Probabilistic Discounting Models: A Comprehensive Comparison

Prioteasa Cristi Andrei
Matrikelnummer: 4740844
qf323@stud.uni-heidelberg.de

Denisa Radu
Matrikelnummer: 4278920
of300@stud.uni-heidelberg.de

March 2, 2025

Both students require a full grading.

Contents

1 Project Overview

This project explores probabilistic discounting in decision-making through hierarchical Bayesian modeling using PyMC, focusing on predicting both choice behavior and reaction times in tasks involving certain versus uncertain rewards. The model architecture incorporates parameters at group, participant, and trial levels, with the core choice model employing hyperbolic discounting complemented by loss aversion (λ) to account for differential processing of gains and losses. Key findings include: (1) incorporating loss aversion significantly improved choice prediction accuracy from 0.55 to 0.612; (2) data standardization was essential for model convergence but resulted in small β_1 parameters, leading to minimal variations in reaction time predictions; (3) absolute value difference ($DD_{absolute} = |V_{uncertain} - V_{certain}|$) outperformed squared value difference for measuring decision difficulty, resolving numerical instabilities; (4) participant-specific variance in reaction times (σ_{RT}) showed minimal trial-to-trial variability, supporting a simpler model where σ_{RT} is participant-specific; and (5) model comparison using BIC and AIC indicated marginal improvements from the quadratic RT model over the linear model, but our hypothesis testing indicated that the quadratic model is not significantly better than the linear model.

1.1 Project Structure

The project has the following structure:

- **/src/**: Main code directory
  - **probabilistic_discounting.ipynb**: Core implementation of choice and RT models
  - **probabilistic_discounting_prediction.ipynb**: Model predictions and part of evaluation
  - **probabilistic_validation.ipynb**: Model comparison and validation using BIC and AIC and hyptohesis testing
  - **probabilistic_discounting_debugging.ipynb**: Debugging and parameter analysis
  - **utils.py**: Helper functions for data processing and analysis (including K-fold cross-validation)
  - **/models/**: Saved model traces
- **/data/**: Contains probabilistic discounting experimental data
- **/pdfs/**: Documentation and supplementary materials

2 Data Preprocessing and Standardization

Section written by Denisa Radu

2.1 Data Source

This experiment is a probability discounting task, commonly used in decision-making research to study how individuals weigh certain vs. uncertain rewards. Participants must choose between:

- **certain**: smaller certain reward (guaranteed)
- **uncertain**: larger uncertain reward (which they might receive with some probability)

2.2 Preprocessing

Our `read_dd_data` function processes and standardizes probability discounting data through the following steps (see table 1):

Preprocessing Steps	**Output Variables**
1. Load & Extract Data	`rcert`: Certain reward magnitude
2. Standardize Column Names	`runcert`: Uncertain reward magnitude
3. Clean Data (remove incomplete)	`event_prob`: Probability of uncertain reward
4. Recode Choices (0=certain, 1=uncertain)	`choice`: Decision (0=certain, 1=uncertain)
5. Index Participants	`condition`: Scenario type (1=gain, 2=loss)
6. Reduce Dataset (optional):	`rt`: Response time (ms)
• 10 participants	`participant_idx`: Subject identifier
• 100 trials/participant	
7. Standardize Values	

Table 1: Data Preprocessing Pipeline and Output Variables

2.3 Data Standardization

Data standardization is a crucial step in preprocessing that improves model performance by ensuring variables have comparable scales and distributions.

1. **Log-Transform Reaction Times** Reaction times (rt) tend to be **right-skewed**, meaning some participants take significantly longer than others to respond. To normalize the distribution and reduce skewness, we apply a **natural logarithm transformation**:

$$rt' = \log(rt)$$

   This transformation **compresses extreme values**, making the reaction time distribution closer to normal, which improves statistical analysis and model performance.

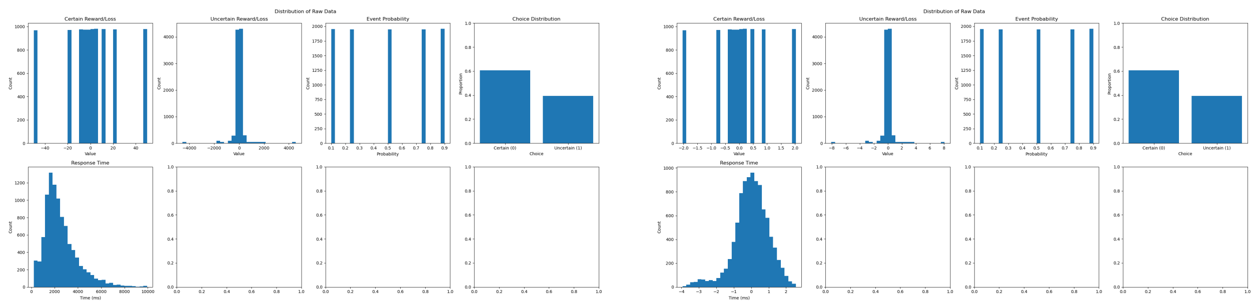
2. **Z-Score Standardization** After log-transforming reaction times, we apply **Z-score standardization** to ensure that all continuous variables have **zero mean and unit variance**. Standardization is computed using:

$$X' = \frac{X - \mu}{\sigma}$$

where:

- X is the original value,
- μ is the **mean** of the variable,
- σ is the **standard deviation**.

