GSERM 2024
Regression for Publishing

June 18, 2024

Link To The “Friday Poll”

https://bit.ly/RegForPub-2024

Implicit in

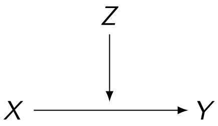
$$Y = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

is that

$$\frac{\partial E(Y)}{\partial X_k} = \beta_k \; \forall \text{ values of } X_k, X_\ell, k \neq \ell.$$

Conceptually: *The marginal association between Y and every X is identical for all values of* $\mathbf{X}$.

Moderating variable Z:

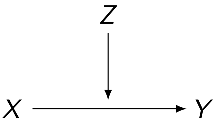


Intuition: *The marginal association between* X *and* Y *varies with / depends on the value(s) of* Z.

Moderating variables imply <u>interactive</u> models.

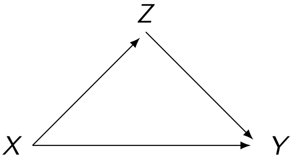
Don't Get Confused

Don't mistake a moderator for a mediator...



Z is a Moderator...

vs.



Z is a Mediator...

Multiplicative interaction:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i$$

So:

$$\begin{aligned} \mathrm{E}(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} + (\beta_1 + \beta_3 X_{2i}) X_{1i} \\ &= \beta_0 + \beta_2 X_{2i} + \psi_1 X_{1i} \end{aligned}$$

where $\psi_1 = \beta_1 + \beta_3 X_{2i}$. This means that the *marginal effect*:

$$\frac{\partial \mathrm{E}(Y_i)}{\partial X_1} = \beta_1 + \beta_3 X_{2i}.$$

Similarly:

$$\begin{aligned} \mathrm{E}(Y_i) &= \beta_0 + \beta_1 X_{1i} + (\beta_2 + \beta_3 X_{1i}) X_{2i} \\ &= \beta_0 + \beta_1 X_{1i} + \psi_2 X_{2i} \end{aligned}$$

which implies:

$$\frac{\partial \mathrm{E}(Y_i)}{\partial X_2} = \beta_2 + \beta_3 X_{1i}.$$

"Direct Effects"

Note that if $X_2 = 0$, then:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2(0) + \beta_3 X_{1i}(0) \\ &= \beta_0 + \beta_1 X_{1i}. \end{aligned}$$

Similarly, for $X_1 = 0$:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1(0) + \beta_2 X_{2i} + \beta_3(0) X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} \end{aligned}$$

In most instances, the quantities we care about are not β_1 and β_2, but rather ψ_1 and ψ_2.

Inference

Point estimates:

$$\hat{\psi}_1 = \hat{\beta}_1 + \hat{\beta}_3 X_2$$

and

$$\hat{\psi}_2 = \hat{\beta}_2 + \hat{\beta}_3 X_1.$$

For variance, recall that:

$$\mathrm{Var}(a + bZ) = \mathrm{Var}(a) + Z^2 \mathrm{Var}(b) + 2Z\mathrm{Cov}(a, b)$$

Means that:

$$\widehat{\mathrm{Var}(\hat{\psi}_1)} = \widehat{\mathrm{Var}(\hat{\beta}_1)} + X_2^2\widehat{\mathrm{Var}(\hat{\beta}_3)} + 2X_2\widehat{\mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_3)}.$$

and

$$\widehat{\mathrm{Var}(\hat{\psi}_2)} = \widehat{\mathrm{Var}(\hat{\beta}_2)} + X_1^2\widehat{\mathrm{Var}(\hat{\beta}_3)} + 2X_1\widehat{\mathrm{Cov}(\hat{\beta}_2, \hat{\beta}_3)}.$$

Types of Interactions: Dichotomous Xs

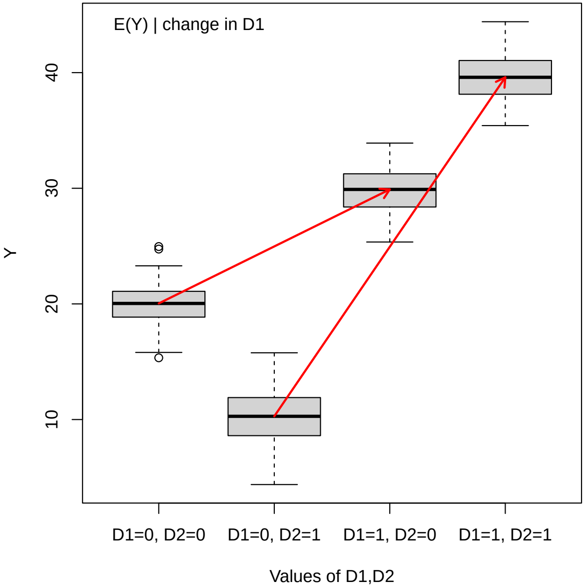
For

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{1i} D_{2i} + u_i$$

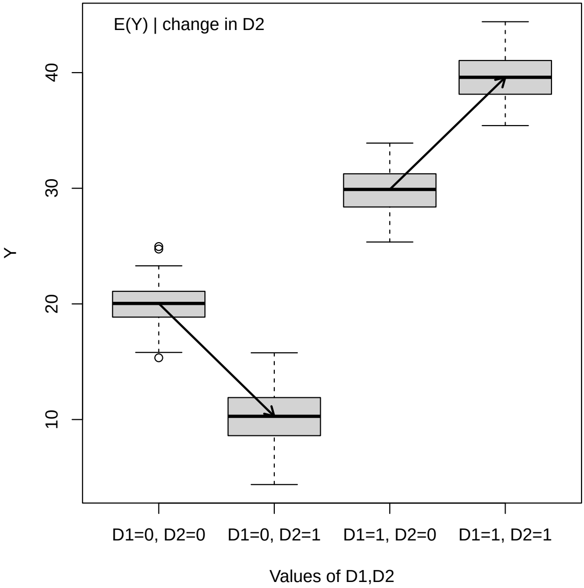
we have:

$$\begin{aligned}
\text{E}(Y|D_1 = 0, D_2 = 0) &= \beta_0 \\
\text{E}(Y|D_1 = 1, D_2 = 0) &= \beta_0 + \beta_1 \\
\text{E}(Y|D_1 = 0, D_2 = 1) &= \beta_0 + \beta_2 \\
\text{E}(Y|D_1 = 1, D_2 = 1) &= \beta_0 + \beta_1 + \beta_2 + \beta_3
\end{aligned}$$

Values of $E(Y)$ for Changes in D_1



Values of $E(Y)$ for Changes in D_2



The model:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 X_i D_i + u_i$$

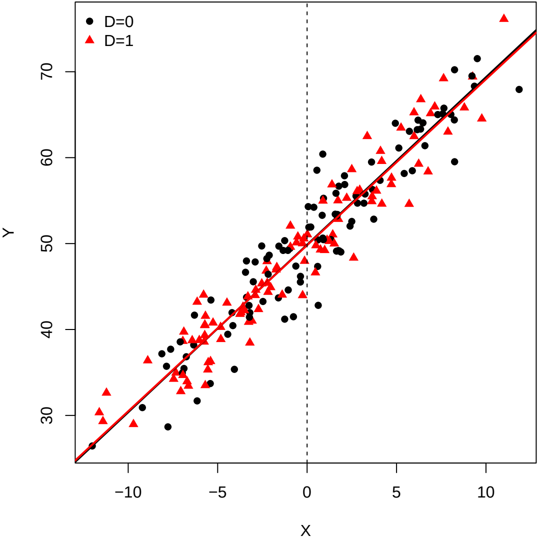
gives:

$$\begin{aligned} \mathrm{E}(Y|X, D=0) &= \beta_0 + \beta_1 X \\ \mathrm{E}(Y|X, D=1) &= (\beta_0 + \beta_2) + (\beta_1 + \beta_3) X \end{aligned}$$

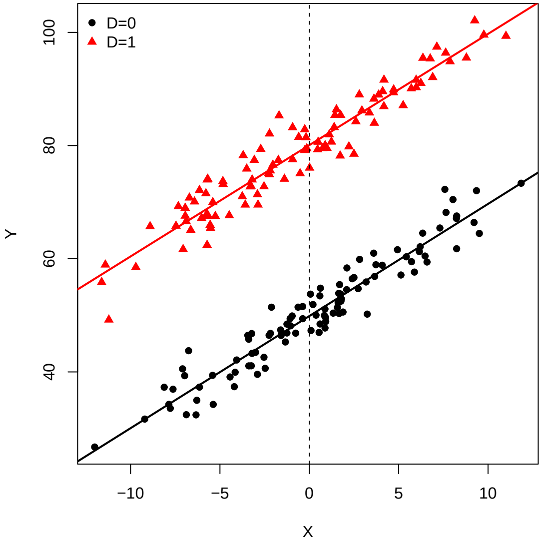
Four possibilities:

- $\beta_2 = \beta_3 = 0$
- $\beta_2 \neq 0$ and $\beta_3 = 0$
- $\beta_2 = 0$ and $\beta_3 \neq 0$
- $\beta_2 \neq 0$ and $\beta_3 \neq 0$

