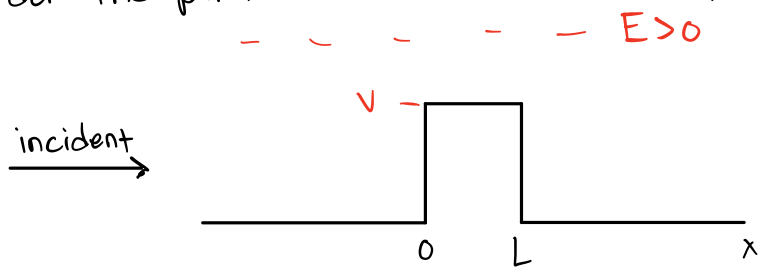
Christopher Williams, 3/15/24

1) Consider the particles incident from the left on a potential barrier:



take case $E > V_0$. Derive expression for transmission coefficient

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = \underbrace{\frac{-2mE}{\hbar^2}}_{k^2}\psi \Rightarrow \psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ \psi(x) & 0 < x < L \\ Fe^{ika} & x > L \end{cases}$$

Transmission Coefficient

Probability that a particle, incident from one side, makes it to the other side of well/barrier.

$$T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$

$R + T = 1,\ T = 1 - R$

$$T = \frac{1}{1 + (ma^2/2\hbar\ E)}$$

For $E < V_0$

I will skip some steps, but can provide them if needed

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = \underbrace{(V_0 - E)\left(\frac{2m}{\hbar^2}\right)}_{\ell^2}\psi$$

$$\psi(x) = Ce^{\ell x} + De^{-\ell x}$$

$$\psi_1(<0) = \psi_2(<0) \Rightarrow Ae^{-ika} + Be^{ika} = Ce^{-\ell a} + De^{\ell a} \quad (1)$$

$$\frac{d\psi_1(<0)}{dx} = \frac{d\psi_2(<0)}{dx} \Rightarrow ikAe^{-ika} - ikBe^{ika} = \ell Ce^{-\ell a} - \ell De^{\ell a} \quad (2)$$

continunity at $x = <0$

$$\psi_2(0) = \psi_3(0) \Rightarrow Ce^{\ell a} + De^{-\ell a} = Fe^{ika} \quad (3)$$

$$\frac{d\psi_2(0)}{dx} = \frac{d\psi_3(0)}{dx} \Rightarrow \ell Ce^{\ell a} - \ell De^{-\ell a} = ikFe^{ika} \quad (4)$$

Continunity at $x = 0$

Goal: $\left|\frac{F}{A}\right|^2 = T$

... Lots of systems of eqs stuff inbound...

$$① \to Be^{ika} = Ce^{-la} + De^{la} - Ae^{-ika} \to ② = 2Ae^{-ika} = C\left[1-\frac{il}{k}\right]e^{-la} + D\left[1+\frac{il}{k}\right]e^{la} \quad ⑤$$

Skipped steps here

$$③ \cdot l = lCe^{la} + lDe^{-la} = Fle^{ika} \quad ③$$

$$③ + ④ \qquad 2lCe^{la} = F[ik + l]e^{ika} \quad ⑥$$

$$④ - ③ \qquad 2De^{-la} = F\left[\frac{-ik}{l} + 1\right]e^{ika} \quad ⑦$$

$$⑥, ⑦ \to ⑤ \Rightarrow 2Ae^{-ika} = \frac{F}{2}\left[1+\frac{ik}{l}\right]e^{ika}e^{-2la}\left[1-\frac{il}{k}\right] + \frac{F}{2}\left[1-\frac{ik}{l}\right]\left[1+\frac{il}{k}\right]e^{2la}e^{ika}$$

Lots of simplifying... $\Rightarrow 2Ae^{-2ika} = \frac{F}{2}\left[2(e^{2la} + e^{-2la}) + i\left(\frac{l^2-k^2}{kl}\right)(e^{2la} - e^{-2la})\right]$

More $\Rightarrow 2Ae^{-2ika} = \frac{F}{2}\left[4\cosh(2la) + 2i\left(\frac{l^2-k^2}{kl}\right)\sinh(2la)\right]$

$$T = \frac{F}{A} = \frac{4e^{-2ika}}{4\cosh(2la) + 2i\left(\frac{l^2-k^2}{kl}\right)\sinh(2la)} \to \left|\frac{F}{A}\right|^2 = \frac{1}{\cosh^2(2la) + \frac{1}{4}\left(\frac{l^2-k^2}{kl}\right)^2\sinh^2(2la)}$$

$$T^{-1} = \cosh^2(2la) + \frac{1}{4}\left(\frac{l^2-k^2}{kl}\right)^2\sinh^2(2la) \to \text{Simplifying} \to = 1 + \frac{V_0^2}{E(V_0-E)}$$

In total, for $E < V_0$, it is $= 1 + \frac{V_0^2}{E(V_0+E)}$

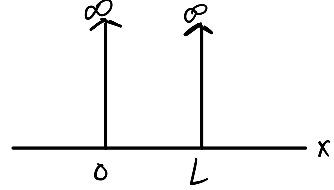
For $E > 0$ $= 1 + \frac{V_0^2}{E(V_0-E)}$

b) let $E = 15.0\,eV$ (electrons), $V = 5\,eV$, $L = 20\,nm$. What is value of T?

