GSERM - St. Gallen 2024
Analyzing Panel Data

June 14, 2024

Logit/Probit Redux

Start with:

$$Y_i^* = \mathbf{X}_i\boldsymbol{\beta} + u_i$$

$$\begin{aligned} Y_i &= 0 \text{ if } Y_i^* < 0 \\ Y_i &= 1 \text{ if } Y_i^* \geq 0 \end{aligned}$$

So:

$$\begin{aligned} \Pr(Y_i = 1) &= \Pr(Y_i^* \geq 0) \\ &= \Pr(\mathbf{X}_i\boldsymbol{\beta} + u_i \geq 0) \\ &= \Pr(u_i \geq -\mathbf{X}_i\beta) \\ &= \Pr(u_i \leq \mathbf{X}_i\beta) \\ &= \int_{-\infty}^{\mathbf{X}_i\boldsymbol{\beta}} f(u)du \end{aligned}$$

Logit

"Standard logistic" PDF:

$$\Pr(u) \equiv \lambda(u) = \frac{\exp(u)}{[1+\exp(u)]^2}$$

CDF:

$$\begin{aligned}
\Lambda(u) &= \int \lambda(u) du \\
&= \frac{\exp(u)}{1+\exp(u)} \\
&= \frac{1}{1+\exp(-u)}
\end{aligned}$$

Logistic $\rightarrow$ "Logit"

$$
\begin{aligned}
\Pr(Y_i = 1) &= \Pr(Y_i^* > 0) \\
&= \Pr(u_i \leq \mathbf{X}_i\boldsymbol{\beta}) \\
&= \Lambda(\mathbf{X}_i\boldsymbol{\beta}) \\
&= \frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1 + \exp(\mathbf{X}_i\boldsymbol{\beta})}
\end{aligned}
$$

$$
\text{(equivalently)} = \frac{1}{1 + \exp(-\mathbf{X}_i\boldsymbol{\beta})}
$$

Likelihoods

$$L_i = \left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)^{Y_i}\left[1-\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right]^{1-Y_i}$$

$$L = \prod_{i=1}^{N}\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)^{Y_i}\left[1-\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right]^{1-Y_i}$$

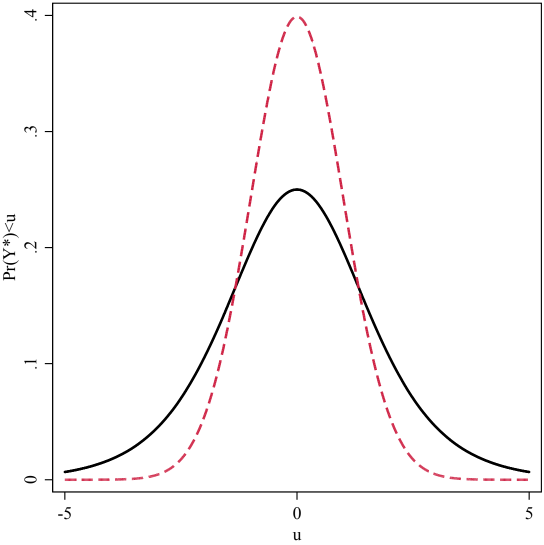
$$\begin{aligned}\ln L \quad = \quad & \sum_{i=1}^{N} Y_i \ln\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right) + \\ & (1-Y_i)\ln\left[1-\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right]\end{aligned}$$

Probit...

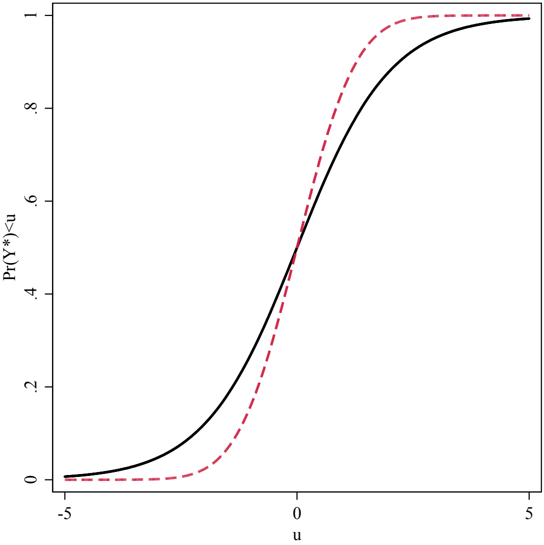
$$\Pr(u) \equiv \phi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$$

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

Standard Normal and Logistic PDFs



Standard Normal and Logistic CDFs



Probit (continued...)

$$\begin{aligned}\Pr(Y_i = 1) &= \Phi(\mathbf{X}_i\boldsymbol{\beta}) \\ &= \int_{-\infty}^{\mathbf{X}_i\boldsymbol{\beta}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\mathbf{X}_i\boldsymbol{\beta})^2}{2}\right) d\mathbf{X}_i\boldsymbol{\beta}\end{aligned}$$

$$L = \prod_{i=1}^{N} \left[\Phi(\mathbf{X}_i\boldsymbol{\beta})\right]^{Y_i} \left[1 - \Phi(\mathbf{X}_i\boldsymbol{\beta})\right]^{(1-Y_i)}$$

$$\ln L = \sum_{i=1}^{N} Y_i \ln\Phi(\mathbf{X}_i\boldsymbol{\beta}) + (1 - Y_i) \ln[1 - \Phi(\mathbf{X}_i\boldsymbol{\beta})]$$

One-way unit effects:

$$Y_{it} = f(\mathbf{X}_{it}\beta + \alpha_i + u_{it})$$

So, think about logit first:

$$\Pr(Y_{it} = 1) = \frac{\exp(\mathbf{X}_{it}\beta + \alpha_i)}{1 + \exp(\mathbf{X}_{it}\beta + \alpha_i)} \equiv \Lambda(\mathbf{X}_{it}\beta + \alpha_i)$$

Incidental Parameters:

- Nonlinearity $\rightarrow$ inconsistency in both $\hat{\alpha}$s and $\hat{\boldsymbol{\beta}}$.
- Anderson's *unconditional* estimator:

$$L^U = \prod_{i=1}^{N} \prod_{t=1}^{T} \Lambda(\mathbf{X}_{it} + \alpha_i)^{Y_{it}} [1 - \Lambda(\mathbf{X}_{it} + \alpha_i)]^{1 - Y_{it}}$$

- Chamberlain's *conditional* estimator:

$$L^C = \prod_{i=1}^{N} \Pr\left(Y_{i1} = y_{i1}, Y_{i2} = y_{i2}, ... Y_{iT} = y_{iT} \mid \sum_{t=1}^{T} Y_{it} \right)$$

Fixed-Effects (continued)

Intuition: Suppose we have $T = 2$. That means that:

- $\Pr(Y_{i1} = 0 \text{ and } Y_{i2} = 0 \mid \sum_T Y_{it} = 0) = 1.0$
- $\Pr(Y_{i1} = 1 \text{ and } Y_{i2} = 1 \mid \sum_T Y_{it} = 2) = 1.0$

and:

$$\Pr\left(Y_{i1} = 0 \text{ and } Y_{i2} = 1 \mid \sum_T Y_{it} = 1\right) = \frac{\Pr(0,1)}{\Pr(0,1) + \Pr(1,0)}$$

with a similar statement for $\Pr(Y_{i1} = 1 \text{ and } Y_{i2} = 0 \mid \sum_T Y_{it} = 1)$.

The Point:

$\sum_{t=1}^{T} Y_{it}$ **is a sufficient statistic for** α_i, so conditioning on it $\equiv$ "fixed effects."