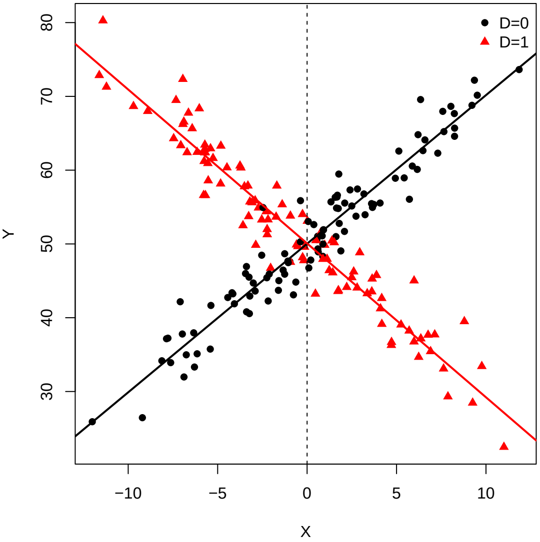
No Slope or Intercept Differences ($\beta_2 = \beta_3 = 0$)



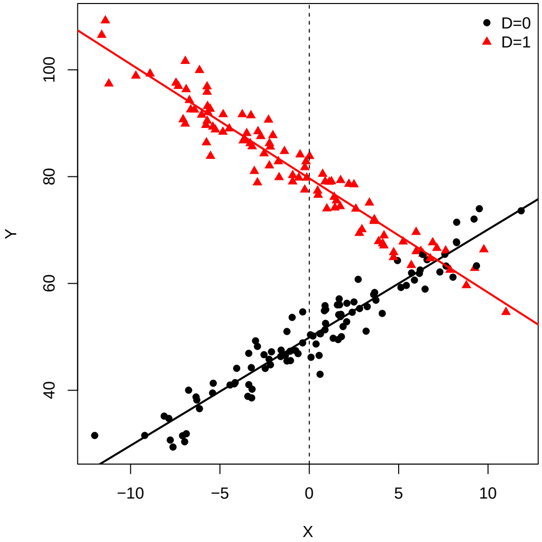
Intercept Shift ($\beta_2 \neq 0$, $\beta_3 = 0$)



Slope Change ($\beta_2 = 0$, $\beta_3 \neq 0$)



Slope and Intercept Change ($\beta_2 \neq 0$, $\beta_3 \neq 0$)



Two Continuous Xs

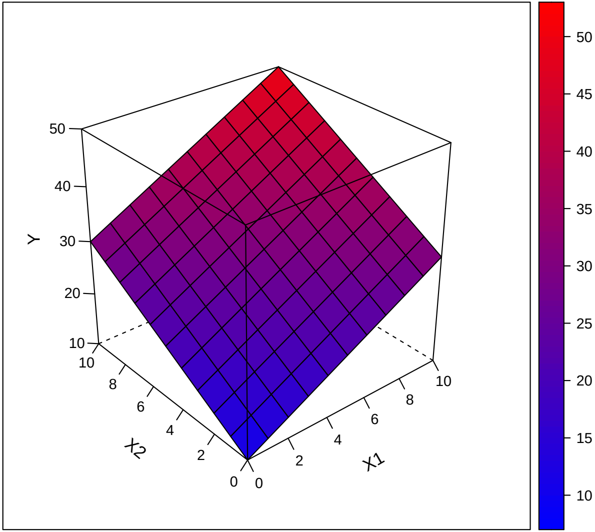
The model:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i$$

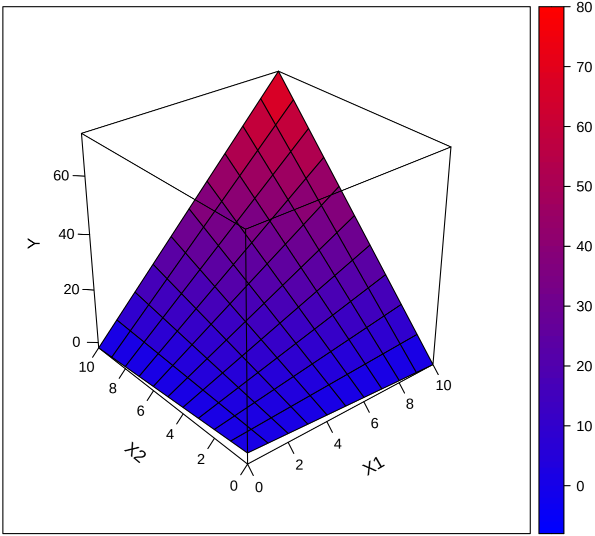
means that:

$$\beta_3 = 0 \;\rightarrow\; \frac{\partial E(Y)}{\partial X_1} = \beta_1 \,\forall\, X_2 \text{ and } \frac{\partial E(Y)}{\partial X_2} = \beta_2 \,\forall\, X_1$$

Two Continuous Variables: No Interactive Effects



Two Continuous Variables: Interaction Present



Quadratic, Cubic, and Other Polynomial Effects

For the polynomial-in-X model:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + ... + \beta_j X_i^j + u_i$$

in general:

$$\frac{\partial \mathrm{E}(Y)}{\partial X} = \beta_1 + 2\beta_2 X + 3\beta_3 X^2 + ... + j\beta_j X^{j-1}.$$

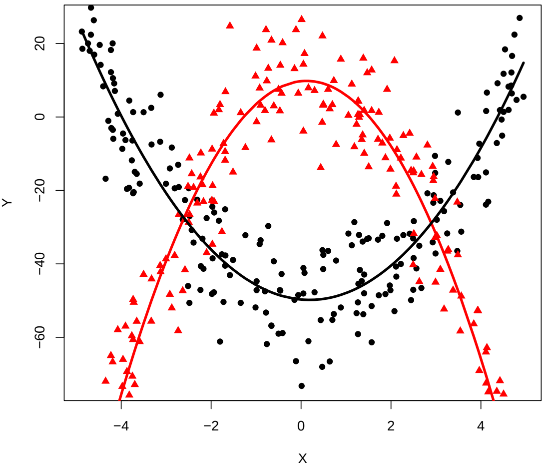
The quadratic case ($j = 2$):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i$$

implies

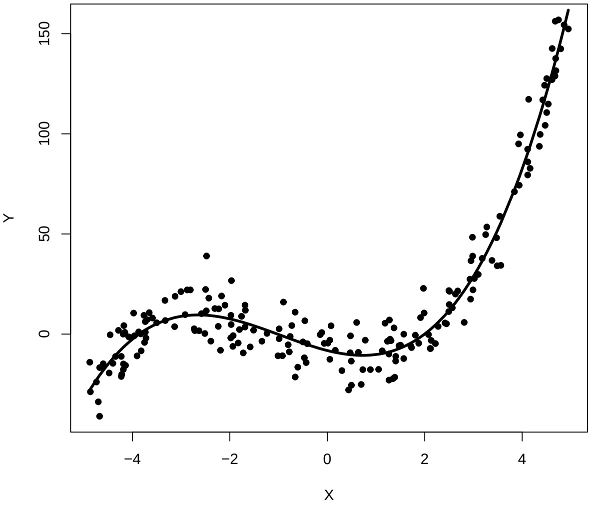
$$\frac{\partial \mathrm{E}(Y)}{\partial X} = \beta_1 + 2\beta_2 X.$$

Two Quadratic Relationships



Note: Red line is $Y_i = 10 + 1X_i - 5X_i^2 + u_i$; black line is $Y_i = -50 - 1X_i + 3X_i^2 + u_i$.

Example of a Cubic Relationship



Note: Solid line is $Y_i = -1 + 1X_i - 8X_i^2 + 5X_i^3 + u_i$.

