Optimisation

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Question 1

The first step is to convert $min(x_1 - 3x_2)$ to a maximisation problem. This is trivial:

$$min(x_1 - 3x_2) = max(-x_1 + 3x_2)$$

and can be rewritten as:

$$x_1 - 3x_2 + P = 0$$

Constraints must also be converted to equalities, with the addition of the slack variables s_1, s_2, and s_3:

$$\begin{gathered} x_1 - x_2 \leq 1 \Rightarrow x_1 - x_2 + s_1 = 1 \\ x_1 - x_2 \geq -1 \Rightarrow -x_1 + x_2 + s_2 = 1 \\ 2x_1 - x_2 \leq 3 \Rightarrow 2x_1 - x_2 + s_3 = 3 \end{gathered}$$

The first tableau can now be constructed:

$$T_1 = \left(\begin{array}{c|ccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & \text{b} & \text{t} \\ \hline 1 & 1 & \text{-3} & 0 & 0 & 0 & 0 & \\ \hline 0 & 1 & \text{-1} & 1 & 0 & 0 & 1 & \text{-1} \\ 0 & \text{-1} & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & \text{-1} & 0 & 0 & 1 & 3 & \text{-3} \end{array}\right)$$

As column x_2 contains the smallest negative number in the first row (-3), we calculate pivot values, in column t, using x_2. As 1 is the smallest non-negative value in t, we pivot around the third element of column x_2, 1. Using the following row operations:

$$\begin{gathered} R_1 = R_1 + 3R_3 \\ R_2 = R_2 + R_3 \\ R_4 = R_4 + R_3 \end{gathered}$$

the next tableau can be constructed:

$$
T_2 = \left(\begin{array}{c|ccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & \text{b} & \text{t} \\
\hline
1 & -2 & 0 & 0 & 3 & 0 & 3 & \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 2 & - \\
0 & -1 & 1 & 0 & 1 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 & 1 & 1 & 4 & 4
\end{array}\right)
$$

The next pivot is the fourth element of column x_1, 1, as 4 is the smallest non-negative pivot value in t. Using the following row operations:

$$
\begin{aligned}
R_1 &= R_1 + 2R_4 \\
R_3 &= R_3 + R_4
\end{aligned}
$$

the final tableau can be constructed.

$$
T_3 = \left(\begin{array}{c|ccccc|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & \text{b} \\
\hline
1 & 0 & 0 & 0 & 5 & 2 & 11 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 & 2 & 1 & 5 \\
0 & 1 & 0 & 0 & 1 & 1 & 4
\end{array}\right)
$$

As there are no remaining negative values in the top row, an optimal result has been calculated. Non-unit columns are non-basic and are therefore 0. Reading from the tableau, the following solution can be obtained:

$$(4, 5, 2, 0, 0)$$

Plugging this back into the original minimisation problem thus gives an optimal value:

$$x_1 - 3x_2 = 4 - 3(5) = -11$$

Question 2

As the LP is provided in a canonical form, we can construct a tableau immediately:

$$T_1 = \left(\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b & t \\ \hline 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & \\ \hline 0 & -2 & 1 & 1 & 1 & 0 & 0 & 1 & -0.5 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & -3 & -1 & 0 & 0 & 1 & 6 & 3 \end{array}\right)$$

By Blend's rule, we use the column containing the smallest negative number in the first row (-2), in x_1, to calculate pivot values in t. As 2 is the smallest non-negative value, we pivot around the third element of column x_1, 1. Using the following row operations:

$$R_1 = R_1 + 2R_3$$
$$R_2 = R_2 + 2R_1$$
$$R_4 = R_4 - 2R_3$$

the next tableau can be constructed:

$$T_2 = \left(\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b & t \\ \hline 1 & 0 & -3 & 1 & 0 & 2 & 0 & 4 & \\ \hline 0 & 0 & -1 & 1 & 1 & 2 & 0 & 5 & -5 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & -1 & 0 & -2 & 1 & 2 & -2 \end{array}\right)$$

As there are no positive pivot values in t, this LP is unbounded.