Notes On Fixed-Effects

Things to bear in mind:

- Fixed effects = no estimates for $\boldsymbol{\beta}_b$
- Interpretation: per logit, but $\mid \hat{\alpha}_i$.
- Everything above is for **logit**...
  - For FE probit, there is no conditional model
  - Unconditional / "brute force" FE probit is biased (see here and here)
- BTSCS in international relations: Green et al. (2001) vs. Beck & Katz (2001) ("Dirty Pool" debate)

Random Effects

Model is:

$$
\begin{aligned}
Y_{it}^{*} &= \mathbf{X}_{it}\beta + u_{it} \\
Y_{it} &= 0 \text{ if } Y_{it}^{*} \leq 0 \text{ ;} \\
&= 1 \text{ if } Y_{it}^{*} > 0
\end{aligned}
$$

with:

$$u_{it} = \alpha_i + \eta_{it}$$

with $\eta_{it} \sim$ i.i.d. N(0,1), and $\alpha_i \sim N(0, \sigma_\alpha^2)$. This implies:

$$\text{Var}(u_{it}) = 1 + \sigma_\alpha^2$$

and so:

$$\text{Corr}(u_{it}, u_{is},\ t \neq s) \equiv \rho = \frac{\sigma_\alpha^2}{1 + \sigma_\alpha^2}$$

which means that we can write $\sigma_\alpha^2 = \left(\frac{\rho}{1-\rho}\right)$.

Probit:

$$\begin{aligned} L_i &= \text{Prob}(Y_{i1} = y_{i1},\ Y_{i2} = y_{i2},\ ...Y_{iT} = y_{iT}) \\ &= \int_{-\infty}^{X_{i1}\beta} \int_{-\infty}^{X_{i2}\beta} ... \int_{-\infty}^{X_{iT}\beta} \phi(u_{i1}, u_{i2}...u_{iT}) du_{iT}...du_{i2}\ du_{i1} \end{aligned}$$

Logit:

$$\begin{aligned} L_i &= \text{Prob}(Y_{i1} = y_{i1},\ Y_{i2} = y_{i2},\ ...Y_{iT} = y_{iT}) \\ &= \int_{-\infty}^{X_{i1}\beta} \int_{-\infty}^{X_{i2}\beta} ... \int_{-\infty}^{X_{iT}\beta} \lambda(u_{i1}, u_{i2}...u_{iT}) du_{iT}...du_{i2}\ du_{i1} \end{aligned}$$

Solution?

$$\phi(u_{i1}, u_{i2}, ...u_{iT}) = \int_{-\infty}^{\infty} \phi(u_{i1}, u_{i2}, ...u_{iT} \mid \alpha_i)\phi(\alpha_i) d\alpha_i$$

- $\hat{\rho} =$ proportion of the variance due to the α_is.
- Implementation: Gauss-Hermite quadrature or MCMC.
- Best with N large and T small.
- Critically requires $\text{Cov}(\mathbf{X}, \alpha) = 0$ (see notes re: Chamberlain's CRE Estimator).

Unit Effects in Practice - Some Simulations

Start with:

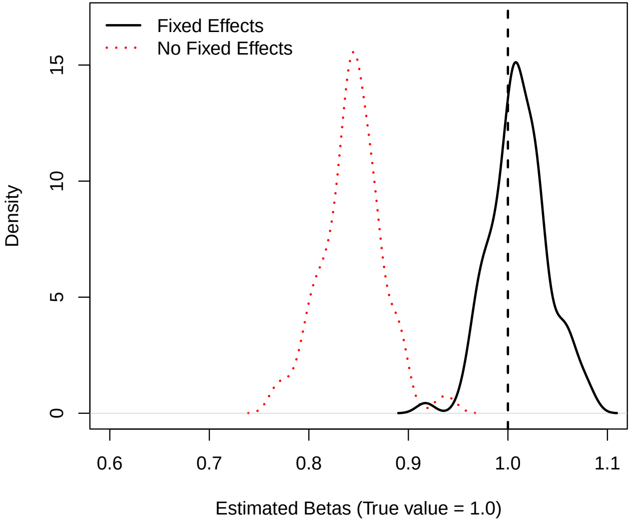
$$\begin{aligned} Y_{it}^* &= 0 + (1 \times X_{it}) + (1 \times D_{it}) + (1 \times \alpha_i) + u_{it} \\ Y_{it} \in \{0,1\} &= f(Y_{it}^*) \end{aligned}$$

where:

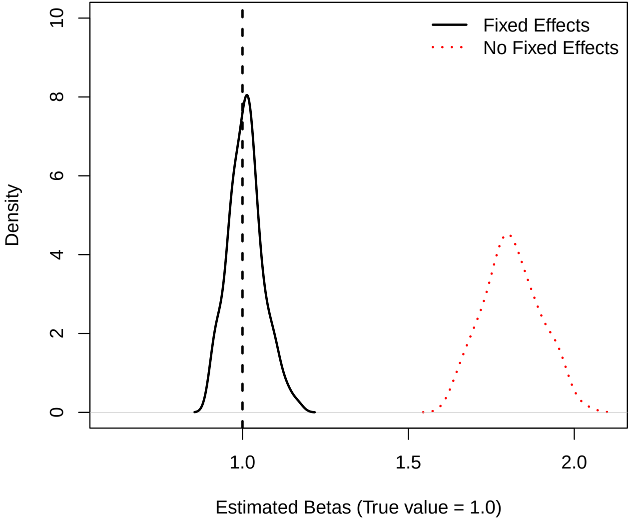
- $\alpha_i \sim N(0,1)$
- $X_{it} \sim N(0, \sigma_X^2)$
- $D_{it} \in \{0,1\}$
- $\text{Cov}(X_{it}, \alpha_i) = \{0, 0.69\}$
- $\text{Cov}(D_{it}, \alpha_i) = 0$
- $f(\cdot) = \{\text{logit, probit}\}$ (as appropriate)

and $N = T = 100$.

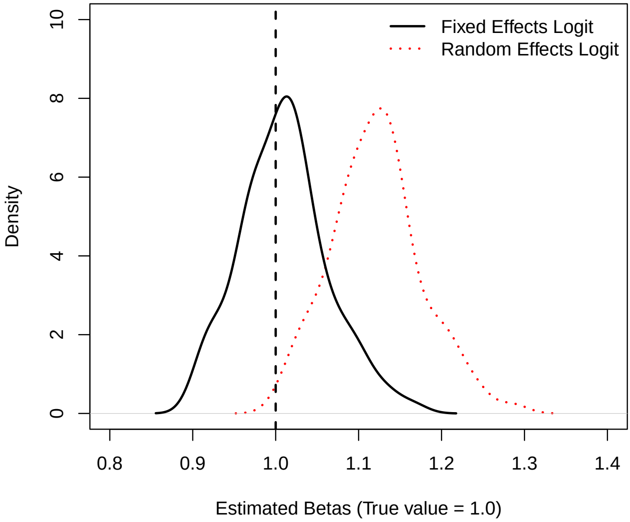
Logit $\hat{\boldsymbol{\beta}}_X$s for $\text{Cov}(X_{it}, \alpha_i) = 0$



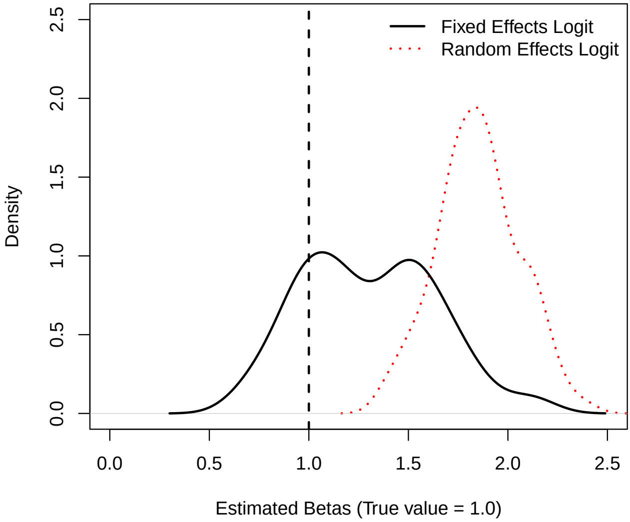
Logit $\hat{\boldsymbol{\beta}}_X$s for $\text{Cov}(X_{it}, \alpha_i) \approx 0.69$



Logit $\hat{\beta}_X$s for $\mathrm{Cov}(X_{it}, \alpha_i) \approx 0.69$



Same Plot, but with $T = 5$...