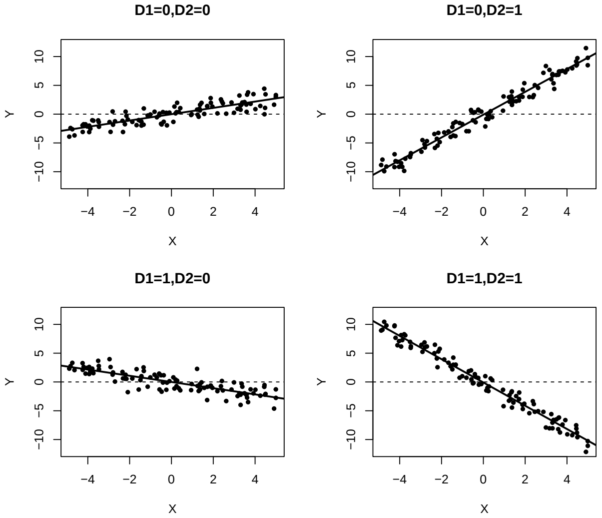
The three-way interaction:

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \\ & \quad \beta_4 X_{1i} X_{2i} + \beta_5 X_{1i} X_{3i} + \beta_6 X_{2i} X_{3i} + \beta_7 X_{1i} X_{2i} X_{3i} + u_i \end{aligned}$$

...is complex; consider the special case of dichotomous X_1, X_2:

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + \beta_2 D_{1i} + \beta_3 D_{2i} + \\ & \quad \beta_4 X_i D_{1i} + \beta_5 X_i D_{2i} + \beta_6 D_{1i} D_{2i} + \beta_7 X_i D_{1i} D_{2i} + u_i \end{aligned}$$

Three-Way Interaction: Two Dummy + One Continuous Covariates



Example: President Clinton's 1996 "Thermometer Score"

Details:

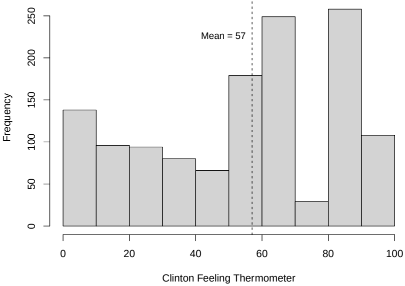
- Source: 1996 American National Election Study
- $N \approx 1300$
- "Feeling Thermometer":

  > *"In the list that follows, rate that person/party using something we call the feeling thermometer. Ratings between 50 degrees and 100 degrees mean that you feel favorable and warm toward the person/party. Ratings between 0 and 50 degrees mean that you don't feel favorable toward the person/party and that you don't care too much for that person. You would rate the person at the 50-degree mark if you don't feel particularly warm or cold toward the person."*

- Ideology measures:
  - Seven-point scale (1 = "very left / liberal," 7 = "very right / conservative")
  - Each respondent places themself + President Clinton
- Party ID: Republican (GOP=1) vs. Democrat/Independent (GOP=0)

Clinton "Feeling Thermometer" Data

Distribution:



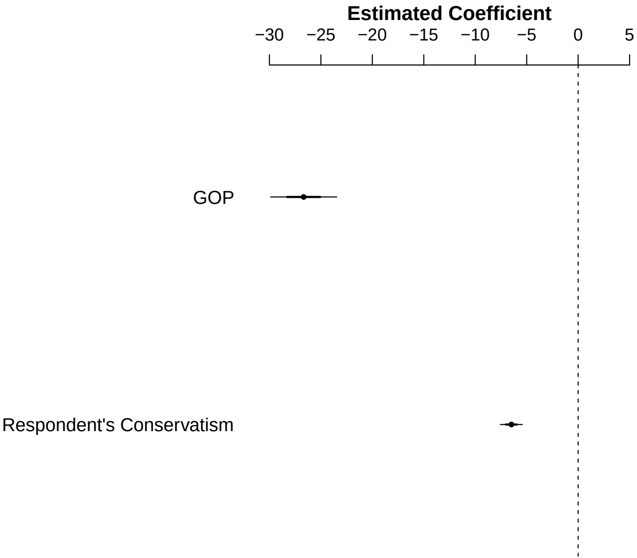
Summary statistics:

```
> describe(ClintonTherm,skew=FALSE)
               vars    n    mean     sd median  min  max range    se
caseid            1 1297 2000.70 718.42   1854 1001 3403  2402 19.95
ClintonTherm      2 1297   57.00  30.00     60    0  100   100  0.83
RConserv          3 1297    4.32   1.39      4    1    7     6  0.04
ClintonConserv    4 1297    2.98   1.37      3    1    7     6  0.04
PID               5 1297    2.06   1.05      2    1    5     4  0.03
GOP               6 1297    0.32   0.47      0    0    1     1  0.01
```

A Basic Regression

```
> fit0<-lm(ClintonTherm~RConserv+GOP,data=CT)
> summary(fit0)

Call:
lm(formula = ClintonTherm ~ RConserv + GOP, data = CT)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   93.476      2.228    42.0   <2e-16 ***
RConserv      -6.487      0.537   -12.1   <2e-16 ***
GOP          -26.670      1.606   -16.6   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 23.6 on 1294 degrees of freedom
Multiple R-squared:  0.38, Adjusted R-squared:  0.379
F-statistic:  396 on 2 and 1294 DF,  p-value: <2e-16
```

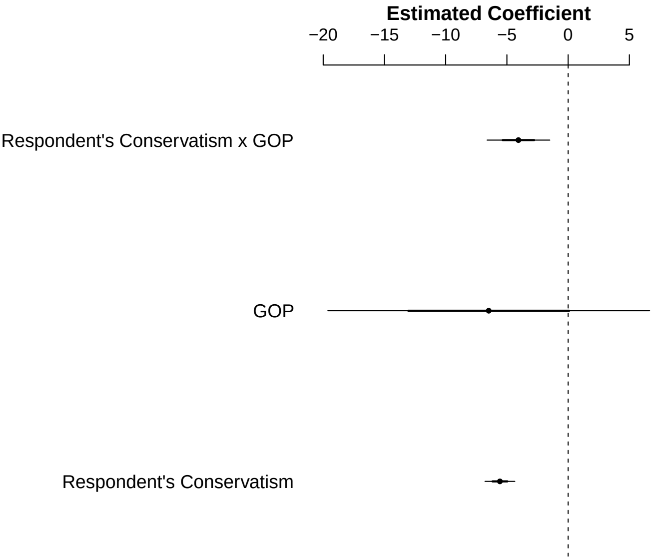
Coefficient Plot: Non-Interactive Model



An Interactive Model

```
> fit1<-lm(ClintonTherm~RConserv+GOP+RConserv*GOP,data=CT)
> summary(fit1)

Call:
lm(formula = ClintonTherm ~ RConserv + GOP + RConserv * GOP,
    data = CT)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    89.927      2.487   36.16   <2e-16 ***
RConserv       -5.571      0.609   -9.15   <2e-16 ***
GOP            -6.484      6.569   -0.99   0.3238
RConserv:GOP   -4.058      1.281   -3.17   0.0016 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 23.6 on 1293 degrees of freedom
Multiple R-squared:  0.384, Adjusted R-squared:  0.383
F-statistic:  269 on 3 and 1293 DF,  p-value: <2e-16
```

Coefficient Plot: Interactive Model



Different ways to specify the (mathematically identical, but sometimes computationally different) interactive model in R :

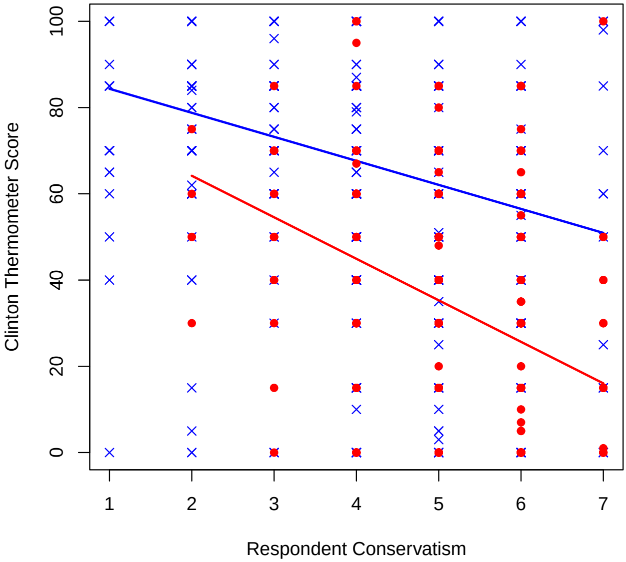
```
> summary(lm(ClintonTherm~RConserv+GOP+RConserv*GOP,data=CT))
> summary(lm(ClintonTherm~RConserv+GOP+RConserv:GOP,data=CT))
> summary(lm(ClintonTherm~RConserv*GOP,data=CT))
> summary(lm(ClintonTherm~(RConserv+GOP)^2,data=CT))
>
> CT$GOPxRC<-CT$GOP * CT$RConserv
> summary(lm(ClintonTherm~RConserv+GOP+GOPxRC,data=CT))
```

Interpretation: Two Regressions, Sort Of

$$\begin{aligned}
\text{E(Thermometer} \mid \text{Non-GOP)}_i &= 89.9 - 6.5(0) - 5.6(\text{R's Conservatism}_i) \\
&\quad -4.0(0 \times \text{R's Conservatism}_i) \\
&= \mathbf{89.9 - 5.6}(\text{R's Conservatism}_\mathbf{i})
\end{aligned}$$

$$\begin{aligned}
\text{E(Thermometer} \mid \text{GOP)}_i &= [89.9 - 6.5(1)] + [-5.6 - 4.0(1 \times \text{R's Conservatism}_i)] \\
&= \mathbf{83.4 - 9.6}(\text{R's Conservatism}_\mathbf{i})
\end{aligned}$$