This ensures that all variables have comparable scales, preventing any single feature from dominating the model due to its larger numerical range.



(a) Data before Standardization

(b) Data after Standardization

Figure 1: Comparison of Data Distribution Before and After Standardization

2.4 Analysis of Standardization

The graphs show **data distributions before and after standardization**.

1. **Certain Reward/Loss Distribution:** The values of the certain reward (`rcert`) are widely spread before standardization, ranging from large negative losses to high positive rewards. After standardization, the values are centered around zero with unit variance. The gaps in the histogram represent discrete reward values that were originally far apart but have been scaled.

   Effect of Standardization:

   - **Distribution Shape:** Standardization preserves the shape but rescales it to have a mean of 0 and standard deviation of 1.
   - **Model Impact:** This ensures that `rcert` does not dominate due to having larger absolute values.

2. **Uncertain Reward/Loss Distribution:** The uncertain reward (`runcert`) shows extreme values before standardization. After standardization, the distribution has been compressed and brought to a standard scale.

Effect of Standardization:

- **Training Balance:** Standardization prevents extreme values from skewing model training.
- **Impact Control:** Large uncertain rewards no longer have disproportionate impact.

3. **Response Time Distribution:** Response time (`rt`) distribution is right-skewed before standardization. After standardization, log transformation reduced skewness, followed by Z-score standardization centering values around 0.

   Effect of Standardization:

   - **Distribution Normality:** The transformed reaction times follow a more normal distribution.
   - **Weight Balance:** This prevents the model from incorrectly weighing long reaction times too heavily.

3 Model Overview

Section written by Cristi Andrei Prioteasa

This section provides a comprehensive comparison of probabilistic discounting models for choice behavior and reaction time prediction. The models are hierarchical, with parameters at the group, participant, and trial levels. We start by implementing simpler models, then extend them to more complex models and compare them using BIC and AIC. We finally conduct a hyptohesis testing on the influence of the quadratic model over the linear model.

3.1 Choice and Reaction Time Models

Table 2: Comparison of Model Components, Parameters, and Specifications

Model Component	Model Variants	Group Parameters	Participant Parameters	Trial-level Parameters
Choice Models				
Default	Hyperbolic Discounting	μ_k, σ_k, μ_β, σ_β	k, β	$V_{uncertain} = \frac{r_{uncertain}}{1+kP}$ $V_{certain} = r_{certain}$
Loss-Aversion	Hyperbolic Discounting with Loss Aversion	μ_k, σ_k, μ_β, σ_β, μ_λ, σ_λ	k, β, λ (loss_aversion)	For gains: $V_{uncertain} = \frac{r_{uncertain}}{1+kP}$ $V_{certain} = r_{certain}$ For losses: $V_{uncertain} = \frac{\lambda r_{uncertain}}{1+kP}$ $V_{certain} = \lambda r_{certain}$
Reaction Time Models				
Linear	Linear RT based on dd	μ_{β_0}, σ_{β_0}, μ_{β_1}, σ_{β_1}, $\sigma_{\sigma_{RT}}$	β_0, β_1, σ_{RT}	$\mu_j = \beta_0 + \beta_1 \text{dd}$
Quadratic	Quadratic RT based on dd	μ_{β_0}, σ_{β_0}, μ_{β_1}, σ_{β_1}, μ_{β_2}, σ_{β_2}, $\sigma_{\sigma_{RT}}$	β_0, β_1, β_2, σ_{RT}	$\mu_j = \beta_0 + \beta_1 \text{dd} + \beta_2 \text{dd}^2$
Sigma-per-trial	Linear RT with trial-specific variability	μ_{β_0}, σ_{β_0}, μ_{β_1}, σ_{β_1}, $\sigma_{\sigma_{RT}}$, γ	β_0, β_1, σ_{RT}	$\mu_j = \beta_0 + \beta_1 \text{dd}$ $\sigma_{RT_{trial}} = \sigma_{RT} \cdot \exp(\gamma \cdot \text{dd})$
Decision Difficulty Models				
Default	Squared value difference	N/A	N/A	$(SV_{uncertain} - SV_{certain})^2$
Absolute-value	Absolute value difference	N/A	N/A	$\lvert SV_{uncertain} - SV_{certain}\rvert$

Note: 'dd' refers to decision difficulty, calculated using either squared value difference or absolute value difference between subjective values of uncertain and certain options.

