1. a) $R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$, $R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$

b) $R_x(\frac{\pi}{2}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ c) $R_y(\frac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ d) $R_z(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $R_x(\frac{\pi}{3}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ 0 & \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\frac{\sqrt{3}}{2} \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$

f) $R_y(\frac{\pi}{3}) = \begin{bmatrix} \cos(\frac{\pi}{3}) & 0 & \sin(\frac{\pi}{3}) \\ 0 & 1 & 0 \\ -\sin(\frac{\pi}{3}) & 0 & \cos(\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 1/2 \end{bmatrix}$

g) $R_z(\frac{\pi}{3}) = \begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) & 0 \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

h) $R_x(\frac{\pi}{4}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ 0 & \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

i) $R_y(\frac{\pi}{4}) = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ j) $R_z(\frac{\pi}{4}) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ k) $R_x(\frac{\pi}{6}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$

L) $R_y(\frac{\pi}{6}) = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$ m) $R_z(\frac{\pi}{6}) = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ n) $R_x(-\frac{\pi}{3}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}$

o) $R_y(-\frac{\pi}{3}) = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$ p) $R_z(-\frac{\pi}{3}) = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ q) $R_x(-\frac{\pi}{6}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}$

r) $R_y(-\frac{\pi}{6}) = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$ s) $R_z(-\frac{\pi}{6}) = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Robot arrives at $(x, y, \theta) = (3.5, -0.25, \theta = \frac{\pi}{6}) = {}^W T_B$

LiDAR mounted $(0.1, 0.0, \theta = 0) = {}^B T_S$

LiDAR senses object at $(x,y) = (1.5, 2.0)$ in Sensor Frame (P)

Where is object in world frame?

${}^W T_S = {}^W T_R \cdot {}^R T_S$

$= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 3.5 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

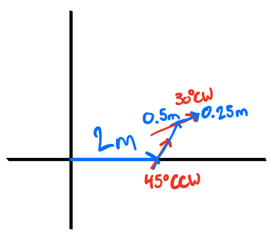
${}^W T_R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{bmatrix}$ ${}^R T_S = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$ {}^{W}T_{S} = {}^{W}T_{R} \cdot {}^{R}T_{S} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 3.5 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6})\cdot 0.1 + 3.5 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6})\cdot 0.1 - 0.25 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{W}T_{S} = \begin{bmatrix} 0.866 & -0.5 & 3.5866 \\ 0.5 & 0.866 & -0.2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^{W}T_{S} \cdot P = {}^{W}T_{S} \cdot \begin{bmatrix} 1.5 \\ 2.0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3.8856 \\ 2.282 \\ 1 \end{bmatrix}} $$

3. $ {}^{W}T_{B} = T_{W} $

Pose A (2 m forward x-axis)
Pose B (Rotates 45° CCW)
Pose C (0.5m forward)
Pose D (Rotates 30° CW)
Pose E (0.25 m forward)



$$ {}^{W}T_{E} = {}^{W}T_{A} \cdot {}^{A}T_{B} \cdot {}^{B}T_{C} \cdot {}^{C}T_{D} \cdot {}^{D}T_{E} $$

$$ = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos -\frac{\pi}{6} & -\sin -\frac{\pi}{6} & 0 \\ \sin -\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0.25 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = $$

$$ \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ E \cdot D = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 1/4 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot C = \begin{bmatrix} \sqrt{3}/2 & 1/2 & \frac{\sqrt{3}+1}{4} \\ -1/2 & \sqrt{3}/2 & -1/4 \\ 0 & 0 & 1 \end{bmatrix} \cdot B = \begin{bmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} & \frac{\sqrt{3}+1}{4} \\ \frac{-\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}+\sqrt{6}}{4} & -1/4 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = \boxed{\begin{bmatrix} 0.9659 & -0.2588 & 2.6148 \\ 0.2588 & 0.9659 & 0.2676 \\ 0 & 0 & 1 \end{bmatrix} = (2.6148, 0.2676, \tfrac{\pi}{12})} $$

4. Robot moves & arrives at $(x, y, \theta) = (3.5, -0.25, \frac{\pi}{6})$ in World frame

LiDAR mounted at $(0.1, 0, 0)$ relative to body frame

LiDAR senses object at $r = 0.5\text{m}$ and $\varphi = \frac{\pi}{10}$ w.r.t x-axis of sensor frame

Where is object in world frame?

$${}^{W}T_{R} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 3.5 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{R}T_{S} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{W}T_{S} = {}^{W}T_{R} \cdot {}^{R}T_{S}$$

$${}^{W}T_{S} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 3.5 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 3.5866 \\ 0.5 & 0.866 & -0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{W}T_{P} = {}^{W}T_{S} \cdot {}^{S}T_{P} = \begin{bmatrix} 0.866 & -0.5 & 3.5866 \\ 0.5 & 0.866 & -0.2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5\cos\frac{\pi}{10} \\ 0.5\sin\frac{\pi}{10} \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3.921 \\ 0.172 \\ 1 \end{bmatrix}}$$

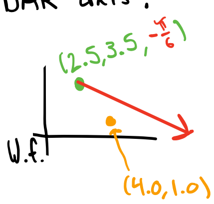
5. Robot moves across floor and arrives at $(x, y, \theta) = (2.5, 3.5, -\frac{\pi}{6})$ in w.f

