\rightarrow$ Let $3x+2 = e^t \Rightarrow t = \log(3x+2)$ and $x = \left(\frac{e^t-2}{3}\right)$

$\rightarrow$ $\therefore$ From $\frac{dy}{dx}(ax+b) = a \cdot D \cdot y$ and

$$(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 D(D-1)\cdot y \; ; \; D = \frac{d}{dt}$$

we can get,

$$(3x+2)^2 \cdot \frac{d^2y}{dx^2} = 9D(D-1) \cdot y \quad \text{and}$$

$$(3x+2) \cdot \frac{dy}{dx} = \frac{3}{\cancel{8}} \cdot D \cdot y \; ; \quad D = \frac{d}{dt}$$

$\Rightarrow$ Substituting these in eq^n (1),

$$\left[\overset{9D^2-9D}{\cancel{\quad}} + 3 \cdot 3D - 36\right] y = 3\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1.$$

$$\therefore [9D^2 - 36] \cdot y = \frac{3}{9}(e^{2t} - 4e^t + 4) + \frac{4}{3} \cdot e^t - \frac{8}{3} + 1$$

$$\therefore 9(D^2-4) \cdot y = \cancel{8}\,\frac{1}{3} \cdot e^{2t} - \cancel{\frac{4}{3} \cdot e^t} + \frac{4}{3} + \cancel{\frac{4}{3} \cdot e^t} - \frac{8}{3} + 1$$

$$\therefore 9(D^2-4) \cdot y = \frac{1}{3} \cdot e^{2t} \cancel{+ \frac{2}{3} e^t} - \frac{1}{3}.$$

$$\therefore 9(D^2-4)y = \frac{1}{3}(e^{2t} \cancel{+ 2e^t} - 1).$$

$$\therefore (D^2-4)y = \frac{1}{27}(e^{2t}-1) \rightarrow R(t). \quad \text{--- (2)}$$

$\therefore$ The A.E. is $m^2 - 4 = 0$

$\therefore m = 2, -2.$

$\therefore \; y_c = C_1 e^{2t} + C_2 \cdot e^{-2t}$; C_1, C_2 = constants

→ Since, the R(t) of the eq^n ② is $1/27\,(e^{2t} - 1)$ the choice of $y_p = A \cdot e^{2t} + B$ but e^{2t} is in the y_c, therefore we have to modify the choice of y_p as follows :

$$y_p = At \cdot e^{2t} + B$$

$$\therefore y_p' = At \cdot e^{2t} (2) + e^{2t} \cdot A(1) + 0$$

$$= A \cdot e^{2t} [2t + 1]$$

$$\therefore y_p'' = 2At \cdot e^{2t} (2) + e^{2t} \, 2A (1) + e^{2t} \cdot (2) \cdot A$$

$$= A \cdot e^{2t} [4t + 2 + 2]$$

$$= A \cdot e^{2t} [4t + 4]$$

→ Now, replacing y by y_p in eq^n ②, we get,

$$y_p'' - 4 \cdot y_p = 1/27 \, (e^{2t} - 1)$$

$$\therefore A \cdot e^{2t} [4t + 4] - 4A \cdot e^{2t} \cdot t - 4B = 1/27 \, (e^{2t} - 1)$$

$$\therefore A \cdot e^{2t} [\cancel{4t} + 4 - \cancel{4t}] - 4B = \frac{1}{27} \cdot e^{2t} - \frac{1}{27}$$

$\therefore 4A = \frac{1}{27}$ and $4B = \frac{1}{27}$

$\therefore \boxed{A = \frac{1}{108}}$ $\qquad \therefore \boxed{B = \frac{1}{108}}$

$\therefore \; y_p = \frac{1}{108} \cdot t \cdot e^{t} + \frac{1}{108}$

$\therefore$ The g. solⁿ of (2) is,

$y = y_c + y_p$

$\therefore \; y = c_1 \cdot e^{2t} + c_2 \cdot e^{-2t} + \frac{1}{108} \cdot t \cdot e^{2t} + \frac{1}{108}$

$\therefore$ The g. solⁿ of (1) is,

$\therefore \; y = c_1 \cdot e^{2\log(3x+2)} + c_2 \cdot e^{-2 \cdot \log(3x+2)}$