4 Modeling Choices and Justifications

4.1 Choice Model Decisions

Section written by Denisa Radu

Our modeling approach encompassed three key components: choice models, reaction time models, and decision difficulty models. The initial implementation utilized a basic hyperbolic discounting model, which was subsequently enhanced with a loss aversion component to incorporate the condition variable present in the dataset. This extension yielded substantial improvements in model performance, increasing mean prediction accuracy from **0.55** to **0.61**.

Hyperbolic Discounting: We selected this model for its established success in capturing probability discounting behavior. It allows for diminishing effect of probability on subjective values, and its parameter k represents individual differences in discounting rates. We experiment with different group level priors for the discounting parameter k and the choice sensitivity parameter β, which we will discuss in the experimental results section.

Subjective Values: Subjective values are computed as:

$$SV_{certain} = r_{cert} \tag{1}$$

$$SV_{uncertain} = \frac{r_{uncert}}{1 + k_p \times p_{event}} \tag{2}$$

where r_{cert} is the certain reward, r_{uncert} is the uncertain reward, k_p is the participant-specific discounting parameter, and p_{event} is the probability of the uncertain event.

Probability of Choosing the Uncertain Option: The probability of choosing the uncertain option is modeled with a logistic function:

$$P(uncertain) = \frac{1}{1 + e^{-\beta_p(SV_{uncertain} - SV_{certain})}} \tag{3}$$

where β_p is the participant-specific choice sensitivity parameter.

Loss Aversion Extension: We added this component to capture asymmetric valuation of gains and losses, incorporating prospect theory principles (i.e. people are more sensitive to losses than gains). The loss aversion parameter λ typically exceeds 1, indicating greater sensitivity to losses, making it crucial for modeling condition effects (gain vs. loss framing). By solely implementing a loss aversion parameter, we can see that the model is able to better capture the observed behavior of the participants (in terms of choice prediction accuracy), as we will see in the experimental results section.

$$V_{certain} = \begin{cases} r_{cert} & \text{if gain condition} \\ -\lambda_p \times r_{cert} & \text{if loss condition} \end{cases} \tag{4}$$

$$V_{uncertain} = \begin{cases} \frac{r_{uncert}}{1+k_p \times p_{event}} & \text{if gain condition} \\ -\lambda_p \times \frac{r_{uncert}}{1+k_p \times p_{event}} & \text{if loss condition} \end{cases} \tag{5}$$

where λ_p is the participant-specific loss aversion parameter.

Hierarchical Structure: This modeling approach accounts for individual differences while leveraging group-level trends. It improves parameter estimation for participants with fewer trials and allows partial pooling of information across participants. We experiment with different group level priors are analyse the in-group variance of the individual parameters.

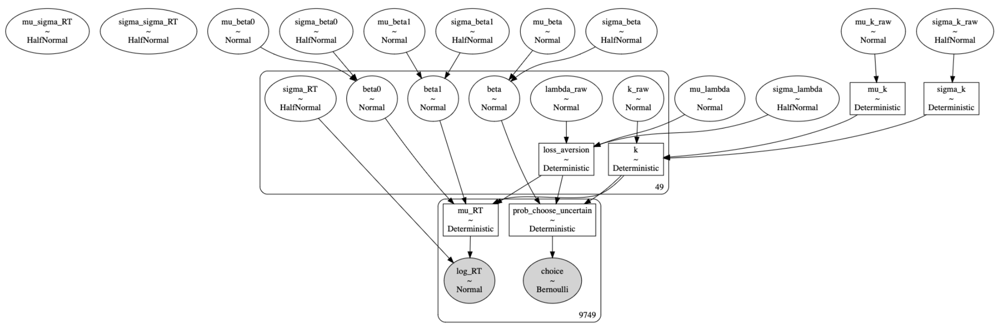


Figure 2: Example of used configuration for hierachical modeling.

4.2 Reaction Time Model Decision

For the next section, we use a standardized log version of the reaction time data (see Hick's Law 1952), as it results in more reasonable parameter estimates. We detail further on our decision in the experimental results section. Our approach to reaction time modeling evolved based on observations and model comparison:

Section written by Cristi Andrei Prioteasa

Linear RT Model (Baseline): Our initial model was built on evidence that decision difficulty affects response time. We modeled log-RT to account for the right-skewed nature of RT distributions. The model includes participant-specific intercepts and slopes to capture individual differences in response patterns. Formally, for each trial j:

$$\log(RT_j) \sim \mathcal{N}(\mu_j, \sigma_{RT})$$
$$\mu_j = \beta_0 + \beta_1 \cdot \mathrm{dd}_j$$

where dd_j is the decision difficulty for trial j, β_0 is the participant-specific intercept, β_1 is the participant-specific slope, and σ_{RT} is the residual standard deviation. Note that for this version we model the standard deviation as participant specific and not trial specific. Moreover, our framework allows for both uncertainty estimation (sampling from the posterior of this distribution), but in our results we only report the mean.

In order to assess the quality of prediction, we exponentiate and back-transform the predicted log-RT to the original RT scale.