LiDAR mounted at $(0.1, 0, 0)$ relative to robot

Object at $(x, y) = (4.0, 1.0)$ in World frame

- Where is object in sensor frame? (Show work)
- What is it's range & bearing relative to LiDAR axis?

$${}^{W}T_{R} = \begin{bmatrix} \cos(-\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 2.5 \\ \sin(-\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & 3.5 \\ 0 & 0 & 1 \end{bmatrix}$$



$${}^{R}T_{S} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{W}T_{P} = \begin{bmatrix} 4.0 \\ 1.0 \\ 1 \end{bmatrix}$$

a)

$${}^{W}T_{P} \begin{bmatrix} \cos(-\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 2.5 \\ \sin(-\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & 3.5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & \frac{\sqrt{3}+40}{20} \\ -1/2 & \sqrt{3}/2 & \frac{10\sqrt{3}+69}{20} \\ 0 & 0 & 1 \end{bmatrix}$$

$$ {}^{W}T_P^{-1} \cdot q_w = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} $$

$$ \begin{bmatrix} \sqrt{3} & 1 & \frac{-25\sqrt{3}-36}{10} \\ 1 & \sqrt{3} & \frac{-7\sqrt{3}-7}{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4.0 \\ 1.0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{15\sqrt{3}-26}{10} \\ \frac{-5\sqrt{3}+1}{2} \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -0.0019 \\ -3.8301 \\ 1 \end{bmatrix}} $$

b) $r = \sqrt{x^2 + y^2} = \sqrt{(-0.0019)^2 + (-3.8301)^2} = 3.8301 \text{ m}$

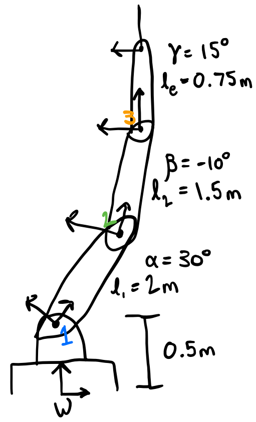
$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-3.8301}{-0.0019}\right) = 89.97^\circ$

$\boxed{(r, \theta) = (3.8301 \text{ m}, 89.97^\circ)}$

Not to scale,
Not correct angles

6) 3-link planar arm

given $q = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ w/ values $q = \begin{bmatrix} 30° \\ -10° \\ 15° \end{bmatrix}$



a) Where is origin of each joint relative to world frame?

b) The homogeneous transform from end effector to world frame?

c) Given point at $(x,y)_w = (2.5, 3.0)$m, Where is point in end effector frame?

a) $l_1 = (0, 0.5)$m

$l_2 = (0 + 2\cos(30), 0.5 + 2\sin(30))$
$= (1.73, 1.5)$

$l_3 = (1.73 + 1.5\cos(30-10), 1.5 + 1.5\sin(30-10))$
$= (3.21, 1.24)$

$= (3.14, 2.01)$

$$q_w = {}^{W}T_1 \cdot {}^{1}T_2 \cdot {}^{2}T_3 \cdot {}^{3}T_{ee} \cdot q_{ee} = {}^{W}T_{ee} \cdot q_{ee}$$

b) $$l_1 = \begin{bmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \quad l_2 = \begin{bmatrix} \cos(-10) & -\sin(-10) \\ \sin(-10) & \cos(-10) \end{bmatrix} \begin{bmatrix} 2.0 \\ 0 \end{bmatrix}$$

$$l_3 = \begin{bmatrix} \cos(15) & -\sin(15) \\ \sin(15) & \cos(15) \end{bmatrix} \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} \quad q_{ee} = \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$R_z(\alpha) \cdot R_z(\beta) = \begin{bmatrix} \cos(30) & -\sin(30) & 0 \\ \sin(30) & \cos(30) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(-10) & -\sin(-10) & 0 \\ \sin(-10) & \cos(-10) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.940 & -0.342 & 0 \\ 0.342 & 0.940 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$q = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} l_1\cos(\alpha) + l_2\cos(\alpha+\beta) + l_3\cos(\alpha+\beta+\gamma) \\ l_1\sin(\alpha) + l_2\sin(\alpha+\beta) + l_3\sin(\alpha+\beta+\gamma) + y_1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = P$$

$$\begin{bmatrix} 30 \\ -10 \\ 15 \end{bmatrix} = \begin{bmatrix} 2\cos(30) + 1.5\cos(30-10) + 0.5\cos(30-10+15) \\ 2\sin(30) + 1.5\sin(30-10) + 0.5\sin(30-10+15) + 0.5 \end{bmatrix} = \begin{bmatrix} 20.64 \\ 2.30 \end{bmatrix} = P$$

$${}^{W}T_{ee} = \begin{bmatrix} 0.940 & -0.342 & 20.64 \\ 0.342 & 0.940 & 2.30 \\ \vec{0} & & 1 \end{bmatrix}$$

c) $$(x,y)_{ee} = {}^{W}T_{ee}^{-1} \cdot (x,y)_w = \begin{bmatrix} 0.9394 & 0.3418 & -20.17 \\ -0.3418 & 0.9394 & 4.894 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2.5 \\ 3.0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -16.80 \\ 6.858 \\ 1 \end{bmatrix}}$$