An algorithm to detect a Hurwitz continued fraction expansion

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Hello Christopher!

Here comes a proposal to terminate your algorithm and to detect the Hurwitz continued fraction expansion of a generalized Zopf number.

Assumptions.

a) You can compute all the elements of a regular continued fraction expansion of a given number ξ:

$$\xi = \big[a_0, a_1, a_2, a_3, \dots \big], \tag{1}$$

where $a_0 \geqslant 0$ and $a_n \geqslant 1$ for all $n \geqslant 1$.

b) The expansion in (1) is a Hurwitz continued fraction expansion, namely,

$$\xi = \big[a_0, a_1, \dots, a_{\rho-1}, \overline{P_1(k), P_2(k), \dots, P_m(k)} \big]_{k=1}^{\infty}, \tag{2}$$

where $\rho \geqslant 0$, and $P_1(k), \dots, P_m(k)$ are polynomials of degree 0 or 1 taking positive integer values for all $k \geqslant 1$. We want to find ρ, m, and the polynomials $P_1, \dots, P_m$.

c) There is an algorithm to decide whether a given Hurwitz continued fraction expansion,

$$\xi' := \big[a_0, a_1, \dots, a_{\rho'-1}, \overline{Q_1(k), Q_2(k), \dots, Q_{m'}(k)} \big]_{k=1}^{\infty} \tag{3}$$

with given numbers ρ', m' and polynomials $Q_1(k), Q_2(k), \dots, Q_{m'}(k)$ of degree 0 or 1, represents the number ξ (i.e., $\xi' = \xi$) or not (i.e., $\xi' \neq \xi$). If such an algorithm is not known, we compare the first N partial denominators of ξ and ξ'. We choose the natural number N as a parameter. In order to check $\xi = \xi'$, the partial denominators with the indices $0, 1, \dots, N$ of the two Hurwitz continued fractions are compared with each other. If the test is negative, ξ and ξ' definitely do not match; if the test is positive, the probability increases for a larger N that the correct continued fraction expansion was found.

Our algorithm to terminate the computation of the elements in (1) works as follows:

Let $g \geqslant 1$ and $1 \leqslant h \leqslant g$ be integers, where g takes the values $1, 2, \dots,$ and h takes the values $1, 2, \dots, g$ for every g until our algorithm terminates. We decide by the following procedure, whether a_g is the first element of the second primitive period with length h of the Hurwitz continued fraction expansion (HCFE) of ξ:

1) Set $g = 1$.

2) Set $h = 0$.

3) Set $h = h+1$ and go to **4)**, if $h+1 \leqslant g$. Otherwise, set $g = g+1$ and go to **2)**.

4) Compute the finite sequences $S_1(g,h)$ and $S_2(g,h)$ of length h:

$$\begin{aligned} S_1(g,h) &:= \big(a_g - a_{g-h}, a_{g+1} - a_{g-h+1}, \ldots, a_{g+h-1} - a_{g-1}\big)\,, \\ S_2(g,h) &:= \big(a_{g+h} - a_g, a_{g+h+1} - a_{g+1}, \ldots, a_{g+2h-1} - a_{g+h-1}\big)\,. \end{aligned}$$

Since the partial denominators in the main period increase at most linearly, three consecutive main periods are required to detect the formulas for the partial denominators and two main periods to test them.

5) We distinguish two cases:

Case 1. $S_1(g,h) \neq S_2(g,h)$
Then a_g is not the first element of the second primitive period with length h of the HCFE of ξ. Go to **3)**.