Reason for choosing the linear model: one of the most well-established findings in cognitive psychology and decision science is that harder decisions take longer (e.g. see Ratcliff and Rouder, 1998). Humans differ in their baseline speed of responding (intercept β_0). They also differ in how much decision difficulty affects their RT (slope β_1). By using a hierarchical framework, we account for general processing speed differences between individuals and variability in sensitivity to decision difficulty (some people slow down more than others for difficult choices).

Section written by Denisa Radu

Quadratic RT Model (Extension): We extended the linear model to capture potential non-linear effects of decision difficulty on RT. The theoretical motivation was that very easy and very difficult decisions might both lead to fast responses, with moderate difficulty resulting in slower responses. This model adds one additional parameter per participant (β_2) to capture the quadratic relationship.

$$\log(RT_j) \sim \mathcal{N}(\mu_j, \sigma_{RT})$$
$$\mu_j = \beta_0 + \beta_1 \cdot \mathrm{dd}_j + \beta_2 \cdot \mathrm{dd}_j^2$$

where β_2 is the participant-specific quadratic effect parameter.

Reason for choosing the quadratic model: The linear model assumes a linear relationship between decision difficulty and RT, which may not fully capture the observed data. The quadratic model allows for a nonlinear relationship, potentially capturing the idea that very easy or very difficult decisions might both lead to faster responses, with moderate difficulty resulting in slower responses.

Section written by Cristi Andrei Prioteasa

Sigma-per-trial Model (Variance Extension): This model was developed based on the hypothesis that RT variability might depend on decision difficulty. It allows the standard deviation parameter to vary systematically across trials. A single global parameter (γ) modulates the effect of difficulty on variability, providing a more flexible approach to modeling response time distributions.

$$\mu_{RT} = \beta_{0p} + \beta_{1p} \times \mathrm{dd}_j \tag{6}$$
$$\sigma_{RT_{trial}} = \sigma_{RTp} \times (1 + \gamma \times \mathrm{dd}_j) \tag{7}$$
$$\log(RT) \sim \mathcal{N}(\mu_{RT}, \sigma_{RT_{trial}}) \tag{8}$$

where γ is a global parameter that modulates how decision difficulty affects RT variability.

In our experiments, we show little variation in the standard deviation across trials (see figure 3), and we therefore decided to model it as participant specific.

Reason for choosing the sigma-per-trial model: The sigma-per-trial model allows for a more flexible approach to modeling response time distributions, potentially capturing the idea that RT variability might depend on decision difficulty and not being constant across trials. This turned out to carry little information, and we therefore decided to model it as participant specific (see (Bogacz et al., 2006).

Although DDMs and LBA models are mechanistic, empirical studies show that a linear approximation of RT vs. difficulty often fits well.

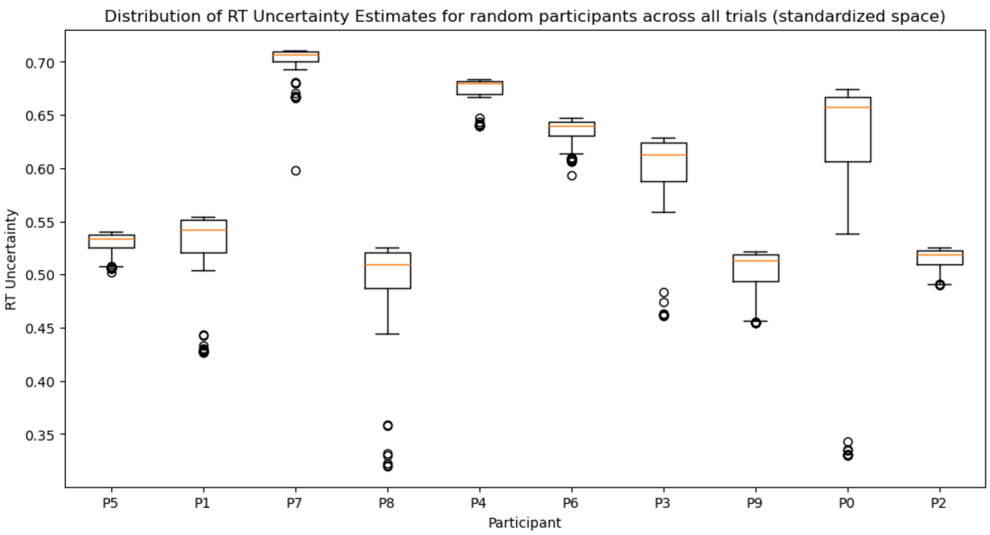


Figure 3: The plot shows the distribution of RT uncertainty modeled per trial.

