GSERM 2023
Regression for Publishing

June 22, 2023

Things that are not counts:

- Ordinal scales/variables
- Grouped Binary Data
  - $\frac{N \text{ of ``successes''}}{N \text{ of ``trials''}}$
  - Binomial data
  - $=$ counts only if Pr(“success”) is small

Count properties:

- Discrete / integer-values
- Non-negative
- “Cumulative”

Count Data: Motivation

$$\text{Arrival Rate} = \lambda$$

$$\Pr(\text{Event})_{t,t+h} = \lambda h$$

$$\Pr(\text{No Event})_{t,t+h} = 1 - \lambda h$$

$$\begin{aligned}\Pr(Y_t = y) &= \frac{\exp(-\lambda h)\lambda h^y}{y!} \\ &= \frac{\exp(-\lambda)\lambda^y}{y!}\end{aligned}$$

Poisson Assumptions

- No Simultaneous Events
- Constant Arrival Rate
- Independent Event Arrivals

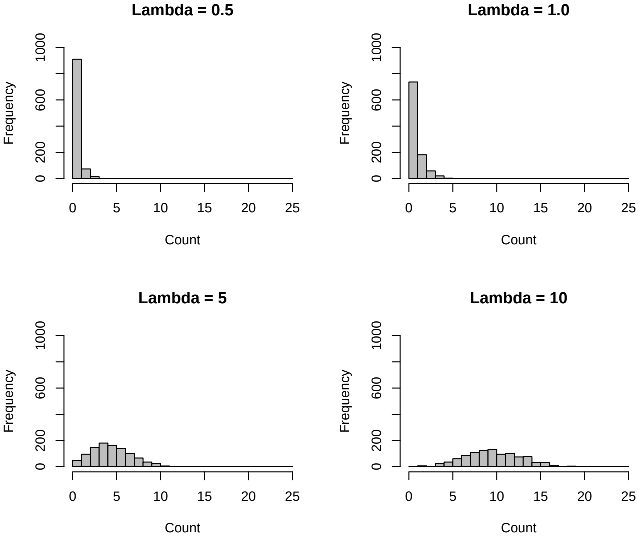
For M independent Bernoulli trials with (sufficiently small) probability of success π and where $M\pi \equiv \lambda > 0$,

$$\begin{aligned}\Pr(Y_i = y) &= \lim_{M\to\infty}\left[\binom{M}{y}\left(\frac{\lambda}{M}\right)^{y}\left(1-\frac{\lambda}{M}\right)^{M-y}\right] \\ &= \frac{\lambda^{y}\exp(-\lambda)}{y!}\end{aligned}$$

Poisson: Characteristics

- Discrete
- $E(Y) = \text{Var}(Y) = \lambda$
- Is not preserved under affine transformations...
- For $X \sim \text{Poisson}(\lambda_X)$ and $Y \sim \text{Poisson}(\lambda_Y)$,
  $Z = X + Y \sim \text{Poisson}(\lambda_{X+Y})$ *iff* X and Y are *independent* but
- ...same is not true for differences.
- $\lambda \to \infty \iff Y \sim N$

Poissons: Examples



Suppose

$$E(Y_i) \equiv \lambda_i = \exp(\mathbf{X}_i\boldsymbol{\beta})$$

then

$$\Pr(Y_i = y|\mathbf{X}_i, \boldsymbol{\beta}) = \frac{\exp[-\exp(\mathbf{X}_i\boldsymbol{\beta})][\exp(\mathbf{X}_i\boldsymbol{\beta})]^y}{y!}$$

Poisson Likelihood

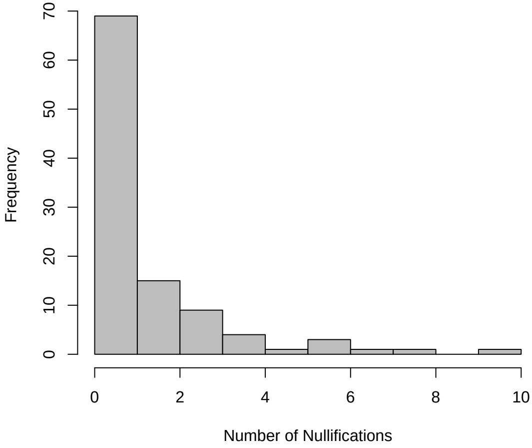
$$L = \prod_{i=1}^{N} \frac{\exp[-\exp(\mathbf{X}_i\beta)][\exp(\mathbf{X}_i\beta)]^{Y_i}}{Y_i!}$$

$$\ln L = \sum_{i=1}^{N} [-\exp(\mathbf{X}_i\beta) + Y_i\mathbf{X}_i\beta - \ln(Y_i!)]$$

- $Y_i =$ number of Acts of Congress overturned by the Supreme Court in each Congress,
- The *mean tenure* (`tenure`) of the Supreme Court's justices ($\bar{X} = 10.4, \sigma = 3.4, \mathrm{E}(\hat{\beta}) > 0$).
- Whether (1) or not (0) there was *unified government* (`unified`) ($\bar{X} = 0.83, \mathrm{E}(\hat{\beta}) < 0$).

Supreme Court Nullifications, 1789-1996

```
> hist(Nulls$nulls,main="",xlab="Number of Nullifications",
    col="grey")
```



Estimation

```
> nulls.poisson<-glm(nulls~tenure+unified,family="poisson",data=Nulls)
> summary(nulls.poisson)

Call:
glm(formula = nulls ~ tenure + unified, family = "poisson", data = Nulls)

Deviance Residuals:
   Min      1Q  Median      3Q     Max
-2.367  -1.503  -0.623   0.561   4.153

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -0.8776     0.3713   -2.36  0.01809 *
tenure        0.0959     0.0256    3.74  0.00018 ***
unified       0.1435     0.2327    0.62  0.53747
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 251.80  on 103  degrees of freedom
Residual deviance: 237.52  on 101  degrees of freedom
AIC: 385.1

Number of Fisher Scoring iterations: 6
```

Interpretation: Incidence Rate Ratios

$$\begin{aligned}\frac{\hat{\lambda}|X_D=1}{\hat{\lambda}|X_D=0} &= \frac{\exp(\hat{\beta}_0+\bar{\mathbf{X}}\hat{\boldsymbol{\beta}}+(X_D=1)\hat{\beta}_{X_D})}{\exp(\hat{\beta}_0+\bar{\mathbf{X}}\hat{\boldsymbol{\beta}}+(X_D=0)\hat{\beta}_{X_D})}\\ &= \exp(\hat{\beta}_{X_D})\end{aligned}$$

- Like ORs
- `unified`: IRR=exp(0.143) = 1.15

Incidence Rate Ratios, continued

$$\text{IRR}_{X_k, X_{k+\delta}} = \exp(\delta \hat{\beta}_k)$$

So, a ten-year difference in `tenure`:

$$\begin{aligned}
\text{IRR} &= \exp(10 \times 0.096) \\
&= \exp(0.96) \\
&= 2.61
\end{aligned}$$

