Night4: Optimization and Gradient Ascent

Quantitative Engineering Analysis

Spring 2019

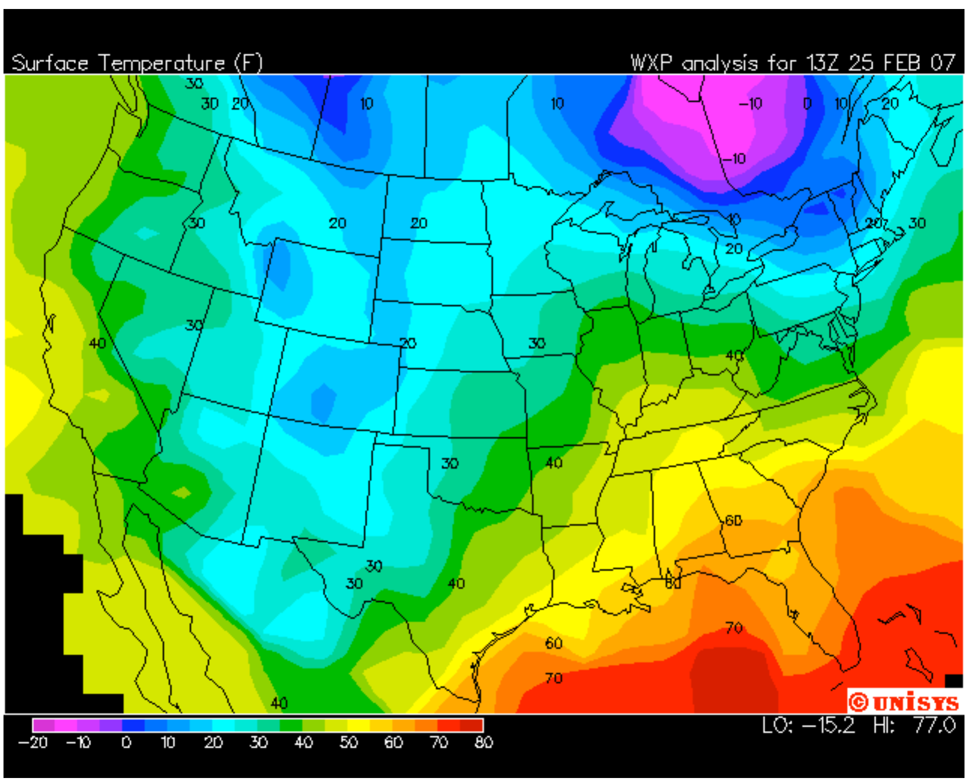
1 Learning Goals

By the end of this assignment, you should feel confident with the following:

- Extending the idea of optimization
- Partial derivatives and the gradient
- Unconstrained optimization: concept and the method of steepest ascent

2 Conceptual Exercise: The Leisure Seeker [1 Hr]

The map below gives the temperature across the United States on a certain winter day. Regions of the same color have the same temperature: violet represents the coldest areas, and temperatures rise as the colors traverse the spectrum from indigo to blue to green to yellow to orange to red.



Locate Chicago on the map and mark it with a dot. The weather in Chicago is freezing in winter, so a resident of the city decides to embark on a journey in search of the sun. From Chicago, she wants to travel in the direction in which temperature rises most quickly.

Exercise (1) In which direction should she go? Draw an arrow, starting at Chicago, that indicates this direction.

As her journey proceeds, she decides to keep traveling in the direction in which the weather warms up most quickly: wherever she is at any moment, she moves in the direction of fastest temperature rise.

Exercise (2) Make a rough sketch of the route she takes.

Exercise (3) Where does she end up, assuming that she doesn't leave the United States?

Exercise (4) What would happen if her friend started in Billings, Montana? Where would he end up?

3 *Readings, Videos, and Conceptual Questions - Partials and Gradients [3 Hrs]*

At this link you will find a set of readings and videos about partial derivatives, the gradient, and the Hessian. Some of this is stuff you already dealt with in the boats and faces modules; some of it is new. Read the text and/or watch these videos, and then write short qualitative answers to the following questions:

Exercise (5) What is meant by f_x? By $\frac{\partial^2 f}{\partial x^2}$? By $D_u f$? By ∇f?

Exercise (6) In Stewart, the idea of gradient is discussed primarily in two dimensional and three dimensional "physical" spaces, using $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$, and so forth. But more generally, you can have a gradient of a function of any number of variables. Give a real-world example of a gradient for a situation that involves more than 3 variables.