$+ \frac{1}{108} \cdot \log(3x+2) \, [3x+2]^{2} + \frac{1}{108}$

$\therefore \; y = c_1 \cdot (3x+2)^{2} + c_2 \cdot (3x+2)^{-2}$

$+ \frac{(3x+2)^{2}}{108} \cdot \log(3x+2) \cdot + \frac{1}{108}$

$\therefore \; y = c_1 (3x+2)^{2} + c_2 (3x+2)^{-2}$

$+ \frac{1}{108} \left[(3x+2)^{2} \cdot \log(3x+2) + 1 \right]$

c_1, c_2 = constants

Ex: Solve $[(x+1)^2 D^2 + (x+1)D]y = (2x+3)(2x+4)$

→ Here, $[(x+1)^2 D'^2 + (x+1) D'] y = (2x+3)(2x+4)$

∴ $[(x+1)^2 \cdot D'^2 + (x+1) D']y = 4x^2 + 10x + 12$ —— (1).

→ $(x+1)^2 \cdot \dfrac{d^2y}{dx^2} = D(D-1) \cdot y.$

and $(x+1) \cdot \dfrac{dy}{dx} = D \cdot y.$

let $(x+1) = e^t$

$\Rightarrow t = \log(x+1)$

and $x = e^t - 1$

(from $(ax+b)^2 \cdot \dfrac{d^2y}{dx^2} = a^2 \cdot D(D-1) \cdot y.$ | $D = \dfrac{d}{dt}$.

and $(ax+b) \cdot \dfrac{dy}{dx} = a \cdot D \cdot y.$)

→ Substituting these in eqn (1),

$$[D^2 - D + D]y = 4(e^t - 1)^2 + 14(e^t - 1) + 12$$

$$\cancel{= 4x^2 + 10x + 12}$$

$$= 4(e^{2t} - 2 \cdot e^t + 1) + 14 \cdot e^t - 2$$

∴ $D^2 \cdot y =$ [illegible] —— (2).

→ The A.E. is $m^2 = 0$

∴ $m = 0, 0$.

∴ $y_c = c_1 \cdot e^0 + c_2 \cdot t \cdot e^0$

∴ $y_c = c_1 + c_2 \cdot t$; c_1, c_2 = constants.

→ Since, $R(x)$ of (2) is,

$$4 \cdot e^{2t} - 8e^t + 4 + 14e^t - 2$$

$$= 4 \cdot e^{2t} + 6e^t + 2$$

the choice of $y_p = Ae^{2t} + Be^t + c$ but c is in the y_c, therefore we have to modify the choice of y_p as follows.

[$ct \to$ ✗ ($\because$ ct is also in y_c)]

$$y_p = A\cdot e^{2t} + B\cdot e^{t} + C\cdot t^2.$$

$$\therefore y_p' = 2A\cdot e^{2t} + B\cdot e^{t} + 2C\cdot t$$

$$\therefore y_p'' = 4A\cdot e^{2t} + B\cdot e^{t} + 2C$$

→ Now, replacing y by y_p in ②, we get,

$$y_p'' = 4\cdot e^{2t} + 6\cdot e^{t} + 2.$$

$$\therefore 4A\cdot e^{2t} + B\cdot e^{t} + 2C = 4\cdot e^{2t} + 6e^{t} + 2.$$

$$\Rightarrow 4A\cdot e^{2t} = 4\cdot e^{2t} \;\&\; B\cdot e^{t} = 6\cdot e^{t} \;\&\; 2C = 2$$

$$\Rightarrow \boxed{A = 1} \quad \boxed{B = 6} \quad \boxed{C = 1}$$

→ $$\therefore y_p = e^{2t} + 6e^{t} + t^2$$

→ The g. soln of ② is,

$$y = y_c + y_p$$

$$\therefore y = C_1 + C_2\cdot t + e^{2t} + 6\cdot e^{t} + t^2.$$

→ The g. soln of ① is,

$$\therefore y = C_1 + C_2\cdot \log(x+1) + (x+1)^2 + 6\cdot(x+1) + [\log(x+1)]^2$$