Case 2. $S_1(g,h) = S_2(g,h)$
Let

$$\xi' := \big[\, a_0, a_1, \ldots, a_{g-h-1}, \overline{Q_1(k), Q_2(k), \ldots, Q_h(k)}\,\big]_{k=1}^{\infty}\,, \tag{4}$$

where we set for every μ with $1 \leqslant \mu \leqslant h$:

$$Q_\mu(k) \;:=\; \begin{cases} a_{g-h+\mu-1} & \text{if} \;\; a_{g-h+\mu-1} = a_{g+\mu-1} = a_{g+h+\mu-1}\,, \\ (a_{g+\mu-1} - a_{g-h+\mu-1})k + 2a_{g-h+\mu-1} - a_{g+\mu-1} & \text{if} \;\; a_{g-h+\mu-1} < a_{g+\mu-1} < a_{g+h+\mu-1}\,, \\ \text{stop the computations in Case 2 and go to } \mathbf{3)} & \;\;\;\;\;\text{otherwise}\,. \end{cases}$$

Go to **6)**.

6) We can check by assumption c) whether $\xi = \xi$ or $\xi \neq \xi$:

$$\text{Go to } \mathbf{7)} \text{ if } \;\xi' = \xi\,,$$
$$\text{Go to } \mathbf{3)} \text{ if } \;\xi' \neq \xi\,.$$

7) End: Output: The HCFE of ξ in (2) is given by the right-hand side of (4) with

$$\begin{aligned} \rho &= \rho' = g-h\,, \\ m &= m' = h\,, \\ P_\mu &= Q_\mu \quad (1 \leqslant \mu \leqslant h)\,. \end{aligned}$$

The algorithm terminates by our assumptions. ◊

Example 1: Let

$$\begin{aligned} \xi &:= \big[2,3,4,\overline{5,2k+3,7}\,\big]_{k=1}^{\infty} \\ &= \big[\;\underbrace{2,3,4}_{\text{(preperiod)}}\;,\;\underbrace{5,5,7}_{\text{(first period)}}\;,\;\underbrace{5,7,7}_{\text{(second period)}}\;,\;\underbrace{5,9,7}_{\text{(third period)}}\;,\ldots\,\big] \\ &= \big[\,a_0, a_1, a_2, \ldots\,\big]\,. \end{aligned}$$

So we have

n	0	1	2	3	4	5	6	7	8	9	10	11
a_n	2	3	4	5	5	7	5	7	7	5	9	7

The first element in the second period is $a_6 = 7$. The period length is 3. We do not know this structure at the beginning. What we know are the numerical values of the partial denominators of ξ.

At the beginning of the eighteenth step[1], our algorithm has reached the values $g = 6$ and $h = 3$. Then, we obtain the sequences

$$\begin{aligned} S_1(6,3) &= \big(a_6 - a_3, a_7 - a_4, a_8 - a_5\big) \\ &= \big(5-5, 7-5, 7-7\big) \\ &= \big(0, 2, 0\big)\,, \\ S_2(6,3) &= \big(a_9 - a_6, a_{10} - a_7, a_{11} - a_8\big) \\ &= \big(5-5, 9-7, 7-7\big) \\ &= \big(0, 2, 0\big)\,. \end{aligned}$$

Therefore, Case 2 occurs in **5)**: $S_1(6,3) = S_2(6,3)$. The continued fraction expansion of ξ' in (4) is given by

$$\begin{aligned} \xi' &= \big[a_0, a_1, a_2, \overline{Q_1(k), Q_2(k), Q_3(k)}\,\big]_{k=1}^{\infty} \\ &= \big[2, 3, 4, \overline{Q_1(k), Q_2(k), Q_3(k)}\,\big]_{k=1}^{\infty}\,, \end{aligned}$$

where

$$Q_1(k) \;=\; a_3 \;=\; 5 \;=\; const_k\,,$$

since $a_3 = a_6 = a_9 = 5$;

$$Q_2(k) \;=\; (a_7 - a_4)k + 2a_4 - a_7 \;=\; (7-5)k + 2\cdot 5 - 7 \;=\; 2k+3\,,$$

since $a_4 = 5 < a_7 = 7 < a_{10} = 9$;

$$Q_3(k) \;=\; a_5 \;=\; 7 \;=\; const_k\,,$$

since $a_5 = a_8 = a_{11} = 7$.