Either I messed up bad, or L doesn't matter

$$T^{-1} = 1 + \frac{(5)}{(15)^2(5-15)} \Rightarrow \frac{449}{450} \Rightarrow \frac{1}{\frac{449}{450}} = \frac{450}{449} = \boxed{1.022\,eV}$$

2) Consider the particle in a box



What is the probability of finding the particle between $x = 0.5L$ and $0.51L$ in

a) the ground state   b) the 3rd excited state

a) $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \Rightarrow \psi(x) = A\sin(kx) + B\cos(kx)$$

probability of finding of finding particle at $x=0$ or $x=L$ is zero.
$x=0$, $\sin(0)=0$ & $\cos(0)=1$, B must equal zero

$$\psi(x) = A\sin(kx) \Rightarrow \frac{d\psi}{dx} = kA\cos(kx) \Rightarrow \frac{d^2\psi}{dx^2} = -k^2A\sin(kx) \quad \frac{d^2\psi}{dx^2} = -k^2\psi$$

$$k = \left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} \Rightarrow \psi = A\sin\left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} x$$

$$0 = A\sin\left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} L \Rightarrow \left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} L = n\pi \Rightarrow \psi = A\sin\frac{n\pi}{L}x$$

$$\int_0^L \psi^2 dx = 1 = \sum A^2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = 1$$

$$\int_{0.5L}^{0.51L} \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)dx, \quad n=3 \Rightarrow \int_{0.5L}^{0.51L} \frac{2}{L}\sin^2\left(\frac{3\pi x}{L}\right)dx \Rightarrow$$

Mathematica

$$\frac{2}{L}\left(\left(\frac{0.51L}{2} - \frac{L\sin\left(\frac{3\pi(0.51L)}{L}\right)}{4\pi(3)}\right) - \left(\frac{0.5L}{2} - \frac{L\sin\left(\frac{3\pi(0.5L)}{L}\right)}{4\pi(3)}\right)\right) = \boxed{0.976\%}$$

b) the ground state

$$\int_{0.5L}^{0.51L} \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)dx \quad , \quad n=1 \Rightarrow \int_{0.5L}^{0.51L} \frac{2}{L}\sin^2\left(\frac{1\pi x}{L}\right)dx \Rightarrow$$

$$\frac{2}{L}\left(\left(\frac{0.51L}{2} - \frac{L\sin\left(\frac{1\pi(0.51L)}{L}\right)}{4\pi(3)}\right) - \left(\frac{0.5L}{2} - \frac{L\sin\left(\frac{1\pi(0.5L)}{L}\right)}{4\pi(3)}\right)\right) = \boxed{1.00262\%}$$

3) Given $\psi(x) = Ax^{-\alpha x^3 - i\delta}$ (A, α, δ are real)

Normalize ψ and determine the probability of finding the object between $x=1$ and $x=3$

Normalize $= \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \psi^*\psi\, dx = 1$

$$= \int_{-\infty}^{\infty} (Ax^{-\alpha x^3 i\delta})(Ax^{-\alpha x^3 - i\delta})\,dx$$

$$= A^2\int_{-\infty}^{\infty} (x^{-\alpha x^3 i\delta})(x^{-\alpha x^3 - i\delta})\,dx = A^2\int_{-\infty}^{\infty} x^{-x^3\alpha - i\delta - ix^3\alpha\delta}$$

I don't believe this is correct

I sadly could not figure how to normalize this. Defeated. Sorry

I figure the next part is along the lines of this: $\int_1^3 (\,)\, x^{-\alpha x^3 - i\delta} dx$

But your email confused me, so I'm sure now.

4) The Hermitian conjugate (or adjoint) of an operator $\hat{Q}$ is the operator $\hat{Q}^{+}$ Such that

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}^{+}f|g\rangle \qquad (\text{for all } f \text{ and } g)$$

(A Hermitian Operator, then, is equal to its Hermitian conjugate: $\hat{Q}=\hat{Q}^{+}$)

a) Find the Hermitian conjugates of x, i, and $\frac{d}{dx}$.

$$\langle f|xg\rangle = \langle f|x|g\rangle \Rightarrow \int_{-\infty}^{\infty} f^*(x)\,x\,g(x)\,dx$$

$$= \int_{-\infty}^{\infty} x f^*(x) g(x)\,dx \Rightarrow \int_{-\infty}^{\infty} x^* f^*(x) g(x)\,dx$$

$$= \int_{-\infty}^{\infty} [x f(x)]^* g(x)\,dx$$

$$= \langle xf|g\rangle \Rightarrow x^{+} = x$$

$$\langle f|ig\rangle = \langle f|i|g\rangle = \int_{-\infty}^{\infty} f^*(x)\,i\,g(x)\,dx = \int_{-\infty}^{\infty} i f^*(x) g(x)\,dx$$