C_1, C_2 = constants

Ex :- Solve $x^2 \cdot \frac{d^2y}{dx^2} - 3x \cdot \frac{dy}{dx} + 4y = x^2 \cdot \log x$ —— (1)

→ Let $x = e^t \Rightarrow t = \log x$

→ From $ax^2 \cdot \frac{d^2y}{dx^2} = a \cdot D \cdot (D-1) \cdot y$ and

$ax \cdot \frac{dy}{dx} = a \cdot D \cdot y$; $D = \frac{d}{dt}$

we can get,

$x^2 \cdot \frac{d^2y}{dx^2} = D(D-1)y$ and

$x \cdot \frac{dy}{dx} = D \cdot y$.

→ Substituting these in (1),

$[D^2 - D - 3D + 4]\, y = e^{2t} \cdot t$

$\therefore (D^2 - 4D + 4) \cdot y = e^{2t} \cdot t$ —— (2)

← R(t)

$\therefore$ The A.E. is $m^2 - 4m + 4 = 0$.

$\therefore (m-2)(m-2) = 0$.

$\therefore m = 2, 2$.

$\therefore y_c = c_1 \cdot e^{2t} + c_2 \cdot e^{2t} \cdot t$

→ Since, the R(t) of eq^n (2) is $e^{2t} \cdot t$
the choice of $y_p = (At + B)e^{2t}$
but $B \cdot e^{2t}$ is in the y_c, therefore we

have to modify the choice of y_p as follows:

$t \cdot e^{2t}$

$\downarrow$

$At + B \cdot c \cdot e^{2t}$

$(At + B) \cdot e^{2t}$

$$y_p = t\,(At + B) \cdot e^{2t}$$

but again $B \cdot t \cdot e^{2t}$ is in y_c. So, again modify the choice of y_p,

$$y_p = t^2\,(At + B) \cdot e^{2t}$$

$$\therefore y_p = A \cdot t^3 \cdot e^{2t} + B \cdot t^2 \cdot e^{2t}$$

$$\therefore y_p' = At^3 \cdot e^{2t}(2) + e^{2t}\, 3At^2 + 2Bt \cdot e^{2t} + Bt^2 \cdot e^{2t}(2)$$

$$\therefore y_p' = 2At^3 \cdot e^{2t} + 3At^2 \cdot e^{2t} + 2Bt \cdot e^{2t} + 2Bt^2 \cdot e^{2t}$$

$$\therefore y_p'' = 2At^3\, e^{2t}(2) + e^{2t} \cdot 6At^2 + 3At^2 \cdot e^{2t}(2) + e^{2t} \cdot (6At) + 2Bt \cdot e^{2t}(2) + e^{2t} \cdot (2B) + 2Bt^2 \cdot e^{2t}(2) + e^{2t}(4Bt)$$

$$\therefore y_p'' = 4At^3 \cdot e^{2t} + 12At^2 \cdot e^{2t} + 6At \cdot e^{2t} + 8Bt \cdot e^{2t} + 4Bt^2 \cdot e^{2t} + 2B \cdot e^{2t}$$

$$\therefore y_p'' = e^{2t}\left[4At^3 + t^2(12A + 4B) + t(6A + 8B) + 2B\right]$$

→ Substituting these in eqn. (2), let replace y by y_p in (2),

$$y_p'' - 4y_p' + 4y_p = e^{2t} \cdot t$$

$$\therefore e^{2t}\left[4At^3 + t^2(12A+4B) + t(6A+8B) + 2B\right] - 4\left[e^{2t}(2At^3 + 3At^2 + 2Bt + 2Bt^2)\right] + 4e^{2t}\cdot t^2(At+B) = e^{2t}\cdot t$$

$$\therefore e^{2t}\left[\{4At^3 + t^2(12A+4B) + t(6A+8B) + 2B\} - 4\{2At^3 + t^2(3A+2B) + 2Bt\} + 4At^3 + 4Bt^2\right] = e^{2t}\cdot t$$

$$\therefore \cancel{e^{2t}\left[8At^3 + t^2(12A+4B) + t(6A+12B) + 2B - 8At^3 - t^2(12A+8B) - 8Bt\right] = e^{2t}\cdot t}$$

