GSERM 2024
Regression for Publishing

June 17, 2024

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- Class: June 17-21, 2024, 09:15 - 15:15 CET, at the University of St. Gallen SQUARE building, room 11-0101, "Gallus."

- The course outline / syllabus is here.

- More important: The syllabus, slides, readings, code, data, etc. are all available on the course github repo (viewable at https://github.com/PrisonRodeo/GSERM-RFP-2024).

Assessment

Evaluation at GSERM isn't easy... the plan:

- One "homework exercise"
  - Practical exercise – "real" data analysis and discussion
  - Assigned Tuesday (June 18); due Friday (June 21)
  - Worth 300 possible points
- Final Examination
  - Multiple essay-style questions + "real" data analysis
  - Some choice of questions to answer
  - Assigned Friday (June 23)
  - Due either Friday, June 21, 2024 ("in-class" alternative) or Friday, June 28, 2024 ("take-home" alternative)
  - Worth 700 possible points
- Total course = 1000 possible points
- Grades assessed on Swiss (1 - 6) scale

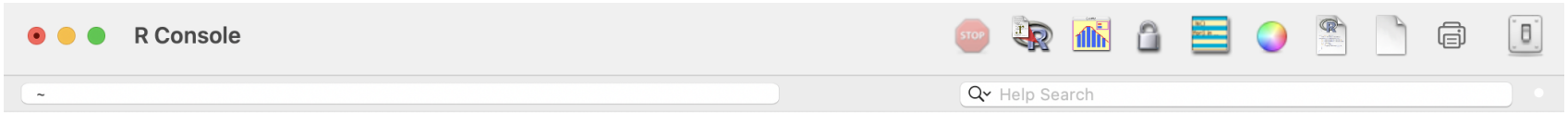
Software

R

- All examples, plots, etc. are generated using R
- Current version is 4.4.0
- Desktop: Be sure to get the RStudio / Posit IDE...
- Alternatively: Can be run in a browser, using Posit Cloud
- The course Github repo contains a bit of introductory code for people who may never have used R, and a list of R resources.

Stata

- Current version is 18
- Main linear regression command is `-regress-` (or `reg`)




```
R version 4.4.0 (2024-04-24) -- "Puppy Cup"
Copyright (C) 2024 The R Foundation for Statistical Computing
Platform: aarch64-apple-darwin20

R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.

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R is a collaborative project with many contributors.
Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[R.app GUI 1.80 (8376) aarch64-apple-darwin20]

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> 
```
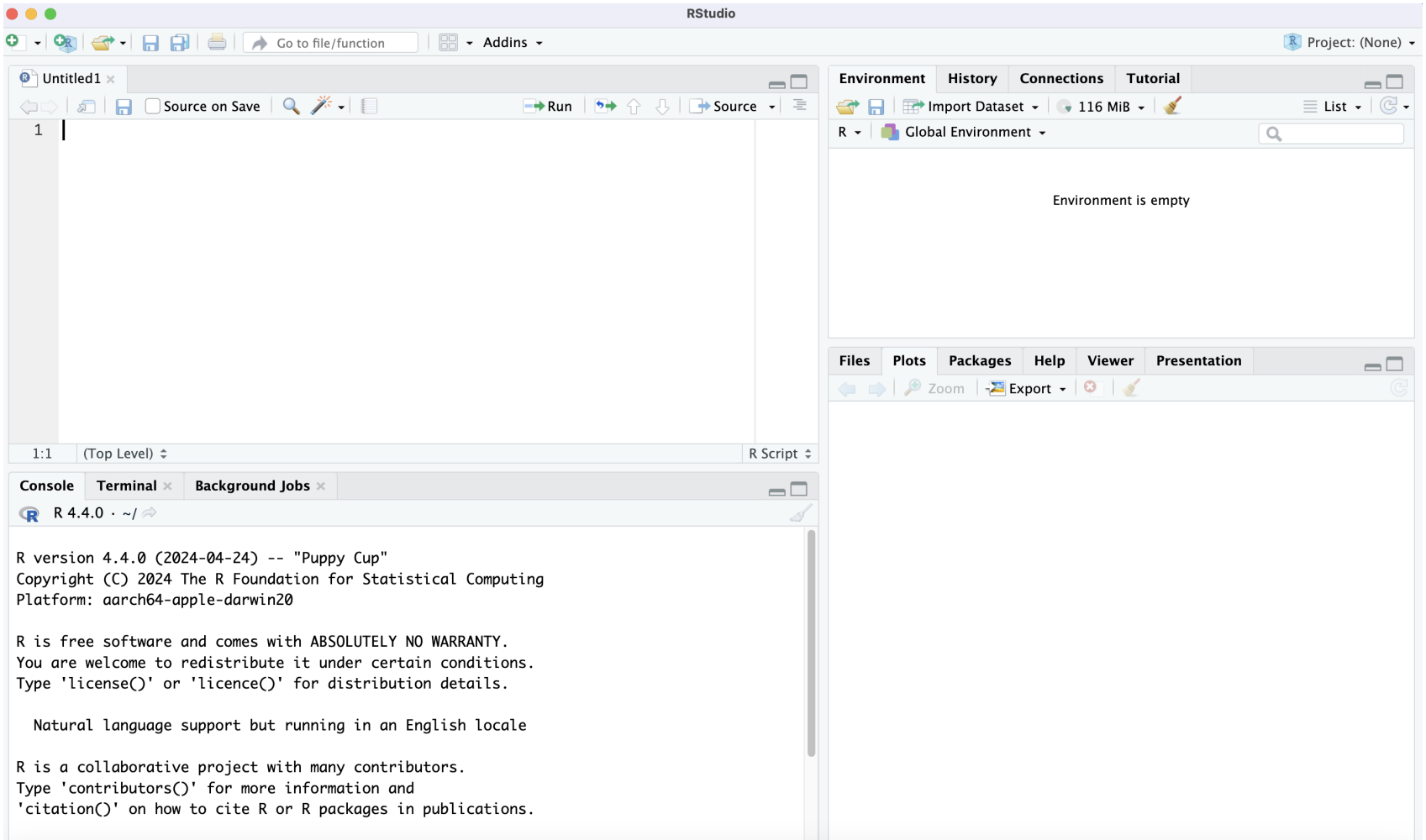