We obtain the final result

$$\xi' \;=\; \big[2, 3, 4, \overline{5, 2k+3, 7}\,\big]_{k=1}^{\infty} \;=\; \xi\,.$$

No pattern was found in the previous iteration steps of the algorithm:

Step 1:

$$S_1(1,1) \;=\; \big(a_1 - a_0\big) \;=\; \big(1\big) \;=\; \big(a_2 - a_1\big) \;=\; S_2(1,1)\,,$$

but

$$\xi' \;=\; \big[\,\overline{Q_1(k)}\,\big]_{k=1}^{\infty} \;=\; \big[\,\overline{(a_1 - a_0)k + 2a_0 - a_1}\,\big]_{k=1}^{\infty} \;=\; \big[\,\overline{k+1}\,\big]_{k=1}^{\infty} \;\neq\; \xi\,.$$

Step 2:

$$S_1(2,1) \;=\; \big(a_2 - a_1\big) \;=\; \big(1\big) \;=\; \big(a_3 - a_2\big) \;=\; S_2(2,1)\,,$$

but

$$\xi' \;=\; \big[2, \overline{Q_1(k)}\,\big]_{k=1}^{\infty} \;=\; \big[2, \overline{(a_2 - a_1)k + 2a_1 - a_2}\,\big]_{k=1}^{\infty} \;=\; \big[2, \overline{k+2}\,\big]_{k=1}^{\infty} \;\neq\; \xi\,.$$

[1]We investigate the first five steps of the algorithm below.

Step 3:

$$S_1(2,2) \;=\; \big(\,a_2 - a_0, a_3 - a_1\,\big) \;=\; \big(\,2,2\,\big) \;\neq\; \big(\,1,2\,\big) \;=\; \big(\,a_4 - a_2, a_5 - a_3\,\big) \;=\; S_2(2,2)\,.$$

Step 4:

$$S_1(3,1) \;=\; \big(\,a_3 - a_2\,\big) \;=\; \big(\,1\,\big) \;\neq\; \big(\,0\,\big) \;=\; \big(\,a_4 - a_3\,\big) \;=\; S_2(3,1)\,.$$

Step 5:

$$S_1(3,2) \;=\; \big(\,a_3 - a_1, a_4 - a_2\,\big) \;=\; \big(\,2,1\,\big) \;\neq\; \big(\,2,0\,\big) \;=\; \big(\,a_5 - a_3, a_6 - a_4\,\big) \;=\; S_2(3,2)\,.$$

$\Diamond$

Example 2: Let

$$\xi \;:=\; \big[\,2, \overline{3k+2, 1, 7k-2}\,\big]_{k=1}^{\infty} \;=\; \big[\,2,5,1,5,8,1,12,11,1,19,\ldots\,\big]\,.$$

The first partial denominators are given by the following table.

n	0	1	2	3	4	5	6	7	8	9
a_n	2	5	1	5	8	1	12	11	1	19

In the first eight steps of the algorithm, Case 1 never occurs in **5)**, so there is no need to check $\xi' = \xi$ in these eight cases.

Step 1:

$$S_1(1,1) \;=\; \big(\,a_1 - a_0\,\big) \;=\; \big(\,3\,\big) \;\neq\; \big(\,-4\,\big) \;=\; \big(\,a_2 - a_1\,\big) \;=\; S_2(1,1)\,;$$

Step 2:

$$S_1(2,1) \;=\; \big(\,a_2 - a_1\,\big) \;=\; \big(\,-4\,\big) \;\neq\; \big(\,4\,\big) \;=\; \big(\,a_3 - a_2\,\big) \;=\; S_2(2,1)\,;$$

Step 3:

$$S_1(2,2) \;=\; \big(\,a_2 - a_0, a_3 - a_1\,\big) \;=\; \big(\,-1,0\,\big) \;\neq\; \big(\,7,-4\,\big) \;=\; \big(\,a_4 - a_2, a_5 - a_3\,\big) \;=\; S_2(2,2)\,;$$

Step 4:

$$S_1(3,1) \;=\; \big(\,a_3 - a_2\,\big) \;=\; \big(\,4\,\big) \;\neq\; \big(\,3\,\big) \;=\; \big(\,a_4 - a_3\,\big) \;=\; S_2(3,1)\,;$$