$$\therefore e^{2t}\left[4At^3 + (12A+4B)t^2 + (6A+8B)t + 2B - 8At^3 - (12A+4B)\cdot t^2 - 8Bt + 4At^3 + 4Bt^2\right] = e^{2t}\cdot t$$

$$\therefore e^{2t}\left[(6A+8B-8B)t\right] = e^{2t}\cdot t$$

$$\Rightarrow 6A = 1$$

$$\Rightarrow \boxed{A = \frac{1}{6}} \quad \text{and} \quad \boxed{B = 0}$$

$$\therefore y_p = (At^3 + Bt^2)e^{2t}$$

$$= \left(\frac{1}{6}t^3 + 0\right)e^{2t} = \frac{1}{6}t^3\cdot e^{2t}$$

$\therefore$ The g. sol^n of (2) is,

$$y = y_c + y_p$$

$$\therefore y = c_1\cdot e^{2t} + c_2\cdot e^{2t}\cdot t + \frac{1}{6}t^3\cdot e^{2t}$$

Date
Page

$\therefore$ The g. solⁿ of (1) is given by,

$$y = c_1 \cdot e^{2\log x} + c_2 \cdot e^{2\cdot\log x} \cdot \log x$$

$$+ \frac{1}{6} (\log x)^3 \cdot e^{2\log x}$$

$$\therefore y = c_1 \cdot x^2 + \log x \cdot c_2 \cdot x^2 + \frac{x^2}{6} (\log x)^3.$$

Ex:- Solve $x^2 y'' - 2.5 xy' - 2.0 y = 0$

→ Given $x^2 y'' - 2.5 xy' - 2.0 y = 0$ is Cauchy - Euler's diff. eqⁿ. —— (1)

Let $x = e^t \Rightarrow t = \log x$

→ Now, $x^2 \cdot y'' = D(D-1) \cdot y$
and $xy' = D \cdot y$, where, $D = \frac{d}{dt}$.

→ Substituting these in (1),

$$[D^2 - D - 2.5D - 2.0]\, y = 0.$$

$$\therefore [D^2 - 3.5D - 2]\, y = 0.$$

$\therefore$ A.E. is $m^2 - 3.5m - 2 = 0$.

$a = 1, \quad b = -3.5, \quad c = -2.$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3.5 \pm \sqrt{12.25 - 4(1)(-2)}}{2(1)}.$$

$\therefore m = \frac{3.5 + 4.5}{2}, \frac{3.5 - 4.5}{2}$

$\therefore m = 4, -1/2.$

$\therefore y_c = c_1 \cdot e^{4t} + c_2 \cdot e^{-\frac{1}{2}t}$

$\therefore y(x) = c_1 \cdot e^{4t} + c_2 \cdot e^{-\frac{1}{2}t}$

$= c_1 \cdot e^{4 \log x} + c_2 \cdot e^{-\frac{1}{2} \log x}$

$= c_1 \cdot x^4 + c_2 \cdot x^{-1/2}$

$\therefore y(x) = c_1 \cdot x^4 + c_2 \cdot \frac{1}{\sqrt{x}}$; $c_1, c_2 =$ constant

Ex:- Solve $(x^2 D^2 - 3xD + 4)y = 0$; $y(1) = 0$, $y'(1) = 3$

→ Given $[x^2 D'^2 - 3xD' + 4]y = 0.$ — (1)