Incidence Rate Ratios

```
> library(mfx)
> nulls.poisson.IRR<-poissonirr(nulls~tenure+unified,
                     data=Nulls)
> nulls.poisson.IRR

Call:
poissonirr(formula = nulls ~ tenure + unified, data = Nulls)

Incidence-Rate Ratio:
           IRR Std. Err.    z   P>|z|
tenure  1.1006    0.0282 3.74 0.00018 ***
unified 1.1543    0.2686 0.62 0.53747
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Predicted Values ($\hat{Y}$s)

Mean predicted Y:

$$\begin{aligned} E(Y|\bar{\mathbf{X}}_i) &= \exp[-0.878 + (0.096 \times 10) + (0.143 \times 1)] \\ &= \exp(0.225) \\ &= 1.25 \end{aligned}$$

In-Sample

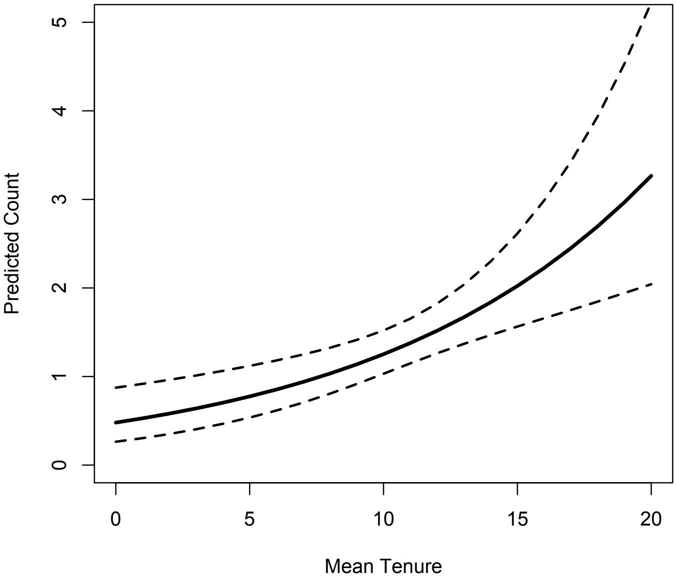
- R : in `$fitted.values`
- Stata : use `predict`

Out-of-Sample: use `predict`

Example: Out-Of-Sample Predicted Values

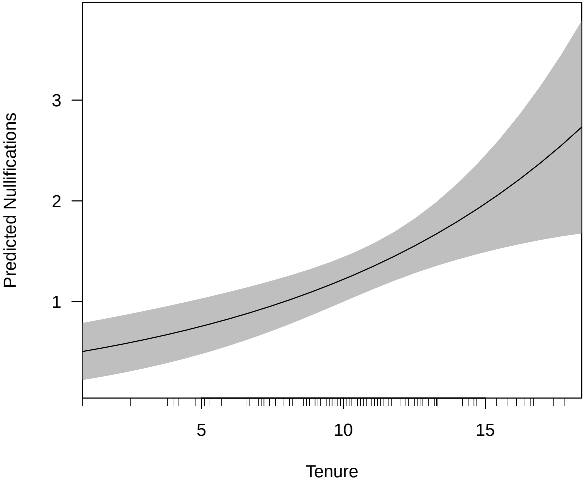
```
> tenure<-seq(0,20,1)
> unified<-1
> simdata<-as.data.frame(cbind(tenure,unified))
> nullhats<-predict(nulls.poisson,newdata=simdata,se.fit=TRUE)
>
> # NOTE: These are XBs, not predicted counts.
> # Transforming:
> nullhats$Yhat<-exp(nullhats$fit)
> nullhats$UB<-exp(nullhats$fit + 1.96*(nullhats$se.fit))
> nullhats$LB<-exp(nullhats$fit - 1.96*(nullhats$se.fit))
> plot(simdata$tenure,nullhats$Yhat,t="l",lwd=3,ylim=c(0,5),ylab=
+            "Predicted Count", xlab="Mean Tenure")
> lines(simdata$tenure,nullhats$UB,lwd=2,lty=2)
> lines(simdata$tenure,nullhats$LB,lwd=2,lty=2)
>
> plot(simdata$tenure,nullhats$Yhat,t="l",lwd=3,ylim=c(0,5),ylab=
+            "Predicted Count", xlab="Mean Tenure")
> lines(simdata$tenure,nullhats$UB,lwd=2,lty=2)
> lines(simdata$tenure,nullhats$LB,lwd=2,lty=2)
```

Plotting Out-Of-Sample Predicted Values



Same Thing, using -margins-

```
> cplot(nulls.poisson,"tenure",xlab="Tenure",
      ylab="Predicted Nullifications")
```



Predicted Probabilities

$$\widehat{\Pr(Y_i = y}|\mathbf{X}_i, \hat{\beta}) = \frac{\exp[-\exp(\mathbf{X}_i\hat{\beta})][\exp(\mathbf{X}_i\hat{\beta})]^y}{y!}$$

$$\begin{aligned}
\rightarrow \quad \widehat{\Pr(Y_i = 0}|\bar{\mathbf{X}}_i, \hat{\boldsymbol{\beta}}) &= \frac{[\exp(-1.25)](1.25)^0}{0!} \\
&= \frac{(0.287)(1)}{1} \\
&= 0.287
\end{aligned}$$

$$\begin{aligned}
\widehat{\Pr(Y_i = 1}|\bar{\mathbf{X}}_i, \hat{\boldsymbol{\beta}}) &= \frac{[\exp(-1.25)](1.25)^1}{1!} \\
&= \frac{(0.287)(1.25)}{1} \\
&= 0.359
\end{aligned}$$

Predicted Probabilities

$$
\begin{aligned}
\widehat{\Pr(Y_i = 2|\bar{\mathbf{X}}_i, \hat{\boldsymbol{\beta}})} &= \frac{[\exp(-1.25)](1.25)^2}{2!} \\
&= \frac{(0.287)(1.563)}{2} \\
&= 0.224
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\Pr(Y_i = 3|\bar{\mathbf{X}}_i, \hat{\boldsymbol{\beta}})} &= \frac{[\exp(-1.25)](1.25)^3}{3!} \\
&= \frac{(0.287)(1.953)}{6} \\
&= 0.093
\end{aligned}
$$

“Exposure” and “Offsets”

$$E(Y_i|\mathbf{X}_i, M_i) = \lambda_i M_i$$

Same as including $\ln(M_i)$ in $\mathbf{X}$ with $\beta_{\ln M} = 1$.

Example: Data on numbers of interstate disputes by country, 1950-1985...

- $N = 102$, but
- `Ndyads` = number of dyad-years which were aggregated to create each observation, ranging from five to 3249
- `disputes` = number of (interstate) dispute-years that country experienced during 1950-1985
- `allies` = number of (dyadic) ally-years each country had during 1950-1985
- `openness` = $\frac{1}{36}\left(\frac{\text{Imports}_t + \text{Exports}_t}{\text{GDP}_t}\right)$ across all 36 years in the data.

“Exposure” and “Offsets”: Data

```
# Data are aggregated dyadic data, 1950-1985...

