

# Optimisation

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## Question 1

The first step is to convert  $\min(x_1 - 3x_2)$  to a maximisation problem. This is trivial:

$$\min(x_1 - 3x_2) = \max(-x_1 + 3x_2)$$

and can be rewritten as:

$$x_1 - 3x_2 + P = 0$$

Constraints must also be converted to equalities, with the addition of the slack variables  $s_1$ ,  $s_2$ , and  $s_3$ :

$$\begin{aligned}x_1 - x_2 &\leq 1 \Rightarrow x_1 - x_2 + s_1 = 1 \\x_1 - x_2 &\geq -1 \Rightarrow -x_1 + x_2 + s_2 = 1 \\2x_1 - x_2 &\leq 3 \Rightarrow 2x_1 - x_2 + s_3 = 3\end{aligned}$$

The first tableau can now be constructed:

$$T_1 = \left( \begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & b & t \\ \hline 1 & 1 & -3 & 0 & 0 & 0 & 0 & \\ \hline 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 & 3 & -3 \end{array} \right)$$

As column  $x_2$  contains the smallest negative number in the first row ( $-3$ ), we calculate pivot values, in column  $t$ , using  $x_2$ . As 1 is the smallest non-negative value in  $t$ , we pivot around the third element of column  $x_2$ , 1. Using the following row operations:

$$\begin{aligned}R_1 &= R_1 + 3R_3 \\R_2 &= R_2 + R_3 \\R_4 &= R_4 + R_3\end{aligned}$$

the next tableau can be constructed:

$$T_2 = \left( \begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & b & t \\ \hline 1 & -2 & 0 & 0 & 3 & 0 & 3 & \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 & - \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 & 4 \end{array} \right)$$

The next pivot is the fourth element of column  $x_1$ , 1, as 4 is the smallest non-negative pivot value in  $t$ . Using the following row operations:

$$\begin{aligned} R_1 &= R_1 + 2R_4 \\ R_3 &= R_3 + R_4 \end{aligned}$$

the final tableau can be constructed.

$$T_3 = \left( \begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & b \\ \hline 1 & 0 & 0 & 0 & 5 & 2 & 11 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \end{array} \right)$$

As there are no remaining negative values in the top row, an optimal result has been calculated. Non-unit columns are non-basic and are therefore 0. Reading from the tableau, the following solution can be obtained:

$$(4, 5, 2, 0, 0)$$

Plugging this back into the original minimisation problem thus gives an optimal value:

$$x_1 - 3x_2 = 4 - 3(5) = -11$$

## Question 2

As the LP is provided in a canonical form, we can construct a tableau immediately:

$$T_1 = \left( \begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b & t \\ \hline 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & \\ \hline 0 & -2 & 1 & 1 & 1 & 0 & 0 & 1 & -0.5 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & -3 & -1 & 0 & 0 & 1 & 6 & 3 \end{array} \right)$$

By Blend's rule, we use the column containing the smallest negative number in the first row ( $-2$ ), in  $x_1$ , to calculate pivot values in  $t$ . As 2 is the smallest non-negative value, we pivot around the third element of column  $x_1$ , 1. Using the following row operations:

$$\begin{aligned} R_1 &= R_1 + 2R_3 \\ R_2 &= R_2 + 2R_1 \\ R_4 &= R_4 - 2R_3 \end{aligned}$$

the next tableau can be constructed:

$$T_2 = \left( \begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b & t \\ \hline 1 & 0 & -3 & 1 & 0 & 2 & 0 & 4 & \\ \hline 0 & 0 & -1 & 1 & 1 & 2 & 0 & 5 & -5 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & -1 & 0 & -2 & 1 & 2 & -2 \end{array} \right)$$

As there are no positive pivot values in  $t$ , this LP is *unbounded*.

### Question 3

For this problem, 9 variables are necessary. Each relates to the amount of paint of one colour (cyan  $C$ , magenta  $M$ , or yellow  $Y$ ) used in the creation of paint of another colour (red  $R$ , green  $G$ , blue  $B$ , or black  $K$ ), e.g.  $C_K$  denotes the number of gallons of cyan ( $C$ ) paint used in the production of black ( $K$ ) paint. The objective function is therefore:

$$\max\left(\frac{10 \cdot (Y_R + M_R)}{2} + \frac{15 \cdot (Y_G + C_G)}{2} + \frac{25 \cdot (M_B + C_B)}{2} + \frac{25 \cdot (C_K + M_K + Y_K)}{3}\right)$$

Each term is the amount of paint of a certain colour produced multiplied by the value of a gallon of paint of that colour: 10 for  $R$ , 15 for  $G$ , 25 for  $B$ , and 25 for  $K$ . The following constraints capture the limited quantities of paint available: 11 gallons of  $Y$ , 10 of  $C$ , and 5 of  $M$ :

$$\begin{aligned} Y_R + Y_G + Y_K &\leq 11 \\ C_G + C_B + C_K &\leq 10 \\ M_B + M_K + M_R &\leq 5 \end{aligned}$$

Also necessary are constraints maintaining the correct ratios of paints used:

$$\begin{aligned} Y_R &== M_R \\ Y_G &== C_G \\ M_B &== C_B \\ C_K &== M_K == Y_K \end{aligned}$$

and constraints ensuring that the input paint volume equals the output paint volume:

$$\begin{aligned} Y_R + M_R &== R \\ Y_G + C_G &== G \\ M_B + C_B &== B \\ C_K + M_K + Y_K &== K \end{aligned}$$

An optimal solution produces 10 gallons of  $G$  paint and 15 gallons of  $K$  paint, using 10 gallons of  $C$  paint, 5 gallons of  $M$  paint, and 10 gallons of  $Y$  paint, for a total value of £525.

## Question 4

### Part A

This is a simple knapsack problem. Each variable  $A, B, C, D, E, F$  is binary: whether or not the item was taken. This leads to the following function:

$$\max(60A + 70B + 40C + 70D + 16E + 100F)$$

in which the constants are the values (£) of each item. The only constraint is equally simple: that the weight of the taken items does not exceed 20kg:

$$6A + 7B + 4C + 9D + 3E + 8F \leq 20$$

in which the constants are the weights (kg) of each item. An optimal solution is to take items  $B, C$ , and  $F$ , resulting in a total weight of 19kg and a total value of £210.

### Part B

This part adds a new constraint: that taking  $C$  only makes sense if  $D$  is also taken, but not vice versa. This can be elegantly expressed as:

$$D - C \geq 0$$

This condition is only unsatisfied if  $D = 0$  and  $C = 1$ . With this constraint, an optimal solution is to take items  $D, E$ , and  $F$ , resulting in a total weight of 20kg and a total value of £186.

### Part C

This part adds a further modification. It is now possible to exceed the 20kg limit, but with a penalty of £15 for each kg over. A new variable,  $w$ , is necessary. The objective function is modified to:

$$\max(60A + 70B + 40C + 70D + 16E + 100F - 15w)$$

to capture the cost of exceeding the weight limit. An additional constraint is also required:

$$w \geq 6A + 7B + 4C + 9D + 3E + 8F - 20$$

to set  $w$  to number of kg over the weight limit the solution is. An optimal solution is to take items  $A, B$ , and  $F$ , resulting in a total weight of 21kg and a total value of £215.