CSS 215: DISCRETE MATHEMATICS

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SOKOINE UNIVERSITY OF AGRICULTURE

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CLASS

BSc. ICTM & BSc. ITS

GENERATING FUNCTIONS

1. Principle of Inclusion-Exclusion

2. Generating Functions

Principle of Inclusion-Exclusion

The principle of inclusion-exclusion can be seen as a generalisation of the sum rule. Suppose that there are $n(A)$ ways to perform task A and $n(B)$ ways to perform task B, how many ways are there to perform task A or B, if the methods to perform these tasks are not distinct? This is a cardinality problem and therefore, we begin by defining the cardinality of a set.

Definition (Cardinality of a set)

If $\mathcal{S}$ is a set, then the number of elements present in the set $\mathcal{S}$ is known as **cardinality** of $\mathcal{S}$ denoted by $|\mathcal{S}|$. Mathematically, if

$$\mathcal{S} = \{s_1, s_2, \cdots, s_k\}, \quad \text{then} \quad |\, \mathcal{S} \,| = k, \;\; k \in \mathbb{N}. \tag{1}$$

The cardinality of an empty set ϕ, is zero, that is, $|\, \phi \,| = 0$.
Now, suppose that A and B are any two finite sets. How is $|A \cup B|$ related to $|A|$ and $|B|$?

Principle of Inclusion-Exclusion...

So, $\mid A \cup B \mid= 6$, $\mid A \cap B \mid= 2$, $\mid A \mid= 3$, and $\mid B \mid= 5$. Clearly, $\mid A \cup B \mid=\mid A \mid + \mid B \mid - \mid A \cap B \mid$. Thus, we have the following results:

Theorem (Two Set Inclusion-Exclusion Principle)

Let A and B are two finite sets, then

$$\mid A \cup B \mid=\mid A \mid + \mid B \mid - \mid A \cap B \mid . \tag{2}$$

Definition (Cardinality of a set)

If $\mathcal{S}$ is a set, then the number of elements present in the set $\mathcal{S}$ is known as **cardinality** of $\mathcal{S}$ denoted by $|\mathcal{S}|$. Mathematically, if

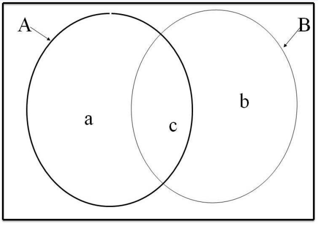
$$\mathcal{S} = \{s_1, s_2, \cdots, s_k\}, \quad \text{then} \quad \mid \mathcal{S} \mid= k, \ k \in \mathbb{N}. \tag{3}$$

The cardinality of an empty set $\emptyset$, is zero, that is, $\mid \emptyset \mid= 0$.
Now, suppose that A and B are any two finite sets. How is $|A \cup B|$ related to $|A|$ and $|B|$?

Principle of Inclusion-Exclusion...

Proof.

Suppose $\mid A \cap B \mid = c$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then we have $\mid A \mid = a + c$ and $\mid B \mid = b + c$ for $a,\ b$, and c non-negative.



Therefore,

$$\begin{aligned} \mid A \cup B \mid &= a + c + b = (a + c) + (b + c) - c \\ &= \mid A \mid + \mid B \mid - \mid A \cap B \mid . \end{aligned}$$

If A and B are disjoints, then $\mid A \cap B \mid = \mid \phi \mid = 0$, and hence, $\mid A \cup B \mid = \mid A \mid + \mid B \mid$.

Principle of Inclusion-Exclusion...

Example

Find the number of positive integers ≤ 300 and divisible by 2 or 3.

Solution

Let

$$A = \{x \in \mathbb{N} \mid x \leq 300 \text{ and divisible by } 2\}$$
$$B = \{x \in \mathbb{N} \mid x \leq 300 \text{ and divisible by } 3\}.$$

Then, $A \cap B$ consists of positive integers ≤ 300 that are divisible by 2 and 3, i.e., divisible by 6.
Thus, $\mid A \mid = \left[\frac{300}{2}\right] = 150, \ \mid B \mid = \left[\frac{300}{3}\right] = 100, \ \mid A \cap B \mid = \left[\frac{300}{6}\right] = 50.$
Hence,

$$\begin{aligned} \mid A \cup B \mid &= \mid A \mid + \mid B \mid - \mid A \cap B \mid \\ &= 150 + 100 - 50 \\ &= 200. \end{aligned}$$

Thus, there are 200 positive integers ≤ 300 which are divisible by 2 or 3.