Thermometer Scores by Conservatism, GOP and Non-GOP



Interactive Results ≈ Separate Regressions

```
> NonReps<-subset(CT,GOP==0)
> Reps<-subset(CT,GOP==1)
> split<-list("Interactive" = lm(ClintonTherm~RConserv*GOP,data=CT),
+             "Non-GOP" = lm(ClintonTherm~RConserv,data=NonReps),
+             "GOP" = lm(ClintonTherm~RConserv,data=Reps))
> modelsummary(split,output="SplitModels-24.tex",
+              gof_omit="DF|Deviance|LogLik|AIC|BIC")
```

Feeling Thermometer Models

	Interactive	Non-GOP	GOP
(Intercept)	89.927	89.927	83.443
	(2.487)	(2.469)	(6.170)
RConserv	−5.571	−5.571	−9.629
	(0.609)	(0.604)	(1.144)
GOP	−6.484		
	(6.569)		
RConserv × GOP	−4.058		
	(1.281)		
Num.Obs.	1297	887	410
R2	0.384	0.088	0.148
R2 Adj.	0.383	0.087	0.146
F	269.011	84.961	70.878
RMSE	23.53	23.38	23.86

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

For `RConserv`:

$$\begin{aligned} \text{Clinton Thermometer}_i &= \beta_0 + (\beta_1 + \beta_3 \text{GOP}_i)\text{R's Conservatism}_i + \\ &\quad \beta_2 \text{GOP}_i + u_i \\ &= \beta_0 + \psi_{1i}\text{R's Conservatism}_i + \beta_2 \text{GOP}_i + u_i. \end{aligned}$$

So:

$$\hat{\psi}_{1i} = \hat{\beta}_1 + \hat{\beta}_3 \times \text{GOP}_i$$

and

$$\hat{\sigma}_{\psi_1} = \sqrt{\widehat{\text{Var}(\hat{\beta}_1)} + (\text{GOP})^2 \widehat{\text{Var}(\hat{\beta}_3)} + 2(\text{GOP})\widehat{\text{Cov}(\hat{\beta}_1, \hat{\beta}_3)}}.$$

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

For `GOP`:

$$\begin{aligned}
\text{Clinton Thermometer}_i &= \beta_0 + (\beta_2 + \beta_3 \times \text{R's Conservatism}_i)\text{GOP}_i + \\
&\quad \beta_1(\text{R's Conservatism}_i) + u_i \\
&= \beta_0 + \psi_{2i}\text{GOP}_i + \beta_1(\text{R's Conservatism}_i) + u_i.
\end{aligned}$$

So:

$$\hat{\psi}_{2i} = \hat{\beta}_2 + \hat{\beta}_3 \times (\text{R's Conservatism}_i).$$

and

$$\hat{\sigma}_{\psi_2} = \sqrt{\widehat{\text{Var}(\hat{\beta}_2)} + (\text{R's Conservatism}_i)^2 \widehat{\text{Var}(\hat{\beta}_3)} + 2k\widehat{\text{Cov}(\hat{\beta}_2, \hat{\beta}_3)}}.$$

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

```
> Psi1<-fit1$coeff[2]+fit1$coeff[4]

> Psi1
 RConserv
-9.628577

> SPsi1<-sqrt(vcov(fit1)[2,2] + (1)^2*vcov(fit1)[4,4] + 2*1*vcov(fit1)[2,4])
> SPsi1
[1] 1.127016

> Psi1 / SPsi1 # <-- t-statistic
 RConserv
-8.543422
```

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

```
> # psi_2 | RConserv = 1
> fit1$coeff[3]+(1 * fit1$coeff[4])
      GOP
-10.54208

> sqrt(vcov(fit1)[3,3] + (1)^2*vcov(fit1)[4,4] + 2*1*vcov(fit1)[3,4])
[1] 5.335847

# Implies t is approximately 2



> # psi_2 | RConserv = 7
> fit1$coeff[3]+(7 * fit1$coeff[4])
      GOP
-34.89045

> sqrt(vcov(fit1)[3,3] + (7)^2*vcov(fit1)[4,4] + 2*7*vcov(fit1)[3,4])
[1] 3.048302

# t is approximately 11
```

An Easier Way: `linearHypothesis()`

```
> library(car)
> linearHypothesis(fit1,"RConserv+RConserv:GOP")
Linear hypothesis test

Hypothesis:
RConserv  + RConserv:GOP = 0

Model 1: restricted model
Model 2: ClintonTherm ~ RConserv + GOP + RConserv * GOP

  Res.Df    RSS Df Sum of Sq  F Pr(>F)
1   1294 758714
2   1293 718173  1     40541 73 <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
>
> # Note: Same as t-test:
> sqrt(73)
[1] 8.544
```

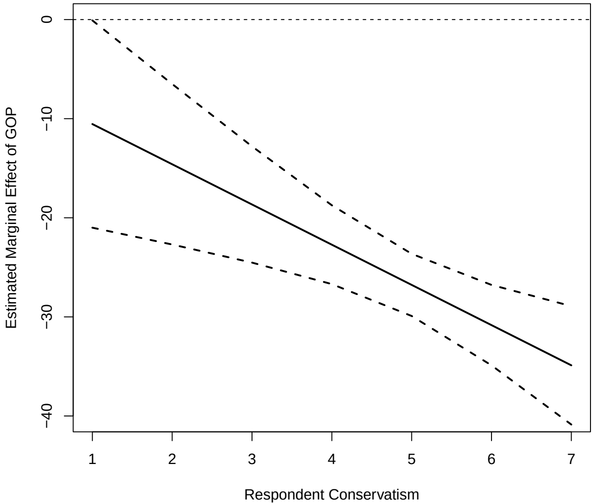
An Easier Way: `linearHypothesis()`

```
> # psi_2 | RConserv = 7:
> linearHypothesis(fit1,"GOP+7*RConserv:GOP")
Linear hypothesis test

Hypothesis:
GOP  + 7 RConserv:GOP = 0

Model 1: restricted model
Model 2: ClintonTherm ~ RConserv + GOP + RConserv * GOP

  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1   1294 790938
2   1293 718173  1     72766 131.01 < 2.2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Marginal Effects Plots, I

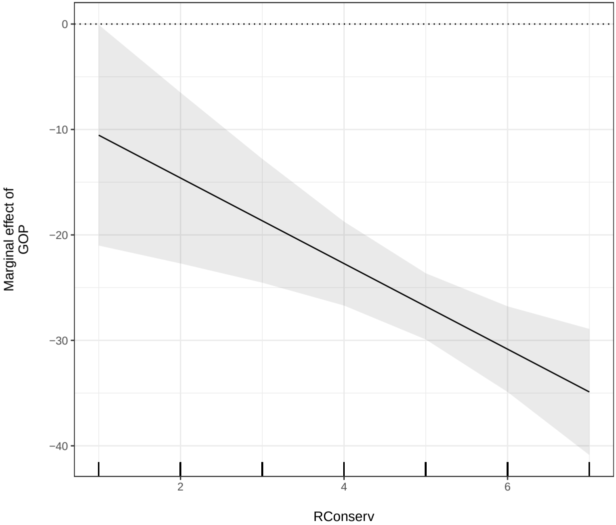
```
> ConsSim<-seq(1,7,1)
> psis<-fit1$coeff[3]+(ConsSim * fit1$coeff[4])
> psis.ses<-sqrt(vcov(fit1)[3,3] +
  (ConsSim)^2*vcov(fit1)[4,4] + 2*ConsSim*vcov(fit1)[3,4])

> plot(ConsSim,psis,t="l",lwd=2,xlab="Respondent Conservatism",
  ylab="Estimated Marginal Effect",ylim=c(-40,0))
> lines(ConsSim,psis+(1.96*psis.ses),lty=2,lwd=2)
> lines(ConsSim,psis-(1.96*psis.ses),lty=2,lwd=2)
> abline(h=0,lwd=1,lty=2)
```

The Plot...



0
−10
−20
−30
−40
Estimated Marginal Effect of GOP
1
2
3
4
5
6
7
Respondent Conservatism

Same, Using `plot_me` (`plotMElm` package)



Interacting Two Continuous Covariates

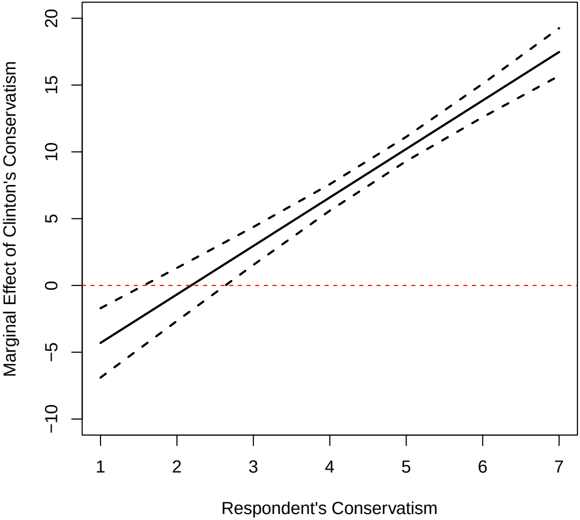
```
> fit2<-lm(ClintonTherm~RConserv+ClintonConserv+RConserv*ClintonConserv,data=CT)
> summary(fit2)

