CSS215: DISCRETE MATHEMATICS

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CLASS

BSc. ICTM & BSc. ITS

Propositional Logic

Definition

Logic is the study of reasoning. It is also defined as the science of the correctness or incorrectness reasoning, or the study of methods and principles used to distinguish good (correct) from bad (incorrect) reasoning.

Definition

A statement or proposition is a declarative sentence which is either true or false but not both. We use the lower case letters of alphabet (such as p, q, r) to represent statements.

Examples

- *Mzumbe University is in Morogoro.* *Proposition*
- *It is raining today.* *Proposition*
- *Get up and do your homework.* *Command*
- *What is the atomic weight of carbon?* *Question*
- *Let us go to town today.* *Proposal*
- *What a beautiful morning!* *Exclamation*

Propositinal Logic . . .

Definition

Truth Value: The truth value of a statement is the truthfulness or falsity of a statement or proposition. A statement has a truth value of true T if it is true and a truth value of false F if it is false.

Definition

Compound statement: A compound statement is a statement formed by combining two or more simple statements called components. For example, "If it rains outside, then I will not go to class" is a compound statement. The components are: "It rains outside" and "I will not go to class."

Definition

Primitive statement: A primitive statement is a statement that cannot be broken down into simpler statements. For example, "Morogoro is in Tanzania" is a primitive statement.

Definition

Truth table: A truth table is a table that represent the relationships between truth values of propositions and compound propositions formed from these propositions.

Propositinal Logic ...

Logical connectives

They are used to connect several statements into a single statement. The fundamental logical connectives are:

- Negation ($\neg$, NOT)
- Conjunction ($\wedge$, AND)
- Disjunction ($\vee$, OR)
- Exclusive or ($\oplus$)
- Implication ($\rightarrow$, conditional)
- Biconditional ($\leftrightarrow$)

Negation

Definition

Let p be a proposition. The **negation of p**, denoted by $\neg p$, is the statement

"It is not the case that p."

The proposition $\neg p$ read as "**not p**." The truth value of $\neg p$, is the opposite of the truth value of p.

Examples

Find the negation for each of the followuing statement:

1). *SUA is in Morogoro.*

2). *Today is Friday.*

3). $2 + 5 = 8$.

4). *Daladala stops running at 9:00pm.*

Solution

1). *It is not the case that SUA is in Morogoro. (SUA is not in Morogoro)*
2). *It is not the case that today is Friday. (Today is not Friday)*
3). $2 + 5 \neq 8.$
4). *It is not the case that daladala stops running at 9:00pm.*

Truth Table for Negation of Proposition

p	$\neg p$
T	F
F	T

Conjuction

Definition

Let p and q be propositions. The **conjuction** of p and q, denoted by $p \wedge q$, is the proposition "p and q". The proposition $p \wedge q$ is true when both p and q are true and false otherwise.

Examples

1. *SUA is in Morogoro and* $5 + 2 = 8$.
2. *The sun is shining and it is raining.*
3. *13 is a perfect square and 9 is a prime.*

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Definition

Let p and q be propositions. The **disjuction** of p and q, denoted by $p \vee q$, is the proposition "p or q". The proposition $p \vee q$ is false when both p and q are false and true otherwise.

Examples

1. *SUA is in Morogoro or* $5 + 2 = 8$.
2. *The sun is shining or it is raining.*
3. *13 is a perfect square or 9 is a prime.*

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive OR

NB:

$p \vee q$ (the or is used inclusively, i.e.,$p \vee q$ is true when either p or q or both are true).

Definition

Let p and q be propositions. The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Truth Table for $p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

Definition

Let p and q be propositions. The proposition "p **implies** q" denoted by $\mathbf{p} \rightarrow \mathbf{q}$ is called **implication**. It is false when p is true and $q$is false and is true otherwise.

In $p \rightarrow q$, p is called hypothesis and q is called conclusion.

Truth Table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \rightarrow q$ is read in a variety of equivalent ways:

- "if p then q".
- "p only if q".
- "p is sufficient for q".
- "q whenever p".
- "q neccessary for p".
- "q follows from p".

Examples

What are truth values of the following conditional statement.