Exercise (7) "The gradient always points in the direction of fastest increase." Can you think of physical examples where there is *more than one* direction of fastest increase? What's going on here?

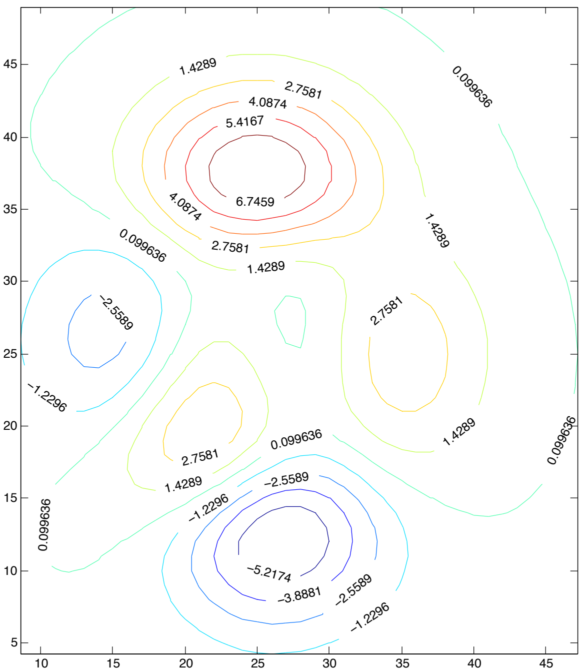
Exercise (8) "The gradient is always normal to level curves/surfaces." Explain.

Exercise (9) "The directional derivative in the direction of $\mathbf{u}$ is given by $\nabla f \cdot \hat{\mathbf{u}}$." Why does this make sense?

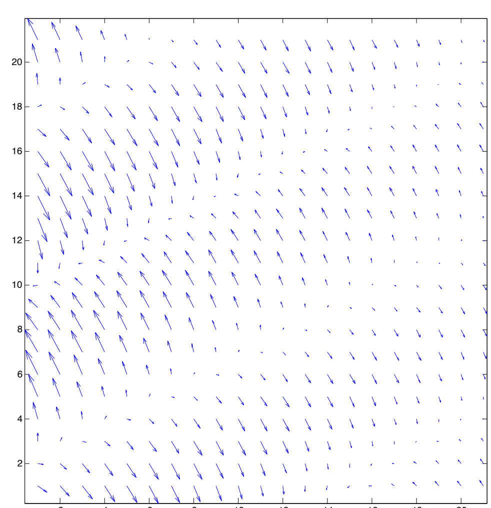
Exercise (10) If the gradient is zero, does that imply that you are at a max or a min? Why or why not?

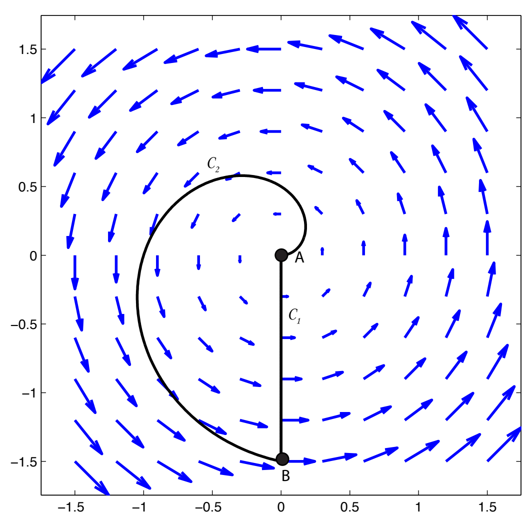
Exercise (11) What does it mean for $\frac{\partial^2 f}{\partial x^2} > 0$? What about $\frac{\partial^2 f}{\partial x \partial y} > 0$?

Exercise (12) The diagram below shows a contour plot for a function. Sketch in the gradient field.