This is not in of itself a *certificate* of unboundedness, however. A certificate comprises $x' = \overline{x} + td$ where $\overline{x}$ is a feasible solution to the LP and d is a vector such that:

1. $Ad = \mathbb{0}$
2. $d \leq \mathbb{0}$
3. $c^T d \leq \mathbb{0}$

Examining T_2 produces a feasible solution: $\overline{x}^T = (2, 0, 0, 5, 0, 2)$.

$$x' = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 5 \\ 0 \\ 2 \end{bmatrix} + d = \begin{bmatrix} 2 + d_1 \\ d_2 \\ d_3 \\ 5 + d_4 \\ d_5 \\ 2 + d_6 \end{bmatrix} \tag{1}$$

Plugging this in to the original constraints:

$$-2d_1 + d_2 + d_3 + d_4 = 0$$
$$d_1 - d_2 + d_5 = 0$$
$$2d_1 - 3d_2 - d_3 + d_6 = 0$$

Through trial and error, a solution to this system of equations may be found: $d^T = (1, 1, 1, 0, 0, 2)$. The certificate of unboundedness for this LP is therefore:

$$x' = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 5 \\ 0 \\ 2 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \tag{2}$$

Question 3

For this problem, 9 variables are necessary. Each relates to the amount of paint of one colour (cyan C, magenta M, or yellow Y) used in the creation of paint of another colour (red R, green G, blue B, or black K), e.g. C_K denotes the number of gallons of cyan (C) paint used in the production of black (K) paint. The objective function is therefore:

$$max\left(\tfrac{10\cdot(Y_R+M_R)}{2} + \tfrac{15\cdot(Y_G+C_G)}{2} + \tfrac{25\cdot(M_B+C_B)}{2} + \tfrac{25\cdot(C_K+M_K+Y_K)}{3}\right)$$

Each term is the amount of paint of a certain colour produced multiplied by the value of a gallon of paint of that colour: 10 for R, 15 for G, 25 for B, and 25 for K. The following constraints capture the limited quantities of paint available: 11 gallons of Y, 10 of C, and 5 of M:

$$Y_R + Y_G + Y_K <= 11$$
$$C_G + C_B + C_K <= 10$$
$$M_B + M_K + M_R <= 5$$

Also necessary are constraints maintaining the correct ratios of paints used:

$$Y_R == M_R$$
$$Y_G == C_G$$
$$M_B == C_B$$
$$C_K == M_K == Y_K$$

and constraints ensuring that the input paint volume equals the output paint volume:

$$Y_R + M_R == R$$
$$Y_G + C_G == G$$
$$M_B + C_B == \mathrm{B}$$
$$C_K + M_K + Y_K == K$$

An optimal solution produces 10 gallons of G paint and 15 gallons of K paint, using 10 gallons of C paint, 5 gallons of M paint, and 10 gallons of Y paint, for a total value of £525.

Question 4

Part A

This is a simple knapsack problem. Each variable A, B, C, D, E, F is binary: whether or not the item was taken. This leads to the following function:

$$max(60A + 70B + 40C + 70D + 16E + 100F)$$

in which the constants are the values (£) of each item. The only constraint is equally simple: that the weight of the taken items does not exceed 20kg:

$$6A + 7B + 4C + 9D + 3E + 8F <= 20$$

in which the constants are the weights (kg) of each item. An optimal solution is to take items B, C, and F, resulting in a total weight of $19kg$ and a total value of £210.

Part B

This part adds a new constraint: that taking C only makes sense if D is also taken, but not vice versa. This can be elegantly expressed as:

$$D - C >= 0$$

This condition is only unsatisfied if $D = 0$ and $C = 1$. With this constraint, an optimal solution is to take items D, E, and F, resulting in a total weight of $20kg$ and a total value of £186.

Part C

This part adds a further modification. It is now possible to exceed the $20kg$ limit, but with a penalty of £15 for each kg over. A new variable, w, is necessary. The objective function is modified to:

$$max(60A + 70B + 40C + 70D + 16E + 100F - 15w)$$

to capture the cost of exceeding the weight limit. An additional constraint is also required:

$$w >= 6A + 7B + 4C + 9D + 3E + 8F - 20$$

to set w to number of kg over the weight limit the solution is. An optimal solution is to take items A, B, and F, resulting in a total weight of $21kg$ and a total value of £215.