4.3 Decision Difficulty Formulations

Section written by Denisa Radu

We explored two approaches to quantifying decision difficulty:

Squared Value Difference: Default approach based on the idea that any deviation from equal values increases difficulty. This formulation provides symmetric treatment of SV differences, regardless of which option has higher value. This leads to problematic numerical issues with the model, and we therefore decided to use the absolute value difference instead.

$$DD_{default} = (V_{uncertain} - V_{certain})^2 \tag{9}$$

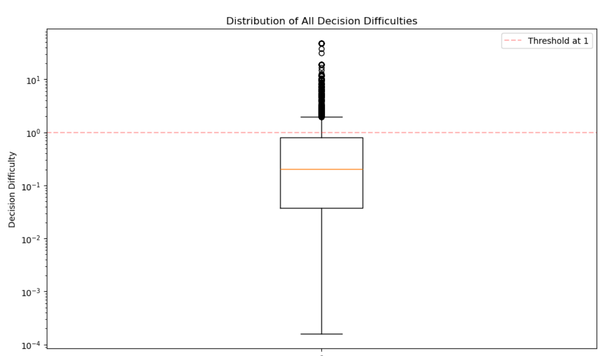


Figure 4: Distribution of decision difficulty values using squared value difference (note the log scale).

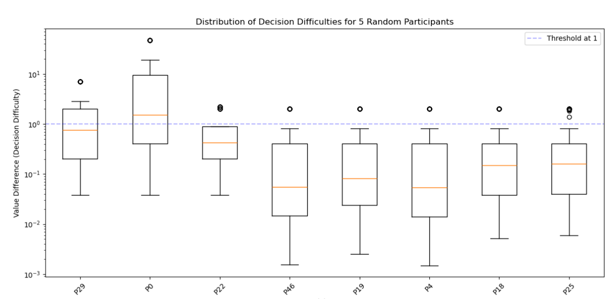


Figure 5: Distribution of decision difficulty across random participants.

Absolute Value Difference: Alternative formulation that gives linear rather than quadratic scaling with value differences. This approach was found to provide better fit in our experimental testing, solving numerical issues with the squared value difference and leading to more spread parameter estimates.

$$DD_{absolute} = |V_{uncertain} - V_{certain}| \tag{10}$$

5 Parameter Details

Table 3: Model Parameters and Their Descriptions

Group-level Parameters	
μ_k & σ_k	Mean and standard deviation of the hyperbolic discounting parameter across participants
μ_β & σ_β	Mean and standard deviation of the choice sensitivity parameter
μ_λ & σ_λ	Mean and standard deviation of the loss aversion parameter
μ_{β_0} & σ_{β_0}	Mean and standard deviation of the RT intercept
μ_{β_1} & σ_{β_1}	Mean and standard deviation of the RT slope (linear effect of difficulty)
μ_{β_2} & σ_{β_2}	Mean and standard deviation of the quadratic RT effect
$\sigma_{\sigma_{RT}}$	Group-level variability in the RT standard deviation
γ	Global parameter relating decision difficulty to RT variability (in sigma-per-trial model)
Participant-level Parameters	
k	Individual discounting parameter (how much each participant discounts uncertain outcomes)
β	Individual choice sensitivity parameter
λ	Individual loss aversion parameter (loss_aversion)
β_0	Individual RT intercept
β_1	Individual RT slope (linear effect of difficulty)
β_2	Individual quadratic effect parameter (in quadratic model)
σ_{RT}	Individual RT variability
Trial-level Parameters	
Value difference	$V_{uncertain} - V_{certain}$ (computed for each trial)
Decision difficulty	Absolute or squared value difference (per trial)
$\sigma_{RT_{trial}}$	Trial-specific RT variability (in sigma-per-trial model)

6 Experimental Results

In this section, we present the results of our experiments. We start by describing a series of experiments that lead to the current findings, that are not included in the submitted version of the code. The results of the sigma-per-trial model are not included in this section, as it did not provide any additional insights.

6.1 Numerical instability issues

Section written by Cristi Andrei Prioteasa

Numerical instability issues: The first and most problematic issue encountered was the skewness of the parameters. By using non standardized data, the parameters were very skewed and the model was not able to converge. We therefore decided to use a standardized version of the data, which resulted in more reasonable parameter estimates. This leads to our beta1 parameter being very small, which in turn creates almost identical RT predictions for all trials of a participant (see figure 10).

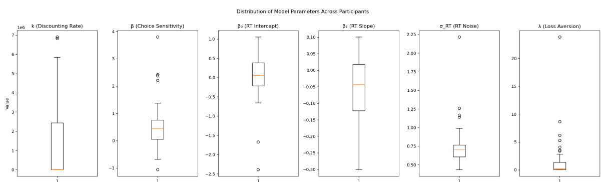


Figure 6: Distribution of model parameters for linear model.

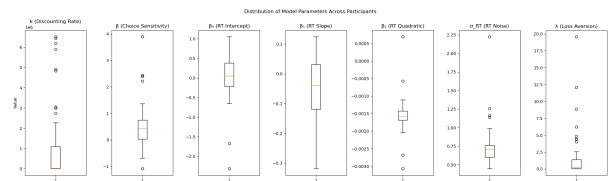


Figure 7: Distribution of model parameters for quadratic model.