Step 5:

$$S_1(3,2) \;=\; \big(\,a_3 - a_1, a_4 - a_2\,\big) \;=\; \big(\,0,7\,\big) \;\neq\; \big(\,-4,4\,\big) \;=\; \big(\,a_5 - a_3, a_6 - a_4\,\big) \;=\; S_2(3,2)\,;$$

Step 6:

$$\begin{aligned} S_1(3,3) \;&=\; \big(\,a_3 - a_0, a_4 - a_1, a_5 - a_2\,\big) \;=\; \big(\,3,3,0\,\big) \\ &\neq\; \big(\,7,3,0\,\big) \;=\; \big(\,a_6 - a_3, a_7 - a_4, a_8 - a_5\,\big) \;=\; S_2(3,3)\,; \end{aligned}$$

Step 7:

$$S_1(4,1) \;=\; \big(\,a_4 - a_3\,\big) \;=\; \big(\,3\,\big) \;\neq\; \big(\,-7\,\big) \;=\; \big(\,a_5 - a_4\,\big) \;=\; S_2(4,1)\,;$$

Step 8:

$$S_1(4,2) \;=\; \big(\,a_4 - a_2, a_5 - a_3\,\big) \;=\; \big(\,7,-4\,\big) \;\neq\; \big(\,4,10\,\big) \;=\; \big(\,a_6 - a_4, a_7 - a_5\,\big) \;=\; S_2(4,2)\,;$$

Step 9:

$$\begin{aligned} S_1(4,3) &= \big(a_4-a_1, a_5-a_2, a_6-a_3\big) = \big(3,0,7\big) \\ &= \big(3,0,7\big) = \big(a_7-a_4, a_8-a_5, a_9-a_6\big) = S_2(4,3)\,. \end{aligned}$$

Therefore, we obtain

$$\xi' = \big[a_0, \overline{Q_1(k), Q_2(k), Q_3(k)}\,\big]_{k=1}^{\infty} = \big[2, \overline{Q_1(k), Q_2(k), Q_3(k)}\,\big]_{k=1}^{\infty}$$

with

$$Q_1(k) = (a_4-a_1)k + 2a_1 - a_4 = 3k+2\,,$$

since $a_1 < a_4 < a_7$;

$$Q_2(k) = a_2 = 1 = const_k\,,$$

since $a_2 = a_5 = a_8 = 1$;

$$Q_3(k) = (a_6-a_3)k + 2a_3 - a_6 = 7k-2\,,$$

since $a_3 < a_6 < a_9$. ♢

A variant of the algorithm

If upper bounds are known for the lengths of the preperiod $a_0, a_1, \ldots, a_{\rho-1}$ and the first main period $a_\rho, a_{\rho_1}, \ldots, a_{\rho+m-1}$, then $g = \rho+m$, the index of the first element of the second main period, satisfies the inequality

$$g \leqslant P+M\,, \tag{5}$$

where $\rho \leqslant P$ and $m \leqslant M$. While g runs from 1 to (at most) $P+M$, the number h runs from 1 to g for every g. This means that our algorithm terminates after

$$1+2+\cdots+(P+M) = \frac{(P+M)(P+M+1)}{2} = \mathcal{O}\big((P+M)^2\big) \tag{6}$$

steps at the latest. Step **6** in the algorithm is now carried out as follows:

6a) Let $s \geqslant 1$ be the smallest integer such that

$$(s+1)h \geqslant M\,, \tag{7}$$

and let $t \geqslant 0$ be the smallest integer such that

$$g+th \geqslant P\,. \tag{8}$$

Due to the definition of a_g as the first element of the second main period, we have

$$g+rh \leqslant n \leqslant g+(r+1)h-1 \qquad \Longleftrightarrow \qquad a_n = Q_{1+(n-g \bmod h)}(r+2)\,. \tag{9}$$

For example, if $r=0$ and $n=g$, then $a_g = Q_1(2)$ follows.