Principle of Inclusion-Exclusion...

Theorem (Three Set Inclusion-Exclusion Principle)

Let A, B *and* C *are three finite sets, then*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap \quad (4)$$

Proof.

Consider $|A \cup B \cup C| = |A \cup (B \cup C)|$ and the applying the Two Set Inclusion-Exclusion Principle, the desired result is obvious. □

Example

How many integers between 1 and 600 inclusive are divisible by neither 3, nor 5, nor 7?

Principle of Inclusion-Exclusion...

Solution

Let $A,\ B$ *and Let* A_k *denote the numbers which are divisible by* $k = 3, 5, 7$. *Define*

$$\begin{aligned}
A_3 &= \{x \mid 1 \leq x \leq 600 \text{ and divisible by } 3\} \\
A_5 &= \{x \mid 1 \leq x \leq 600 \text{ and divisible by } 5\} \\
A_7 &= \{x \mid 1 \leq x \leq 600 \text{ and divisible by } 7\} \\
A_{15} &= \{x \mid 1 \leq x \leq 600 \text{ and divisible by both } 3 \text{ and } 5\} \\
A_{21} &= \{x \mid 1 \leq x \leq 600 \text{ and divisible by both } 3 \text{ and } 7\} \\
A_{35} &= \{x \mid 1 \leq x \leq 600 \text{ and divisible by both } 5 \text{ and } 7\} \\
A_{105} &= \{x \mid 1 \leq x \leq 600 \text{ and divisible by both } 3,\ 5 \text{ and } 7\}.
\end{aligned}$$

Solution...

Then,

$$| A_3 | = \left[\frac{600}{3}\right] = 200, \quad | A_5 |= \left[\frac{600}{5}\right] = 120, \quad | A_7 |= \left[\frac{600}{7}\right] = 85,$$

$$| A_{15} | = \left[\frac{600}{15}\right] = 40, \quad | A_{21} |= \left[\frac{600}{21}\right] = 28, \quad | A_{35} |= \left[\frac{600}{35}\right] = 17,$$

$$| A_{105} | = \left[\frac{600}{105}\right] = 5,$$

Applying the Three Set Inclusion-Exclusion Principle, we have

$$\begin{aligned} | A_3 \cup A_5 \cup A_7 | &=| A_3 | + | A_5 | + | A_7 | - | A_{15} | - | A_{21} | - | A_{35} | + | A_1 \\ &= 200 + 120 + 85 - 40 - 28 - 17 + 5 \\ &= 325 \end{aligned}$$

Principle of Inclusion-Exclusion...

Solution...

Now, number of integers not divisible by 3, 5, 7 will be given by

$$\begin{aligned} \mid (A_3 \cup A_5 \cup A_7)' \mid &= \mid \mathcal{S} \mid - \mid A_3 \cup A_5 \cup A_7 \mid \\ &= 600 - 325 \\ &= 275 \end{aligned}$$

Example

How many positive integers ≤ 100 are multiples of either 2 or 5?

Solution

$$\begin{aligned} A_2 &= \{x \mid x \leq 100 \text{ and multiple of } 2\} \\ A_5 &= \{x \mid x \leq 100 \text{ and multiple of } 5\} \\ A_2 \cap A_5 &= \{x \mid x \leq 100 \text{ and multiple of both } 2 \text{ and } 5\}. \end{aligned}$$

Solution...

Then,

$$\mid A_2 \mid = \left[\frac{100}{2}\right] = 50, \quad \mid A_5 \mid = \left[\frac{100}{5}\right] = 20, \quad \mid A_2 \cap A_5 \mid = \left[\frac{100}{2 \times 5}\right] = 10,$$

Applying the Inclusion-Exclusion principle, the multiples of either 2 or 5 are given by

$$\begin{aligned} \mid A_2 \cup A_5 \mid &= \mid A_2 \mid + \mid A_5 \mid - \mid A_2 \cap A_5 \mid \\ &= 50 + 20 - 10 \\ &= 60. \end{aligned}$$

Exercise

1. *Find the number of positive integers* ≤ 3000 *and not divisible by 7 or 8.*
2. *Find the number of positive integers* ≤ 2076 *and divisible by 3, 5, and 7 respectively.*
3. *Find the number of positive integers not exceeding 100 that are not divisible by 5 or by 7.*
4. *Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.*
5. *Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.*
6. *How many bit strings of length eight do not contain six consecutive 0's?*
7. *How many permutations of the 26 letters of the English alphabet do not contain any of the strings fish, rat or bird?*
8. *How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?*

Generating Functions

Definition (Generating Functions)

Let $a_0, a_1, a_2, \cdots$, be a sequence of real numbers. The function

$$g(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots = \sum_{n=0}^{\infty} a_nX^n \qquad (8)$$

is the **generating function** for the sequence $\{a_n\}$.