6.2 Modeling the priors and hyperprior distributions

Section written by Cristi Andrei Prioteasa

Modeling the priors and hyperprior distributions: We experimented with different priors and hyperprior distributions, leading to the current configuration. These priors were chosen to be weakly informative while maintaining numerical stability. The normal distributions for mean parameters allow for both positive and negative effects, while the half-normal distributions ensure positive standard deviations. The specific scale parameters were selected based on the standardized data and initial model explorations (see table 5).

Note: The choice of reaction time model (linear vs. quadratic) did not significantly impact choice accuracy, as the choice component was modeled identically in both cases, sharing only one common parameter with the reaction time component (k).

6.3 Reward and loss experimental condition

Section written by Denisa Radu

We experimented with two reward discounting models, one which does not take into account the condition (we let the sensitivity parameter β take negative values which correspond to the loss regime) and one model in which we explicitly define a loss aversion parameter λ for the loss regime,

Table 4: **Hierarchical Model Parameters and Prior Distributions**

Level	Parameter	PyMC Distribution	Description
Group	μ_k	Deterministic(exp(μ_{k_raw}))	Mean discounting rate
	μ_{k_raw}	Normal($\mu = 0, \sigma = 1$)	Log-space mean discounting rate
	σ_k	Deterministic(exp(σ_{k_raw}))	SD of discounting rate
	σ_{k_raw}	HalfNormal($\sigma = 1$)	Log-space SD of discounting rate
	μ_β	Normal($\mu = 0, \sigma = 1$)	Mean choice consistency
	σ_β	HalfNormal($\sigma = 0.5$)	SD of choice consistency
	μ_{β_0}	Normal($\mu = 0, \sigma = 1$)	Mean RT intercept
	σ_{β_0}	HalfNormal($\sigma = 0.5$)	SD of RT intercept
	μ_{β_1}	Normal($\mu = 0, \sigma = 5$)	Mean RT slope
	σ_{β_1}	HalfNormal($\sigma = 2$)	SD of RT slope
	μ_λ	Normal($\mu = 1, \sigma = 0.5$)	Mean loss aversion
	σ_λ	HalfNormal($\sigma = 0.5$)	SD of loss aversion
Participant	k_i	Deterministic(exp($\mu_k + \sigma_k \cdot k\_raw_i$))	Discounting rate
	k_raw_i	Normal($\mu = 0, \sigma = 1$)	Standardized discounting rate
	β_i	Normal($\mu = \mu_\beta, \sigma = \sigma_\beta$)	Choice consistency
	β_{0i}	Normal($\mu = \mu_{\beta_0}, \sigma = \sigma_{\beta_0}$)	RT intercept
	β_{1i}	Normal($\mu = \mu_{\beta_1}, \sigma = \sigma_{\beta_1}$)	RT slope for difficulty
	σ_{RT_i}	HalfNormal($\sigma = \sigma_{\sigma_{RT}}$)	RT variability
	λ_i	Deterministic(exp($\mu_\lambda + \sigma_\lambda \cdot \lambda\_raw_i$))	Loss aversion
Trial	$\mathrm{SV}_{uncertain}$	Deterministic	Uncertain option subjective value
	$\mathrm{SV}_{certain}$	Deterministic	Certain option subjective value
	σ_{RT_trial}	Deterministic($\sigma_{RT_i} \cdot (1 + \gamma \cdot \text{difficulty})$)	Trial-specific RT variability

Table 5: Parameters in the probabilistic discounting model organized by hierarchical level. Group-level parameters characterize population distributions, participant-level parameters capture individual differences, and trial-level parameters model specific trial outcomes. The "sigma-per-trial" configuration incorporates trial-specific RT variability based on decision difficulty.

defined as in equation 4. We note an accuracy improvement of the loss aversion model over the no loss aversion model of 0.12.

We note that this analysis was only performed for the choice component of the model, not for reaction times. The reaction time model used the same discounting parameter k regardless of whether the trial involved gains or losses. Future work could explore whether having separate discounting parameters for gains and losses in the reaction time component would improve RT predictions.

6.4 Model Comparison and hypothesis testing

Section written by Cristi Andrei Prioteasa

We compare our models using the BIC score and AIC score (we leave the sigma-per-trial model out of this comparison as it did not provide any additional insights).

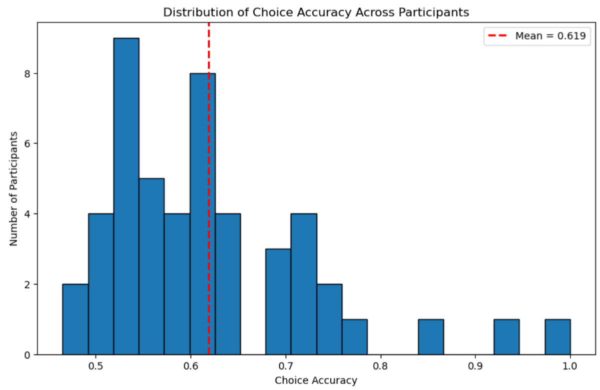


Figure 8: Accuracy histogram for the linear/quadratic model.