> IR<-read.csv("Data/offsetIR.csv")
> summary(IR)
     ccode          Ndyads         disputes          allies          openness        exposure  
 Min.   :  2   Min.   :   5   Min.   : 0.00   Min.   :   0.0   Min.   :0.032   Min.   :1.61  
 1st Qu.:214   1st Qu.:  44   1st Qu.: 0.00   1st Qu.:   0.0   1st Qu.:0.185   1st Qu.:3.79  
 Median :436   Median :  92   Median : 1.00   Median :  26.0   Median :0.296   Median :4.52  
 Mean   :418   Mean   : 179   Mean   : 3.55   Mean   :  63.9   Mean   :0.392   Mean   :4.42  
 3rd Qu.:598   3rd Qu.: 146   3rd Qu.: 4.00   3rd Qu.:  81.0   3rd Qu.:0.535   3rd Qu.:4.98  
 Max.   :900   Max.   :3249   Max.   :52.00   Max.   :1283.0   Max.   :1.659   Max.   :8.09  
                                                                NA's   :12                   

> cor(IR,use="complete.obs")
            ccode   Ndyads disputes  allies openness exposure
ccode     1.00000 -0.29623  -0.1399 -0.3983  0.02744  -0.6544
Ndyads   -0.29623  1.00000   0.8626  0.9200 -0.07511   0.6988
disputes -0.13989  0.86257   1.0000  0.8255 -0.16819   0.6335
allies   -0.39826  0.92004   0.8255  1.0000 -0.12548   0.7003
openness  0.02744 -0.07511  -0.1682 -0.1255  1.00000  -0.1433
exposure -0.65442  0.69878   0.6335  0.7003 -0.14325   1.0000
```

Ignoring Exposure

```
> IR.fit1<-glm(disputes~allies+openness,data=IR,family="poisson")
> summary(IR.fit1)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  1.1559498  0.1117581  10.343  < 2e-16 ***
allies       0.0025184  0.0001159  21.734  < 2e-16 ***
openness    -1.1144132  0.2773631  -4.018 5.87e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 732.68  on 101  degrees of freedom
Residual deviance: 392.97  on  99  degrees of freedom
  (12 observations deleted due to missingness)
AIC: 588.29

Number of Fisher Scoring iterations: 6
```

Correcting for Exposure

```
> IR.fit2<-glm(disputes~allies+openness,data=IR,family="poisson",
  offset=log(Ndyads))
> summary(IR.fit2)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -3.2906055  0.1194616 -27.545  < 2e-16 ***
allies      -0.0006058  0.0001333  -4.544 5.52e-06 ***
openness    -1.6040587  0.3167415  -5.064 4.10e-07 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

    Null deviance: 320.19  on 101  degrees of freedom
Residual deviance: 277.79  on  99  degrees of freedom
  (12 observations deleted due to missingness)
AIC: 473.11

Number of Fisher Scoring iterations: 5
```

Correcting for Exposure (continued)

```
> IR.fit3<-glm(disputes~allies+openness+log(Ndyads),data=IR,
+              family="poisson")
> summary(IR.fit3)

Call:
glm(formula = disputes ~ allies + openness + log(Ndyads), family = "poisson",
    data = IR)

Deviance Residuals:
   Min      1Q  Median      3Q     Max
-2.838  -1.390  -0.758   0.605   4.731

Coefficients:
               Estimate  Std. Error z value             Pr(>|z|)
(Intercept) -2.42656676  0.34345252   -7.07      0.0000000000016 ***
allies      -0.00000948  0.00025687   -0.04                 0.97
openness    -1.44462460  0.31193821   -4.63      0.0000036368547 ***
log(Ndyads)  0.81097748  0.07095243   11.43 < 0.0000000000000002 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 732.68  on 101  degrees of freedom
Residual deviance: 270.59  on  98  degrees of freedom
  (12 observations deleted due to missingness)
AIC: 467.9

Number of Fisher Scoring iterations: 5
```

Test $\beta_{\texttt{exposure}} = 1.0$

```
> # z-test:

> 2*pnorm((0.811-1)/.071)
[1] 0.007768438
```

```
> # Wald test:

> wald.test(b=coef(IR.fit3),Sigma=vcov(IR.fit3),Terms=4,H0=1)