1. *If Samia Suluhu is president of Tanzania, then 4 is prime number.*
2. *If today is Friday, then $2 \times 3 = 6$.*

Converse, Contrapositive and Inverse

Definition

Let $p \to q$ be a conditional statement. Then

- The **converse** of $p \to q$ is the statement $q \to p$.
- The **contrapositive** of $p \to q$ is the statement $\neg q \to \neg p$.
- The **inverse** of $p \to q$ is the statement $\neg p \to \neg q$.

Example

What are the contrapositive, the converse and the inverse of the conditional statement. "The home team wins whenever it is raining".

Solution

We can rewrite the given statement as "If it is raining, then the home team wins."

Solution....

- *The contrapositive of this conditional statement is*
  "If the home team does not win, then it is not raining."
- *The converse is*
  "If the home team win, then it is raining."
- *The inverse is*
  "If it is not raining, then the home team does not win."

Biconditional

Definition

Let p and q be propositions. The **biconditional** $p \leftrightarrow q$ (**read** p **if and only if** q), is true when p and q have the same truth values and is false otherwise.

Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Constructing the truth table

Example

Construct a truth table for $(p \to q) \wedge (\neg p \leftrightarrow q)$

Solution

p	q	$\neg p$	$p \to q$	$\neg p \leftrightarrow q$	$(p \to q) \wedge (\neg p \leftrightarrow q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Tautology and Contradiction

Definition

A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.

Example

$p \vee \neg p$ is tautology.

Definition

A compound proposition that is always false for all possible truth values of the propositions is called a **contradiction**.

Example

$p \wedge \neg p$ is contradiction.

Equivalence

Definition

Two propositions are **equivalent** if their truth values in the truth table are the **same**.

Example

$p \to q$ is equivalent to $\neg q \to \neg p$ (contrapositive).

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Equivalent statements are important for logical reasoning since they can be substituted and can help us to make a logical argument.

Logical equivalence

Definition

The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table).

The notation $p \Leftrightarrow q$ denotes p and q are logically equivalent.

Important Logical Equivalences

- Identity
  - $p \wedge T \Leftrightarrow p$
  - $p \vee F \Leftrightarrow p$
- Domination
  - $p \vee T \Leftrightarrow T$
  - $p \wedge F \Leftrightarrow F$
- Idempotent
  - $p \vee p \Leftrightarrow p$
  - $p \wedge p \Leftrightarrow p$
- Double negation
  - $\neg(\neg p) \Leftrightarrow p$

- Commutative
  - $p \vee q \Leftrightarrow q \vee p$
  - $p \wedge q \Leftrightarrow q \wedge p$
- Associtative
  - $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
  - $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
- Distributive
  - $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- De Morgan's
  - $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
  - $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- Negation
  - $p \vee \neg p \Leftrightarrow T$
  - $p \wedge \neg p \Leftrightarrow F$
- Implication: $p \rightarrow q \Leftrightarrow \neg p \vee q$

Logical equivalence. . .

Example

Show $(p \wedge q) \rightarrow p$ is a tautology. In other words $((p \wedge q) \rightarrow p \Leftrightarrow T)$

Solution

By using truth table

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	*T*	*T*	*T*
T	*F*	*F*	*T*
F	*T*	*F*	*T*
F	*F*	*F*	*T*

Solution....

Using logical identities:

$$
\begin{aligned}
(p \wedge q) \rightarrow p &\Leftrightarrow \neg(p \wedge q) \vee p, \text{ by equivalence} \\
&\Leftrightarrow (\neg p \vee \neg q) \vee p, \text{ by De Morgan's law} \\
&\Leftrightarrow (\neg q \vee \neg p) \vee p, \text{ by commutative law} \\
&\Leftrightarrow \neg q \vee (\neg p \vee p), \text{ by associative law} \\
&\Leftrightarrow \neg q \vee T \\
&\Leftrightarrow T, \text{ by domination}
\end{aligned}
$$

Rules of Inference for Propositional Logic

A **Proof** in mathematics is valid argument that establish the truth of a mathematical statement. Rules of inference are templates for building valid arguments.

Definition

An **argument** is a sequence of propositions (called **premises**) followed by proposition (called **conclusion**). In short, an **argument** is a sequence of statements that end with a conclusion.

Definition

An **argument** is valid if the truth of all its premises implies that the conclusion is **true**. In other words, an argument is valid if the conclusion (final statement) follows from the truth of the preceding statements (premises).