$$= \int_{-\infty}^{\infty} (-i)^* f^*(x) g(x)\,dx = \int_{-\infty}^{\infty} [-i f(x)]^* g(x)\,dx$$

$$= \langle -if|g\rangle \Rightarrow i^{+} = -i$$

$$\langle f|\tfrac{d}{dx}g\rangle = \langle f|\tfrac{d}{dx}|g\rangle \Rightarrow \int_{-\infty}^{\infty} f^*(x)\frac{d}{dx}g(x)\,dx = \int_{-\infty}^{\infty} f^*(x)\frac{dg}{dx}\,dx = \underbrace{f^*(x)g(x)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \frac{df^*}{dx} g(x)\,dx$$

$$= -\int_{-\infty}^{\infty} \frac{d}{dx}\left(u(x) + iv(x)\right)^* g(x)\,dx = -\int_{-\infty}^{\infty} \left(\frac{du}{dx} - i\frac{dv}{dx}\right) g(x)\,dx$$

$$= -\int_{-\infty}^{\infty}\left(\frac{d}{dx}\left(u(x)+iv(x)\right)\right)^* g(x)dx = -\int_{-\infty}^{\infty}\left(\frac{df}{dx}\right)^* g(x)\,dx = \int_{-\infty}^{\infty}\left(-\frac{df}{dx}\right)^* g(x)dx$$

$$= \langle -\frac{d}{dx} f | g \rangle \Rightarrow \left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$$

c) Show that $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger \hat{Q}^\dagger$

$$\langle f | \hat{Q}\hat{R} g \rangle = \langle f | \hat{Q}\hat{R} | g \rangle$$

$$= \int_{-\infty}^{\infty} f^*(x)\hat{Q}\hat{R} g(x)dx = \int_{-\infty}^{\infty} f^*(x)\left(\hat{Q}\left(\hat{R}g(x)\right)\right)dx = \int_{-\infty}^{\infty}\left(\hat{Q}^\dagger f(x)\right)^*\left(\hat{R}g(x)\right)dx$$

$$= \int_{-\infty}^{\infty}\left(\hat{R}^\dagger\left(\hat{Q}^\dagger f(x)\right)\right)^* g(x)dx = \int_{-\infty}^{\infty}\left(\hat{R}^\dagger \hat{Q}^\dagger f(x)\right)^* g(x)dx \Rightarrow \langle \hat{R}^\dagger \hat{Q}^\dagger f | g \rangle$$

$$\langle f | (\hat{Q}+\hat{R}) g \rangle = \langle f | (\hat{Q}+\hat{R} | g \rangle$$

$$= \int_{-\infty}^{\infty} f^*(x)(\hat{Q}+\hat{R})g(x)dx = \int_{-\infty}^{\infty}\left(f^*(x)\hat{Q}g(x) + f^*(x)\hat{R}g(x)\right)dx$$

$$= \int_{-\infty}^{\infty} f^*(x)\hat{Q}g(x)dx + \int_{-\infty}^{\infty} f^*(x)\hat{R}g(x)dx = \int_{-\infty}^{\infty}\left(\left[\hat{Q}^\dagger f(x)\right]^* g(x) + \left[\hat{R}^\dagger f(x)\right]^* g(x)\right)dx$$

$$= \int_{-\infty}^{\infty}\left(\left[\hat{Q}^\dagger + \hat{R}^\dagger\right] f(x)\right)^* g(x)dx = \langle (\hat{Q}^\dagger + \hat{R}^\dagger) f | g \rangle \therefore (\hat{Q}+\hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger$$

b) Construct hermitian conjugate of the harmonic oscillator raising operator, a_+

$$a_+^\dagger = \left(\frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})\right)^\dagger = \left(\frac{1}{\sqrt{2\hbar m\omega}}\right)^* (-i\hat{p}+m\omega\hat{x})^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left((-i\hat{p})^\dagger + (m\omega\hat{x})^\dagger\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left((-i)^*\hat{p}^\dagger + (m\omega)^*\hat{x}^\dagger\right) = \frac{1}{\sqrt{2\hbar m\omega}}\left(i\left(-i\hbar\frac{d}{dx}\right)^\dagger + m\omega\hat{x}\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(i(i\hbar)\left(-\frac{d}{dx}\right)+m\omega\hat{x}\right) = \frac{1}{\sqrt{2\hbar m\omega}}\left(i\left(-i\hbar\frac{d}{dx}\right)+m\omega\hat{x}\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{p}+m\omega\hat{x}\right) = \hat{a}_-$$

5) Is the ground state of the infinite square well an eigenfunction of momentum? Is so, what is its momentum? If not, why not?

Ground state: $\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$

momentum operator: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$$\hat{p}\psi_1(x) = -i\hbar\frac{\partial}{\partial x}\left(\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\right) = -i\hbar\left(\frac{\pi}{a}\right)\left(\sqrt{\frac{2}{a}}\cos\frac{\pi x}{a}\right) \neq \hat{p}\psi_1(x)$$

$\hat{p}\psi_1(x) \neq$ to constant times $\psi_1(x)$, the ground state of the infinite square well is not an eigenfunction of the momentum operator.