→ let $x = e^t \Rightarrow t = \log x$

Now, $x^2 \cdot D'^2 = x^2 \cdot \frac{d^2y}{dx^2} = D(D-1) \cdot y.$

and $x \cdot D' = x \cdot \frac{dy}{dx} = D \cdot y.$; $D = \frac{d}{dt}$

→ Substituting these in eq. (1),

$[D^2 - D - 3D + 4]y = 0.$

$\therefore [D^2 - 4D + 4]y = 0$

$\therefore$ The A.E. is $m^2 - 4m + 4 = 0$

$\therefore (m-2)(m-2) = 0.$

$\therefore m = 2, 2.$

$\therefore y_c = e^{2t} [c_1 + c_2 \cdot t]$; c_1, c_2 = constants

$\therefore y(t) = c_1 \cdot e^{2t} + c_2 \cdot t \cdot e^{2t}.$

$\therefore y(x) = c_1 \cdot e^{2\log x} + c_2 \cdot \log x \cdot e^{2\log x}$

$= c_1 \cdot x^2 + c_2 \cdot \log x \cdot x^2$

$\therefore y(x) = x^2 [c_1 + c_2 \cdot \log x]$

$\therefore y'(x) = 2x \cdot [c_1 + c_2 \log x] + x^2 [0 + c_2 \cdot 1/x]$

$\therefore y'(x) = 2x [c_1 + c_2 \log x] + x \cdot c_2.$

→ Now, $y(1) = 0$

$\therefore (1)^2 [c_1 + c_2 \log(1)] = 0$

$\therefore \boxed{c_1 = 0}$

and $y'(1) = 3$

$\therefore 2(1) [c_1 + c_2 \cdot \log(1)] + 1 \cdot c_2 = 3$

$\therefore 2 [0 + 0] + c_2 = 3$

$\therefore \boxed{c_2 = 3}$

$\therefore$ P. solⁿ of given diff. eqⁿ. is,

$y = x^2 [3 \log x]$

$\therefore \boxed{y = 3 \log x \cdot x^2.}$

Date
Page

* System of simultaneous first order linear differential equations :-

Ex :- Solve the simultaneous differential equations :-

$\frac{dx}{dt} + 2y + \sin t = 0$; $\frac{dy}{dt} - 2x - \cos t = 0$

$x(t)$ $y(t)$

given that $x = 0$ and $y = 1$ when $t = 0$.

→ Let $\frac{d}{dt} = D$ ∴ the given d. eqⁿ becomes,

$$Dx + 2y = -\sin t \quad \text{—— (1)}$$
$$Dy - 2x = \cos t \quad \text{—— (2)}$$

→ Now, multiplying (1) with 2 and (2) with D, we get,

D = operator
it has to come before function
$2Dx$ ✓
$2xD$ (✗)

$$\cancel{2Dx} + 4y = -2\sin t$$
$$\cancel{-2Dx} + D^2y = D.\cos t$$

$$D^2y + 4y = -2\cdot\sin t + D.\cos t$$
$$\therefore D^2y + 4y = -2\sin t - \sin t$$
$$\left(\because D\cos t = \frac{d}{dt}\cos t = -\sin t\right)$$

$$\therefore D^2y + 4y = -3\sin t \quad \text{—— (3)}$$

$R(t)$

→ The A.E. of (3) is, $(m^2 + 4) = 0$.

$$\therefore m^2 = -4$$
$$\therefore m = \pm 2i$$

Date ______
Page ______

$\therefore\ y_c = C_1 \cdot \cos 2t + C_2 \cdot \sin 2t$; $C_1, C_2 =$ constants

$\rightarrow$ Let, $y_1(t) = \cos 2t$, $y_2(t) = \sin 2t$.

then,

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2 \neq 0.$$

and $$W_1 = \begin{vmatrix} 0 & \sin 2t \\ 1 & 2\cos 2t \end{vmatrix} = -\sin 2t$$

and $$W_2 = \begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & 1 \end{vmatrix} = \cos 2t$$

$\therefore$ By the method of variation of parameters,

$$y_p = y_1(t) \cdot u_1 + y_2(t) \cdot u_2$$

$$= y_1(t) \int \frac{R(t) \cdot W_1}{W} \cdot dt + y_2(t) \cdot \int \frac{R(t) \cdot W_2}{W} \cdot dt$$

$$= \cos 2t \int \frac{(-3\sin t) \cdot (-\sin 2t)}{2} \cdot dt +$$

$$\sin 2t \int \frac{(-3\sin t) \cdot \cos 2t}{2} \cdot dt$$

$$= \frac{3\cos 2t}{4} \int 2\sin 2t \cdot \sin t \cdot dt$$

$$- \frac{3}{4} \cdot \sin 2t \int 2\sin t \cdot \cos 2t \cdot dt$$

$$= \frac{3}{4} \cos 2t \int \left[\cos t - \cos 3t\right] \cdot dt$$

$$- \frac{3}{4} \sin 2t \int \left[\sin 3t + \sin(-t)\right] \cdot dt$$

$\therefore\ y_p = \frac{3}{4}\cos 2t \int \cos t \cdot dt - \frac{3}{4}\cos 2t \int \cos 3t \cdot dt$