Software

R

- `pglm` (panel GLMs) (maximum likelihood + quadrature)
- `bife` (fixed-effects logit / probit only)
- `glmer` (general mixed-effects models, including RE)
- `glmmML` (via Gauss-Hermite quadrature)
- `MCMCpack` (`MCMChlogit`)
- Various user-generated functions (e.g., here).
- Interpretation via `modelsummary` and `marginaleffects`

Stata

- `xtprobit`, `xtlogit`, `xtcloglog`
- Plus `xttrans` (transition probabilities), `quadchk` (quadrature checking), `xtrho` / `xtrhoi` (estimation of within-unit covariances)

Example: WDI "Plus"

Data from the WDI, plus POLITY and the UCDP:

- `ISO3` - The country's International Standards Organization (ISO) three-letter identification code.
- `Year` - The year that row of data applies to (1960=1).
- `CivilWar` - Civil conflict indicator: 1 if there was a civil conflict in that country in that year; 0 otherwise. From the UCDP.
- `OnsetCount` - The sum of new conflict episodes in that country / year. From UCDP.
- `LandArea` - Land area (sq. km).
- `PopMillions` - Popluation (in millions).
- `PopGrowth` - Population Growth (percent).
- `UrbanPopulation` - Urban Population (percent of total).
- `GDPPerCapita` - GDP per capita (constant 2010 $US).
- `GDPPerCapGrowth` - GDP Per Capita Growth (percent annual).
- `PostColdWar` - 1 if `Year` > 1989, 0 otherwise.
- `POLITY` - The POLITY score of democracy/autocracy. Scaled so that 0 = most autocratic, 10 = most democratic.

$N = 215$, $\bar{T} = 64$, NT varies (due to missingness).

Model:

$$\begin{aligned}\texttt{Civil War}_{it} \quad = \quad & f[\beta_0 + \beta_1 ln(\texttt{Land Area}_{it}) + \beta_2 ln(\texttt{Population}_{it}) + \\ & +\beta_3\texttt{Urban Population}_{it} + +\beta_4 ln(\texttt{GDP}_{it}) + +\beta_5\texttt{GDP Growth}_{it} + \\ & +\beta_6\texttt{Post-Cold War}_{it} + +\beta_7\texttt{POLITY}_{it} + +\beta_5\texttt{POLITY}^2_{it} + u_{it}]\end{aligned}$$

```
> describe(DF,skew=FALSE)
                vars     n      mean         sd    median     min        max       range       se
ISO3*              1 13822    108.49      62.35    108.00    1.00      216.0      215.00     0.53
Year*              2 13822     32.50      18.47     32.00    1.00       64.0       63.00     0.16
YearNumeric        3 13760   1991.50      18.47   1991.50 1960.00     2023.0       63.00     0.16
country*           4 13760    108.00      62.07    108.00    1.00      215.0      214.00     0.53
CivilWar           5  9052      0.13       0.34      0.00    0.00        1.0        1.00     0.00
OnsetCount         6  9394      0.05       0.24      0.00    0.00        4.0        4.00     0.00
LandArea           7 11941 605302.93 1639812.91 107160.00    2.03 16389950.0 16389947.97 15006.31
PopMillions        8 13515     25.11     104.75      4.29    0.00     1417.2     1417.17     0.90
UrbanPopulation    9 13482     51.72      25.74     50.92    2.08      100.0       97.92     0.22
GDPPerCapita      10 10099  12088.98   19130.23   3891.83  122.88   228667.9   228545.05   190.36
GDPPerCapGrowth   11 10074      1.95       6.21      2.11  -64.43      140.5      204.91     0.06
PostColdWar       12 13760      0.53       0.50      1.00    0.00        1.0        1.00     0.00
POLITY            13  8279      5.55       3.71      6.50    0.00       10.0       10.00     0.04
POLITYSquared     14  8279     44.57      40.24     42.25    0.00      100.0      100.00     0.44
```

Variation

Variable	Dim	Mean	SD	Min	Max	Observations
Year	overall	1991.5	18.474	1960	2023	N = 13760
	between		0	1991.5	1991.5	n = 215
	within		18.474	1960	2023	T = 64
CivilWar	overall	0.134	0.341	0	1	N = 9052
	between		0.221	0	1	n = 172
	within		0.255	-0.783	1.117	T = 52.628
OnsetCount	overall	0.049	0.242	0	4	N = 9394
	between		0.083	0	0.597	n = 172
	within		0.227	-0.548	3.92	T = 54.616
LandArea	overall	605302.933	1639812.915	2.027	16389950	N = 11941
	between		1756124.731	2.028	16379341.333	n = 215
	within		14364.404	180581.622	688581.622	T = 55.54
PopMillions	overall	25.109	104.751	0.003	1417.173	N = 13515
	between		100.983	0.009	1104.338	n = 215
	within		28.291	-436.87	534.349	T = 62.86
UrbanPopulation	overall	51.717	25.74	2.077	100	N = 13482
	between		24.154	6.912	100	n = 214
	within		9.043	5.953	86.692	T = 63
GDPPerCapita	overall	12088.981	19130.234	122.885	228667.935	N = 10099
	between		21061.499	330.723	167809.27	n = 210
	within		6930.784	-37755.586	118168.443	T = 48.09
GDPPerCapGrowth	overall	1.952	6.215	-64.426	140.48	N = 10074
	between		1.782	-8.078	9.247	n = 212
	within		6.017	-66.481	133.991	T = 47.519
PostColdWar	overall	0.531	0.499	0	1	N = 13760
	between		0	0.531	0.531	n = 215
	within		0.499	0	1	T = 64
POLITY	overall	5.551	3.708	0	10	N = 8279
	between		2.985	0	10	n = 165
	within		2.229	-1.431	12.319	T = 50.176

Pooled Logit

```
> Logit<-glm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+              GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared,data=DF,family="binomial")

> summary(Logit)

Call:
glm(formula = CivilWar ~ log(LandArea) + log(PopMillions) + UrbanPopulation +
    log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar + POLITY +
    POLITYSquared, family = "binomial", data = DF)

Coefficients:
                  Estimate Std. Error z value      Pr(>|z|)
(Intercept)       -0.81201    0.52665   -1.54       0.12312
log(LandArea)      0.00183    0.03230    0.06       0.95487
log(PopMillions)   0.66738    0.03685   18.11       < 2e-16 ***
UrbanPopulation    0.01195    0.00335    3.57       0.00036 ***
log(GDPPerCapita) -0.52155    0.06110   -8.54       < 2e-16 ***
GDPPerCapGrowth   -0.04027    0.00651   -6.19 0.00000000061 ***
PostColdWar       -0.32313    0.08606   -3.75       0.00017 ***
POLITY             0.66857    0.06140   10.89       < 2e-16 ***
POLITYSquared     -0.06460    0.00581  -11.12       < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 5829.3  on 6984  degrees of freedom
Residual deviance: 4615.6  on 6976  degrees of freedom
  (6837 observations deleted due to missingness)
AIC: 4634

Number of Fisher Scoring iterations: 6
```