Call:
lm(formula = ClintonTherm ~ RConserv + ClintonConserv + RConserv *
    ClintonConserv, data = CT)

Residuals:
   Min     1Q Median     3Q    Max
-95.48 -13.48   0.48  15.48 100.15

Coefficients:
                        Estimate Std. Error t value  Pr(>|t|)
(Intercept)              119.351      5.163   23.11   < 2e-16 ***
RConserv                 -19.567      1.036  -18.88   < 2e-16 ***
ClintonConserv            -7.931      1.648   -4.81 0.0000017 ***
RConserv:ClintonConserv    3.629      0.339   10.69   < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 22 on 1293 degrees of freedom
Multiple R-squared:  0.462, Adjusted R-squared:  0.461
F-statistic:  370 on 3 and 1293 DF,  p-value: <2e-16
```

Hypothesis Tests ("by hand")

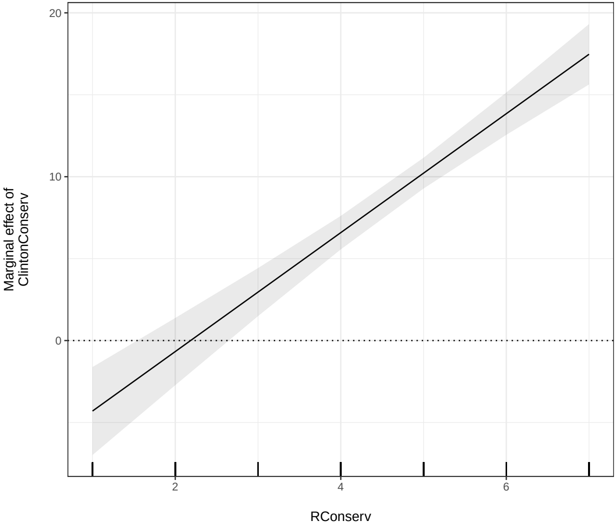
```
> psi<-fit2$coef[2]+(1*fit2$coef[4])
> psi
RConserv
  -15.94

> varpsi<-sqrt(vcov(fit2)[2,2] + (1)^2*vcov(fit2)[4,4] + 2*1*vcov(fit2)[2,4])
> varpsi
[1] 0.744

> psi/varpsi # t-test
RConserv
  -21.42

> linearHypothesis(fit2,"RConserv+1*RConserv:ClintonConserv")
Linear hypothesis test

Hypothesis:
RConserv  + RConserv:ClintonConserv = 0

Model 1: restricted model
Model 2: ClintonTherm ~ RConserv + ClintonConserv + RConserv * ClintonConserv

  Res.Df    RSS Df Sum of Sq   F Pr(>F)
1   1294 850442
2   1293 627658  1    222784 459 <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> (psi/varpsi)^2
RConserv
   458.9
```

Marginal Effect Plot, II

```
> psis2<-fit2$coef[3]+(ConsSim*fit2$coef[4])
> psis2.ses<-sqrt(vcov(fit2)[3,3] + (ConsSim)^2*vcov(fit2)[4,4]
  + 2*ConsSim*vcov(fit2)[3,4])

> plot(ConsSim,psis2,t="l",lwd=2,xlab="Respondent's
  Conservatism",ylab="Marginal Effect of Clinton's
  Conservatism",ylim=c(-10,20))
> lines(ConsSim,psis2+(1.96*psis2.ses),lty=2,lwd=2)
> lines(ConsSim,psis2-(1.96*psis2.ses),lty=2,lwd=2)
> abline(h=0,lty=2,lwd=1,col="red")
```

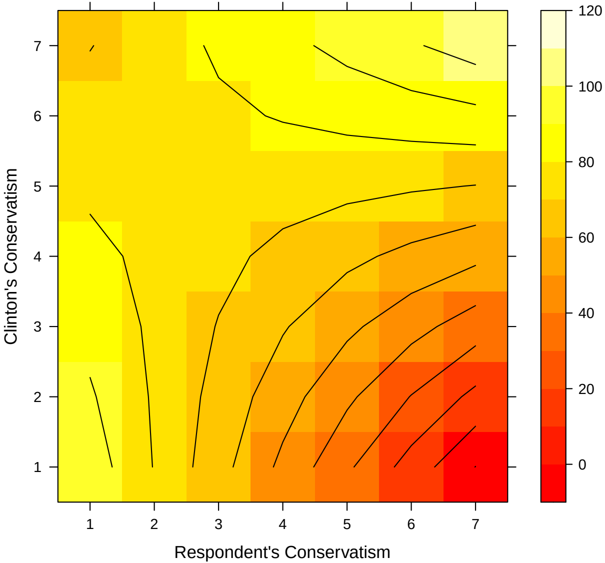

Marginal Effect of Clinton's Conservatism
Respondent's Conservatism
−10
−5
0
5
10
15
20
1
2
3
4
5
6
7

Same, Using `plot_me`



Predicted Values: A Contour Plot

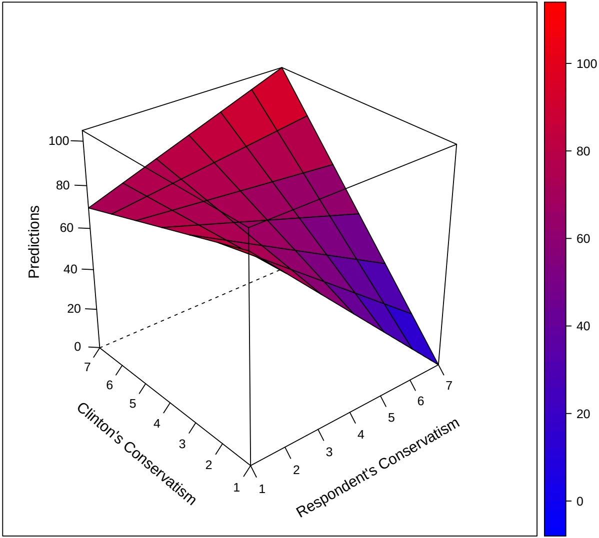
```
> library(lattice)
> grid<-expand.grid(RConserv=seq(1,7,1),
+       ClintonConserv=seq(1,7,1))
> hats<-predict(fit2,newdata=grid)

> levelplot(hats~grid$RConserv*grid$ClintonConserv,
+             contour=TRUE,cuts=12,pretty=TRUE,
+             xlab="Respondent's Conservatism",
+             ylab="Clinton's Conservatism",
+             col.regions=heat.colors)
```



Clinton's Conservatism
Respondent's Conservatism
1
2
3
4
5
6
7
120
100
80
60
40
20
0

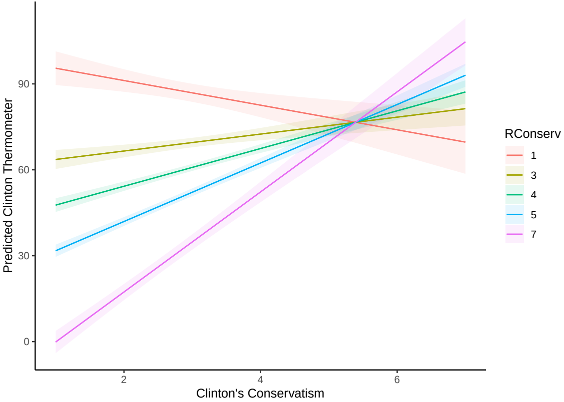
Predicted Values: A Wireframe Plot

```
> trellis.par.set("axis.line",list(col="transparent"))

> wireframe(hats~grid$RConserv*grid$ClintonConserv,
  drape=TRUE,
  xlab=list("Respondent's Conservatism",rot=30),
  ylab=list("Clinton's Conservatism",
  rot=-40),zlab=list("Predictions",rot=90),
  scales=list(arrows=FALSE,col="black"),
  zoom=0.85,pretty=TRUE),
  col.regions=colorRampPalette(c("blue","red"))(100))
```


Predictions
0
20
40
60
80
100
Clinton's Conservatism
1
2
3
4
5
6
7
Respondent's Conservatism
1
2
3
4
5
6
7
0
20
40
60
80
100

Predictions Using `marginaleffects`