$\quad - \frac{3}{4}\sin 2t \int \sin 3t \cdot dt + \frac{3}{4}\sin 2t \int \sin t \cdot dt$

$= \frac{3}{4}\Big[\cos 2t \cdot \sin t - \cos 2t \cdot \frac{\sin 3t}{3}$

$\quad + \sin 2t \cdot \frac{\cos 3t}{3} - \sin 2t \cdot \cos t\Big]$

$=$ ---- → Use undetermined co-efficient

$=$ ----- method

$\therefore\ y_p = -\sin t$

→ Now, $y = y_c + y_p$

$\therefore \boxed{y(t) = C_1 \cdot \cos 2t + C_2 \cdot \sin 2t - \sin t.}$; $C_1, C_2 =$ constants

* → From eq^n ②, $2x = Dy - \cos t$.

$\therefore\ x = \frac{1}{2}\left[\frac{d}{dt}(y) - \cos t\right].$

where, $\frac{d}{dt}(y) = \frac{d}{dt}\left[C_1 \cos 2t + C_2 \sin 2t - \sin t\right]$

$= -2C_1 \cdot \sin 2t + 2C_2 \cdot \cos 2t - \cos t$

$\therefore\ x = \frac{1}{2}\left[-2C_1 \sin 2t + 2C_2 \cdot \cos 2t - \cos t - \cos t\right]$

$\therefore\ x = \frac{1}{2}\left\{-2C_1 \cdot \sin 2t + 2C_2 \cos 2t - 2\cos t\right\}$

when $t=0$, $x=0$ & $y=1$.

$\therefore$ $x = -\sin 2t + \cos 2t - \cos t$

& $y = \cos 2t + \sin 2t - \sin t$.

$\therefore x(t) = c_2 \cdot \cos 2t - c_1 \cdot \sin 2t - 2\cos t$.

Ans

Ex:: Solve the simultaneous differential equations

$\frac{dx}{dt} = 5x + y$; $\frac{dy}{dt} = y - 4x$. — (1)

→ Let $\frac{d}{dt} = D$. $\therefore$ the given d. eqⁿ becomes,

$\therefore \begin{cases} Dx - 5x - y = 0 \text{ and} \\ Dy + 4x - y = 0. \end{cases}$

$\therefore$ $x(D-5) - y = 0$ (2) and

$4x + y(D-1) = 0$ (3)

→ Multiplying (2) with 4 and (3) with $(D-5)$ we get,

$4x(D-5) - 4y = 0$.

$4x(D-5) + y(D-1)(D-5) = 0$

$-\quad - \quad -$

$-4y - y[D^2 - 5D - D + 5] = 0$.

$\therefore [-D^2 + 6D - 5 - 4]y = 0$.

$\therefore (D^2 - 6D + 9)y = 0$.

$\therefore$ The A.E. is $m^2 - 6m + 9 = 0$.

$\therefore m = 3, 3$.

$\therefore$ $\boxed{y(t) = c_1 \cdot e^{3t} + c_2 \cdot e^{3t} \cdot t}$; $c_1, c_2 =$ constants

$= y_c$

Ans (1).

→ From eqⁿ (3), $x = \frac{(1-D)y}{4} = \frac{1}{4}(y - Dy)$

$\therefore \quad x = \frac{1}{4}\Big[\left(c_1 e^{3t} + c_2 \cdot e^{3t} \cdot t \right) - \big(3 c_1 \cdot e^{3t} +$

$c_2 \cdot e^{3t}(1) + 3 \cdot c_2 e^{3t} \cdot t\big)\Big]$

$\therefore \quad x = \frac{e^{3t}}{4}\left[c_1 + c_2 \cdot t - 3 \cdot c_1 - c_2 - 3c_2 \cdot t \right]$

$\therefore \quad \boxed{x = \frac{1}{4} \cdot e^{3t}\left(-2c_1 - 2c_2 t - c_2 \right)}$ — Ans (2)

Ex:- Solve the simultaneous diff. eqⁿ.