Fixed Effects

```
> FELogit<-bife(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+               GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared|ISO3,data=DF,model="logit")

> summary(FELogit)
binomial - logit link

CivilWar ~ log(LandArea) + log(PopMillions) + UrbanPopulation +
    log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar + POLITY +
    POLITYSquared | ISO3

Estimates:
                  Estimate Std. error z value Pr(> |z|)
log(LandArea)     -13.81071    8.22081   -1.68     0.093 .
log(PopMillions)    0.66428    0.29457    2.26     0.024 *
UrbanPopulation     0.01785    0.01239    1.44     0.150
log(GDPPerCapita)  -0.33280    0.17452   -1.91     0.057 .
GDPPerCapGrowth    -0.05128    0.00845   -6.07   1.3e-09 ***
PostColdWar        -0.21860    0.17884   -1.22     0.222
POLITY              0.71094    0.09360    7.60   3.1e-14 ***
POLITYSquared      -0.07364    0.00890   -8.27   < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

residual deviance= 2835,
null deviance= 4408,
n= 3962, N= 83

( 6837 observation(s) deleted due to missingness )
( 3023 observation(s) deleted due to perfect classification )

Number of Fisher Scoring Iterations: 6

Average individual fixed effect= 172.4
```

Alternative Fixed Effects (using `feglm`)

```
> FELogit2<-feglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+             GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared|ISO3,data=DF,family="binomial")

NOTES: 6,837 observations removed because of NA values (LHS: 4,770, RHS: 6,837).
       77 fixed-effects (3,023 observations) removed because of only 0 (or only 1) outcomes.

> summary(FELogit2)
GLM estimation, family = binomial, Dep. Var.: CivilWar
Observations: 3,962
Fixed-effects: ISO3: 83
Standard-errors: Clustered (ISO3)
                 Estimate Std. Error z value    Pr(>|z|)
log(LandArea)   -13.81263    9.31735 -1.4825 0.138216796
log(PopMillions)  0.66427    0.76041  0.8736 0.382352433
UrbanPopulation   0.01785    0.03673  0.4860 0.626988502
log(GDPPerCapita) -0.33280   0.41724 -0.7976 0.425097622
GDPPerCapGrowth  -0.05128    0.01266 -4.0508 0.000051037 ***
PostColdWar      -0.21860    0.48166 -0.4538 0.649940396
POLITY            0.71095    0.24874  2.8582 0.004260535 **
POLITYSquared    -0.07364    0.02453 -3.0018 0.002683538 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Log-Likelihood: -1,417.4   Adj. Pseudo R2: 0.316066
           BIC:  3,588.7     Squared Cor.: 0.400104
```

Random Effects

```
> RELogit<-pglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+                GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared|ISO3,data=DF,family=binomial
+                effect="individual",model="random")

> summary(RELogit)
--------------------------------------------
Maximum Likelihood estimation
Newton-Raphson maximisation, 8 iterations
Return code 3: Last step could not find a value above the current.
Boundary of parameter space?
Consider switching to a more robust optimisation method temporarily.
Log-Likelihood: -1640
10  free parameters
Estimates:
                 Estimate Std. error t value   Pr(> t)
(Intercept)      -3.34254    2.34543   -1.43      0.15
log(LandArea)    -0.01437    0.09710   -0.15      0.88
log(PopMillions)  1.17645    0.08545   13.77   < 2e-16 ***
UrbanPopulation   0.00686    0.02895    0.24      0.81
log(GDPPerCapita) -0.39269   0.25443   -1.54      0.12
GDPPerCapGrowth  -0.05400    0.01200   -4.50 0.0000068 ***
PostColdWar      -0.31271    0.20671   -1.51      0.13
POLITY            0.76234    0.07210   10.57   < 2e-16 ***
POLITYSquared    -0.07769    0.00663  -11.72   < 2e-16 ***
sigma             2.21027    0.12320   17.94   < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
--------------------------------------------
```

Models of Civil War

	Logit	FE Logit	FEs+Robust	RE Logit
Intercept	-0.81			-3.34
	(0.53)			(2.35)
ln(Land Area)	0.00	-13.81	-13.81	-0.01
	(0.03)	(8.22)	(9.32)	(0.10)
ln(Population)	0.67^{*}	0.66^{*}	0.66	1.18^{*}
	(0.04)	(0.29)	(0.76)	(0.09)
Urban Population	0.01^{*}	0.02	0.02	0.01
	(0.00)	(0.01)	(0.04)	(0.03)
ln(GDP Per Capita)	-0.52^{*}	-0.33	-0.33	-0.39
	(0.06)	(0.17)	(0.42)	(0.25)
GDP Growth	-0.04^{*}	-0.05^{*}	-0.05^{*}	-0.05^{*}
	(0.01)	(0.01)	(0.01)	(0.01)
Post-Cold War	-0.32^{*}	-0.22	-0.22	-0.31
	(0.09)	(0.18)	(0.48)	(0.21)
POLITY	0.67^{*}	0.71^{*}	0.71^{*}	0.76^{*}
	(0.06)	(0.09)	(0.25)	(0.07)
POLITY Squared	-0.06^{*}	-0.07^{*}	-0.07^{*}	-0.08^{*}
	(0.01)	(0.01)	(0.02)	(0.01)
Estimated Sigma				2.21^{*}
				(0.12)
AIC	4633.63			3299.59
BIC	4695.30			
Log Likelihood	-2307.82	-1417.42	-1417.42	-1639.80
Deviance	4615.63	2834.83	2834.83	
Num. obs.	6985	3962	3962	
Num. groups: ISO3			83	
Pseudo R^2			0.32	

$^{*}p < 0.05$

Models For Event Counts

Event Counts

Properties:

- Discrete / integer-values
- Non-negative
- "Cumulative"

Motivation:

$$\text{Arrival Rate} = \lambda$$

$$\Pr(\text{Event})_{t,t+h} = \lambda h$$

$$\Pr(\text{No Event})_{t,t+h} = 1 - \lambda h$$

$$\begin{aligned} \Pr(Y_t = y) &= \frac{\exp(-\lambda h)\lambda h^y}{y!} \\ &= \frac{\exp(-\lambda)\lambda^y}{y!} \end{aligned}$$

Poisson: Assumptions and Motivations

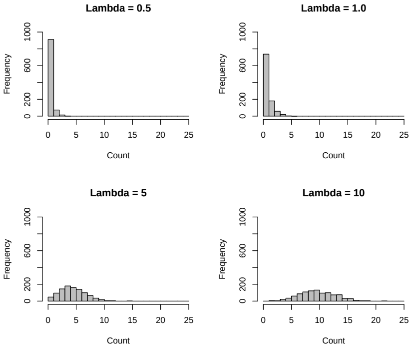
- No Simultaneous Events
- Constant Arrival Rate
- Independent Event Arrivals

Another motivation: For M independent Bernoulli trials with (sufficiently small) probability of success π and where $M\pi \equiv \lambda > 0$,

$$\begin{aligned}\Pr(Y_i = y) &= \lim_{M \to \infty} \left[\binom{M}{y} \left(\frac{\lambda}{M} \right)^y \left(1 - \frac{\lambda}{M} \right)^{M-y} \right] \\ &= \frac{\lambda^y \exp(-\lambda)}{y!}\end{aligned}$$

Poisson: Characteristics

- Discrete
- $\mathrm{E}(Y) = \mathrm{Var}(Y) = \lambda$
- Is not preserved under affine transformations...
- For $X \sim \text{Poisson}(\lambda_X)$ and $Y \sim \text{Poisson}(\lambda_Y)$, $Z = X + Y \sim \text{Poisson}(\lambda_{X+Y})$ iff X and Y are independent but
- ...same is not true for differences.
- $\lambda \to \infty \iff Y \sim N$



Poisson Regression

Suppose

$$\mathrm{E}(Y_i) \equiv \lambda_i = \exp(\mathbf{X}_i\boldsymbol{\beta})$$

then

$$\Pr(Y_i = y|\mathbf{X}_i, \boldsymbol{\beta}) = \frac{\exp[-\exp(\mathbf{X}_i\boldsymbol{\beta})][\exp(\mathbf{X}_i\boldsymbol{\beta})]^y}{y!}$$

with likelihood:

$$L = \prod_{i=1}^{N} \frac{\exp[-\exp(\mathbf{X}_i\boldsymbol{\beta})][\exp(\mathbf{X}_i\boldsymbol{\beta})]^{Y_i}}{Y_i!}$$

and log-likelihood:

$$\ln L = \sum_{i=1}^{N}[-\exp(\mathbf{X}_i\boldsymbol{\beta}) + Y_i\mathbf{X}_i\boldsymbol{\beta} - \ln(Y_i!)]$$

Event Counts: Unit Effects

The Poisson model:

$$Y_{it} \sim \text{Poisson}(\mu_{it} = \alpha_i \lambda_{it})$$

with $\lambda_{it} = \exp(\mathbf{X}_{it}\boldsymbol{\beta})$ implies:

$$\begin{aligned} \text{E}(Y_{it} \mid \mathbf{X}_{it}, \alpha_i) &= \mu_{it} \\ &= \alpha_i \exp(\mathbf{X}_{it}\boldsymbol{\beta}) \\ &= \exp(\delta_i + \mathbf{X}_{it}\boldsymbol{\beta}) \end{aligned}$$

where $\delta_i = \ln(\alpha_i)$.