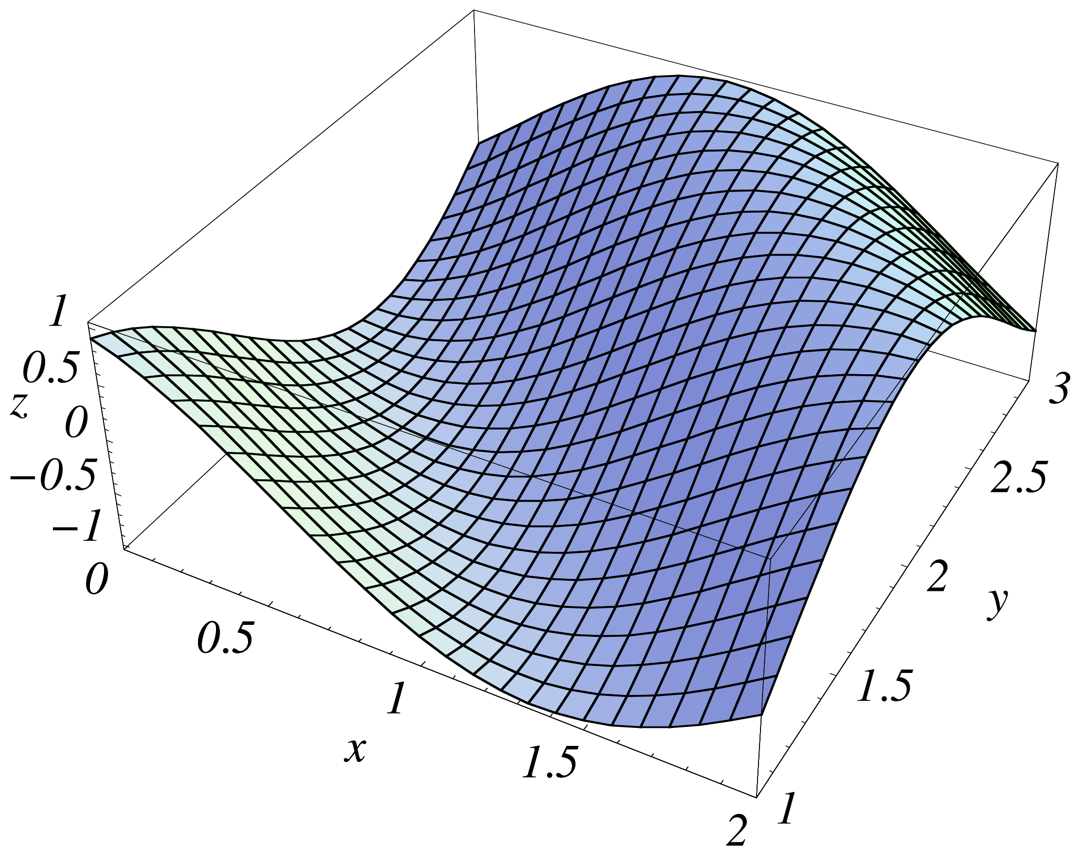


Exercise (13) The diagrams below show two vector fields. One is the gradient of a function of two variables; the other is not. Which one could be a gradient? Why? Sketch in the level curves for the one that works.





Exercise (14) A function f has the graph shown below. The grid lines on the graph are lines along which either the x- or the y- coordinate is held constant.



Determine the sign of each of the following:

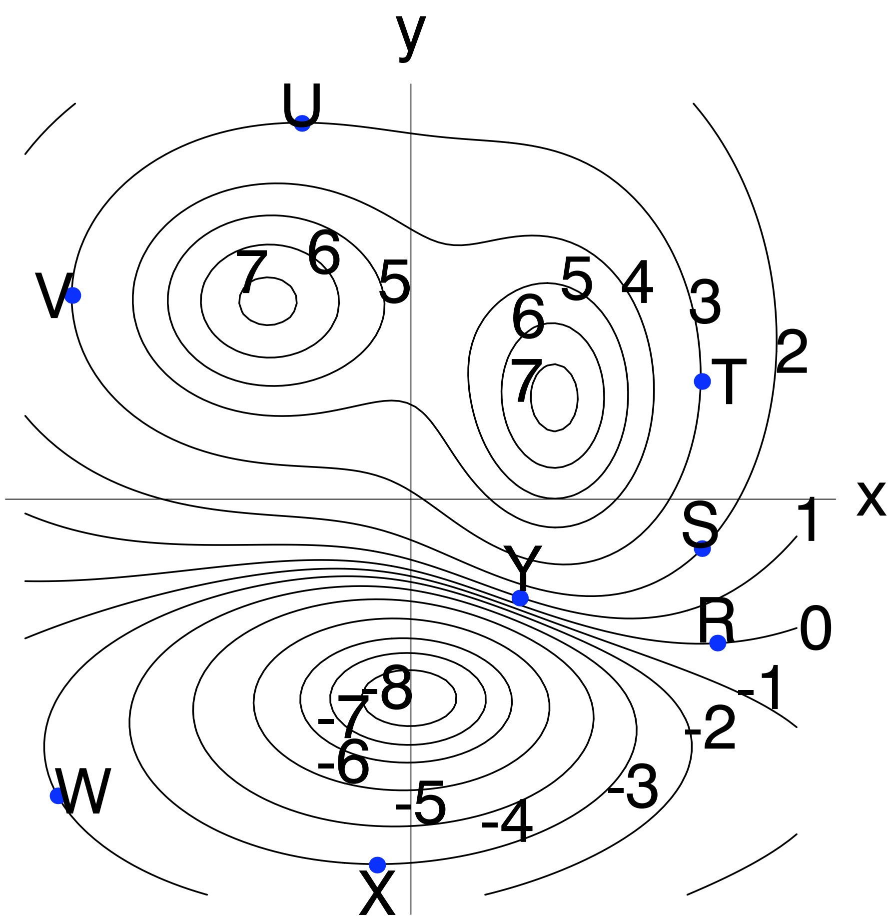
(a) $f_x(1,2)$

(b) $f_y(1,2)$

(c) $f_{xx}(1,2)$

(d) $f_{yy}(1,2)$

Exercise (15) The diagram below shows some level curves of a function $g(x,y)$. The numbers indicate the g-values of these level curves, and the letters indicate points on the level curves. Note that point Y is on the level curve $g = 1$ and point W is on the level curve $g = -2$.



(a) What are the signs of the partial derivatives $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at each of the points marked? Why?

(b) At which of the points marked does the gradient vector ∇g have the greatest magnitude? Explain.

(c) Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The directional derivative $D_{\hat{\mathbf{u}}} g$ is zero at exactly one of the points marked. Which point is it?

4 *Readings, Videos, and Conceptual Questions - Optimization with Gradient Ascent [2 Hrs]*

At this link you will *also* find a set of readings and videos about gradient ascent (or descent - same thing, but for a sign!).

Read the text from Giordano and watch the videos. If you are interested, you can also read the stuff about conjugate gradient ascent, which is cool and leverages eigenstuff - but is optional!

Then answer the following questions:

Exercise (16) At the origin, a given function has a gradient of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and a Hessian of

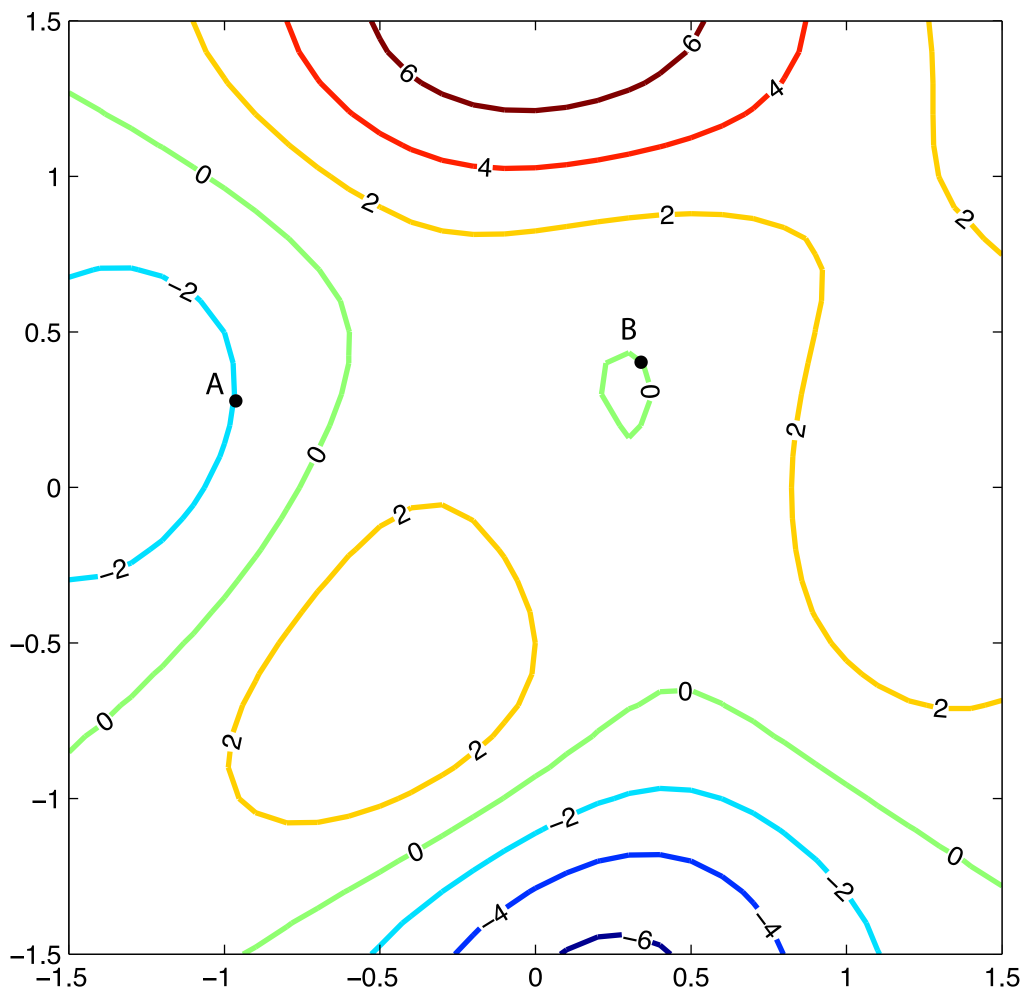
$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Sketch a contour plot of the function in the vicinity of the origin.