RStudio


RStudio
Go to file/function
Addins
Project: (None)
Untitled1
Source on Save
Run
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1
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R Script
Environment
History
Connections
Tutorial
Import Dataset
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Presentation
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R version 4.4.0 (2024-04-24) -- "Puppy Cup"
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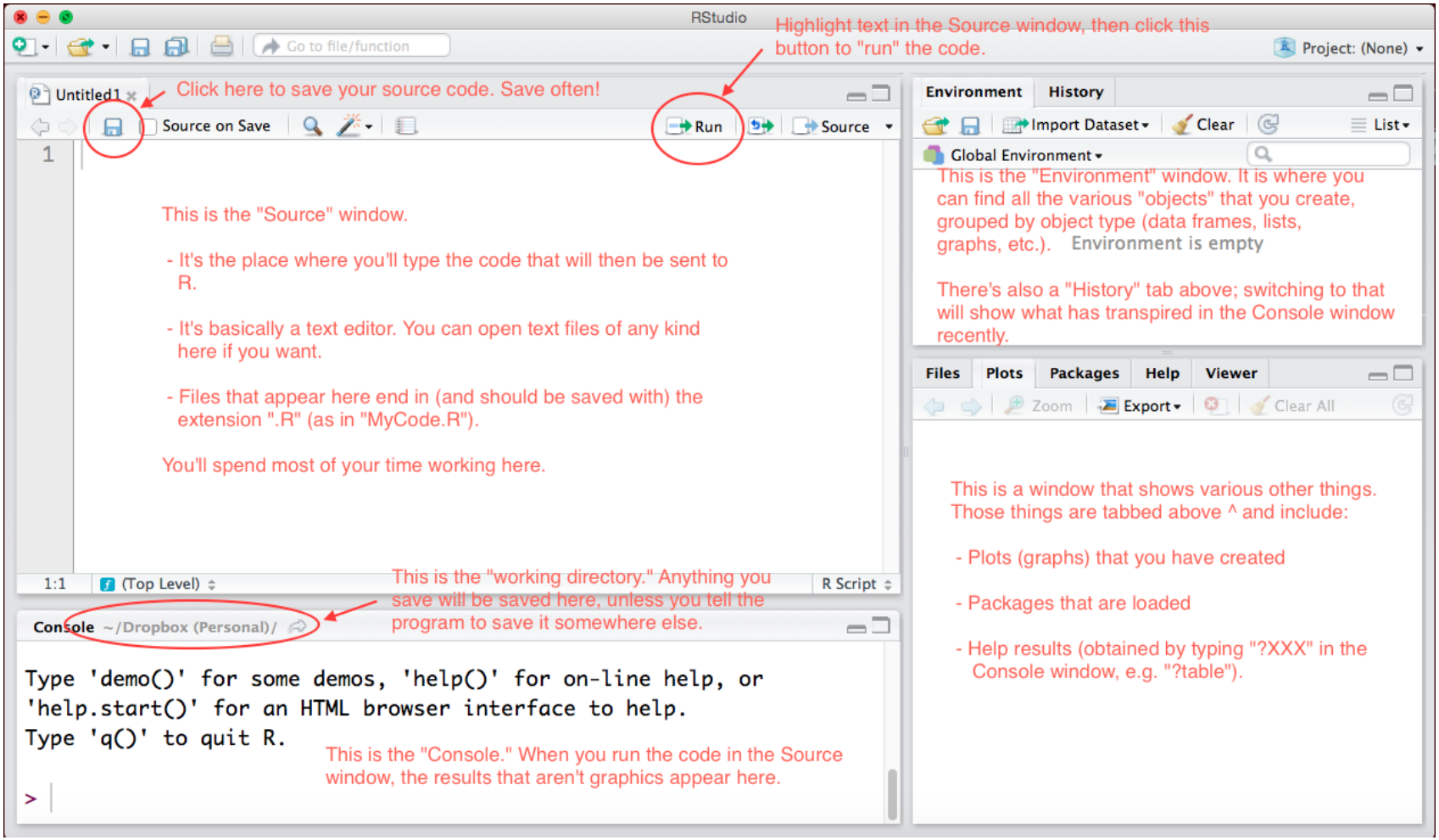
RStudio (annotated)