Wald test:
----------
Chi-squared test:
X2 = 7.1, df = 1, P(> X2) = 0.0077
```

Contagion, Heterogeneity, and Dispersion


Lisco
26
Oshkosh
27
Lewellen
26


Seattle
WASHINGTON
MONTANA
NORTH DAKOTA
MINNESOTA
SOUTH DAKOTA
WISCONSIN
MICHIGAN
Ottawa
Toronto
MAINE
NB
VT
NH
OREGON
IDAHO
WYOMING
NEBRASKA
IOWA
Chicago
NEW YORK
MA
Boston
CT
RI
ILLINOIS
OHIO
PENN
New York
NEVADA
United States
UTAH
COLORADO
KANSAS
MISSOURI
INDIANA
WEST VIRGINIA
MD
DE
NJ
San Francisco
KENTUCKY
VIRGINIA
CALIFORNIA
Las Vegas
OKLAHOMA
TENNESSEE
NORTH CAROLINA
Los Angeles
ARIZONA
NEW MEXICO
ARKANSAS
MISSISSIPPI
SOUTH CAROLINA
San Diego
ALABAMA
GEORGIA
TEXAS
LOUISIANA
Houston
Gulf of California
FLORIDA
Gulf of Mexico
Mexico

Cats (daily values):

$$
\begin{aligned}
Y_{cats} &= \{0, 1, 1, 0, 2, 0, 1, 0, 3, 1, 2, 1, 0, 2\} \\
\bar{Y}_{cats} &= 1.0, \\
\sigma_{cats} &= 0.92.
\end{aligned}
$$

$$E(Y_{cats}) = \lambda_{cats}$$

Poisson assumes:

- $Y = 0$ at $t = 0$,
- Exclusive events
- $t_j = t_k \, \forall \, j \neq k$
- *Constant, independent* Pr(Event) over t

Antelope

Daily values:

$$
\begin{aligned}
Y_{antelope} &= \{0,0,0,0,0,0,0,0,0,0,0,0,7,7\} \\
\bar{Y}_{antelope} &= 1.0, \\
\sigma_{antelope} &= 6.46.
\end{aligned}
$$

Positive contagion $\rightarrow$ *overdispersion*.

Daily values:

$$\begin{aligned} Y_{foxes} &= \{1,0,1,1,1,1,1,2,1,1,1,1,1,1\} \\ \bar{Y}_{foxes} &= 1.0, \\ \sigma_{foxes} &= 0.15. \end{aligned}$$

Negative contagion $\rightarrow$ *underdispersion*.

Aggregated two-day measures:

$$\begin{aligned} Y_{cats} &= \{1, 1, 2, 1, 4, 3, 2\} \\ Y_{antelope} &= \{0, 0, 0, 0, 0, 0, 14\} \\ Y_{foxes} &= \{1, 2, 2, 3, 2, 2, 2\} \end{aligned}$$

Poisson requires:

- Correct specification
- Correct distribution for ϵ
- Constant $\mathrm{E}(Y|\mathbf{X}, \boldsymbol{\beta})$

Omitted variables $\rightarrow$ overdispersion:

$$\lambda_i \equiv \mathrm{E}(Y_i) = f[\mathbf{X}_i\boldsymbol{\beta} + Z_i\theta]$$

Overdispersion: A Test

Examine:

$$\hat{u}_i = \delta \hat{\lambda}_i + \epsilon_i$$

where

$$\hat{u}_i = \frac{(Y_i - \hat{\lambda}_i)^2 - Y_i}{\hat{\lambda}_i \sqrt{2}}$$

- Estimate a Poisson regression of Y_i on $\mathbf{X}_i$, and generate predicted counts $\hat{\lambda}_i$.
- Calculate $\hat{u}_i$ according to the equation above.
- Estimate δ using OLS, and test $H_0 : \hat{\delta} = 0$.

$$\begin{aligned} \mathrm{E}(Y_i) \equiv \lambda_i &= \exp(\mathbf{X}_i\boldsymbol{\beta} + u_i) \\ &= \exp(\mathbf{X}_i\boldsymbol{\beta})\exp(u_i) \\ &= \lambda_i\nu_i \end{aligned}$$

$$\nu_i \sim \text{gamma}\left(1, \frac{1}{\alpha}\right)$$

$$\Pr(Y_i = y|\lambda_i, \alpha) = \left(\frac{\Gamma(\alpha^{-1} + Y_i)}{\Gamma(\alpha^{-1})\Gamma(Y_i + 1)}\right)\left(\frac{\alpha^{-1}}{\alpha^{-1} + \lambda_i}\right)^{\alpha^{-1}}\left(\frac{\lambda_i}{\lambda_i + \alpha^{-1}}\right)^{Y_i}$$

where

$$\Gamma(a) = \int_0^\infty \exp(-t)t^{a-1}dt$$

Negative Binomial

Basis:

$$\lambda_i = \exp(\mathbf{X}_i \boldsymbol{\beta})$$

Model has

$$\mathsf{E}(Y) = \lambda$$

$$\mathsf{Var}(Y) = \lambda(1 + \alpha\lambda), \ \alpha > 0$$

Negative Binomial (log-)Likelihood

$$\begin{aligned}\ln L_{NB} &= \sum_{i=1}^{N}\Bigg\{\left(\sum_{j=0}^{Y_i-1}\ln(j+\alpha^{-1})\right) - \ln Y_i! - \\ &\quad (Y_i-\alpha^{-1})\ln[1+\alpha\exp(\mathbf{X}_i\boldsymbol{\beta})] + Y_i\ln\alpha + Y_i\mathbf{X}_i\boldsymbol{\beta}\Bigg\}\end{aligned}$$

So:

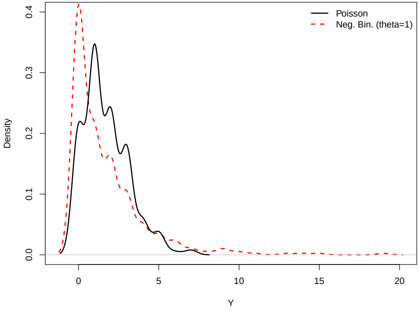
- $\alpha = 0 \iff \mathrm{E}(Y) = \mathrm{Var}(Y)$
- LR test for overdispersion:

$$-2\times(\widehat{\ln L_{Poisson}} - \widehat{\ln L_{NB}}) \sim \chi^2_1$$

- $\widehat{E(Y_i)} \equiv \hat{\lambda}_i = \exp(\mathbf{X}_i\hat{\boldsymbol{\beta}})$

What Difference Does It Make?

```
> N<-400
> set.seed(7222009)
> X <- runif(N,min=0,max=1)
> YPois <- rpois(N,exp(0+1*X))           # Poisson
> YNB <- rnbinom(N,size=1,mu=exp(0+1*X)) # NB with theta=1.0
>
> describe(cbind(YPois,YNB))
      vars   n mean   sd median trimmed  mad min max range skew kurtosis   se
YPois    1 400 1.72 1.41      1    1.56 1.48   0   7     7 0.92     0.84 0.07
YNB      2 400 1.71 2.44      1    1.22 1.48   0  19    19 2.76    11.15 0.12
```



What Difference Does It Make (cont'd)?

```
> # Regressions:
>
> summary(glm(YPois~X,family="poisson")) # Poisson

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.009637   0.085337  -0.113     0.91
X            1.030573   0.131992   7.808 5.82e-15 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 516.06  on 399  degrees of freedom
Residual deviance: 453.55  on 398  degrees of freedom
AIC: 1274.4



> summary(glm.nb(YPois~X))               # NB

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.009629   0.085345  -0.113     0.91
X            1.030557   0.132007   7.807 5.86e-15 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for Negative Binomial(7837.699) family taken to be 1)

    Null deviance: 515.96  on 399  degrees of freedom
Residual deviance: 453.46  on 398  degrees of freedom
AIC: 1276.5

              Theta:  7838
          Std. Err.:  135342
Warning while fitting theta: iteration limit reached

 2 x log-likelihood:  -1270.451
```

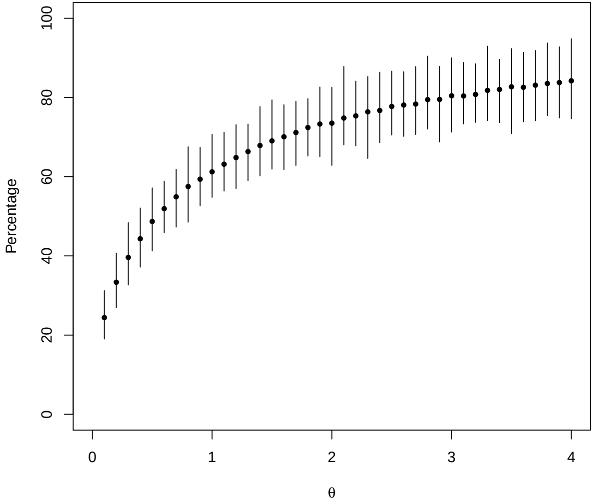
What Difference Does It Make (cont'd)?

```
> # More regressions:
>
> summary(glm(YNB~X,family="poisson"))  # Poisson

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.03170    0.08593  -0.369    0.712
X            1.06109    0.13248   8.009 1.15e-15 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 1118.0  on 399  degrees of freedom
Residual deviance: 1052.1  on 398  degrees of freedom
AIC: 1698.6



> summary(glm.nb(YNB~X))                # NB

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.03525    0.13650  -0.258    0.796
X            1.06773    0.22809   4.681 2.85e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for Negative Binomial(0.8499) family taken to be 1)

    Null deviance: 436.92  on 399  degrees of freedom
Residual deviance: 414.81  on 398  degrees of freedom
AIC: 1407.4

              Theta:  0.850
          Std. Err.:  0.109

 2 x log-likelihood:  -1401.354
```

Poisson Regression Underestimates N.B. Variances