The generating function for the finite sequence $a_0, a_1, \cdots, a_n$ can be defined by letting $a_i = 0$ for $i > n$. Thus,

$$g(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$$

is the generating function for the finite sequence $a_0, a_1, \cdots, a_n$,

$$g(X) = 1 + 2X + 3X^2 + \cdots + (n+1)X^n + \cdots$$

is the generating function for the sequence of positive integers,

Generating Functions...

$$g(X) = 1 + 3X + 6X^2 + \cdots + \frac{n(n+1)}{2}X^{n-1} + \cdots$$

is the generating function for the sequence of triangular numbers, and since

$$\frac{X^n - 1}{X - 1} = 1 + X + X^2 + \cdots + X^{n-1}, \quad \text{then } g(X) = \frac{X^n - 1}{X - 1}$$

is the generating function for the sequence of n-ones.

Remark

The letter X does not represent anything. The various powers X^n of X are simply used to keep track of the corresponding terms a_n of the sequence. In other words, we think of the powers X^n as placeholders.

Example

The generating function of the sequence $\underbrace{1, \cdots, 1}_{k}, 0, 0, 0, 0, \cdots$ is given by

$1 + X + X^2 + X^3 + \cdots + X^{k-1} = \frac{1-X^k}{1-X}$.

Generating Functions...

Example

The generating function of the sequence $1, 1, 1, 1, \cdots$ is given by

$$1 + X + X^2 + X^3 + \cdots = \sum_{n=0}^{\infty} X^n = \frac{1}{1-X}.$$

Example

The generating function of the sequence $2, 4, 1, 1, 1, \cdots$ is given by

$$\begin{aligned} 2 + 4X + X^2 + X^3 + X^4 + X^5 + \cdots &= (1 + 3X) + (1 + X + X^2 + X^3 + X^4 \\ &= (1 + 3X) + \sum_{n=0}^{\infty} X^n \\ &= 1 + 3X + \frac{1}{1-X}. \end{aligned}$$

Generating Functions...

Example...

A frequently used generating function is

$$\frac{1}{1-aX} = 1 + aX + a^2X^2 + \cdots + a^nX^n + \cdots = \sum_{n=0}^{\infty} a^nX^n. \tag{10}$$

Suppose that
$g(X) = \frac{1}{1-aX} = 1 + aX + a^2X^2 + \cdots + a^nX^n + \cdots = \sum_{n=0}^{\infty} a^nX^n$.
Then,

$$\begin{aligned}
ag(X) &= \frac{a}{1-aX} = a + a^2X + a^3X^2 + \cdots + a^{n+1}X^n + \cdots = \sum_{n=0}^{\infty} a^{n+1}X^n \\
g'(X) &= \frac{a}{(1-aX)^2} = a + 2a^2X + 3a^3X^2 + \cdots + na^nX^{n-1} + \cdots = \sum_{n=0}^{\infty} (n+1)a^{n+1}X^n \\
Xg'(X) &= \frac{aX}{(1-aX)^2} = aX + 2a^2X^2 + 3a^3X^3 + \cdots + na^nX^n + \cdots = \sum_{n=0}^{\infty} na^nX^n.
\end{aligned}$$

Generating Functions...

For example,

$$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \cdots + X^n = \sum_{n=0}^{\infty} X^n.$$

Example

Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \cdots, m$. What is the generating function for the sequence $a_0, a_1, \cdots, a_m$?

Solution

The generating function for this sequence is

$$g(X) = C(m, 0) + C(m, 1)X + C(m, 2)X^2 + \cdots + C(m, m)X^m.$$

The binomial theorem shows that $g(X) = (1 + X)^m$.