6b) We consider the finite sequence

$$\overline{\big(a_{g+(t+\nu)h}, \ldots, a_{g+(t+\nu+1)h-1}\big)}_{\nu=0}^{s} = \big(a_{g+th}, \ldots, a_{g+(t+s)h}, \ldots, a_{g+(t+s+1)h-1}\big)\,, \tag{10}$$

which consisits of $(s+1)h \geqslant 2h$ elements. Due to $P > \rho-1$ and (8), all elements of the sequence in (10) are in the range of the main periods. Because the sequence in (10) consists of at least M elements due to the inequality (7), it is longer than two (primitive) main periods. Note the comment at the end of step 4).

6c) Due to the properties of the sequence in (10) shown in 6b), $\xi = \xi'$ (with ξ' given by (4)) holds **if and only if**

$$\begin{aligned} & \overline{\big(Q_1(t+\nu+2), \ldots, Q_h(t+\nu+2)\big)}_{\nu=0}^{s} && (11) \\ \overset{(9)}{=} \; & \overline{\big(a_{g+(t+\nu)h}, \ldots, a_{g+(t+\nu+1)h-1}\big)}_{\nu=0}^{s} \\ \overset{(10)}{=} \; & \big(a_{g+th}, \ldots, a_{g+(t+s)h}, \ldots, a_{g+(t+s+1)h-1}\big)\,. && (12) \end{aligned}$$

It is therefore tested whether the two finite sequences in (11) and (12) match. If so, the Hurwitz continued fraction in (1) has definitely been found (and not just with a certain probability):

Go to **7)**

If the sequences in (11) and (12) do not match, the situation $\xi \neq \xi'$ is given:

Go to **3)**

6d) However, due to (6), the Hurwitz continued fraction is found after $1+2+3+\cdots+(P+M)$ iteration steps at the latest.

Introduction of additional search functions

In the previous version of the algorithm, two search functions $S_1(g,h)$ and $S_2(g,h)$ were used. Three possible consecutive intervals are tested as main periods. Each such interval contains h integers:

$$\begin{aligned} I_0 &:= \big[\, g-h, \ldots, g-1 \,\big]\,, \\ I_1 &:= \big[\, g, \ldots, g+h-1 \,\big]\,, \\ I_2 &:= \big[\, g+h, \ldots, g+2h-1 \,\big]\,. \end{aligned}$$

In the following, m denotes the number of search functions that will be used now. Here, $m \geqslant 3$ is assumed. These m search functions scan $m+1$ consecutive intervals, each containing h integers:

$$I_\mu := \big[\, g+(\mu-1)h, \ldots, g+\mu h-1 \,\big] \qquad (0 \leqslant \mu \leqslant m)\,. \tag{13}$$

Then, the search functions $S_\mu(g,h)$ are defined as follows:

$$\begin{aligned} S_\mu(g,h) &:= \big(\, a_{g+(\mu-1)h} - a_{g+(\mu-2)h}\,,\, \ldots\,,\, a_{g+\mu h-1} - a_{g+(\mu-1)h-1} \,\big) \\ &= \big(\, a_{g+(\mu-1)h+\nu} - a_{g+(\mu-2)h+\nu} \,\big)_{\nu=0}^{h-1} \qquad (1 \leqslant \mu \leqslant m)\,. \end{aligned} \tag{14}$$

For $\mu = 1$ and $\mu = 2$ we obtain the two search functions used in the former version of the algorithm. The case distinction in step 5 of the algorithm is now:

Case 1. There are indices $1 \leqslant \mu_1 < \mu_2 \leqslant m$ satisfying $S_{\mu_1}(g,h) \neq S_{\mu_2}(g,h)$.