```
Sims <- 250 # (250 sims each)
theta <- seq(0.1,4,by=0.1) # values of theta
diffs<-matrix(nrow=Sims,ncol=length(theta))

set.seed(7222009)
for(j in 1:length(theta)) {
  for(i in 1:Sims) {
    X<-runif(N,min=0,max=1)
    Y<-rnbinom(N,size=theta[j],mu=exp(0+1*X))
    p<-glm(Y~X,family="poisson")
    nb<-glm.nb(Y~X)
    diffs[i,j]<- ((sqrt(vcov(p))[2,2]) / sqrt(vcov(nb))[2,2])*100
  }
}
```

Percentage of True (Negative Binomial) S.E. From Fitted Poisson Model, by $\theta = \frac{1}{\alpha}$



Negative Binomial In Practice

Model fitting (in R):

- `glm.nb` (in `MASS`)
- `negbinomial` (in `VGAM`)
- `negbin` (in `aod`)
- `glmnb.fit` (in `statmod`)
- Probably others...

Model interpretation + diagnostics:

- `fitNBP` (in `statmod`) (dispersion parameter estimation)
- `negbinirr` (in `mfx`) (IRRs)
- `negbinmfx` (in `mfx`) (marginal effects)

Underdispersion: COM Poisson

Underdispersion:

- ... is implied by *negative contagion*
- $\rightarrow$ limiting effect on $\max(Y)$.

Conway-Maxwell-Poisson: Allows for either over- or underdispersion:

$$Pr(Y_i = y) = \frac{\lambda^y}{(y!)^\nu} \frac{1}{Z(\lambda, \nu)}$$

where:

$$Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu}.$$

$\nu \in [0, \infty)$, with:

- $\nu = 1 \rightarrow$ Poisson
- $\nu \rightarrow \infty \rightarrow$ Bernoulli$\left(\frac{\lambda}{1+\lambda}\right)$
- $\nu = 0 \rightarrow$ geometric (with $0 < \lambda < 1$)

And yes, there is an R package...

Example: SCOTUS Amicus Curiae (1953-85)

- $N = 7157$
- `namici` is the number of amicus curiae briefs filed in each case,
- `term` is the term (i.e., year) of the Court,
- `civlibs` is whether (=1) or not (=0) the case involved a civil rights and liberties issue.

```
> summary(amici)
    namici            term          civlibs
 Min.   : 0.00   Min.   :53.0   Min.   :0.000
 1st Qu.: 0.00   1st Qu.:64.0   1st Qu.:0.000
 Median : 0.00   Median :72.0   Median :1.000
 Mean   : 1.03   Mean   :71.1   Mean   :0.501
 3rd Qu.: 1.00   3rd Qu.:79.0   3rd Qu.:1.000
 Max.   :53.00   Max.   :85.0   Max.   :1.000
```

```
> amici.poisson<-glm(namici~term+civlibs,data=amici,family="poisson")
> summary(amici.poisson)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -4.51196    0.11190  -40.32   <2e-16 ***
term         0.06361    0.00147   43.27   <2e-16 ***
civlibs     -0.29797    0.02350  -12.68   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 22875  on 7156  degrees of freedom
Residual deviance: 20675  on 7154  degrees of freedom
  (4 observations deleted due to missingness)
AIC: 26862

Number of Fisher Scoring iterations: 6
```

Overdispersion Test: “By Hand”

```
> Phats<-fitted.values(amici.poisson)
> Uhats<-((amici$namici-Phats)^2 - amici$namici) / (Phats * sqrt(2))
> summary(lm(Uhats~Phats))

Call:
lm(formula = Uhats ~ Phats)

Residuals:
   Min     1Q Median     3Q    Max
  -5.9   -3.0   -2.3   -1.9 1707.0

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    1.579      0.693    2.28    0.023 *
Phats          1.466      0.591    2.48    0.013 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 28.4 on 7155 degrees of freedom
Multiple R-squared:  0.000858,Adjusted R-squared:  0.000718
F-statistic: 6.14 on 1 and 7155 DF,  p-value: 0.0132
```

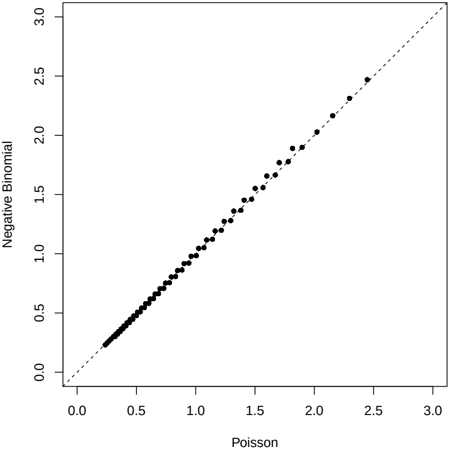
Negative Binomial Regression

```
> library(MASS)
> amici.NB<-glm.nb(namici~term+civlibs,data=amici)
> summary(amici.NB)

Call:
glm.nb(formula = namici ~ term + civlibs, data = amici, init.theta = 0.256657474,
    link = log)

Coefficients:
            Estimate Std. Error z value             Pr(>|z|)
(Intercept) -4.68314    0.22058  -21.23 < 0.0000000000000002 ***
term         0.06573    0.00304   21.60 < 0.0000000000000002 ***
civlibs     -0.26777    0.05403   -4.96           0.00000072 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for Negative Binomial(0.2567) family taken to be 1)

    Null deviance: 5442  on 7156  degrees of freedom
Residual deviance: 4968  on 7154  degrees of freedom
AIC: 17378

Number of Fisher Scoring iterations: 1
              Theta:  0.25666
          Std. Err.:  0.00838
```