Equality, Sum and Products of Generating Functions

Definition

Two generating functions

$$f(X) = \sum_{n=0}^{\infty} a_n X^n \quad \text{and} \quad g(X) = \sum_{n=0}^{\infty} b_n X^n \tag{12}$$

are equal if $a_n = b_n$ for every $n \leq 0$.

Example

Let

$$1 + 3X + 6X^2 + 10X^3 + \cdots \text{ and } g(X) = 1 + \frac{2.3}{2}X + \frac{3.4}{2}X^2 + \frac{4.5}{2}X^3 + \cdots$$

then $f(X) = g(X)$.

Equality, Sum and Products of Generating Functions...

Theorem

Suppose that the sequences $a_0, a_1, a_2, a_3, \cdots$ *and* $b_0, b_1, b_2, b_3, \cdots$ *have the generating function* $f(X) = \sum_{n=0}^{\infty} a_n X^n$ *and* $g(X) = \sum_{n=0}^{\infty} b_n X^n$ *respectively. Then, the generating function of the sequence* $a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots$ *is given by*

$$f(X) + g(X) = \sum_{n=0}^{\infty} a_n X^n + \sum_{n=0}^{\infty} b_n X^n, \tag{13}$$

and the generating function of the sequence $a_0 \times b_0, a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, \cdots$ *is given by*

$$f(X)g(X) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n. \tag{14}$$

Equality, Sum and Products of Generating Functions...

Example

The generating function of the sequence $3, 1, 3, 1, 3, 1, \cdots$ can be obtained by combining the generating functions of the two sequences $1, 1, 1, 1, 1, 1, \cdots$ and $2, 0, 2, 0, 2, 0, \cdots$. Now the generating function for the sequence $2, 0, 2, 0, 2, 0, \cdots$ is given by

$$2 + 2X^2 + 2X^4 + \cdots = 2(1 + X^2 + X^4 + \cdots) = \frac{2}{1 - X^2}$$

and that of $1, 1, 1, 1, 1, 1, \cdots$ is given by

$$1 + X + X^2 + X^3 + \cdots = \sum_{n=0}^{\infty} X^n = \frac{1}{1 - X}.$$

Hence, the generating function of the sequence $3, 1, 3, 1, 3, 1, \cdots$ is given

$$\frac{1}{1 - X} + \frac{2}{1 - X^2}.$$

Equality, Sum and Products of Generating Functions...

Example

The generating function of the sequence $1, 2, 3, 4, \cdots$ is given by

$$
\begin{aligned}
1 + 2X + 3X^2 + 4X^3 + \cdots &= \frac{d}{dX}(1 + X + X^2 + X^3 + \cdots) \\
&= \frac{d}{dX}(\frac{1}{1-X}) \\
&= \frac{1}{(1-X)^2}.
\end{aligned}
$$

The generating function of the sequence $0, 1, 1/2, 1/3, 1/4, \cdots$ is given by

$$
\begin{aligned}
X + \frac{X^2}{2} + \frac{X^3}{3} + \frac{X^4}{4} + \cdots &= \int (1 + X + X^2 + X^3 + \cdots)\,dX \\
&= \int (\frac{1}{1-X})\,dX
\end{aligned}
$$

where c is an absolute constant.

The special case $X = 0$ gives $c = 0$ so that

$$= c - \ln(1 - X)$$

where c is an absolute constant.
The special case $X = 0$ gives $c = 0$ so that

$$X + \frac{X^2}{2} + \frac{X^3}{3} + \frac{X^4}{4} + \cdots = -\ln(1 - X).$$

Theorem

Suppose that the sequence $a_0, a_1, a_2, a_3, \cdots$ has generating function $f(X)$. Then, for every $k \in \mathbb{N}$, the generating function of the (delayed) sequence

$$\underbrace{0, \cdots, 0}_{k}, a_0, a_1, a_2, a_3, \cdots$$

is given by $X^k f(X)$.

Equality, Sum and Products of Generating Functions...

Proof.