Things We Will And Won't Do

Will: "Regression":

$$Y = f(\mathbf{X})$$

Won't: Multivariate regression:

$$\mathbf{Y} = f(\mathbf{X})$$

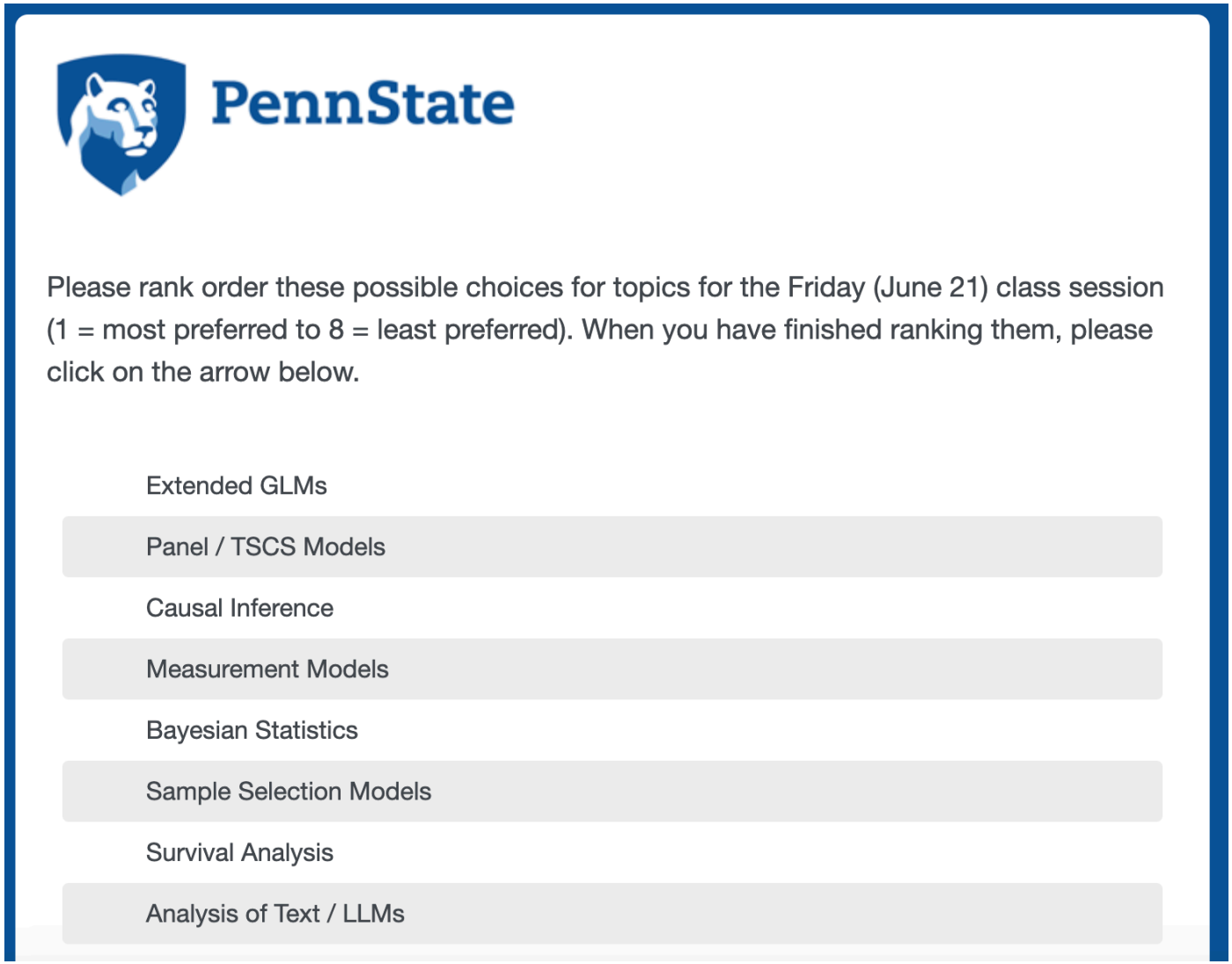
Won't: Measurement (e.g. PCA, factor analysis, IRT, etc.):