```
> library(marginaleffects)
> p <- plot_predictions(fit2,condition=c("ClintonConserv","RConserv")) +
      theme_classic() +
      labs(x = "Clinton's Conservatism",
           y = "Predicted Clinton Thermometer")
> p
```



Interpreting Interactions: Other Tools

A partial list of packages and things:

- All the various model summary and plotting tools in Vincent Arel-Bundock's `modelsummary` and `marginaleffects` packages
- `interactions` (vignette)
- `emmeans` (see, e.g., this vignette)
- `plot_model` in `sjPlot` (tutorial)
- `interactionR`
- `interplot`
- `InteractionPoweR` (power analyses for interactive models)

Variable Transformations

- Induce / conform to linearity
- Induce / conform to additivity
- Induce normality in the u_is
- Facilitate interpretation
- **Make the model fit the theory**

This:

$$Y_i = \beta_0 X_i^{\beta_1} u_i$$

becomes this:

$$\ln(Y_i) = \ln(\beta_0) + \beta_1 X_i + \ln(u_i)$$

And this:

$$\exp(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

becomes this:

$$Y_i = \ln(\beta_0) + \beta_1 \ln(X_i) + \ln(u_i)$$

The "Ladder of Powers":

Transformation	p	$f(X)$	Fox's $f(X)$
Cube	3	X^3	$\frac{X^3-1}{3}$
Square	2	X^2	$\frac{X^2-1}{2}$
(None/Identity)	(1)	(X)	(X)
Square Root	$\frac{1}{2}$	$\sqrt{X}$	$2(\sqrt{X}-1)$
Cube Root	$\frac{1}{3}$	$\sqrt[3]{X}$	$3(\sqrt[3]{X}-1)$
Log	0 (sort of)	$\ln(X)$	$\ln(X)$
Inverse Cube Root	-$\frac{1}{3}$	$\frac{1}{\sqrt[3]{X}}$	$\frac{\left(\frac{1}{\sqrt[3]{X}}-1\right)}{-\frac{1}{3}}$
Inverse Square Root	-$\frac{1}{2}$	$\frac{1}{\sqrt{X}}$	$\frac{\left(\frac{1}{\sqrt{X}}-1\right)}{-\frac{1}{2}}$
Inverse	-1	$\frac{1}{X}$	$\frac{\left(\frac{1}{X}-1\right)}{-1}$
Inverse Square	-2	$\frac{1}{X^2}$	$\frac{\left(\frac{1}{X^2}-1\right)}{-2}$
Inverse Cube	-3	$\frac{1}{X^3}$	$\frac{\left(\frac{1}{X^3}-1\right)}{-3}$

A General Rule

Using higher-order power transformations (e.g., squares, cubes, etc.) “inflates” large values and “compresses” small ones; conversely, using lower-order power transformations (logs, etc.) “compresses” large values and “inflates” (or “expands”) smaller ones.

Power Transformations: Two Issues

1. X must be *positive*; so:

$$X^* = X + (|X_\ell| + \epsilon)$$

with (CZ's Rule of Thumb):

$$\epsilon = \frac{X_{\ell+1} - X_\ell}{2}$$

2. Power transformations generally require that:

$$\frac{X_h}{X_l} > 5 \text{ (or so)}$$

A Note On Logarithms

Note that:

$$\ln(X|X \leq 0) \text{ is undefined.}$$

For $X = 0$, we might:

1. exclude observations,
2. add some arbitrary amount (perhaps 1.0) to *all observations*
3. add some arbitrary amount (perhaps 1.0) to *observations where* $X = 0$
4. add some arbitrary amount (perhaps 1.0) to *observations where* $X = 0$, and include a variable D_i in your regression, where:

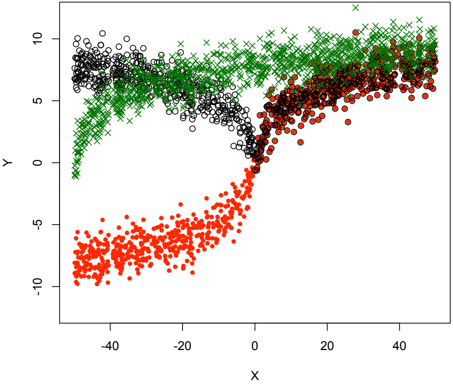
$$D_i = \begin{cases} 1 \text{ if } X_i = 0 \\ 0 \text{ otherwise} \end{cases}$$

The short answer: **Do #4.** Find out more at this poster.

A Note On Logarithms (continued)

For $X < 0$, we should think about how we expect X and Y to covary when $X < 0$:

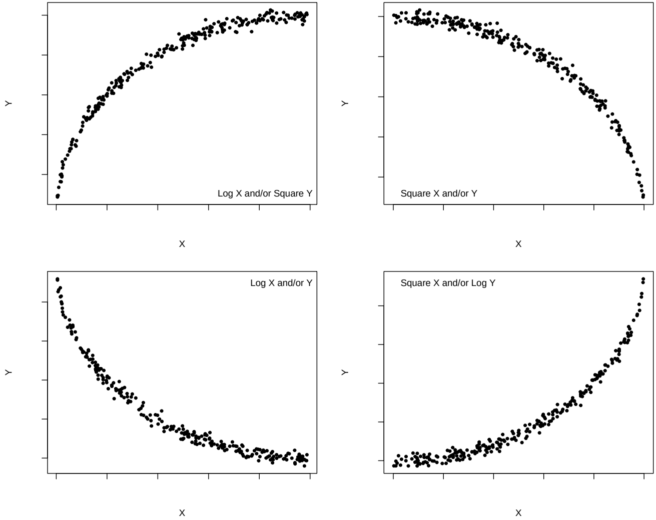
1. a "shift", where the logarithmic form starts at values of X less than zero,
2. a "V-curve," where $E(Y|X = k) = E(Y|X = -k)$, or
3. an "S-curve," where the $X - Y$ relationship for $X < 0$ "mirrors" that for $X > 0$ [so $E(Y|X = k) = -E(Y|X = -k)$]



Which is correct? **It depends on your theory.** Again: find out more at this poster.

Which Transformation?

Mosteller and Tukey's "Bulging Rule":



Transformed Xs: Interpretation

For:

$$\ln(Y_i) = \beta_0 + \beta_1 X_i + u_i,$$

then:

$$\mathrm{E}(Y) = \exp(\beta_0 + \beta_1 X_i)$$

and so:

$$\frac{\partial \mathrm{E}(Y)}{\partial X} = \exp(\beta_1).$$

Similarly, for:

$$Y_i = \beta_0 + \beta_1 \ln(X_i) + u_i$$

we have:

$$\frac{\partial \mathrm{E}(Y)}{\partial \ln(X)} = \beta_1.$$

So doubling X (say, from X_ℓ to $2X_\ell$):

$$\begin{aligned} \Delta \mathrm{E}(Y) &= \mathrm{E}(Y|X = 2X_\ell) - \mathrm{E}(Y|X = X_\ell) \\ &= [\beta_0 + \beta_1 \ln(2X_\ell)] - [\beta_0 + \beta_1 \ln(X_\ell)] \\ &= \beta_1[\ln(2X_\ell) - \ln(X_\ell)] \\ &= \beta_1 \ln(2) \end{aligned}$$

Specifying:

$$\ln(Y_i) = \beta_0 + \beta_1 \ln(X_i) + ... + u_i$$

means:

$$\text{Elasticity}_{YX} \equiv \frac{\%\Delta Y}{\%\Delta X} = \beta_1.$$

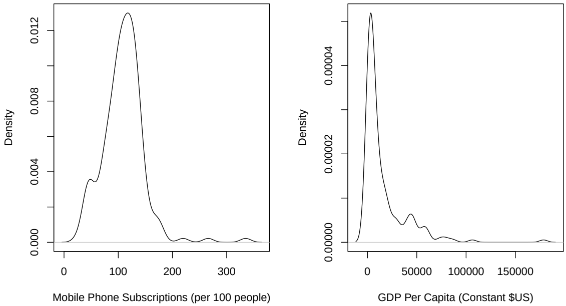
In other words, a one-percent change in X leads to a $\hat{\beta}_1$-percent change in Y.

An Example: Cell Phones and Wealth

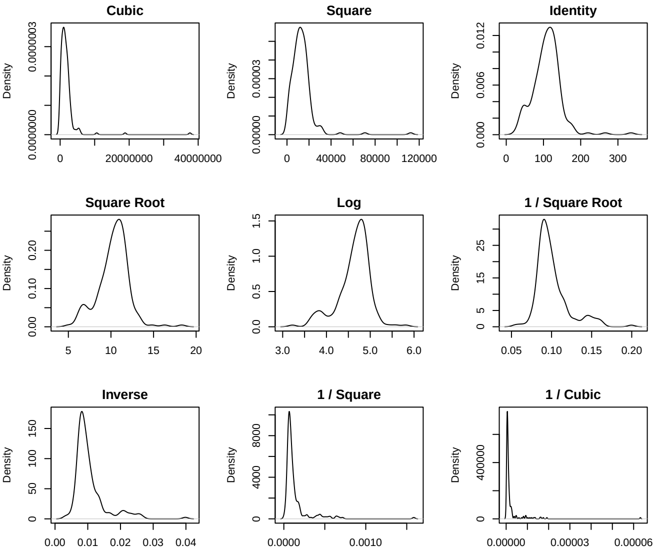
Data are a tiny subset of the World Development Indicators (2018 *only*)...

- `ISO3` - The country's ISO3 (alphanumeric) code.
- `country` - The name of the country.
- `GDPPerCapita` - GDP per capita (constant 2010 $US)
- `MobileCellSubscriptions` - Mobile / cellular subscriptions per 100 people

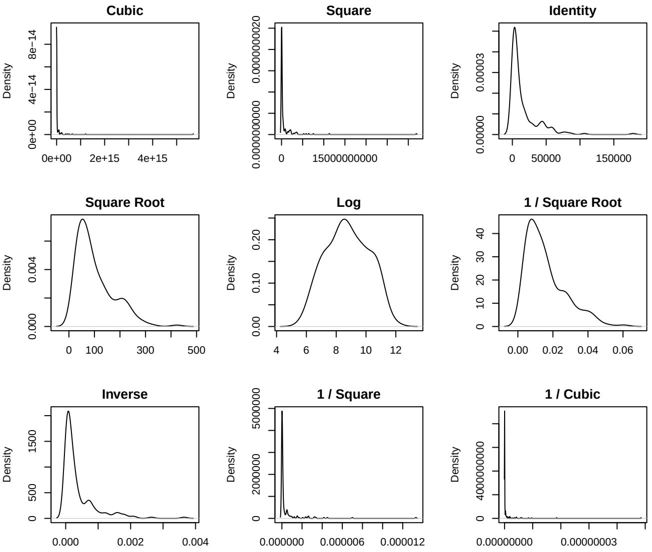
```
> describe(WDI)
                        vars   n    mean       sd median trimmed     mad    min      max  range skew kurtosis      se
ISO3*                      1 186    93.5    53.84   93.5    93.5   68.94   1.00    186.0    185 0.00    -1.22    3.95
Country*                   2 186    93.5    53.84   93.5    93.5   68.94   1.00    186.0    185 0.00    -1.22    3.95
GDPPerCapita               3 186 15769.2 22904.65 6257.3 11146.0 7358.36 274.13 178571.4 178297 3.05    14.09 1679.45
MobileCellSubscriptions    4 186   109.0    38.01  110.1   108.4   30.00  25.11    335.1    310 1.32     7.02    2.79
```



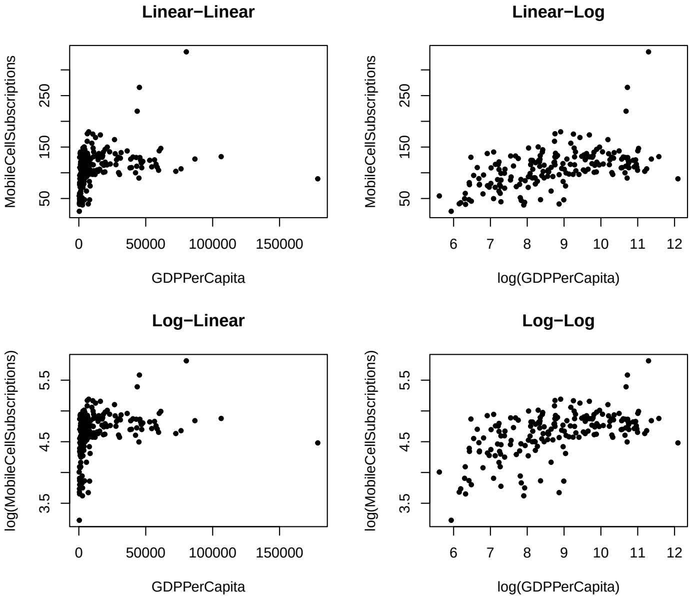
“Ladder of Powers”: Mobile Subscriptions



"Ladder of Powers": Wealth / GDP



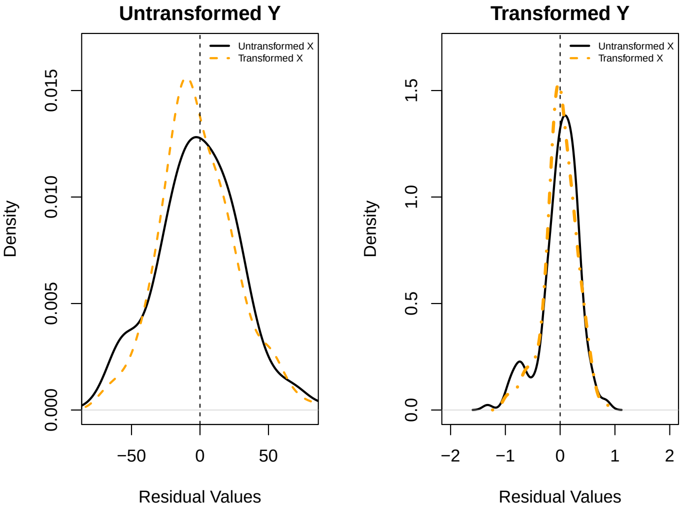
Scatterplots



Models with and Without Transformations

	Linear Y & X	Linear Y, Log X	Log Y, Linear X	Log Y and X
(Intercept)	100.235	82.944	4.546	4.357
	(3.199)	(3.904)	(0.032)	(0.037)
GDP Per Capita (1000s)	0.557		0.005	
	(0.115)		(0.001)	
Logged GDP Per Capita (1000s)		14.047		0.146
		(1.669)		(0.016)
Num.Obs.	186	186	186	186
R2	0.113	0.278	0.103	0.314
R2 Adj.	0.108	0.274	0.098	0.310
F	23.370	70.830	21.142	84.305
RMSE	35.71	32.22	0.35	0.31

Density Plots of $\hat{u}_i$s



(One) simple solution: Polynomials...

- First-order / linear ($P = 1$):

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- Second-order / quadratic ($P = 2$):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i$$

- Third-order / cubic ($P = 3$):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + u_i$$

- ...pth-order ($P = p$):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + ... + \beta_p X_i^p + u_i$$

Read coefficients "left to right." So, for the quadratic:

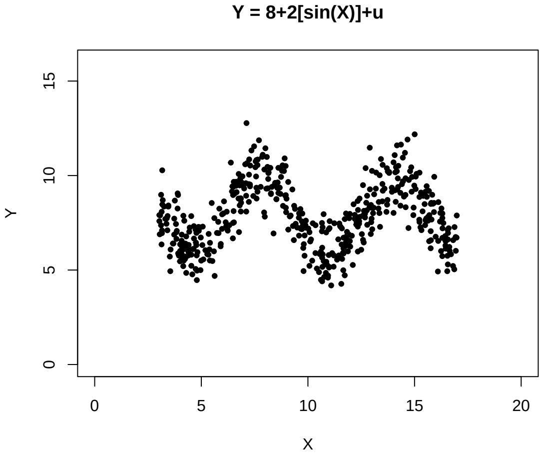
$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i$$

then:

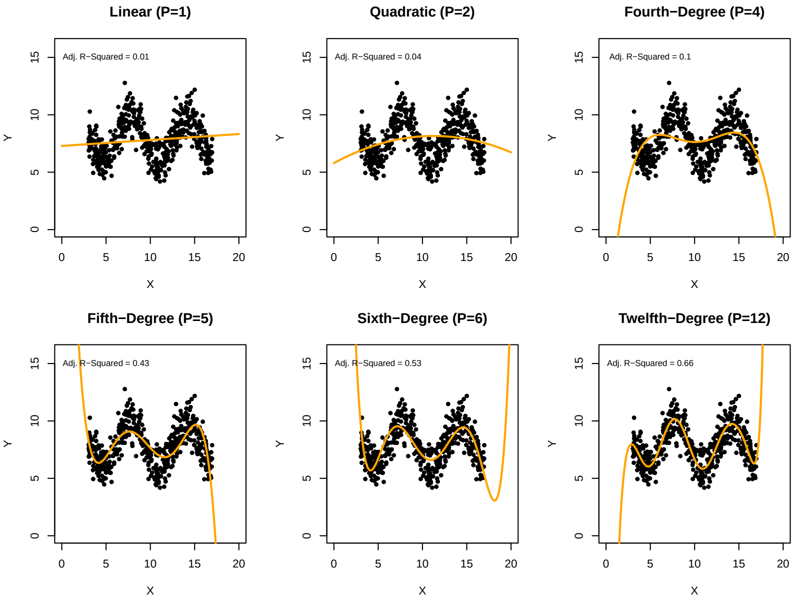
	$\hat{\beta}_2$		
$\hat{\beta}_1$	< 0	$= 0$	> 0
< 0	E(Y) decreases in X at an increasing rate	E(Y) decreases linearly in X	E(Y) decreases in X at low values of X, but increases in X at high values of X
$= 0$	E(Y) decreases in X^2	E(Y) is (quadratically) unrelated to X	E(Y) increases in X^2
> 0	E(Y) increases in X at low values of X, but decreases in X at high values of X	E(Y) increases linearly in X	E(Y) increases in X at an increasing rate

Polynomials: Simulated Example