$\frac{d^2x}{dt^2} + y = \sin t \quad , \quad \frac{d^2y}{dt^2} + x = \cos t$

→ Here, Given, $\frac{d^2x}{dt^2} + y = \sin t$, $\frac{d^2y}{dt^2} + x = \cos t$ —— (1)

let, $\frac{d}{dt} = D$, then (1) becomes,

$D^2x + y = \sin t$ —— (2)

$D^2y + x = \cos t$ —— (3)

Now, multiplying (3) by D^2,
and subatracting it from (2),

$$\begin{aligned} D^2x + y &= \sin t \\ D^2x + D^4y &= D^2\cos t \\ \hline \end{aligned}$$

$D^2 \cdot \cos t = D(-\sin t) = -\cos t$

$y - D^4 y = \sin t - (-\cos t)$

$\therefore -(D^4y - y) = \sin t + \cos t$

$\therefore \quad (D^4 - 1)y = -(\cos t + \sin t)$ —— (4)

$R(t)$

Date
Page

$\therefore$ The A.E. is, $m^4 - 1 = 0$

$\therefore m^4 = 1$

$\therefore m = \pm 1, \pm i$

$\therefore y_c = c_1 \cdot e^t + c_2 \cdot e^{-t} + c_3 \cdot \cos t + c_4 \cdot \sin t$

c_1, c_2, c_3, c_4 = constants

→ Since $R(t) = -(\cos t + \sin t)$, the choice of $y_p = A\cos t + B \cdot \sin t$, but y_p contains the functions of y_c, therefore we have to modify the choice of y_p as follows:

$$y_p = t\,[A\cos t + B \cdot \sin t]$$

$$\therefore y_p' = t\,(-A \cdot \sin t + B\cos t) + (A\cos t + B\sin t)\,(1)$$

$$\therefore y_p' = \cos t\,[B \cdot t + A] + \sin t\,[B - At]$$

$$\therefore y_p'' = \cos t\,[B + 0] + [Bt + A](-\sin t) + \sin t\,[0 - A] + [B - At]\cos t$$

$$\therefore y_p'' = \cos t\,[B + B - At] + \sin t\,[-A - Bt - A]$$

$$\therefore y_p'' = 2B \cdot \cos t - At \cdot \cos t - 2A \cdot \sin t - Bt \cdot \sin t$$

$$\therefore y_p'' = 2B \cdot \cos t - 2A \cdot \sin t - t\,[B \cdot \sin t + A \cdot \cos t]$$

$$\therefore y_p''' = -2B \cdot \sin t - 2A \cdot \cos t - \{t\,[B\cos t - A\sin t] + [B\sin t + A\cos t](1)\}$$

$$\therefore y_p''' = -2B\sin t - 2A\cos t - t\,(B\cos t - A\sin t) - B\sin t - A\cos t$$

$$\therefore y_p''' = -3B\sin t - 3A\cos t - t\,[B\cos t - A\sin t]$$

$\therefore y_p^{iv} = -3B\cos t + 3A\cdot\sin t - \{t\,[-B\sin t - A\cos t] + (B\cos t - A\sin t)(1)\}$

$\therefore y_p^{iv} = -3B\cos t + 3A\sin t + t\,(B\sin t + A\cos t) - B\cos t + A\sin t$

$\therefore y_p^{iv} = -4B\cos t + 4A\sin t + t\,[B\sin t + A\cos t]$

→ Substituting these values in (4),

$y_p^{iv} - y_p = -[\cos t + \sin t]$.

$\therefore -4B\cos t + 4A\sin t + \cancel{Bt\cdot\sin t} + At\cdot\cos t - \cancel{At\cdot\cos t} - \cancel{Bt\cdot\sin t} = -\cos t - \sin t$.

$\therefore \Rightarrow -4B\cos t = -\cos t$ and $4A\cdot\sin t = -\sin t$

$\Rightarrow \boxed{B = \frac{1}{4}}$ $\Rightarrow \boxed{A = \frac{-1}{4}}$

$\therefore y_p = t\left[-\frac{1}{4}\cos t + \frac{1}{4}\cdot\sin t\right]$.