$$\mathbf{Y} = \mathbf{W}^{\mathrm{T}}\mathbf{X}$$

Won't: Classification:

- Cluster Analysis / Network Models / etc.
- Classification and Regression Trees $\rightarrow$ Random Forests.
- Pattern Recognition
- Machine Learning (beyond regression), Support Vector Machines, etc.

Friday = "Participants' Choice"



Why Regression?

	Description	**Explanation**	**Prediction**
Task	Summarize data	Correlation/causation	Forecast OOS / future data
Emphasis	Data	Theory / Hypotheses	Outcomes
Focus	Univariate	Multivariate	Multivariate
Typical Application	Summarize / "reduce" data	Discuss marginal associations between predictors and an outcome of interest	Optimize out-of-sample predictive power / minimize prediction error

Regression

"Regression," conceptually:

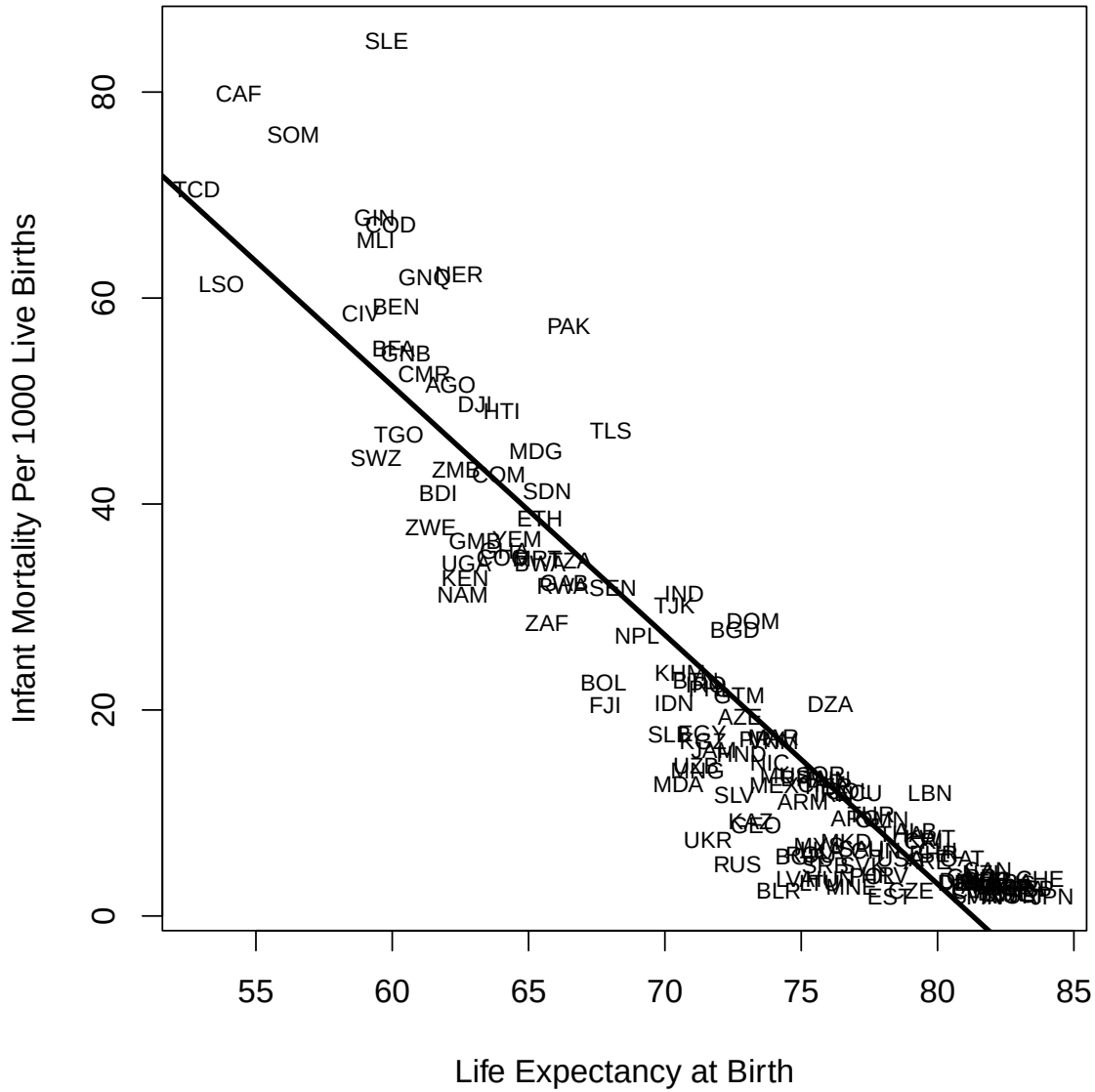
$$\Pr(Y|\mathbf{X}) = f(\mathbf{X})$$

Two important things:

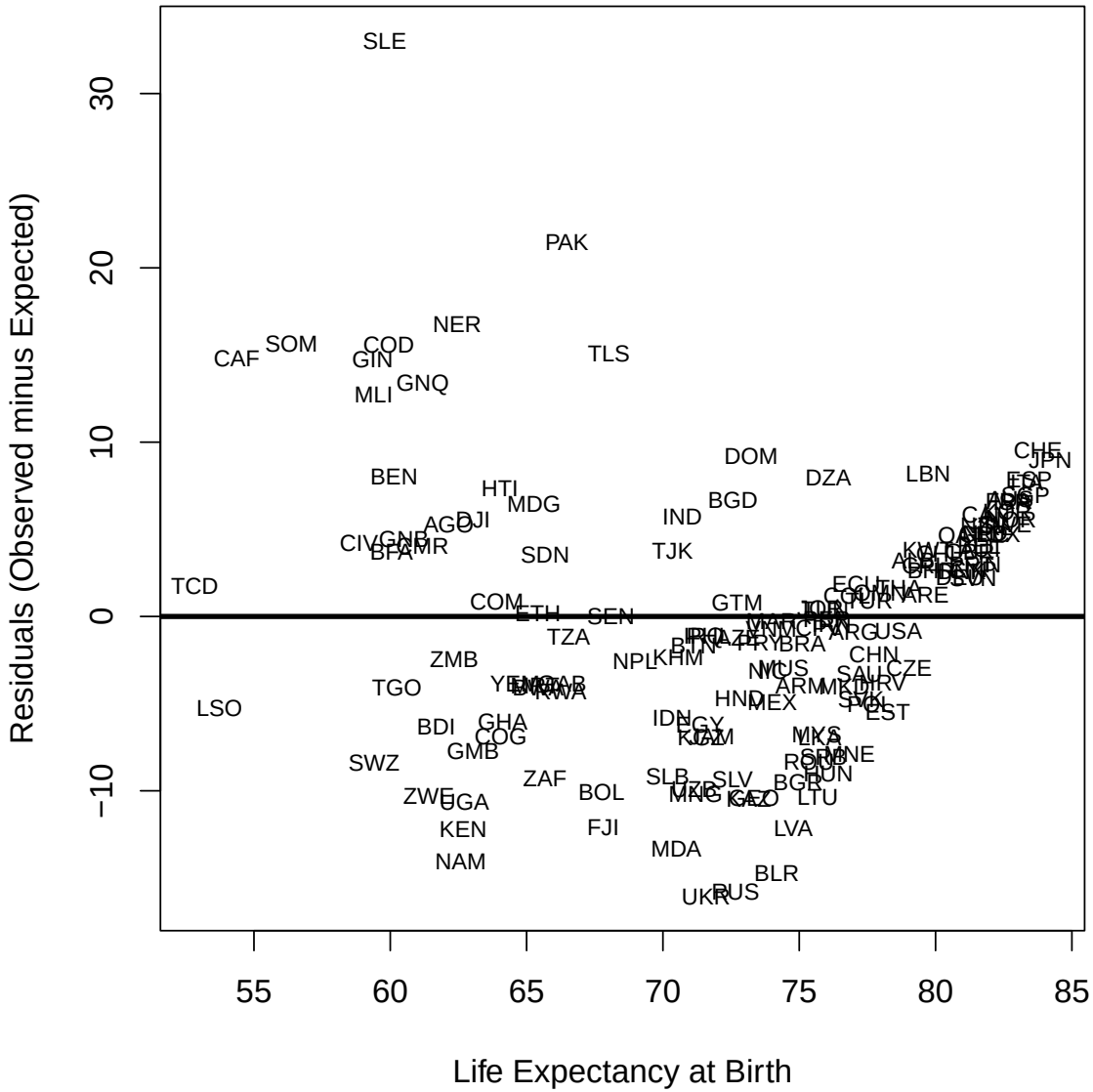
- The distribution of Y is *conditional on all variables in* $\mathbf{X}$, and
- The conditional distribution of Y is conditional on the *joint distribution* of the elements of $\mathbf{X}$.