```
> N<-500
> set.seed(7222009)
> X<-runif(N,3,17)
> Y<-8+2*sin(X)+rnorm(N)
```



Some Polynomial Regressions



Check out the $P = 12$ regression:

```
> summary(R.12)

Call:
lm(formula = Y ~ X + I(X^2) + I(X^3) + I(X^4) + I(X^5) + I(X^6) +
    I(X^7) + I(X^8) + I(X^9) + I(X^10) + I(X^11) + I(X^12))

Coefficients: (1 not defined because of singularities)
                  Estimate      Std. Error t value Pr(>|t|)
(Intercept)  -92.9216900503  816.2489413571   -0.11     0.91
X            149.4055086103 1212.1158531764    0.12     0.90
I(X^2)      -104.6855472388  788.3527710495   -0.13     0.89
I(X^3)        46.9192208616  296.7267961681    0.16     0.87
I(X^4)       -14.5570311719   71.9111574249   -0.20     0.84
I(X^5)         3.1175787785   11.8013895994    0.26     0.79
I(X^6)        -0.4537511003    1.3406553442   -0.34     0.74
I(X^7)         0.0442854569    0.1056218156    0.42     0.68
I(X^8)        -0.0028398562    0.0056659448   -0.50     0.62
I(X^9)         0.0001145568    0.0001974535    0.58     0.56
I(X^10)       -0.0000026340    0.0000040301   -0.65     0.51
I(X^11)        0.0000000263    0.0000000366    0.72     0.47
I(X^12)                  NA              NA      NA       NA

Residual standard error: 0.986 on 488 degrees of freedom
Multiple R-squared:  0.669, Adjusted R-squared:  0.662
F-statistic: 89.7 on 11 and 488 DF,  p-value: <2e-16
```

“Raw” vs. Orthogonal Polynomials (continued)

What’s going on?

- The “raw” polynomial terms are (often strongly) correlated with each other...

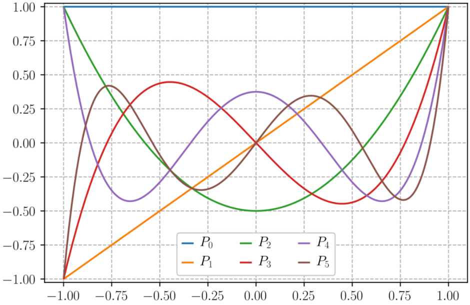
```
> cor(X,I(X^2))
[1] 0.984
```

- $\rightarrow$ large standard errors / imprecision in the estimates
- Can also lead to numerical instability in estimation...

Orthogonal Polynomials

An alternative is to use *orthogonal polynomials*...

- Think of these as orthogonal (uncorrelated) versions of the polynomials above
- There are many of them; probably the most commonly-used are the Legendre polynomials:



- The math is a bit complex; the R command is `poly()`

"Raw" polynomials using poly():

```
> P.12R<-lm(Y~poly(X,degree=12,raw=TRUE))
> summary(P.12R)

Call:
lm(formula = Y ~ poly(X, degree = 12, raw = TRUE))

Residuals:
    Min      1Q  Median      3Q     Max
-2.8041 -0.6935 -0.0245  0.6881  3.0645

Coefficients: (1 not defined because of singularities)
                                      Estimate      Std. Error t value Pr(>|t|)
(Intercept)                        -92.9216900503  816.2489413571  -0.11     0.91
poly(X, degree = 12, raw = TRUE)1  149.4055086103 1212.1158531764   0.12     0.90
poly(X, degree = 12, raw = TRUE)2 -104.6855472388  788.3527710495  -0.13     0.89
poly(X, degree = 12, raw = TRUE)3   46.9192208616  296.7267961681   0.16     0.87
poly(X, degree = 12, raw = TRUE)4  -14.5570311719   71.9111574249  -0.20     0.84
poly(X, degree = 12, raw = TRUE)5    3.1175787785   11.8013895994   0.26     0.79
poly(X, degree = 12, raw = TRUE)6   -0.4537511003    1.3406553442  -0.34     0.74
poly(X, degree = 12, raw = TRUE)7    0.0442854569    0.1056218156   0.42     0.68
poly(X, degree = 12, raw = TRUE)8   -0.0028398562    0.0056659448  -0.50     0.62
poly(X, degree = 12, raw = TRUE)9    0.0001145568    0.0001974535   0.58     0.56
poly(X, degree = 12, raw = TRUE)10  -0.0000026340    0.0000040301  -0.65     0.51
poly(X, degree = 12, raw = TRUE)11   0.0000000263    0.0000000366   0.72     0.47
poly(X, degree = 12, raw = TRUE)12             NA              NA      NA       NA

Residual standard error: 0.986 on 488 degrees of freedom
Multiple R-squared:  0.669, Adjusted R-squared:  0.662
F-statistic: 89.7 on 11 and 488 DF,  p-value: <2e-16
```

Orthogonal polynomials:

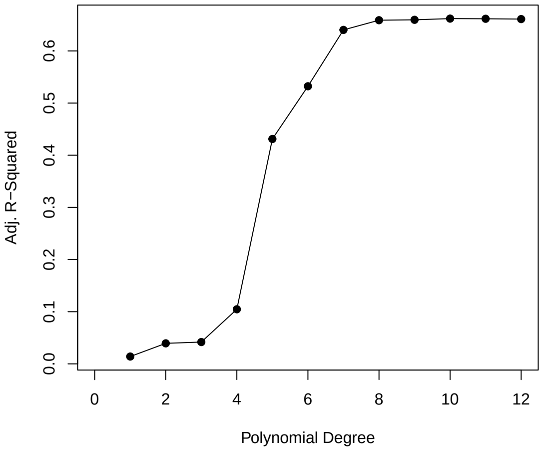
```
> P.12<-lm(Y~poly(X,degree=12))
> summary(P.12)

Call:
lm(formula = Y ~ poly(X, degree = 12))

Residuals:
    Min      1Q  Median      3Q     Max
-2.8255 -0.6953 -0.0286  0.6894  3.0733

Coefficients:
                       Estimate Std. Error t value      Pr(>|t|)
(Intercept)              7.7989     0.0441  176.80       < 2e-16 ***
poly(X, degree = 12)1    4.7901     0.9863    4.86 0.00000161352 ***
poly(X, degree = 12)2   -6.2379     0.9863   -6.32 0.00000000058 ***
poly(X, degree = 12)3    2.5039     0.9863    2.54         0.011 *
poly(X, degree = 12)4   -9.5937     0.9863   -9.73       < 2e-16 ***
poly(X, degree = 12)5  -21.5763     0.9863  -21.87       < 2e-16 ***
poly(X, degree = 12)6   12.0295     0.9863   12.20       < 2e-16 ***
poly(X, degree = 12)7   12.4067     0.9863   12.58       < 2e-16 ***
poly(X, degree = 12)8   -5.2176     0.9863   -5.29 0.00000018541 ***
poly(X, degree = 12)9   -1.4389     0.9863   -1.46         0.145
poly(X, degree = 12)10   2.0529     0.9863    2.08         0.038 *
poly(X, degree = 12)11   0.7097     0.9863    0.72         0.472
poly(X, degree = 12)12  -0.3987     0.9863   -0.40         0.686
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.986 on 487 degrees of freedom
Multiple R-squared:  0.669, Adjusted R-squared:  0.661
F-statistic: 82.1 on 12 and 487 DF,  p-value: <2e-16
```

What Degree Polynomial?

```
> for(degree in 1:12) {
+   fit <- lm(Y~poly(X,degree))
+   assign(paste("P", degree, sep = "."), fit)
+ }
> anova(P.1,P.2,P.3,P.4,P.5,P.6,P.7,P.8,P.9,P.10,P.11,P.12)
Analysis of Variance Table

Model  1: Y ~ poly(X, degree)
Model  2: Y ~ poly(X, degree)
Model  3: Y ~ poly(X, degree)
Model  4: Y ~ poly(X, degree)
Model  5: Y ~ poly(X, degree)
Model  6: Y ~ poly(X, degree)
Model  7: Y ~ poly(X, degree)
Model  8: Y ~ poly(X, degree)
Model  9: Y ~ poly(X, degree)
Model 10: Y ~ poly(X, degree)
Model 11: Y ~ poly(X, degree)
Model 12: Y ~ poly(X, degree)
   Res.Df  RSS Df Sum of Sq      F        Pr(>F)
1     498 1409
2     497 1370  1        39  40.00 0.00000000058 ***
3     496 1364  1         6   6.44         0.011 *
4     495 1272  1        92  94.60       < 2e-16 ***
5     494  807  1       466 478.51       < 2e-16 ***
6     493  662  1       145 148.74       < 2e-16 ***
7     492  508  1       154 158.22       < 2e-16 ***
8     491  481  1        27  27.98 0.00000018541 ***
9     490  479  1         2   2.13         0.145
10    489  474  1         4   4.33         0.038 *
11    488  474  1         1   0.52         0.472
12    487  474  1         0   0.16         0.686
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

What Degree Polynomial?

Plotting R^2_{adj} for different polynomial degrees...



Polynomial Tips

Good things...

- Polynomials are flexible functional forms for nonlinear marginal associations
- They are also easy to fit, and easily interpretable

Cautions...

- Polynomials can be prone to overfitting, which...
- ...can lead to poor out-of-sample generalizability / predictive power
- This is especially true outside the observed values of the data (extrapolation)

- **Theory is valuable**.
- **Try different things**.
- **Look at plots**.
- **It takes practice**.