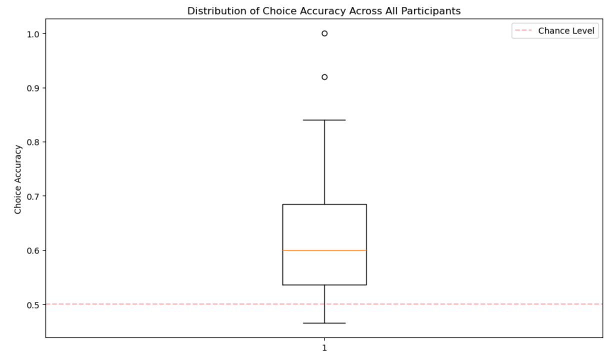


Figure 9: Box plot of accuracy for the linear/quadratic model.

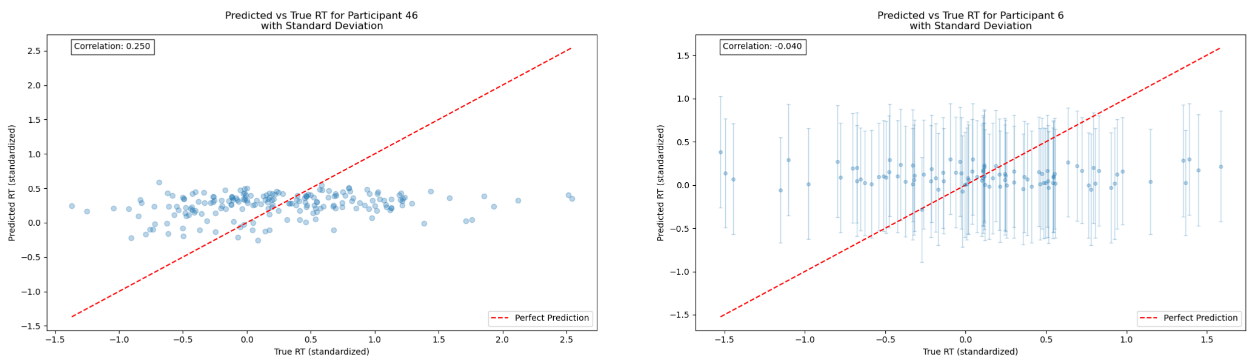


Figure 10: Comparison of predicted vs observed reaction times for both the linear model (left) and sigma-per-trial model (right). The diagonal lines represent perfect prediction.

Model	**BIC**	**AIC**
Linear RT Model	1751.88	1742.06
Quadratic RT Model	1724.16	1709.44

The linear model shows better performance with a lower BIC value, indicating improved fit even when accounting for the additional parameter. This is clear to see when we look at the energy plots (figure 11a) and pair plots figure 11c, bigger overlap means better fit).

Using Bayesian analysis, we examined the significance of the quadratic term (β_2) in improving reaction time predictions. The analysis revealed no substantial evidence supporting the inclusion of the quadratic term, with a group-level mean of -0.0031 and a 95% Highest Density Interval (HDI) of [-0.0096, 0.0015]. Since the HDI contains zero, we cannot reject the null hypothesis that $\beta_2 = 0$ at the group level. Furthermore, only 10% of participants demonstrated a significant individual quadratic effect, suggesting this pattern is rare in our sample. These findings, combined with the BIC and AIC scores, indicate that the linear model provides a sufficient fit for reaction time predictions, and the added complexity of the quadratic term is not empirically justified. The data suggests that reaction times change approximately linearly with decision difficulty, prompting us to favor the simpler linear model for subsequent analyses.

$$H_0 : \text{The quadratic term does not improve RT predictions } (\beta_2 = 0) \tag{11}$$

$$H_1 : \text{The quadratic term significantly improves RT predictions } (\beta_2 \neq 0) \tag{12}$$

6.5 Qualitative Findings

Section written by Denisa Radu

We observed the following:

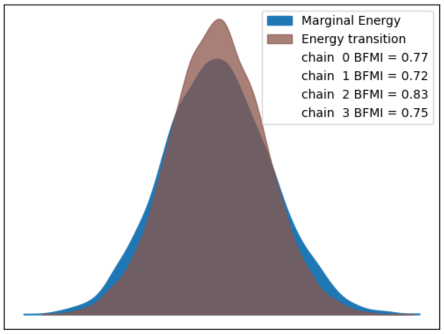
- **Reaction Time Variability**: Sigma for reaction time varied significantly across participants but not across trials. Modeling sigma per trial did not show much variance (approximately 0.6 ± 0.01).
- **Decision Difficulty Range**: Decision difficulty values were concentrated around small values (approximately 0.1, due to standardization of the data), which might limit the model's ability to capture RT differences based on difficulty.
- **RT Prediction Issues**: The RT prediction output is the same value for each trial of a participant, suggesting potential issues with the model sensitivity to trial-level factors (small β_1 parameter combined with small decision difficulty values).