$\rightarrow$ Regression is hard...

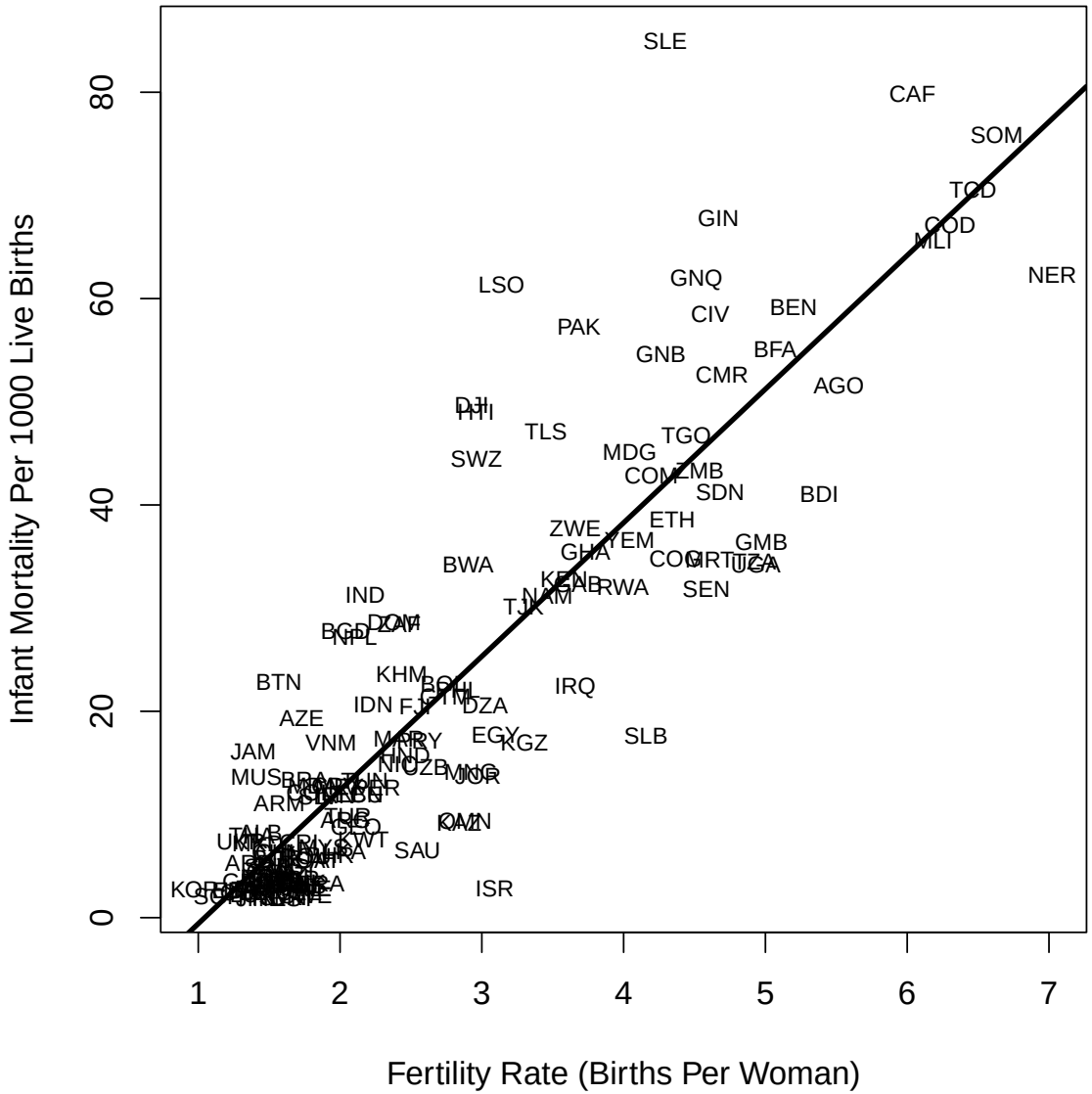
Example: Infant Mortality and Life Expectancy (2018)