Case 2. $S_1(g,h) = S_2(g,h) = \cdots = S_m(g,h)$

The functions $Q_\mu(k)$ for $1 \leqslant \mu \leqslant h$ are as before:

$$Q_\mu(k) \;:=\; \begin{cases} a_{g-h+\mu-1} & \text{if} \;\; a_{g-h+\mu-1} = a_{g+\mu-1} = a_{g+h+\mu-1}\,, \\ (a_{g+\mu-1} - a_{g-h+\mu-1})k + 2a_{g-h+\mu-1} - a_{g+\mu-1} & \text{if} \;\; a_{g-h+\mu-1} < a_{g+\mu-1} < a_{g+h+\mu-1}\,, \\ \text{stop the computations in Case 2 and go to } \mathbf{3)} & \phantom{\text{if}} \;\; \text{otherwise}\,. \end{cases}$$

No further changes are made to the algorithm.

The numerical handling of step 6 in the case of generalized Zopf numbers

We consider the generalized Zopf numbers

$$\mathfrak{z}(a,b,c) := a+c+\cfrac{b}{a+2c+\cfrac{b}{a+3c+\cfrac{b}{a+4c+\ddots}}} = \sqrt{b}\,\frac{I_\alpha(\beta)}{I_{1+\alpha}(\beta)}\,, \tag{15}$$

with

$$c > 0\,, \quad \alpha := \frac{a}{c} \quad \text{and} \quad \beta := \frac{2\sqrt{b}}{c}\,.$$

The parameter b can be negative (as well as a), so that $\beta = 2i\sqrt{-b}/c$. The function $I_\alpha(z)$ denotes the modified Bessel function of the first kind for complex arguments z,

$$I_\alpha(z) := i^{-\alpha}J_\alpha(iz) \quad \text{with} \quad J_\alpha(z) = \Big(\frac{z}{2}\Big)^\alpha \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\alpha+1)}\Big(\frac{z}{2}\Big)^{2k}, \tag{16}$$

so that

$$I_\alpha(z) = \Big(\frac{z}{2}\Big)^\alpha \sum_{k=0}^\infty \frac{1}{k!\Gamma(k+\alpha+1)}\Big(\frac{z}{2}\Big)^{2k}. \tag{17}$$

In any case, the right-hand side in (15) is a real number.

Let us assume, our algorithm has found a candidate ξ' as a Hurwitz continued fraction given by (3). We want to check in step 6 of the above algorithm whether $\xi' = \mathfrak{z}(a,b,c)$. Here comes a necessary condition for this identity. First, we write (3) in the general form

$$\xi' = \big[\,a_0, a_1, a_2, a_3, \dots\,\big]\,. \tag{18}$$

By $Q_\varphi(x)$ with $1 \leqslant \varphi \leqslant m'$, we denote a polynomial from the main period of the continued fraction in (3) having degree 1 and a maximum leading coefficient among all polynomials in (3) of degree 1. Next, using the partial denominators a_ν, we compute for natural numbers $n \geqslant 1$ the matrix

$$\begin{pmatrix} p_{m'n+\rho'+\varphi-2} & p_{m'n+\rho'+\varphi-3} \\ q_{m'n+\rho'+\varphi-2} & q_{m'n+\rho'+\varphi-3} \end{pmatrix} = \prod_{\nu=0}^{m'n+\rho'+\varphi-2} \begin{pmatrix} a_\nu & 1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

cf. [2, (11) in § 5]. Then, using Theorem II.6 in our book [1], we have for every integer $n \geqslant 1$ the two inequalities

$$\big(2+Q_\varphi(n+1)\big)q_{m'n+\rho'+\varphi-2}\left|\sqrt{b}\frac{I_\alpha(\beta)}{I_{1+\alpha}(\beta)}q_{m'n+\rho'+\varphi-2} - p_{m'n+\rho'+\varphi-2}\right| > 1\,, \tag{20}$$

$$Q_\varphi(n+1)q_{m'n+\rho'+\varphi-2}\left|\sqrt{b}\frac{I_\alpha(\beta)}{I_{1+\alpha}(\beta)}q_{m'n+\rho'+\varphi-2} - p_{m'n+\rho'+\varphi-2}\right| < 1\,, \tag{21}$$

provided that $\xi' = \mathfrak{z}(a,b,c)$. Consequently, if one finds an integer $n \geqslant 1$, so that both inequalities (20) and (21) are not fulfilled simultaneously, we have $\xi' \neq \mathfrak{z}(a,b,c)$.