Definition

An **argument form** in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid if no matter which propositions are substituted for the propositional variables in its premises, if the premises are all true, then the conclusion is true.

Rules of Inference...

Example (1)

"If it rains, I drive to school".

"It rains".

Therefore, "I drive to school". The argument belongs to the following form:

$$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

This form is valid no matter what propositions are substituted to the variables.

In other words, an argument form with premises $p_1, p_2, \cdots, p_n$ and the conclusion q is valid if and only if $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is **tautology**.

Example (2)

$((p \to q) \wedge p) \to q$ is tautology.

Rules of Inferences...

Some valid argument forms called **rules of inferences** are derived and can be used to construct complicated argument forms.

Definition

Rules of inferences are templates for constructing valid arguments. Inference means deriving conclusions from evidences.

Common rules of inferences...

- Modus Pones :

  $p \to q$
  $\underline{p \quad\quad}$
  $\therefore q$

- Modus Tollens:

  $\neg q$
  $\underline{p \to q}$
  $\therefore \neg p$

- Hypothetical Syllogism:

  $p \to q$
  $\underline{q \to r}$
  $\therefore p \to r$

- Disjunctive Syllogism:

  $p \vee q$
  $\underline{\neg p \quad\quad}$
  $\therefore q$

- Addition:

  p

  $\therefore p \vee q$

- Simplification:

  $p \wedge q$

  $\therefore p$

- Conjunction:

  p

  q

  $\therefore p \wedge q$

- Resolution:

  $p \vee q$

  $\neg p \vee r$

  $\therefore q \vee r$

Exercise

State which rule of inference is applied in each of the following argument

i) *If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.*
ii) *If you have the current password, then you can log onto the network. You have a current password. Therefore, you can log onto the network.*
iii) *If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.*
iv) *If it snows today, the university will close. The university will not close today. Therefore, it did not snow today.*
v) *Tofu is healthy to eat. Therefore, either tofu or cheeseburger is healthy to eat.*
vi) *I go swimming or eat an ice cream. I did not go swimming. Therefore, I eat an ice cream.*
vii) *It is cloudy and raining. Therefore, it is cloudy.*
viii) *If it is sunny today, then I go swimming today. I do not go swimming today. Therefore, it is not sunny today.*

Formal Proofs: using rules of inference to build arguments

Definition

A **formal proof** of a conclusion q given hypotheses $p_1, p_2, \ldots, p_n$ is a sequence of steps, each of which applies some inference rule to hypotheses or previously proven statements (antecedents) to yield a new true statement (the consequent).

A formal proof demonstrates that if the premises are true, then the conclusion is true.

Example

Show that the premises "If it is Tuesday today, then we do the test or quiz", "If the test venue is occupied, we do not do the test", "It is Tuesday today", and "the test venue is occupied" leads to the conclusion "we do the quiz or assignment".

Main steps:

1. Choose propositional variables.
   Let p: it is Tuesday today
   q: we do the test
   r: we do the quiz
   s: the test venue is occupied
   t: we do the assignment
2. Translation into propositional logic premises: $p \to (q \vee r)$, $s \to \neg q, p, s$
   Conclusion: $r \vee t$
3. Construct the Valid Argument.

Solution

1).	$p \to (q \vee r)$	*premise.*
2).	p	*premise.*
3).	$q \vee r$	*modus ponens of (1) and (2).*
4).	$s \to \neg q$	*premise.*
5).	s	*premise.*
6).	$\neg q$	*modus ponens of (4) and (5).*
7).	r	*disjunctive syllogism of (3) and (6).*
8).	$r \vee t$	*addition of (7).*

Valid Arguments

Example

Prove that each of the following argument is valid

1. $p \to q,\ \neg p \to r,\ r \to s \vdash \neg q \to s$.
2. $\neg p \wedge r,\ r \to p,\ \neg r \to s,\ s \to t \vdash t$
3. $p \to q,\ \neg p \to r,\ q \to s, \vdash \neg r \to s$.

Proof.

1).	$p \to q$	premise
2).	$\neg q \to \neg p$	contrapositive of (1).
3).	$\neg p \to r$	premise.
4).	$\neg q \to r$	hypothetical syllogism of (2) and (3).
5).	$r \to s$	premise.
6).	$\neg q \to s$	hypothetical syllogism of (4) and (5).