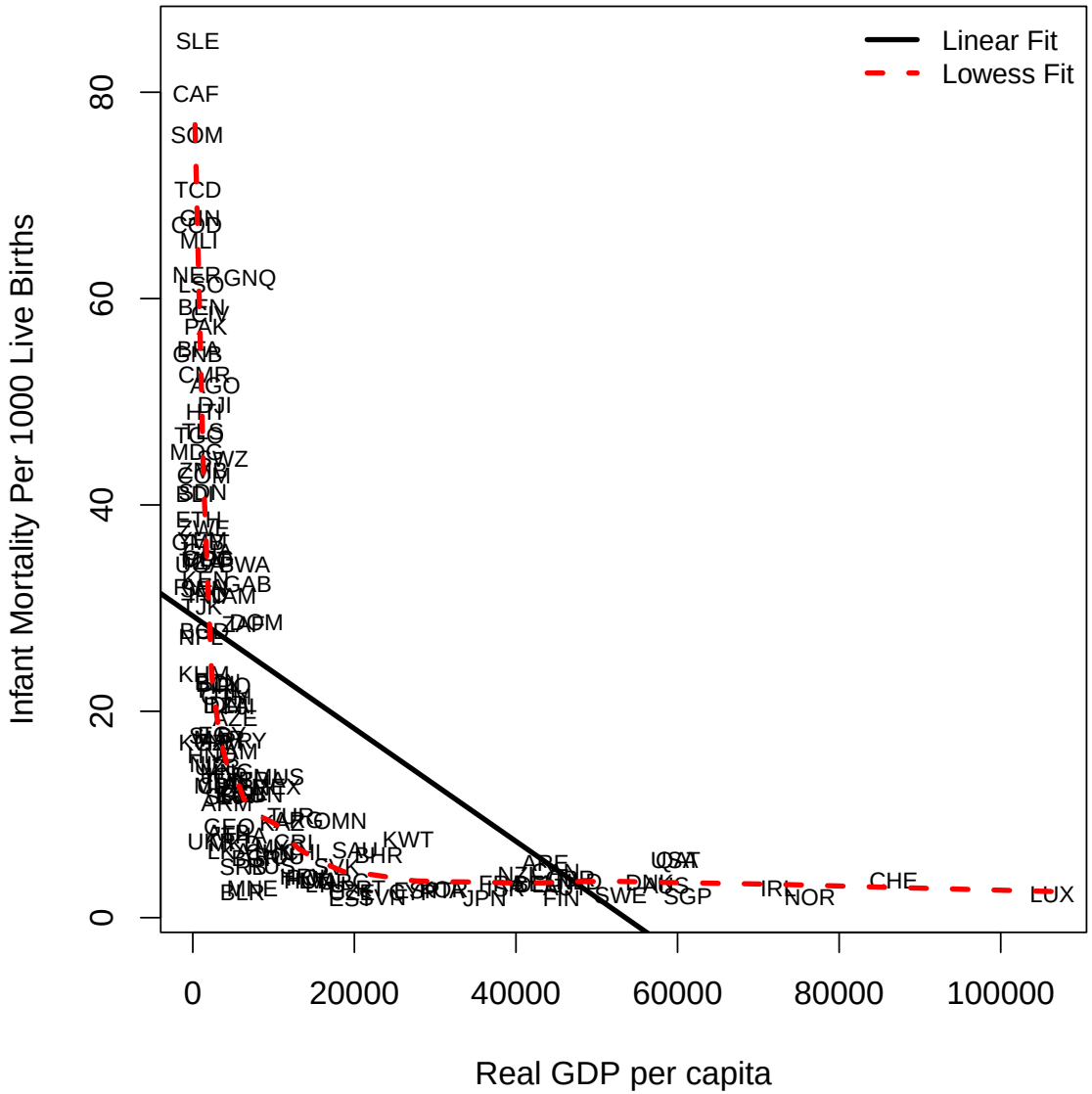
Infant Mortality and Life Expectancy: "Residuals"