7 Implementation Details

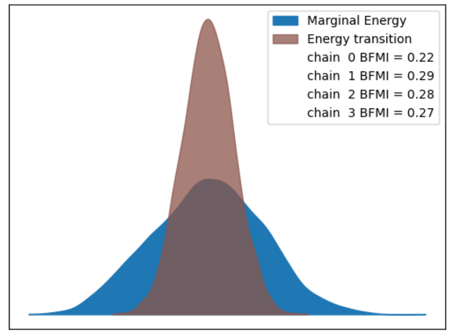
In this study, we implemented hierarchical Bayesian models using PyMC, a probabilistic programming framework that enables flexible specification and efficient sampling of complex statistical models. Our implementation leveraged PyMC's ability to express models as directed acyclic graphs where nodes represent random variables. We defined a three-level hierarchical structure with group-level priors (e.g., $\mu_{k\_raw} \sim \text{Normal}(0, 1)$, $\sigma_\beta \sim \text{HalfNormal}(0.5)$), participant-level parameters (e.g., $k \sim \text{Deterministic}(\exp(\mu_k + \sigma_k \cdot k_{raw}))$, $\beta \sim \text{Normal}(\mu_\beta, \sigma_\beta)$), and trial-level likelihoods. The model incorporated non-centered parameterizations for parameters like discounting rate and loss aversion to improve sampling efficiency. For posterior sampling, we employed PyMC's implementation of the No-U-Turn Sampler (NUTS), a self-tuning variant of Hamiltonian Monte Carlo that automatically adapts step sizes and trajectory lengths to efficiently explore the posterior distribution. We ran 4 parallel chains with 2000 samples each (after 1000 tuning steps) using a target acceptance rate of 0.90, which helped achieve better mixing across complex parameter spaces. PyMC managed the computational complexities behind the scenes, including automatic differentiation through PyTensor (formerly Theano), parallelized sampling, convergence diagnostics, and integration with ArviZ for posterior analysis, allowing us to focus on model specification rather than implementation details.

In order to assess the quality of our convergence, we used the following diagnostics:

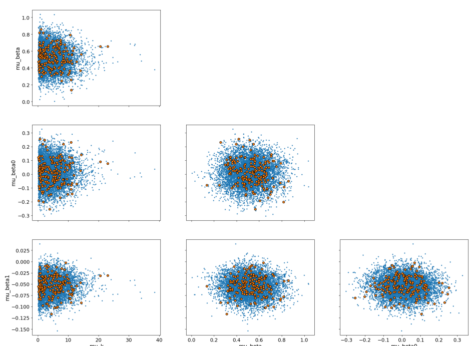
- **R-hat**: All R-hat values were close to 1, indicating good convergence.
- **Effective Sample Size**: All effective sample sizes were greater than 1000, indicating adequate sampling.
- **Traceplots**: All traceplots showed good mixing and no evidence of non-convergence.



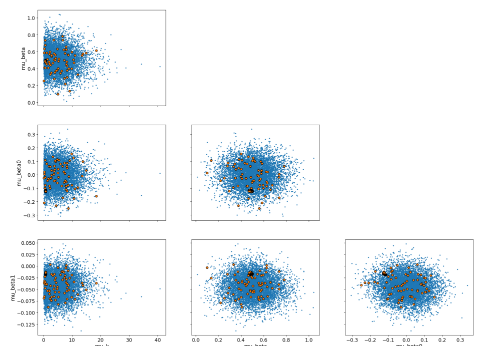
(a) Energy plot for the linear model



(b) Energy plot for the quadratic model



(c) Pair plot for the linear model



(d) Pair plot for the quadratic model

Figure 11: Convergence diagnostics for the linear and quadratic models.

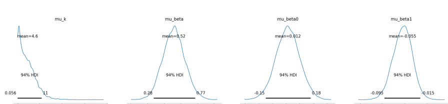


Figure 12

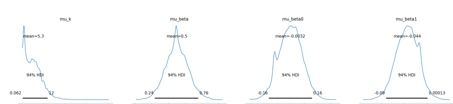


Figure 13

Figure 14: Posterior distributions comparing parameter estimates between linear (left) and quadratic (right) models. The plots show the marginal posterior distributions for key model parameters, illustrating the uncertainty in our parameter estimates.

8 Conclusion

This study provides a comprehensive analysis of probabilistic discounting models incorporating both choice behavior and reaction times. Our key findings include:

- **Model Performance**: The addition of loss aversion to the basic hyperbolic discounting model significantly improved predictive accuracy from 0.55 to 0.619, demonstrating the importance of accounting for gain/loss asymmetries in decision-making.
- **Hierarchical Structure**: Our hierarchical Bayesian approach successfully captured individual differences while leveraging group-level trends, providing robust parameter estimates even for participants with fewer trials.
- **Reaction Time Modeling**: Because of numerical issues and parameter sensitivity, we were not able to obtain a good fit for the reaction time data.