$\square$

Proof.

1).	$\neg p \wedge r$	premise
2).	$\neg p$	simplification of (1).
3).	$r \to p$	premise.
4).	$\neg r$	modus tollens (2) and (3).
5).	$\neg r \to s$	premise.
6).	s	modus pones of (4) and (5).
7).	$s \to t$	premise.
8).	t	modus pones of (6) and (7).

$\square$

Proof.

1).	$p \to q$	premise
2).	$\neg p \vee q$	logical equivalent of (1).
3).	$\neg p \to r$	premise.
4).	$p \vee r$	logical equivalent of (3)
5).	$q \vee r$	resolution of (2) and (4).
6).	$r \vee q$	commutativity of (5).
7).	$\neg r \to q$	logical equivalent of 6.
8).	$q \to s$	premise.
9).	$\neg r \to s$	hypothetical syllogism.

$\square$

Exercise

1. From the single proposition $p \wedge (p \rightarrow q)$. Show that q is a conclusion.
2. Show that the hypotheses: It is not sunny this afternoon and it is colder than yesterday. We will go swimming only if it is sunny. If we do not go swimming, then we will take a canoe trip. If we take a canoe trip, then we will be home by sunset, lead to the conclusion: we will be home by the sunset.
3. Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."
4. Show that the premises: If it is Saturday today, then we play soccer or basketball. If the soccer field is occupied, we dont play soccer. It is Saturday today, and the soccer field is occupied. Leads to the conclusion: "we play basketball or volleyball".

Exercise

1). *Negate the following propositions:*
   a. *It is raining today.*
   b. *There are other life forms on other planets in the universe.*
   c. *The summer in Morogoro is hot and sunny.*
2). *Show that* $(p \wedge q) \rightarrow (p \vee q)$ *is tautology.*
3). *Show that each of the following pair of propositions is logically equivalent.*
   a. $\neg(p \rightarrow q)$ *and* $(p \wedge \neg q)$
   b. $\neg(p \vee (\neg p \wedge q))$ *and* $\neg p \wedge \neg q$.
   c. $\neg(p \leftrightarrow q)$ *and* $\neg p \leftrightarrow q$.
   d. $\neg(p \leftrightarrow q)$ *and* $p \leftrightarrow q$.
4). *Show that* $\neg p \rightarrow (p \rightarrow q)$ *is tautology using:*
   a. *a truth table*
   b. *an equivalence proof.*

Predicate Logic

Definition

A **Predicate** or propositional function is a description of the property (or properties) a variable or subject of the statement may have. A proposition may be created from a propositional function by either assigning a value to the variable or by quantification.

Definition

The independent variable of a propositional function must have a **universe of discourse**, which is a set from which the variable can take values.

- In a statement "x is greater than 3"
  - x is **subject** and "is greater than 3" is **property (predicate)**.
- We can denote "x is greater than 3" by $P(x)$, where P denote the predicate "is greater than 3" and x is variable.
- $P(x)$ is called **propositional function at x.**
- Once a value has been assigned to x, $P(x)$ becomes a proposition and has truth value.

Example

Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution

We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$," is true. However, $P(2)$, which is the statement "$2 > 3$," is false.

A propositional function may associate with two or more variables.

Example

- Given $P(x, y)$ denote the statement "$x = y + 3$", then $P(1, 2)$ is false while $P(3, 0)$ is true.
- If $Q(x, y, z) : x + y = z$, then $Q(1, 2, 3)$ is true while $Q(2, 3, 4)$ is false.

A statement of the form $P(x_1, x_2, \cdots, x_n)$ is the value of the **propositional function** P at the n-tuple $(x_1, x_2, \cdots, x_n)$, and P is also called a n-**place predicate** or a **n-ary predicate**.

Note that: Notice that the universe of discourse must be defined for predicates.

Quantifiers

- A **quantifier** turns a propositional function into a proposition without assigning specific values for the variable.
- **Quantification** expresses the extent to which a predicate is true over a range of elements.

Definition

Universe (Domain) of discourse for predicate variable is a set of all values that may be assigned to the variable.

Definition

Let $P(x)$ be a predicate and U be a domain of x, then the **truth set** of $P(x)$ is a set of all elements x of U such that $P(x)$ is true.