Fixed-Effects Poisson:

- ...has no "incidental parameters" problem (see e.g. Cameron and Trivedi, pp. 281-2)
- This means "brute force" approach works
- Fitted via `glmmML` in R, `xtpoisson` (and `xtnbreg`) in Stata

The Poisson with random effects is:

$$\begin{aligned}\Pr(Y_{i1} = y_{i1}, \ldots Y_{iT} = y_{iT}) &= \int_0^\infty \Pr(Y_{i1} = y_{i1}, \ldots Y_{iT} = y_{iT}) f(\alpha_i) d\alpha_i \\ &= \int_0^\infty \left[\prod_{t=1}^{T} \Pr(Y_{it} \mid \alpha_i)\right] f(\alpha_i) d\alpha_i\end{aligned}$$

- Simplest to assume $\alpha_i \sim \Gamma(\theta)$
- Yields a model with $\mathrm{E}(Y_{it}) = \lambda_{it}$ and $\mathrm{Var}(Y_{it}) = \lambda_{it} + \frac{\lambda_{it}^2}{\theta}$
- Via `glmmML` or `glmer` in R, or `xtpois, re` in Stata
- $\exists$ random effects negative binomial too...

Panel Models: Software

R:

- Tobit = `censReg` (in **censReg**)
- Poisson (random effects) = `glmmML` in **glmmML** or `glmer` in **lme4**
- Poisson (fixed effects) = `glmmML` or "brute force"
- Poisson + negative binomial (FE, RE) = `pglm`

Stata:

- Tobit = `xttobit` (`re` only)
- Poisson / negative binomial = `xtpoisson`, `xtnbreg` (both with `fe`, `re` options)

Conflict Onsets: Pooled Poisson

```
> xtabs(~DF$OnsetCount)
DF$OnsetCount
   0    1    2    3    4
8981  375   30    7    1

> Poisson<-glm(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+              GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared,data=DF,family="poisson")

> summary(Poisson)

Call:
glm(formula = OnsetCount ~ log(LandArea) + log(PopMillions) +
    UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
    POLITY + POLITYSquared, family = "poisson", data = DF)

Coefficients:
                  Estimate Std. Error z value     Pr(>|z|)
(Intercept)       -1.99988    0.72467   -2.76      0.00579 **
log(LandArea)      0.06397    0.04707    1.36      0.17417
log(PopMillions)   0.42562    0.04573    9.31      < 2e-16 ***
UrbanPopulation    0.00804    0.00472    1.70      0.08879 .
log(GDPPerCapita) -0.47969    0.08043   -5.96 0.0000000025 ***
GDPPerCapGrowth   -0.03581    0.00668   -5.36 0.0000000817 ***
PostColdWar        0.25159    0.12064    2.09      0.03703 *
POLITY             0.30534    0.08363    3.65      0.00026 ***
POLITYSquared     -0.03366    0.00799   -4.21 0.0000254512 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 2371.5  on 6984  degrees of freedom
Residual deviance: 1931.5  on 6976  degrees of freedom
  (6837 observations deleted due to missingness)
AIC: 2679

Number of Fisher Scoring iterations: 6
```

Fixed Effects Poisson

```
> FEPoisson<-pglm(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+
+               log(GDPPerCapita)+GDPPerCapGrowth+PostColdWar+POLITY+
+               POLITYSquared,data=DF,family="poisson",effect="individual",
+               model="within")

> summary(FEPoisson)

--------------------------------------------
Maximum Likelihood estimation
Newton-Raphson maximisation, 3 iterations
Return code 8: successive function values within relative tolerance limit (reltol)
Log-Likelihood: -1013
8  free parameters
Estimates:
                  Estimate Std. error t value   Pr(> t)
log(LandArea)     -2.82014    2.86711   -0.98   0.32530
log(PopMillions)   0.61597    0.31877    1.93   0.05332 .
UrbanPopulation   -0.04529    0.01338   -3.38   0.00071 ***
log(GDPPerCapita) -0.10765    0.14544   -0.74   0.45919
GDPPerCapGrowth   -0.02872    0.00686   -4.19 0.0000282 ***
PostColdWar        0.47277    0.19572    2.42   0.01571 *
POLITY             0.51291    0.10812    4.74 0.0000021 ***
POLITYSquared     -0.05185    0.01061   -4.89 0.0000010 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
--------------------------------------------
```

Equivalent Fixed Effects Poisson (using `feglm`)

```
> FEPoisson2<-feglm(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+
+                   log(GDPPerCapita)+GDPPerCapGrowth+PostColdWar+POLITY+
+                     POLITYSquared|ISO3,data=DF,family="poisson")

NOTES: 6,837 observations removed because of NA values (LHS: 4,428, RHS: 6,837).
       67 fixed-effects (2,495 observations) removed because of only 0 outcomes.

> summary(FEPoisson2,cluster="ISO3")

GLM estimation, family = poisson, Dep. Var.: OnsetCount
Observations: 4,490
Fixed-effects: ISO3: 93
Standard-errors: Clustered (ISO3)
                  Estimate Std. Error z value     Pr(>|z|)
log(LandArea)     -2.82014   3.675097 -0.7674 0.4428652862
log(PopMillions)   0.61597   0.347129  1.7745 0.0759873451 .
UrbanPopulation   -0.04529   0.019638 -2.3061 0.0211045803 *
log(GDPPerCapita) -0.10765   0.153162 -0.7029 0.4821364556
GDPPerCapGrowth   -0.02872   0.006582 -4.3638 0.0000127806 ***
PostColdWar        0.47277   0.295534  1.5997 0.1096591753
POLITY             0.51291   0.111671  4.5931 0.0000043681 ***
POLITYSquared     -0.05185   0.011586 -4.4757 0.0000076159 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Log-Likelihood: -1,148.7   Adj. Pseudo R2: 0.093386
           BIC:  3,146.7     Squared Cor.: 0.163105
```

Random Effects Poisson

```
> REPoisson<-pglm(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+
+                 log(GDPPerCapita)+GDPPerCapGrowth+PostColdWar+POLITY+
+                 POLITYSquared,data=DF,family="poisson",effect="individual",
+                 model="random")

> summary(REPoisson)
--------------------------------------------
Maximum Likelihood estimation
Newton-Raphson maximisation, 4 iterations
Return code 8: successive function values within relative tolerance limit (reltol)
Log-Likelihood: -1283
10  free parameters
Estimates:
                  Estimate Std. error t value     Pr(> t)
(Intercept)       -3.47867    1.04013   -3.34     0.00082 ***
log(LandArea)      0.05347    0.07261    0.74     0.46146
log(PopMillions)   0.44330    0.07919    5.60 0.000000022 ***
UrbanPopulation   -0.00471    0.00639   -0.74     0.46120
log(GDPPerCapita) -0.22335    0.10250   -2.18     0.02934 *
GDPPerCapGrowth   -0.03367    0.00683   -4.93 0.000000828 ***
PostColdWar        0.28226    0.12865    2.19     0.02823 *
POLITY             0.45969    0.09613    4.78 0.000001734 ***
POLITYSquared     -0.05071    0.00931   -5.45 0.000000050 ***
sigma              1.77551    0.44025    4.03 0.000055069 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
--------------------------------------------
```

Negative Binomial: Fixed Effects

```
> FENegBin2<-fenegbin(OnsetCount~log(LandArea)+log(PopMillions)+
+                  UrbanPopulation+log(GDPPerCapita)+
+                  GDPPerCapGrowth+PostColdWar+POLITY+
+                  POLITYSquared|ISO3,data=DF)

NOTES: 6,837 observations removed because of NA values (LHS: 4,428, RHS: 6,837).
       67 fixed-effects (2,495 observations) removed because of only 0 outcomes.

Very high value of theta (10000). There is no sign of overdispersion, you may
consider a Poisson model.

Warning message:

[femlm]: The information matrix is singular: presence of collinearity.
```

Table!