Remark 1. *Using the notation in our book [1], we have the following representation for ξ' instead of (3),*

$$\xi' = \big[\, a_0, a_1, \ldots, a_{\rho-1}, \overline{P_0(k), P_1(k), \ldots, P_{\omega-1}(k)}\,\big]_{k=1}^{\infty} . \tag{22}$$

Based on this notation, the following renaming must be carried out within the formulas (19), (20) and (21):

$$\begin{aligned}
m' &\rightarrow \omega\,, \\
\rho' &\rightarrow \rho\,, \\
Q_\varphi(n+1) &\rightarrow P_\varphi(n+1) \quad \textit{with} \quad 0 \leqslant \varphi \leqslant \omega - 1\,, \\
p_{m'n+\rho'+\varphi-2} &\rightarrow p_{\omega n+\rho+\varphi-1}\,, \\
p_{m'n+\rho'+\varphi-3} &\rightarrow p_{\omega n+\rho+\varphi-2}\,, \\
q_{m'n+\rho'+\varphi-2} &\rightarrow q_{\omega n+\rho+\varphi-1}\,, \\
q_{m'n+\rho'+\varphi-3} &\rightarrow q_{\omega n+\rho+\varphi-2}\,.
\end{aligned}$$

Example 3:
We consider the generalized Zopf number

$$\mathfrak{z}(2,-2,1) \;=\; \big[\,2, 2, \overline{2k+1, 1, k, 1}\,\big]_{k=1}^{\infty}\,; \tag{23}$$

see formula (8), page 5, in the appendix of the email dated from June, 2nd. We have $a = 2$, $b = -2$ and $c = 1$. By (15), we obtain

$$\mathfrak{z}(2,-2,1) \;=\; \sqrt{-2}\,\frac{I_2(2\sqrt{-2})}{I_3(2\sqrt{-2})} \;=\; 2.4397493218702328058957069574112274 2515\ldots\,. \tag{24}$$

Comparing the formulas from (3) and (23), it follows that

$$\rho' = 2 \qquad \text{and} \qquad m' = 4\,.$$

Moreover, we have $\varphi = 1$ and $Q_\varphi(x) = Q_1(x) = 2x+1$. We verify the inequalities in (20) and (21) with $n = 1$:

$$\begin{aligned}
p_{m'n+\rho'+\varphi-2} &= p_5 = 61\,, \\
q_{m'n+\rho'+\varphi-2} &= q_5 = 25\,.
\end{aligned}$$

Insert the values into (20) by using the numerical value from (24):

$$\big(2 + Q_1(2)\big)q_5\big|\, q_5\mathfrak{z}(2,-2,1) - p_5,\big| \;=\; 175\big|\, 25\mathfrak{z}(2,-2,1) - 61\,\big| \;=\; 1.0967\ldots > 1\,.$$

Insert the values into (21) by using the numerical value from (24):

$$Q_1(2)q_5\big|\, q_5\mathfrak{z}(2,-2,1) - p_5,\big| \;=\; 125\big|\, 25\mathfrak{z}(2,-2,1) - 61\,\big| \;=\; 0.78337\ldots < 1\,.$$

Remark 2. *For a correct verification of the two inequalities, the knowledge of <u>more than four significant fractional digits</u> of the number $\mathfrak{z}(2,-2,1)$ was required, because the integer*

$$4375 \;=\; 175 \cdot 25 \;=\; \big(2 + Q_\varphi(n+1)\big)q_{m'n+\rho'+\varphi-2}$$

has four decimal places.

References

[1] C.Elsner and Chr.Havens, *Siegel's continued fractions - To be continued via Transducers and Diophantine Analysis* -, (Book in progress).

[2] O.Perron, *Die Lehre von den Kettenbrüchen*, vol. 1, Wissenschaftliche Buchgesellschaft Darmstadt, 1977.