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $P(x) : x$ is even. Then the truth set $P(x)$ is $\{2, 4, 6, 8, 10\}$.

Types of Quantifiers

Definition

The **universal quantification** of $P(x)$, denoted by $\forall x P(x)$, is the statement,

"$P(x)$ for all x in the domain."

- $\forall$ is called **universal quantifier.**
- We read as
  - "**for all** $x, P(x)$"
  - "**for every** $x, P(x)$"
  - "**for every** $x, P(x)$ **is true**"
- An element for which $P(x)$ is false is called a **counter example** of $\forall x P(x)$.

A counter example for $\forall x P(x)$ is an element $t \in U$ such that $P(t)$ is false.

Example

Let x be an apple. The sentence "All apples are green" can be written as "For every x, x is green." Using the universal quantifier $\forall, \forall x(x$ is green$)$ or $\forall x P(x)$ where $P(x) : x$ is green.

- $\forall x P(x)$ is true when $P(x)$ is true for every x in the domain.
- $\forall x P(x)$ is false when there is an x for which $P(x)$ is false.

Example

Let $P(x)$ be the statement "$x+1 > x$". What is the truth value of $\forall x P(x)$, the domain consists of all real numbers?

Solution

Because $P(x)$ is true for all real numbers x, then $\forall x P(x)$ is true.

Example

Let $Q(x)$ be the statement "$x < 2$". What is the truth value of $\forall x Q(x)$, the domain consists of all real numbers?

Solution

$Q(x)$ is not true for every number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$.

Example

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not execeeding 4?

Solution

The statement $\forall x P(x)$ is the same as conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers $1, 2, 3,$ and 4. Because $P(4)$, which is the statement ",$4^2 < 10$" is false, it follows that $\forall x P(x)$ is false.

Example

What is the truth value of $\forall x(x^2 \geq x)$ when the domain

a. consists of all real numbers?
b. consists of integers?

Solution

a. $\forall x(x^2 \geq x)$ *is false, because* $(x^2 \geq x)$ *is false for all real numbers in the range* $0 < x < 1$, *a counterexample* $(0.5)^2 \ngeq 0.5$
b. $\forall x(x^2 \geq x)$ *is true, for all integers* x, $(x^2 \geq x)$.

Types of Quantifiers...

Definition

The **existential quantification** of $P(x)$, denoted by $\exists x P(x)$, is the statement,

"There exists an element x in the domain such that $P(x)$"

- $\exists$ is called **existential quantifier.**
- We read as
  - "**for some** $x, P(x)$"
  - "**there is an** x**, such that** $P(x)$"
  - "**there is at least one** x **such that** $P(x)$ **is true**"
- $\exists x P(x)$ is true when there is an x for which $P(x)$ is true.
- $\exists x P(x)$ is false when $P(x)$ is false for every x in the domain.

Example

Rewrite the sentence "Some chalkboards are black" symbolically.

Solution

Let $U =$ a set of all chalkboards. Let x be an arbitrary chalkboard. Then, the sentence can be written as "There exists an x such that x is black." Using existential quantifier, we have $\exists x P(x)$ where $P(x) : x$ is black.

Example

Let $Q(x) : x > 3$. What is the truth value of $\exists x Q(x)$ when the domain consists of all real numbers?

Solution

$Q(x)$ is sometimes true, for example $Q(4)$ is true. Thus $\exists x Q(x)$ is true.

Definition

The **uniqueness quantification** of $P(x)$, denoted by $\exists! x P(x)$, is the statement,

"There exists a unique x such that $P(x)$."

- $\exists!$ is called **uniqueness quantifier.**
- We read as
  - "**There exists x such that $P(x)$** is true."
  - "**there is exactly one x, such that $P(x)$ is true**"
  - "**there is one and only one x such that $P(x)$ is true**"

Example

What is the truth value of the following quantifications, if the domain consists of all integers:

a. $\exists! x(x > 1)$
b. $\exists! x(x + 3 = 2x)$
c. $\exists! x(x^2 = 1)$
d. $\exists! x(x = x + 1)$