Models of Civil War Onset Counts

	Poisson	FE Poisson	RE Poisson
Intercept	-2.00^{*}		-3.48^{*}
	(0.72)		(1.04)
ln(Land Area)	0.06	-2.82	0.05
	(0.05)	(2.87)	(0.07)
ln(Population)	0.43^{*}	0.62	0.44^{*}
	(0.05)	(0.32)	(0.08)
Urban Population	0.01	-0.05^{*}	-0.00
	(0.00)	(0.01)	(0.01)
ln(GDP Per Capita)	-0.48^{*}	-0.11	-0.22^{*}
	(0.08)	(0.15)	(0.10)
GDP Growth	-0.04^{*}	-0.03^{*}	-0.03^{*}
	(0.01)	(0.01)	(0.01)
Post-Cold War	0.25^{*}	0.47^{*}	0.28^{*}
	(0.12)	(0.20)	(0.13)
POLITY	0.31^{*}	0.51^{*}	0.46^{*}
	(0.08)	(0.11)	(0.10)
POLITY Squared	-0.03^{*}	-0.05^{*}	-0.05^{*}
	(0.01)	(0.01)	(0.01)
Estimated Sigma			1.78^{*}
			(0.44)
AIC	2679.09	2041.78	2585.34
BIC	2740.75		
Log Likelihood	-1330.54	-1012.89	-1282.67
Deviance	1931.52		
Num. obs.	6985		

$^{*}p < 0.05$

Wrap-Up: Some Useful Packages

- `pglm`
  - Workhorse package for panel (FE, RE, BE) GLMs
  - Binary + ordered logit/probit, Poisson / negative binomial
  - Discussed + used extensively in Croissant and Millo (2018) *Panel Data Econometrics with R*
  - The one thing it won't (apparently) do is fixed-effects, binary-response models...
- `fixest`
  - Fast / efficient fitting of FE models
  - Fits linear models, logit, Poisson, and negative binomial
  - Includes easy coefficient plots & tables; simple multi-threading; built-in "robust" S.E.s
- `alpaca`
  - Fast / efficient fitting of GLMs with high-dimensional fixed effects
  - *Includes bias correction for incidental parameters after binary-response models*
  - Also includes useful panel data simulation routines + average partial effects

Generalized Estimating Equations

Quick GLM review

Linear-normal model is:

$$Y_i = \mu_i + u_i$$

with:

$$\mu_i = \mathbf{X}_i \boldsymbol{\beta}.$$

Generalize:

$$g(\mu_i) = \mathbf{X}_i \boldsymbol{\beta}$$

and:

$$Y_i \sim \text{i.i.d.}\, F[\mu_i, \mathbf{V}_i].$$

GLM Estimation

"Score" equations:

$$\mathbf{U}(\beta) = \sum_{i=1}^{N} \mathbf{D}_i' \mathbf{V}_i^{-1} [Y_i - \mu_i] = \mathbf{0}.$$

with:

- $\mathbf{D}_i = \frac{\partial \mu_i}{\partial \beta}$,
- $\mathbf{V}_i = \frac{h(\mu_i)}{\phi}$, and
- $(Y_i - \mu_i) \approx$ a "residual."
- Known as "quasi-likelihood" (e.g. Wedderburn 1974 *Biometrika*).

Now suppose:

$$Y_{it} = \mu_{it} + u_{it}$$

where

- $i \in \{1, ...N\}$ are i.i.d. "units,"
- $t \in \{1, ...T\}$, $T > 1$ are "time points,"
- we want $g(\mu_{it}) = \mathbf{X}_{it}\boldsymbol{\beta}$.

Key issue: Accounting for (conditional) dependence in Y over time.

Full joint distributions over T are hard. But...

Define:

$$\underset{T\times T}{\mathbf{R}_i(\boldsymbol{\alpha})} = \begin{pmatrix} 1.0 & \alpha_{12} & \cdots & \alpha_{1,T} \\ \alpha_{21} & 1.0 & \cdots & \alpha_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{T,1} & \cdots & \alpha_{T,T-1} & 1.0 \end{pmatrix},$$

$\rightarrow$ "working correlation" matrix.

- Completely defined by $\boldsymbol{\alpha}$,
- Structure specified by the analyst.

Liang and Zeger (1986): We can decompose the variance of Y_{it} as:

$$\mathbf{V}_i = \mathrm{diag}(\mathbf{V}_i^{\frac{1}{2}})\,\mathbf{R}_i(\boldsymbol{\alpha})\,\mathrm{diag}(\mathbf{V}_i^{\frac{1}{2}})$$

With a standard GLM assumption about the mean and variance, this is:

$$\mathbf{V}_i = \frac{(\mathbf{A}_i^{\frac{1}{2}})\,\mathbf{R}_i(\boldsymbol{\alpha})\,(\mathbf{A}_i^{\frac{1}{2}})}{\phi}$$

where

$$\underset{T \times T}{\mathbf{A}_i} = \begin{pmatrix} h(\mu_{i1}) & 0 & \cdots & 0 \\ 0 & h(\mu_{i2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & h(\mu_{iT}) \end{pmatrix}$$

What does that mean?

$\mathbf{V}_i = \text{Var}(Y_{it}|\mathbf{X}_{it}, \beta)$ has two parts:

- $\mathbf{A}_i =$ <u>unit-level</u> variation,
- $\mathbf{R}_i(\alpha) =$ within-unit <u>temporal</u> variation.

Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

Independent: $$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & 0 & \cdots & 0 \\ 0 & 1.0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1.0 \end{pmatrix}$$

- Assumes no within-unit temporal correlation.
- Equivalent to GLM on pooled data.

Exchangeable: $$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha & \cdots & \alpha \\ \alpha & 1.0 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \cdots & \alpha & 1.0 \end{pmatrix}$$

- One free parameter in $\mathbf{R}_i(\boldsymbol{\alpha})$ ($\alpha_{ts} = \alpha \; \forall \; t \neq s$)
- Temporal correlation within units is constant across time points.
- Akin (in some respects) to a random-effects model...

Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

AR(p) (e.g., AR(1)):

$$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha & \alpha^2 & \cdots & \alpha^{T-1} \\ \alpha & 1.0 & \alpha & \cdots & \alpha^{T-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{T-1} & \cdots & \alpha^2 & \alpha & 1.0 \end{pmatrix}$$

- One free parameter in $\mathbf{R}_i(\boldsymbol{\alpha})$ ($\alpha_{ts} = \alpha^{|t-s|}\ \forall\ t \neq s$).
- Conditional within-unit correlation an exponential function of the lag.

Stationary(p):

$$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha_1 & \cdots & \alpha_p & 0 & \cdots & 0 \\ \alpha_1 & 1.0 & \alpha_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_p & \cdots & \alpha_1 & 1.0 \end{pmatrix}$$

- AKA "banded," or "*p*-dependent."
- $p \leq T - 1$ free parameters in $\mathbf{R}_i(\boldsymbol{\alpha})$.
- Conditional within-unit correlation an exponential function of the lag, up to lag p, and zero thereafter.

Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

Unstructured: $$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1,T-1} \\ \alpha_{12} & 1.0 & \alpha_{23} & \cdots & \alpha_{2,T-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{1,T-1} & \alpha_{2,T-1} & \cdots & \alpha_{T-1,T-1} & 1.0 \end{pmatrix}$$

- $\frac{T(T-1)}{2}$ free parameters in $\mathbf{R}_i(\boldsymbol{\alpha})$.
- Conditional within-unit correlation is completely data-dependent.

Score equations:

$$\boldsymbol{U}_{GEE}(\boldsymbol{\beta}_{GEE}) = \sum_{i=1}^{N} \mathbf{D}_i' \left[\frac{(\mathbf{A}_i^{\frac{1}{2}})\,\mathbf{R}_i(\boldsymbol{\alpha})\,(\mathbf{A}_i^{\frac{1}{2}})}{\phi} \right]^{-1} [Y_i - \mu_i] = \mathbf{0}$$

Two-step estimation:

- For fixed values of $\boldsymbol{\alpha}_s$ and ϕ_s at iteration s, use Newton scoring to estimate $\hat{\boldsymbol{\beta}}_s$,
- Use $\hat{\boldsymbol{\beta}}_s$ to calculate standardized residuals $(Y_i - \hat{\mu}_i)_s$, from which consistent estimates of $\boldsymbol{\alpha}_{s+1}$ and ϕ_{s+1} can be estimated.

Liang & Zeger (1986):

$$\hat{\boldsymbol{\beta}}_{GEE} \underset{N\to\infty}{\sim} \boldsymbol{N}(\boldsymbol{\beta}, \boldsymbol{\Sigma}).$$

For $\hat{\boldsymbol{\Sigma}}$, two options:

$$\hat{\boldsymbol{\Sigma}}_{\text{Model}} = N\left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right)$$

$$\hat{\boldsymbol{\Sigma}}_{\text{Robust}} = N\left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right)^{-1}\left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{S}}_i \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right)\left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right)^{-1}$$

where $\hat{\boldsymbol{S}}_i = (Y_i - \hat{\mu}_i)(Y_i - \hat{\mu}_i)'$.

Inference (aka, magic!)

- $\hat{\Sigma}_{\text{Model}}$
  - Requires that $\mathbf{R}_i(\boldsymbol{\alpha})$ be "correct" for consistency.
  - Is slightly more efficient than $\hat{\boldsymbol{\Sigma}}_{\text{Robust}}$ if so.

- $\hat{\Sigma}_{\text{Robust}}$
  - Is consistent *even if* $\mathbf{R}_i(\boldsymbol{\alpha})$ *is misspecified*.
  - Is slightly less efficient than $\hat{\boldsymbol{\Sigma}}_{\text{Model}}$ if $\mathbf{R}_i(\boldsymbol{\alpha})$ is correct.

Moral: Use $\hat{\Sigma}_{\textbf{Robust}}$.

Summary

GEEs:

- Are a straightforward variation on GLMs, and so
- Can be applied to a range of data types (continuous, binary, count, proportions, etc.),
- Yield robustly consistent point estimates of βs,
- Account for within-unit correlation in an informed way, but also
- Yield consistent inferences even if the correlation is misspecified.

Practical Issues: Model Interpretation

- In general, GEEs = GLMs.
- GEEs are *marginal* models, so:
  - $\hat{\beta}$s have an interpretation as average / total effects.
  - Estimates / effect sizes generally be smaller than conditional (e.g. fixed/random) effects models.
  - E.g., for logit, $\hat{\boldsymbol{\beta}}_M \approx \frac{\hat{\boldsymbol{\beta}}_C}{\sqrt{1+0.35\sigma_\eta^2}}$, where $\sigma_\eta^2 > 0$ is the variance of the unit effects.
  - See (e.g.) Gardiner et al. (2009) or Koper and Manseau (2009) for expositions.

Practical Issues: Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

- Has been called "more art than science."
- Pointers:
  - Choose based on *substance* of the problem.
  - Remember that $\mathbf{R}_i(\boldsymbol{\alpha})$ is conditional on $\mathbf{X}$, $\hat{\boldsymbol{\beta}}$.
  - Consider unstructured when T is small and N large.
  - Try different ones, and compare.
- In general, it shouldn't matter terribly much...

GEEs: Software

Software	Command(s)/Package(s)
R	`gee` / `geepack` / `geeM` / `multgeeB` / `orth` / `repolr`
Stata	`xtgee` / `xtlogit` / `xtprobit` / `xtpois` / etc.
SAS	`genmod` (w/ `repeated`)

GEEs: Software Tips

- Generally follow GLMs (specify "family" + "link")
- Certain combinations not possible/recommended
- Estimation: Fisher scoring, MLE, etc. (MCMC?)

From the `geepack` manual:

Warning

Use "unstructured" correlation structure only with great care. (It may cause R to crash).

Civil War Redux... GEE: Independence

```
> GEE.ind<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+
+               log(GDPPerCapita)+GDPPerCapGrowth+PostColdWar+POLITY+
+               POLITYSquared,data=DF,id=ISO3,family="binomial",
+               corstr="independence")

> summary(GEE.ind)

Call:
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
    UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
    POLITY + POLITYSquared, family = "binomial", data = DF, id = ISO3,
    corstr = "independence")

 Coefficients:
                  Estimate  Std.err  Wald Pr(>|W|)
(Intercept)       -0.81201  2.02183  0.16  0.68796
log(LandArea)      0.00183  0.12331  0.00  0.98817
log(PopMillions)   0.66738  0.15618 18.26 0.000019 ***
UrbanPopulation    0.01195  0.01405  0.72  0.39489
log(GDPPerCapita) -0.52155  0.25237  4.27  0.03877 *
GDPPerCapGrowth   -0.04027  0.01302  9.57  0.00198 **
PostColdWar       -0.32313  0.26138  1.53  0.21637
POLITY             0.66857  0.21145 10.00  0.00157 **
POLITYSquared     -0.06460  0.01951 10.97  0.00093 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Correlation structure = independence
Estimated Scale Parameters:

            Estimate Std.err
(Intercept)    0.805   0.295
Number of clusters:   160  Maximum cluster size: 57
```

GEE: Exchangeable

```
> GEE.exc<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+GDPPerCapGrowth+
+                  PostColdWar+POLITY+POLITYSquared,data=DF,id=ISO3,family="binomial",corstr="exchangeable")

> summary(GEE.exc)

Call:
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
    UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
    POLITY + POLITYSquared, family = "binomial", data = DF, id = ISO3,
    corstr = "exchangeable")

 Coefficients:
                  Estimate  Std.err  Wald Pr(>|W|)
(Intercept)       -2.67543  2.04945  1.70  0.19174
log(LandArea)      0.03410  0.15498  0.05  0.82585
log(PopMillions)   0.55616  0.16182 11.81  0.00059 ***
UrbanPopulation    0.00542  0.01168  0.22  0.64247
log(GDPPerCapita) -0.22187  0.17520  1.60  0.20538
GDPPerCapGrowth   -0.03599  0.00911 15.62 0.000078 ***
PostColdWar       -0.14495  0.23381  0.38  0.53528
POLITY             0.55143  0.17124 10.37  0.00128 **
POLITYSquared     -0.05620  0.01675 11.26  0.00079 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Correlation structure = exchangeable
Estimated Scale Parameters:

            Estimate Std.err
(Intercept)    0.729   0.178
  Link = identity

Estimated Correlation Parameters:
      Estimate Std.err
alpha    0.342   0.112
Number of clusters:   160  Maximum cluster size: 57
```