$\therefore$ The g. sol^n of (4) is given by,

$y(t) = y_c + y_p$

$\therefore y(t). = c_1 e^{t} + c_2 e^{-t} + c_3\cdot\cos t + c_4\cdot\sin t$. $\left(-\frac{1}{4}\cos t + \frac{1}{4}\sin t\right)t$

$\therefore y(t) = c_1\cdot e^{t} + c_2\cdot e^{-t} + c_3\cdot\cos t + c_4\cdot\sin t + \frac{1}{4}t(\sin t - \cos t)$

Ans (1)

Date
Page

→ From eq. (3), $x = \cos t - D^2 y$
where, for $D^2 y$.

$$Dy = c_1 \cdot e^{t} - c_2 \cdot e^{-t} - c_3 \sin t + c_4 \cdot \cos t + \frac{1}{4}[\cos t + \sin t]t + \frac{1}{4}(\sin t - \cos t)(1)$$

$$D^2 y = c_1 \cdot e^{t} + c_2 \cdot e^{-t} - c_3 \cos t - c_4 \sin t + \frac{1}{4}[-\sin t + \cos t] \cdot t + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t - \sin t)$$

$$\cancel{\therefore x = \cos t - D^2 y}$$

$$\cancel{\therefore x(t) = \cos t - c_1 \cdot e^{t} - c_2 \cdot e^{-t} + c_3 \cos t + c_4 \sin t - \frac{1}{4}[-\sin t + \cos t]}$$

$$\cancel{\therefore x(t) = \cos t - c_1 \cdot e^{t} - c_2 \cdot e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4} \cdot \sin t - \frac{1}{4} \cdot \cos t}$$

$$\cancel{\therefore x(t).}$$

$$\therefore D^2 y = c_1 \cdot e^{t} + c_2 \cdot e^{-t} - c_3 \cos t - c_4 \sin t - \frac{1}{4} \cdot t \sin t + \frac{1}{4} \cdot t \cdot \cos t$$

$$+ \frac{1}{4}\cos t + \cancel{\frac{1}{4}\sin t} + \frac{1}{4}\cos t - \cancel{\frac{1}{4}\sin t}$$

$$\therefore D^2 y = c_1 \cdot e^{t} + c_2 \cdot e^{-t} - c_3 \cos t - c_4 \sin t + \frac{t}{4}[-\sin t + \cos t] + \frac{1}{2}\cos t.$$

From $x = \cos t - D^2 y$.

$$x(t) = \cos t - c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t - \frac{t}{4}[-\sin t + \cos t] - \frac{1}{2}\cos t.$$

$\therefore$ $x(t) = \frac{1}{2}\cos t - c_1 e^{t} - c_2 e^{-t} + c_3 \cos t$
$+ c_4 \sin t - \frac{1}{4} t [\cos t - \sin t]$. — Ans (2)

Ex:- Solve the simultaneous diff. eq^n.
$\frac{dx}{dt} = 2y$ and $\frac{dy}{dt} = 2x$

→ Given, $\frac{dx}{dt} - 2y = 0$ and $\frac{dy}{dt} - 2x = 0$. —— (1)

→ Now, let $\frac{d}{dt} = D$.
$\therefore$ (1) becomes, $Dx - 2y = 0$ and $Dy - 2x = 0$
(2) (3)

→ multiplying (2) with 2 and (3) with D,
we get

$$2\cancel{Dx} - 4y = 0$$
$$-2\cancel{Dx} + D^2y = 0$$

$$(D^2 - 4)y = 0.$$

$\therefore$ → The A. E. is $m^2 - 4 = 0$.
$\therefore m^2 = 4$
$\therefore m = \pm 2$.

$\therefore y_c = c_1 \cdot e^{2t} + c_2 \cdot e^{-2t}$; c_1, c_2 = Constants

$\therefore y(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{-2t}$ —— Ans (1).

→ Now, from eqⁿ 3, $x = \frac{Dy}{2}$

where, $Dy = 2c_1 \cdot e^{2t} - 2c_2 \cdot e^{-2t}$

$\therefore x = c_1 \cdot e^{2t} - c_2 \cdot e^{-2t}$ ——— Ans 2.