```
> 1 / amici.NB$theta
[1] 3.896
```

Predicted Values: Poisson and NB

```
> plot(amici.poisson$fitted.values,amici.NB$fitted.values,xlab="Poisson",
  ylab="Negative Binomial",main="Predicted Counts")
> abline(a=0,b=1,lwd=2)
```



More Things

- Models where Over- / Underdispersion $= f(\mathbf{Z}_i \gamma)$
- Models for Censored / Truncated Counts
- "Zero-Inflated" and "Hurdle" Count Models
- Time Series of Counts
- Panel Data Models for Event Counts
- Spatial Count Models
- etc...

Generalized Linear Models (GLMs)

$$f(z|\psi) = \Pr(Z = z|\psi)$$

Exponential if:

$$f(z|\psi) = r(z)s(\psi)\exp[q(z)h(\psi)]$$

provided that $r(z) > 0$ and $s(\psi) > 0$.

$$f(z|\psi) = \exp[\underbrace{\ln r(z) + \ln s(\psi)}_{\text{“additive”}} + \underbrace{q(z)h(\psi)}_{\text{“interactive”}}]$$

Canonical Forms

$$y = q(z)$$

$$\theta = h(\psi)$$

$$f[y|\theta] = \exp[y\theta - b(\theta) + c(y)].$$

- $b(\theta)$ is a "normalizing constant"
- $c(y)$ is a function solely of y
- $y\theta$ is a multiplicative term

A Familiar Family Member: Poisson

$$f(y|\lambda) = \frac{\exp(-\lambda)\lambda^y}{y!}.$$

$$\begin{aligned} f(y|\lambda) &= \exp\{\ln[\exp(-\lambda)\lambda^y / y!]\} \\ &= \exp[\underbrace{y\ln(\lambda)}_{y\theta} - \underbrace{\lambda}_{b(\theta)} - \underbrace{\ln(y!)}_{c(y)}] \end{aligned}$$

$$f(y|\theta, \phi) = \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right]$$

Familiar Family Member II: Normal

$$f(y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{(y-\mu)^2}{2\sigma^2}\right]$$

$$\begin{aligned}
f(y|\mu,\sigma^2) &= \exp\left[-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y^2 - 2y\mu + \mu^2)\right] \\
&= \exp\left[-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}y^2 + \frac{1}{2\sigma^2}2y\mu - \frac{1}{2\sigma^2}\mu^2\right] \\
&= \exp\left[\frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{y^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)\right] \\
&= \exp\left\{\frac{y\mu - \frac{\mu^2}{2}}{\sigma^2} + \frac{-1}{2}\left[\frac{y^2}{\sigma^2} + \ln(2\pi\sigma^2)\right]\right\}
\end{aligned}$$

$$f(y|\mu,\sigma^2) = \exp\left\{\frac{y\mu - \frac{\mu^2}{2}}{\sigma^2} + \frac{-1}{2}\left[\frac{y^2}{\sigma^2} + \ln(2\pi\sigma^2)\right]\right\}$$

$\theta = \mu$, so:

- $y\theta = y\mu$
- $b(\theta) = \frac{\mu^2}{2}$
- $a(\phi) = \sigma^2$
- $c(y,\phi) = \frac{-1}{2}\left[\frac{y^2}{\sigma^2} + \ln(2\pi\sigma^2)\right]$

Other Family Members

- Binomial ($\supset$ Bernoulli; also Multinomial)
- Exponential
- Gamma
- Logarithmic
- Inverse Gaussian
- Negative Binomial
- others...

$$
\begin{aligned}
\ln L(\theta, \phi | y) &= \ln f(y|\theta, \phi) \\
&= \ln \left\{ \exp \left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi) \right] \right\} \\
&= \frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln L(\theta, \phi | y)}{\partial \theta} \equiv \mathbf{S} &= \frac{\partial}{\partial \theta} \left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi) \right] \\
&= \frac{y - \frac{\partial}{\partial \theta} b(\theta)}{a(\phi)}.
\end{aligned}
$$

Among family members:

- $\mathbf{S}$ is a sufficient statistic for θ.
- $\mathrm{E}(\mathbf{S}) = 0$.
- $\mathrm{Var}(\mathbf{S}) \equiv \mathcal{I}(\theta) = \mathrm{E}[(\mathbf{S})^2|\theta]$

More Estimation

$$E(Y) = \frac{\partial}{\partial\theta} b(\theta)$$

and

$$\text{Var}(Y) = a(\phi) \frac{\partial^2}{\partial\theta^2} b(\theta)$$

Example: Poisson Again

$$\begin{aligned}
\mathsf{E}(Y) &= \frac{\partial}{\partial\theta}\exp(\theta) \\
&= \exp(\theta)|_{\theta=\ln(\lambda)} \\
&= \lambda
\end{aligned}$$

$$\begin{aligned}
\mathsf{Var}(Y) &= 1 \times \frac{\partial^2}{\partial\theta^2}\exp(\theta)|_{\theta=\ln(\lambda)} \\
&= \exp[\ln(\lambda)] \\
&= \lambda
\end{aligned}$$

Example: Normal Again

$$
\begin{aligned}
\mathsf{E}(Y) &= \frac{\partial}{\partial\theta}\left(\frac{\theta^2}{2}\right) \\
&= \theta|_{\theta=\mu} \\
&= \mu
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Var}(Y) &= \sigma^2 \times \frac{\partial^2}{\partial\theta^2}\left(\frac{\theta^2}{2}\right) \\
&= \sigma^2 \times \frac{\partial}{\partial\theta}\theta \\
&= \sigma^2
\end{aligned}
$$

$$Y_i = \mathbf{X}_i \boldsymbol{\beta} + u_i$$

$$\mathrm{E}(Y_i) \equiv \boldsymbol{\mu}_i = \mathbf{X}_i \boldsymbol{\beta}$$

The "Generalized" Part

$$g(\boldsymbol{\mu}_i) = \mathbf{X}_i\boldsymbol{\beta}.$$

$$\begin{aligned}\boldsymbol{\eta}_i &= \mathbf{X}_i\boldsymbol{\beta} \\ &= g(\boldsymbol{\mu}_i)\end{aligned}$$

$$\begin{aligned}\boldsymbol{\mu}_i &= g^{-1}(\boldsymbol{\eta}_i) \\ &= g^{-1}(\mathbf{X}_i\boldsymbol{\beta})\end{aligned}$$

Random component $\sim$ EF($\cdot$) with

$$E(Y_i) = \boldsymbol{\mu}_i.$$

Systematic component:

$$g(\boldsymbol{\mu}_i) = \boldsymbol{\eta}_i$$

or

$$g^{-1}(\boldsymbol{\eta}_i) = \boldsymbol{\mu}_i.$$

$$\begin{aligned}\boldsymbol{\theta}_i &= g(\boldsymbol{\mu}_i)\\ &= \boldsymbol{\eta}_i\\ &= \mathbf{X}_i\boldsymbol{\beta}\end{aligned}$$

$$g^{-1}(\boldsymbol{\theta}_i) = \boldsymbol{\mu}_i$$

GLM Example: Linear-Normal

$$f(y|\mu, \sigma^2) = \mathcal{N}(\mu, \sigma^2)$$

$$\boldsymbol{\mu}_i = \boldsymbol{\eta}_i$$

$$\begin{aligned} \boldsymbol{\mu}_i \equiv \boldsymbol{\theta}_i &= \boldsymbol{\eta}_i \\ Y_i &\sim \mathcal{N}(\boldsymbol{\mu}_i, \sigma^2) \end{aligned}$$

GLM Example: Binary

$$f(y|\pi) = \pi^y(1-\pi)^{1-y}$$

$$\boldsymbol{\theta}_i = \ln\left(\frac{\boldsymbol{\mu}_i}{1-\boldsymbol{\mu}_i}\right)$$

$$\begin{aligned}
\boldsymbol{\mu}_i &= g^{-1}(\boldsymbol{\theta}_i) \\
&= \frac{\exp(\boldsymbol{\eta}_i)}{1+\exp(\boldsymbol{\eta}_i)} \\
Y_i &\sim \text{Bernoulli}(\boldsymbol{\mu}_i)
\end{aligned}$$

GLM Example: Counts (Independent Events)

$$f(y|\lambda) = \frac{\exp(-\lambda)\lambda^{y}}{y!}$$

$$\ln(\lambda_i) = \boldsymbol{\eta}_i$$

$$\begin{aligned}
\boldsymbol{\mu}_i &= g^{-1}(\boldsymbol{\theta}_i) \\
&= \exp(\boldsymbol{\eta}_i) \\
Y_i &\sim \text{Poisson}(\lambda_i)
\end{aligned}$$

Common GLM Flavors

Distribution	Range of Y	Link(s) $g(\cdot)$	Inverse Link $g^{-1}(\cdot)$
Normal	$(-\infty, \infty)$	Identity: $\boldsymbol{\theta} = \boldsymbol{\mu}$ (Canonical)	$\boldsymbol{\theta}$
Binomial	$\{0,...n\}$	Logit: $\boldsymbol{\theta} = \ln\left(\frac{\boldsymbol{\mu}}{1-\boldsymbol{\mu}}\right)$ (Canonical)	$\frac{\exp(\boldsymbol{\theta})}{1+\exp(\boldsymbol{\theta})}$
		Probit: $\boldsymbol{\theta} = \Phi^{-1}(\boldsymbol{\mu})$	$\Phi(\boldsymbol{\theta})$
		C-Log-Log: $\boldsymbol{\theta} = \ln[-\ln(1-\boldsymbol{\mu})]$	$1 - \exp[-\exp(\boldsymbol{\theta})]$
Bernoulli	$\{0,1\}$	(same as Binomial)	(same as Binomial)
Multinomial	$\{0,...J\}$	(same as Binomial)	(same as Binomial)
Poisson	$[0, \infty]$ (integers)	Log: $\boldsymbol{\theta} = \ln(\boldsymbol{\mu})$ (Canonical)	$\exp(\boldsymbol{\theta})$
Gamma	$(0, \infty)$	Reciprocal: $\boldsymbol{\theta} = -\frac{1}{\boldsymbol{\mu}}$ (Canonical)	$-\frac{1}{\boldsymbol{\theta}}$

Note: The Bernoulli is a special case of the Binomial with $n = 1$. The multinomial is the J-outcome variant of the Binomial, and is also related to the Poisson (see, e.g., Agresti 2002).

GLMs: How-To

- Pick $f(Y)$
- Pick $g(\cdot)$
- Specify $\mathbf{X}$
- Estimate

- MLE
- IRLS ($\approx$ MLE):

$$\hat{\boldsymbol{\beta}}^{(t+1)} = [\mathbf{X}'\mathbf{W}^{(t)}\mathbf{X}]^{-1}\mathbf{X}'\mathbf{W}^{(t)}\mathbf{z}^{(t)}$$

with

$$\underset{N \times N}{\mathbf{W}^{(t)}} = \mathrm{diag}\left[\frac{\left(\partial \mu_i^{(t)} / \partial \eta_i^{(t)}\right)^2}{\mathrm{Var}(Y_i)}\right]$$

and

$$\mathbf{z}^{(t)} \quad = \quad \boldsymbol{\eta}^{(t)} + (Y - \boldsymbol{\mu}^{(t)}) \left(\frac{\partial \boldsymbol{\eta}^{(t)}}{\partial \boldsymbol{\mu}}\right).$$

IRLS, Intuitively

At iteration t:

1. Calculate $\mathbf{z}^{(t)}$, $\mathbf{W}^{(t)}$
2. Regress $\mathbf{z}^{(t)}$ on $\mathbf{X}$, using $\mathbf{W}^{(t)}$ as weights, to obtain $\hat{\boldsymbol{\beta}}^{(t+1)}$
3. Generate $\boldsymbol{\eta}^{(t+1)} = \mathbf{X}\hat{\boldsymbol{\beta}}^{(t+1)}$
4. Generate $\boldsymbol{\mu}^{(t+1)} = g^{-1}(\boldsymbol{\eta}^{(t+1)})$
5. Use $\boldsymbol{\eta}^{(t+1)}$ and $\boldsymbol{\mu}^{(t+1)}$ to calculate $\mathbf{z}^{(t+1)}$ and $\mathbf{W}^{(t+1)}$
6. Repeat until convergence.

Residuals

"Response" Residuals:

$$\begin{aligned}\hat{u}_i &= Y_i - \hat{\mu}_i \\ &= Y_i - g^{-1}(\mathbf{X}_i\hat{\beta})\end{aligned}$$

"Pearson" Residuals:

$$\hat{P}_i = \frac{\hat{u}_i}{[\text{Var}(\hat{u}_i)]^{1/2}}$$

"Deviance":

$$
\begin{aligned}
\hat{d}_i &= -2[\ln L_i(\hat{\theta}) - \ln L_i(\theta_S)] \\
&= 2\left\{\left[\frac{Y_i\theta_S - b(\theta_S)}{a(\phi)} + c(Y_i,\phi)\right] - \left[\frac{Y_i\hat{\theta} - b(\hat{\theta})}{a(\phi)} + c(Y_i,\phi)\right]\right\} \\
&= 2\left[\frac{Y_i(\theta_S - \hat{\theta}) - b(\theta_S) + b(\hat{\theta})}{a(\phi)}\right]
\end{aligned}
$$

"Deviance" Residuals:

$$
\hat{r}_{Di} = \left(\frac{\hat{u}_i}{|\hat{u}_i|}\right)\sqrt{\hat{d}_i^2}
$$

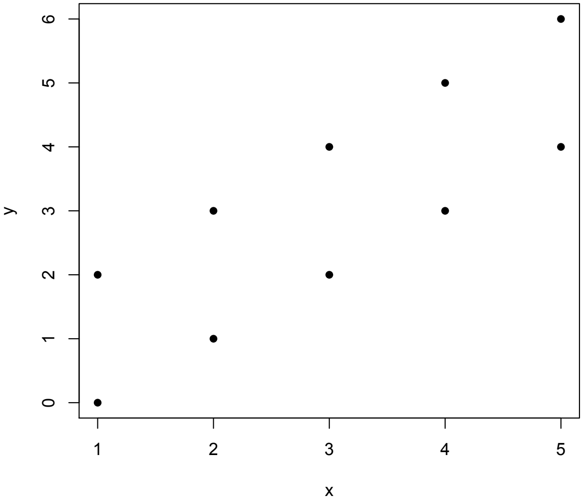
Toy Example: Linear-Normal

$$
\begin{aligned}
X &= \{1,1,2,2,3,3,4,4,5,5\} \\
Y &= \{0,2,1,3,2,4,3,5,4,6\}
\end{aligned}
$$

$$
\begin{aligned}
Y_i &= 0 + 1X_i + u_i \\
\hat{u}_i^2 &= 1 \,\forall\, i
\end{aligned}
$$

$$
\begin{aligned}
\text{“TSS”} \equiv \sum (Y_i - \bar{Y})^2 &= 30 \\
\text{“RSS”} \equiv \sum \hat{u}_i^2 &= 10 \\
\text{“MSS”} \ / \ \text{“ESS”} &= 20
\end{aligned}
$$

Toy Example: Plot



Toy Example: OLS