Exercise (17) The figure below shows a contour plot with two points marked (A and B). For both points,

(a) Draw the path that gradient ascent would use if the step size was small (following the approach in the first video).

(b) Draw the path that gradient ascent would follow if the algorithm is implemented as shown in the second video.



5 *Gradient Ascent [2 Hrs]*

As you've just been learning, Gradient Ascent (or descent) is a technique to determine the maximum or minimum of a function of many variables by taking steps in the direction of the gradient (or negative gradient). If the height of a surface is described by $z = f(\mathbf{r})$, where $\mathbf{r}$ is the position vector in the plane, and we begin at $\mathbf{r}_0$, the points determined by Gradient Ascent are given by

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \lambda_i \nabla f(\mathbf{r}_i), \ i = 0, 1, 2, \ldots$$

where λ_i is the relative size of the step that we take in the direction of the gradient. There are various schemes for choosing these, and one of the simplest is to determine the next step with a simple proportionality

$$\lambda_{i+1} = \delta \lambda_i$$

where both δ and λ_0 are thoughtfully chosen for the problem at hand. We are going to develop a method and implementation to drive your NEATO on the floor of the classroom in a way that physically realizes the method of Steepest Ascent. First, we are going to introduce the mountain you will climb, and you will think through the steps involved.

18. The mountain you will "climb" is defined as follows

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$$

where x, y, and f are measured in feet. You are required to start at $(1, -1)$, and work your way to the top by method of steepest ascent.

(a) Visualize the contours of this function in MATHEMATICA on the domain $(-3, 1) \times (-3, 1)$, and print it out.

(b) Draw the path of steepest ascent if we were moving continuously from a starting point at $(1, -1)$.

(c) Find the gradient of this function.

(d) Assuming $\mathbf{r}_0 = (1, -1)$, what is the initial gradient at $\mathbf{r}_0$? What would be a reasonable choice for λ_0 so that $\mathbf{r}_1$ is not too far from the continuous path? Plot $\mathbf{r}_1$ on your contour plot.

(e) Assuming you place your NEATO at $(1, -1)$ pointing along the y-axis, how much do you have to rotate it in order to align it with the gradient at $\mathbf{r}_0$? What would be a reasonable angular speed?

(f) Assuming that you are going to drive your NEATO at $0.1m/s$, how long would you drive in order to reach $\mathbf{r}_1$? (Careful with unit changes!)

(g) What is the gradient at $\mathbf{r}_1$? What value of δ should you use so that λ_1 and $\mathbf{r}_2$ are reasonable? Plot $\mathbf{r}_2$ on your contour plot.

(h) Assuming your NEATO is now at $\mathbf{r}_1$, how much do you have to rotate it in order to align it with the new gradient? What would be a reasonable angular speed?

(i) Assuming that you are going to drive your NEATO at $0.1 m/s$, how long would you drive in order to reach $\mathbf{r}_2$?