GEE: AR(1)

```
> GEE.ar1<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+GDPPerCapGrowth+
+                   PostColdWar+POLITY+POLITYSquared,data=DF,id=ISO3,family="binomial",corstr="ar1")

> summary(GEE.ar1)

Call:
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
    UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
    POLITY + POLITYSquared, family = "binomial", data = DF, id = ISO3,
    corstr = "ar1")

 Coefficients:
                  Estimate  Std.err Wald Pr(>|W|)
(Intercept)       -1.14508  3.06369 0.14    0.709
log(LandArea)      0.08037  0.21013 0.15    0.702
log(PopMillions)   0.37575  0.21730 2.99    0.084 .
UrbanPopulation   -0.00320  0.01795 0.03    0.858
log(GDPPerCapita) -0.35783  0.27098 1.74    0.187
GDPPerCapGrowth   -0.01643  0.00793 4.30    0.038 *
PostColdWar        0.20467  0.24582 0.69    0.405
POLITY             0.19608  0.12482 2.47    0.116
POLITYSquared     -0.02126  0.01309 2.64    0.104
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Correlation structure = ar1
Estimated Scale Parameters:

            Estimate Std.err
(Intercept)    0.818   0.374
  Link = identity

Estimated Correlation Parameters:
      Estimate Std.err
alpha    0.922   0.039
Number of clusters:   160  Maximum cluster size: 57
```

GEE: Unstructured (2013-2017)

```
> GEE.unstr<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+GDPPerCapGrowth+POLITY+
+                   POLITYSquared,data=DF5,id=ISO3,family="binomial",corstr="unstructured")

> summary(GEE.unstr)

Call:
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
    UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + POLITY +
    POLITYSquared, family = "binomial", data = DF5, id = ISO3,
    corstr = "unstructured")

 Coefficients:
                  Estimate Std.err  Wald Pr(>|W|)
(Intercept)        -1.9922  3.1555  0.40  0.52782
log(LandArea)       0.1442  0.1930  0.56  0.45489
log(PopMillions)    0.8840  0.2488 12.62  0.00038 ***
UrbanPopulation     0.0354  0.0171  4.27  0.03884 *
log(GDPPerCapita)  -0.8469  0.3089  7.52  0.00611 **
GDPPerCapGrowth    -0.0125  0.0297  0.18  0.67372
POLITY              0.5091  0.4071  1.56  0.21111
POLITYSquared      -0.0588  0.0359  2.69  0.10094
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Correlation structure = unstructured
Estimated Scale Parameters:

            Estimate Std.err
(Intercept)    0.675   0.813
  Link = identity

Estimated Correlation Parameters:
          Estimate Std.err
alpha.1:2    0.389   0.490
alpha.1:3    0.409   0.515
alpha.1:4    0.341   0.433
alpha.1:5    0.334   0.423
alpha.2:3    0.728   0.849
alpha.2:4    0.276   0.354
alpha.2:5    0.503   0.588
alpha.3:4    0.396   0.508
alpha.3:5    0.741   0.875
alpha.4:5    0.432   0.548
Number of clusters:   159  Maximum cluster size: 5
```

GEE Model Comparison

GEE Models of Civil War Onset

	Independence	Exchangeable	AR(1)	Unstructured (2013-17)
(Intercept)	−0.812	−2.675	−1.145	−1.992
	(2.022)	(2.049)	(3.064)	(3.155)
ln(Land Area)	0.002	0.034	0.080	0.144
	(0.123)	(0.155)	(0.210)	(0.193)
ln(Population)	0.667***	0.556***	0.376+	0.884***
	(0.156)	(0.162)	(0.217)	(0.249)
Urban Population	0.012	0.005	−0.003	0.035*
	(0.014)	(0.012)	(0.018)	(0.017)
ln(GDP Per Capita)	−0.522*	−0.222	−0.358	−0.847**
	(0.252)	(0.175)	(0.271)	(0.309)
GDP Growth	−0.040**	−0.036***	−0.016*	−0.013
	(0.013)	(0.009)	(0.008)	(0.030)
Post-Cold War	−0.323	−0.145	0.205	
	(0.261)	(0.234)	(0.246)	
POLITY	0.669**	0.551**	0.196	0.509
	(0.211)	(0.171)	(0.125)	(0.407)
POLITY Squared	−0.065***	−0.056***	−0.021	−0.059
	(0.020)	(0.017)	(0.013)	(0.036)
NT	6985	6985	6985	790

Note: + $p < 0.1$, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

GEEs are:

- Robust
- Flexible
- Extensible beyond panel/TSCS context

Appendix: Discrete-Time Survival Models

Survival models:

- ...are models for *time-to-event data*.
- ...have their roots in biostats/epidemiology, plus engineering, sociology, economics.
- Examples...
    - Political careers, confirmation durations, position-taking, bill cosponsorship, campaign contributions, policy innovation/adoption, etc.
    - Cabinet/government durations, length of civil wars, coalition durability, etc.
    - War duration, peace duration, alliance longevity, length of trade agreements, etc.
    - Strike durations, work careers (including promotions, firings, etc.), criminal careers, marriage and child-bearing behavior, etc.

Characteristics:

- *Discrete* events (i.e., not continuous),
- Take place over *time*,
- May not (or never) experience the event (i.e., possibility of *censoring*).

Terminology:

$$\begin{aligned}
Y_i &= \text{the duration until the event occurs,} \\
Z_i &= \text{the duration until the observation is ``censored''} \\
T_i &= \min\{Y_i, Z_i\}, \\
C_i &= 0 \text{ if observation } i \text{ is } \textit{censored}\text{, 1 if it is not.}
\end{aligned}$$

Density:

$$f(t) = \Pr(T_i = t)$$

CDF:

$$\Pr(T_i \leq t) \equiv F(t) = \int_0^t f(t)\, dt$$

Survival function:

$$\begin{aligned} \Pr(T_i \geq t) \equiv S(t) &= 1 - F(t) \\ &= 1 - \int_0^t f(t)\, dt \end{aligned}$$

Hazard:

$$\begin{aligned} \Pr(T_i = t | T_i \geq t) \equiv h(t) &= \frac{f(t)}{S(t)} \\ &= \frac{f(t)}{1 - \int_0^t f(t)\, dt} \end{aligned}$$

Grouped-Data Survival Approaches

Model:

$$\Pr(C_{it} = 1) = f\left(\mathbf{X}_{it}\beta\right)$$

Advantages:

- Easily estimated, interpreted and understood
- Natural interpretations:
  - $\hat{\beta}_0 \approx$ "baseline hazard"
  - Covariates shift this up or down.
- Can incorporate data on time-varying covariates
- Lots of software

Potential Disadvantages:

- Requires time-varying data
- Must deal with time dependence explicitly

Temporal Issues in Grouped-Data Models

(Implicit) "Baseline" hazard:

$$h_0(t) = \frac{\exp(\beta_0)}{1 + \exp(\beta_0)}$$

$\longrightarrow$ No temporal dependence / "flat" hazard

Time trend:

$$\Pr(Y_{it} = 1) = f(\mathbf{X}_{it}\beta + \gamma T_{it})$$

- $\hat{\gamma} > 0 \;\rightarrow$ rising hazard
- $\hat{\gamma} < 0 \;\rightarrow$ declining hazard
- $\hat{\gamma} = 0 \;\rightarrow$ "flat" (exponential) hazard

Variants/extensions: Polynomials...

$$\Pr(Y_{it} = 1) = f(\mathbf{X}_{it}\beta + \gamma_1 T_{it} + \gamma_2 T_{it}^2 + \gamma_3 T_{it}^3 + ...)$$

Temporal Issues in Grouped-Data Models

"Time dummies":

$$\Pr(Y_{it} = 1) = f[\mathbf{X}_{it}\beta + \alpha_1 I(T_{i1}) + \alpha_2 I(T_{i2}) + ... + \alpha_{t_{\max}} I(T_{it_{\max}})]$$

$\longrightarrow$ Beck, Katz, and Tucker's (1998) cubic splines; might also use:

- Fractional polynomials
- Smoothed duration
- Loess/lowess fits
- Other splines (B-splines, P-splines, natural splines, etc.)