Solution

a) *There are some integers* $(x > 1)$ *is true, therefore* $\exists! x(x > 1)$ *is false.*

b) *There is only one integer, which is* $x = 3$ *such that* $(x + 3 = 2x)$ *is true, therefore* $\exists! x(x + 3 = 2x)$ *is true.*

c) *There are two integers, which are* $x = 1$ *and* $x = -1$ *such that* $(x^2 = 1)$ *is true, therefore* $\exists! x(x^2 = 1)$ *is false.*

d) *There is no integer such that* $(x = x + 1)$ *is true, therefore* $\exists! x(x = x + 1)$ *is false.*

Example

Let $D = \{4, 6, 9\}$, $p(x) : 5 \leq x \leq 10$, $q(x)$: is even. Find the true value of each of the following statement:

a. $\exists x\,(p(x) \wedge q(x))$
b. $\forall x\,(p(x) \wedge q(x))$
c. $\forall x\,(p(x) \vee q(x))$
d. $\exists x\,(\neg p(x) \wedge q(x))$
e. $\exists x\,(p(x) \wedge \neg q(x))$

a. $\exists x\,(p(x) \land q(x))$ means there some values of x such that $5 \leq x \leq 10$ and are even.
$x = 6$ $p(6)$ is true and $q(6)$ is true. So $\exists x\,(p(x) \land q(x))$ is true.
b. $\forall x\,(p(x) \land q(x))$ means all values of x satisfy $5 \leq x \leq 10$ and are even. $x = 9$ does not satisfies this property, since $p(9)$ is true and $q(9)$ is false. So $\forall x\,(p(x) \land q(x))$ is false.
c. $\forall x\,(p(x) \lor q(x))$ means x satisfy $5 \leq x \leq 10$ or even.
4 is even and 6 and 9 satisfy $5 \leq x \leq 10$. So $\forall x\,(p(x) \lor q(x))$ is true.
d. $\exists x\,(p(x) \land \neg q(x))$ means there are some values of x such that $5 \leq x \leq 10$ and are odd.
9 satisfies $5 < x < 10$ and is odd. So $\exists x\,(p(x) \land \neg q(x))$ is true.
e. $\exists x\,(\neg p(x) \land q(x))$ means there are some values of x such that $x < 5$ or $x > 10$ and the value of x are even.
$4 < 5$ and is even. So $\exists x\,(\neg p(x) \land q(x))$ is true.

Nested Quantifiers

Example

Rewrite each of the following proposition symbolically where the universe of discourse is the set of all real numbers.

a. For each integer x, there exists an integer y such that $x + y = 0$.
b. There exists an integer x such that $x + y = y$ for every integer y.
c. For all x and y, $xy = yx$.
d. There are integers x and y such that $x + y = 5$.

Solution

a. $\forall x \exists y(x + y = 0)$
b. $\exists x \forall y(x + y = y)$
c. $\forall x \forall y(xy = yx)$
d. $\exists x \exists y(x + y = 5)$

Remarks

(a) The order of variables x and y in $\forall x \forall y$ and $\exists x \exists y$ can be interchanged without affecting the the truth values of predicate. Therefore, for predicate $P(x, y)$
  - $\forall x \forall y P(x, y) = \forall y \forall x P(x, y)$
  - $\exists x \exists y P(x, y) = \exists y \exists x P(x, y)$

(b) The order is important in $\forall x \exists y P(x, y)$ and $\exists y \forall x P(x, y)$

Example

Let $P(x, y) : x < y$ where x and y are integers.

- $\forall x \exists y P(x, y)$ means "for every integer x, there is a suitable y such that $x < y$." $y = x + 1$ is such an integer. Therefore, $\forall x \exists y P(x, y)$ is true, BUT
- $\exists y \forall x P(x, y)$ means "there exists an integer y, say b such that every integer x is less than b." Clearly it is false.

Quantifiers and Connectives

For any prescribed universe any statements $P(x)$, $Q(x)$ in variable x

1. $\forall x[P(x) \wedge Q(x)] \equiv \forall x P(x) \wedge \forall x Q(x)$
2. $\exists x[P(x) \vee Q(x)] \equiv \exists x P(x) \vee \exists x Q(x)$

Negating Quantifiers

For any prescribed universe any statements $P(x)$, $Q(x)$ in variable x

1. $\forall x \neg[\neg P(x)] \equiv \forall x P(x)$
2. $\exists x \neg[\neg P(x)] \equiv \exists x P(x)$
3. $\neg \forall x P(x) \equiv \exists x \neg P(x)$
4. $\neg \exists x P(x) \equiv \forall x \neg P(x)$
5. $\neg[\forall x \neg P(x)] \equiv \exists x P(x)$
6. $\neg[\exists x \neg P(x)] \equiv \forall x P(x)$

Rules of inference for quantified statements

Some valid argument forms, called **rules of inferences**, are derived and can be used to construct complicated argument forms.