```
> linmod<-lm(y~x)
> summary(linmod)

Call:
lm(formula = y ~ x)

Residuals:
       Min         1Q     Median         3Q        Max
-1.000e+00 -1.000e+00  1.110e-16  1.000e+00  1.000e+00

Coefficients:
              Estimate Std. Error   t value Pr(>|t|)
(Intercept) -5.617e-16  8.292e-01 -6.77e-16  1.00000
x            1.000e+00  2.500e-01         4  0.00395 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.118 on 8 degrees of freedom
Multiple R-squared: 0.6667,Adjusted R-squared: 0.625
F-statistic:    16 on 1 and 8 DF,  p-value: 0.00395
```

Toy Example: Linear-Normal GLM

```
> linglm<-glm(y~x,family="gaussian")
> summary(linglm)

Deviance Residuals:
       Min          1Q      Median          3Q         Max
-1.000e+00  -1.000e+00  -5.551e-17   1.000e+00   1.000e+00

Coefficients:
              Estimate Std. Error   t value Pr(>|t|)
(Intercept) -5.617e-16  8.292e-01 -6.77e-16  1.00000
x            1.000e+00  2.500e-01         4  0.00395 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for gaussian family taken to be 1.25)

    Null deviance: 30  on 9  degrees of freedom
Residual deviance: 10  on 8  degrees of freedom
AIC: 34.379

Number of Fisher Scoring iterations: 2
```

Better GLM Example: Political Knowledge

- 2008 NES political knowledge
- Identify Speaker of the House, VP, British PM, and Chief Justice
- $Y_i =$ number of correct answers (out of four)

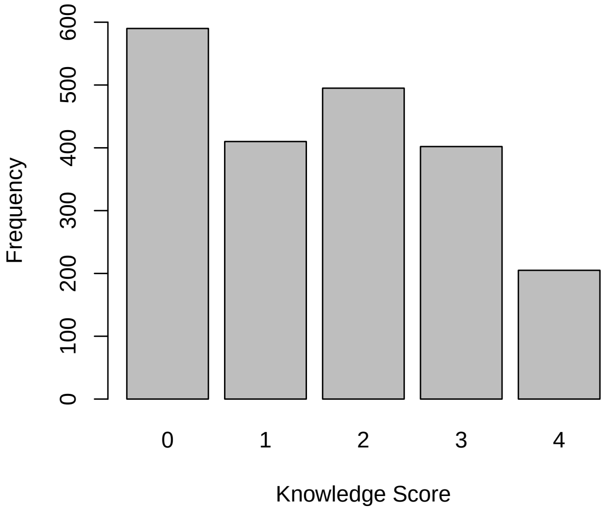
$$f(Y_i, p_i) = \binom{4}{Y_i} p_i^{Y_i} (1 - p_i)^{4 - Y_i}$$

$$\begin{aligned} Y &\sim \text{Binomial}(4, p) \\ \text{E}(Y_i) &= \frac{\exp(\mathbf{X}_i \boldsymbol{\beta})}{1 + \exp(\mathbf{X}_i \boldsymbol{\beta})} \end{aligned}$$

GLM Example Data (2008 NES)

```
> summary(NES08[,4:16])
   knowledge                                sex                                race
 Min.   :0.00   1. Male respondent selected  : 999   1. White                    :1442
 1st Qu.:1.00   2. Female respondent selected:1324   2. Black/African-American: 583
 Median :2.00                                        4. Other race             : 262
 Mean   :2.37                                        5. White and another race:  16
 3rd Qu.:4.00                                        6. Black and another race:   6
 Max.   :4.00                                        (Other)                   :   2
 NA's   :221                                         NA's                      :  12
      age          female          white                  oftenvote    conservative
 Min.   :17   Min.   :0.00   Min.   :0.0000   Seldom          :621   Min.   :1.00
 1st Qu.:33   1st Qu.:0.00   1st Qu.:0.0000   Part of the Time:287   1st Qu.:3.00
 Median :46   Median :1.00   Median :1.0000   Nearly Always   :612   Median :4.00
 Mean   :47   Mean   :0.57   Mean   :0.6207   Always          :788   Mean   :4.14
 3rd Qu.:59   3rd Qu.:1.00   3rd Qu.:1.0000   NA's            : 15   3rd Qu.:5.00
 Max.   :90   Max.   :1.00   Max.   :1.0000                          Max.   :7.00
 NA's   :22                                                          NA's   :697
            prayfreq    heterosexual        married         yrsofschool        income
 Never           :235   Min.   :0.0000   Min.   :0.0000   Min.   : 0.00   Min.   : 1.00
 Once/week       :321   1st Qu.:1.0000   1st Qu.:0.0000   1st Qu.:12.00   1st Qu.: 5.00
 Few times a week:416   Median :1.0000   Median :0.0000   Median :13.00   Median :11.00
 Daily           :525   Mean   :0.9591   Mean   :0.4224   Mean   :13.08   Mean   :10.52
 Several/Day     :806   3rd Qu.:1.0000   3rd Qu.:1.0000   3rd Qu.:15.00   3rd Qu.:15.00
 NA's            : 20   Max.   :1.0000   Max.   :1.0000   Max.   :17.00   Max.   :25.00
                        NA's   :49       NA's   :15       NA's   :11      NA's   :151
```

Political Knowledge (2008 NES)



```
> nes08.binom<-glm(cbind(knowledge,4-knowledge)~age+female+white+oftenvote+conservative
+prayfreq+heterosexual+married+yrsofschool+income,data=nes2008,family=binomial)
> summary(nes08.binom)

Deviance Residuals:
     Min        1Q    Median        3Q       Max
-3.59683  -1.01716   0.03124   1.34899   2.85336

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)   2.097696   0.248976   8.425  < 2e-16 ***
age          -0.010789   0.001910  -5.650 1.60e-08 ***
female        0.213865   0.059534   3.592 0.000328 ***
white        -0.154109   0.064613  -2.385 0.017073 *
oftenvote    -0.097272   0.027511  -3.536 0.000407 ***
conservative  0.019421   0.019317   1.005 0.314704
prayfreq      0.048818   0.022248   2.194 0.028216 *
heterosexual  0.070894   0.138471   0.512 0.608665
married      -0.166501   0.058363  -2.853 0.004333 **
yrsofschool  -0.090790   0.013116  -6.922 4.45e-12 ***
income       -0.009015   0.005259  -1.714 0.086492 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 3181.4  on 1359  degrees of freedom
Residual deviance: 2952.9  on 1349  degrees of freedom
  (963 observations deleted due to missingness)
AIC: 4563.1
Number of Fisher Scoring iterations: 4
```

GLMs: Other Topics + Extensions

Other Topics:

- Generalizations for Overdispersion (binomial)
- Diagnostics (leverage, etc.)
- Joint Mean-Dispersion Models

Extensions:

- Bias-reduced models (a la Firth 1993)
- "Generalized additive models" (GAMs)
- "Generalized estimating equations" (GEEs)
- "Vector" GLMs (Yee and Wild 1996; Yee and Hastie 2003)

GLMs: References

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Dobson, Annette J., and and Adrian G. Barnett. 2008. *An Introduction to Generalized Linear Models*, 3rd Ed. London: Chapman & Hall.

Faraway, Julian J. 2006. *Extending the Linear Model with R: Generalized Linear, Mixed Effects, and Nonparametric Regression Models.* London: Chapman & Hall / CRC.

Dunn, Peter K., and Gordon K. Smyth. 2018. *Generalized Linear Models With Examples in R*. New York: Springer.

Hardin, James W., and Joseph W. Hilbe. 2012. *Generalized Linear Models and Extensions*, 3rd Ed. College Station, TX: Stata Press.