Note that the generating function of the sequence is

$$
\begin{aligned}
a_0X^k + a_1X^{k+1} + a_2X^{k+2} + a_3X^{k+3} + \cdots &= X^k(a_0 + a_1X + a_2X^2 + a_3X^3 \\
&= X^k f(X).
\end{aligned}
$$

$\square$

Example

The generating function of the sequence $0, 1, 2, 3, 4, \cdots$ is given by $\frac{X}{(1-X)^2}$, and the generating function of the sequence $0, 0, 0, 0, 0, 0, 0, 3, 1, 3, 1, 3, 1, \cdots$ is given

$$\frac{X^7}{1-X} + \frac{2X^7}{1-X^2}.$$

Example

Consider the sequence $a_0, a_1, a_2, a_3, \cdots$ where $a_n = n^2 + n$ for every $n \in \mathbb{N} \cup \{0\}$. To find the generating function of this sequence, let $f(X)$ and $g(X)$ denote the generating functions of the sequences $0, 1^2, 2^2, 3^2, 4^2, \cdots$ and $0, 1, 2, 3, 4, \cdots$ respectively. To find $f(X)$, we find that the generating function of the sequence $0, 1^2, 2^2, 3^2, 4^2, \cdots$ is given by

$$\begin{aligned} 1 + 2^2X + 3^2X^2 + 4^2X^3 + \cdots &= \frac{d}{dX}(X + 2X^2 + 3X^3 + 4X^4 + \cdots) \\ &= \frac{d}{dX}[X(1 + 2X + 3X^2 + 4X^3 + \cdots)] \\ &= \frac{d}{dX}(\frac{X}{(1-X)^2}) \\ &= \frac{1+X}{(1-X)^3}. \end{aligned}$$

Equality, Sum and Products of Generating Functions

Example...

Hence, $X^k f(X) = \frac{X(1+X)}{(1-X)^3}$, since $k = 1$.
The generating function of the sequence $0, 1, 2, 3, 4, \cdots$ is given by

$$g(X) = \frac{X}{(1-X)^2}.$$

Hence, the required generating function is given by

$$\begin{aligned} f(X) + g(X) &= \frac{X(1+X)}{(1-X)^3} + \frac{X}{(1-X)^2} \\ &= \frac{2X}{(1-X)^3}. \end{aligned}$$

Equality, Sum and Products of Generating Functions...

Theorem (Extended Binomial Theorem)

Suppose that $k \in \mathbb{N}$. *Formally we have*

$$(1+Y)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} Y^n \tag{18}$$

where for every $n = 0, 1, 2, \cdots$ *the extended binomial coefficient is given by*

$$\binom{-k}{n} = \frac{-k(-k-1)\cdots(-k-n+1)}{n!}. \tag{19}$$

Theorem

Suppose that $k \in \mathbb{N}$. *Then for every* $n = 0, 1, 2, \cdots$ *we have*

$$\binom{-k}{n} = (-1)^n \binom{n+k-1}{n}. \tag{20}$$

Equality, Sum and Products of Generating Functions...

Proof.

$$\begin{aligned}
\binom{-k}{n} &= \frac{-k(-k-1)\cdots(-k-n+1)}{n!} \\
&= (-1)^n \frac{k(k+1)\cdots(k+n-1}{n!}) \\
&= (-1)^n \frac{(n+k-1)(n+k-2)\cdots(n+k-1-n+1)}{n!} \\
&= (-1)^n \frac{(n+k-1)(n+k-2)\cdots k}{n!} \\
&= (-1)^n \binom{n+k-1}{n}.
\end{aligned}$$

$\square$

Equality, Sum and Products of Generating Functions...

Example

We have

$$(1-X)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} (-X)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n \binom{-k}{n} X^n$$

so that the coefficient of X^n in $(1-X)^{-k}$ is given by

$$(-1)^n \binom{-k}{n} = (-1)^n \binom{n+k-1}{n}.$$

Equality, Sum and Products of Generating Functions...

Example

We have

$$(1+2X)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} (2X)^n$$
$$= \sum_{n=0}^{\infty} 2^n \binom{-k}{n} X^n$$

so that the coefficient of X^n in $(1+2X)^{-k}$ is given by

$$2^n \binom{-k}{n} = (-2)^n \binom{n+k-1}{n}.$$

Exercise

1. *Find the generating function of the following sequences*
   1. $0, 0, 0, 1, 0, 1/3, 0, 1/5, 0, 1/7, 0, \cdots$
   2. $a_n = 3n^2 + 7n + 1$ *for* $n \in \mathbb{N} \cup \{0\}$
   3. $a_0, a_1, a_2, \cdots$ *where*

$$a_n = 3^{n+1} + 2n + \binom{3}{n} + 4\binom{-5}{n}.$$

2. *Find the first four terms of the formal power series expression of*

$$(1 - 3X)^{13}.$$