Definition

Rules of inferences are templates for constructing valid arguments.
Inference means deriving conclusions from evidences.

Common rules of inferences for quantified statements

- Universal Instantiation

$$\frac{\forall x P(x)}{\therefore P(c)}$$

- Universal Generalization

$$\frac{P(c)}{\therefore \forall x P(x)}$$

- Existential Instantiation

$$\frac{\exists x P(x)}{\therefore P(c)}$$

- Existential Generalization

$$\frac{P(c)}{\therefore \exists x P(x)}$$

Rules of inference for quantified statements

Exercise

State which rule of inference is applied in each of the following argument

1. *All women are wise. Therefore, Judith is wise.*
2. *Student s has taken calculus. Therefore, all students has taken calculus.*
3. *There is a fish in the pool. Therefore, some fish c is in the pool.*
4. *All dogs are cute. Therefore, his dog is cute.*
5. *There is a person in the store. Therefore, some person c is in the store.*
6. *His dog is playing in the park. Therefore, There is a dog playing in the park.*
7. *Verediana got an A in the class. Therefore, someone got an A in the class*
8. *All students are smart. Therefore, Dani is smart.*
9. *Student c has a personal computer. Therefore, all students has personal computers.*

Exercise

1. Use the rules of inference to construct a valid argument showing that the conclusion "Someone who passed the first exam has not read the book." follows from the premises "A student in this class has not read the book." "Everyone in this class passed the first exam."
2. Using the rules of inference, construct a valid argument to show that "Gauss has two legs" is a consequence of the premises: "Every man has two legs." "Gauss is a man."
3. Show that the following argument is valid. There is a student such that if he knows programming, then he knows Java. All students know programming. Therefore, there is a student that knows either Java or $C++$.
4. Show that the premises "Everyone in this Foundations of Analysis class has taken a course in Basic Mathematics" and "Job is a student in this class" imply the conclusion "Job has taken a course in Basic Mathematics. "
5. Assume that "For all positive integer n, if n is greater than 4, then n^2 is less than 2^n" is true. Show that $100^2 < 2^{100}$.

Solution (1)

1. *Choose propositional functions.*
   Let $C(x)$*:* x *is in the class.*
   $B(x)$*:* x *has read the book.*
   $P(x)$*:* x *passed the first exam.*
2. *Translation into predicate logic premises:*
   $\exists x(C(x) \wedge \neg B(x)), \forall x(C(x) \rightarrow P(x))$
   Conclusion: $\exists x(P(x) \wedge \neg B(x))$
3. *Construct the Valid Argument.*

1.	$\exists x(C(x) \wedge \neg B(x))$	*premise.*
2.	$C(a) \wedge \neg B(a)$	*Existential Instantiation from (1).*
3.	$C(a)$	*Simplification of (2).*
4.	$\forall x(C(x) \rightarrow P(x))$	*premise.*
5.	$C(a) \rightarrow P(a)$	*Universal Instantiation of (4).*
6.	$P(a)$	*Modus ponens of (3) and (5).*
7.	$\neg B(a)$	*Simplification of (2).*
8.	$P(a) \wedge \neg B(a)$	*Conjunction of (6) and (7).*
9.	$\exists x(P(x) \wedge \neg B(x))$	*Existential Generalization of (8).*

Solution (5)

1. *Propositional functions.*
   Let $P(n)$*:* $n > 4$*.*
   $Q(n)$*:* $n^2 < 2^n$*.*
2. *Translation into predicate logic premises:*
   $\forall n(P(n) \rightarrow Q(n)), P(100)$ *Conclusion:* $Q(100)$
3. *Valid Argument.*

1.	$\forall n(P(n) \rightarrow Q(n))$	*Premise.*
2.	$P(100) \rightarrow Q(100)$	*Universal Instantiation of (2).*
3.	$P(100)$	*premise.*
4.	$Q(100)$	*Modus ponens of (2) and (3).*