

Optimisation

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Question 1

The first step is to convert $\min(x_1 - 3x_2)$ to a maximisation problem. This is trivial:

$$\min(x_1 - 3x_2) = \max(-x_1 + 3x_2)$$

and can be rewritten as:

$$x_1 - 3x_2 + P = 0$$

Constraints must also be converted to equalities, with the addition of slack variables s_1 , s_2 , and s_3 :

$$\begin{aligned}x_1 - x_2 \leq 1 &\Rightarrow x_1 - x_2 + s_1 = 1 \\x_1 - x_2 \geq -1 &\Rightarrow -x_1 + x_2 + s_2 = 1 \\2x_1 - x_2 \leq 3 &\Rightarrow 2x_1 - x_2 + s_3 = 3\end{aligned}$$

The first tableau can now be constructed.

x_1	x_2	s_1	s_2	s_3	P	C	t
1	-1	1	0	0	0	1	-1
-1	1	0	1	0	0	1	1
2	-1	0	0	1	0	3	-3
1	-3	0	0	0	1	0	

As column x_2 contains the smallest negative number in the last row (-3), we calculate pivot values, in column t , using x_2 . As 1 is the smallest non-negative value, we pivot around the second element of column x_2 . Using the following row operations:

$$\begin{aligned}R_1 &= R_1 + R_2 \\R_3 &= R_3 + R_2 \\R_4 &= R_4 + 3R_2\end{aligned}$$

the next tableau can be constructed:

x_1	x_2	s_1	s_2	s_3	P	C	t
0	0	1	1	0	0	2	-1
-1	1	0	1	0	0	1	4
1	0	0	1	1	0	4	-1.5
-2	0	0	3	0	1	3	

The next pivot is the third element of column x_1 , 1, as 4 is the smallest non-negative pivot value in t . Using the following row operations:

$$\begin{aligned} R_2 &= R_2 + R_3 \\ R_4 &= R_4 + 2R_3 \end{aligned}$$

the final tableau can be constructed.

x_1	x_2	s_1	s_2	s_3	P	C	t
0	0	1	1	0	0	2	
0	1	0	2	1	0	5	
1	0	0	1	1	0	4	
0	0	0	5	2	1	11	

As there are no remaining negative values in the bottom row, an optimal result has been obtained. Non-unit columns are non-basic and are therefore 0. Reading from the tableau, the following values can be obtained:

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 5 \end{aligned}$$

Plugging these back into the original minimisation problem thus gives an optimal value:

$$x_1 - 3x_2 = 4 - 3(5) = -11$$

Question 2

Question 3

For this problem, 9 variables are necessary. Each relates to the amount of paint of one colour (cyan C , magenta M , or yellow Y) used in the creation of paint of another colour (red R , green G , blue B , or black K), e.g. C_K denotes the number of gallons of cyan (C) paint used in the production of black (K) paint. The objective function is therefore:

$$\max\left(\frac{10 \cdot (Y_R + M_R)}{2} + \frac{15 \cdot (Y_G + C_G)}{2} + \frac{25 \cdot (M_B + C_B)}{2} + \frac{25 \cdot (C_K + M_K + Y_K)}{3}\right)$$

Each term is the amount of paint of a certain colour produced multiplied by the value of a gallon of paint of that colour: 10 for R , 15 for G , 25 for B , and 25 for K . The following constraints capture the limited quantities of paint available: 11 gallons of Y , 10 of C , and 5 of M :

$$\begin{aligned} Y_R + Y_G + Y_K &\leq 11 \\ C_G + C_B + C_K &\leq 10 \\ M_B + M_K + M_R &\leq 5 \end{aligned}$$

Also necessary are constraints maintaining the correct ratios of paints used:

$$\begin{aligned} Y_R &= M_R \\ Y_G &= C_G \\ M_B &= C_B \\ C_K &= M_K = Y_K \end{aligned}$$

These maintain the ratios for R , G , B , and K paint. An optimal solution produces 8 gallons of G paint, 2 gallons of B paint, and 3 gallons of K paint, for a total value of £220.

Question 4

Part A

This is a simple knapsack problem. Each variable A, B, C, D, E, F is binary: whether or not the item was taken. This leads to the following function:

$$\max(60A + 70B + 40C + 70D + 16E + 100F)$$

in which the constants are the values (£) of each item. The only constraint is equally simple: that the weight of the taken items does not exceed 20kg:

$$6A + 7B + 4C + 9D + 3E + 8F \leq 20$$

in which the constants are the weights (kg) of each item. An optimal solution is to take items B, C , and F , resulting in a total weight of 19kg and a total value of £210.

Part B

This part adds a new constraint: that taking C only makes sense if D is also taken, but not vice versa. This can be elegantly expressed as:

$$D - C \geq 0$$

This condition is only unsatisfied if $D = 0$ and $C = 1$. With this constraint, an optimal solution is to take items D, E , and F , resulting in a total weight of 20kg and a total value of £186.

Part C

This part adds a further modification. It is now possible to exceed the 20kg limit, but with a penalty of £15 for each kg over. A new variable, w , is necessary. The objective function is modified to:

$$\max(60A + 70B + 40C + 70D + 16E + 100F - 15w)$$

to capture the cost of exceeding the weight limit. An additional constraint is also required:

$$w = 6A + 7B + 4C + 9D + 3E + 8F - 20$$

to set w to number of kg over the weight limit the solution is. An optimal solution is to take items A, B , and F , resulting in a total weight of 21kg and a total value of £215.