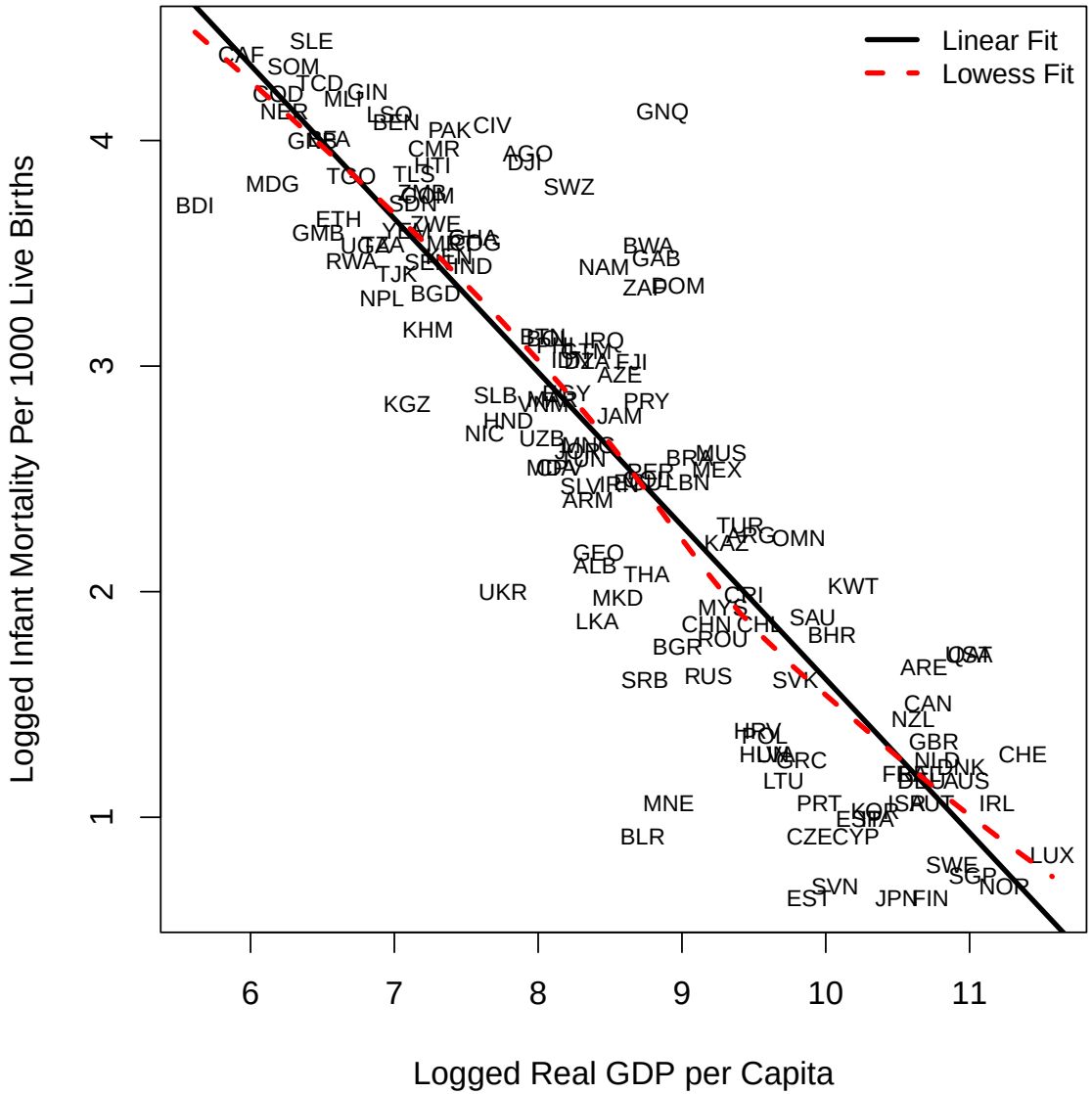
Infant Mortality and Fertility



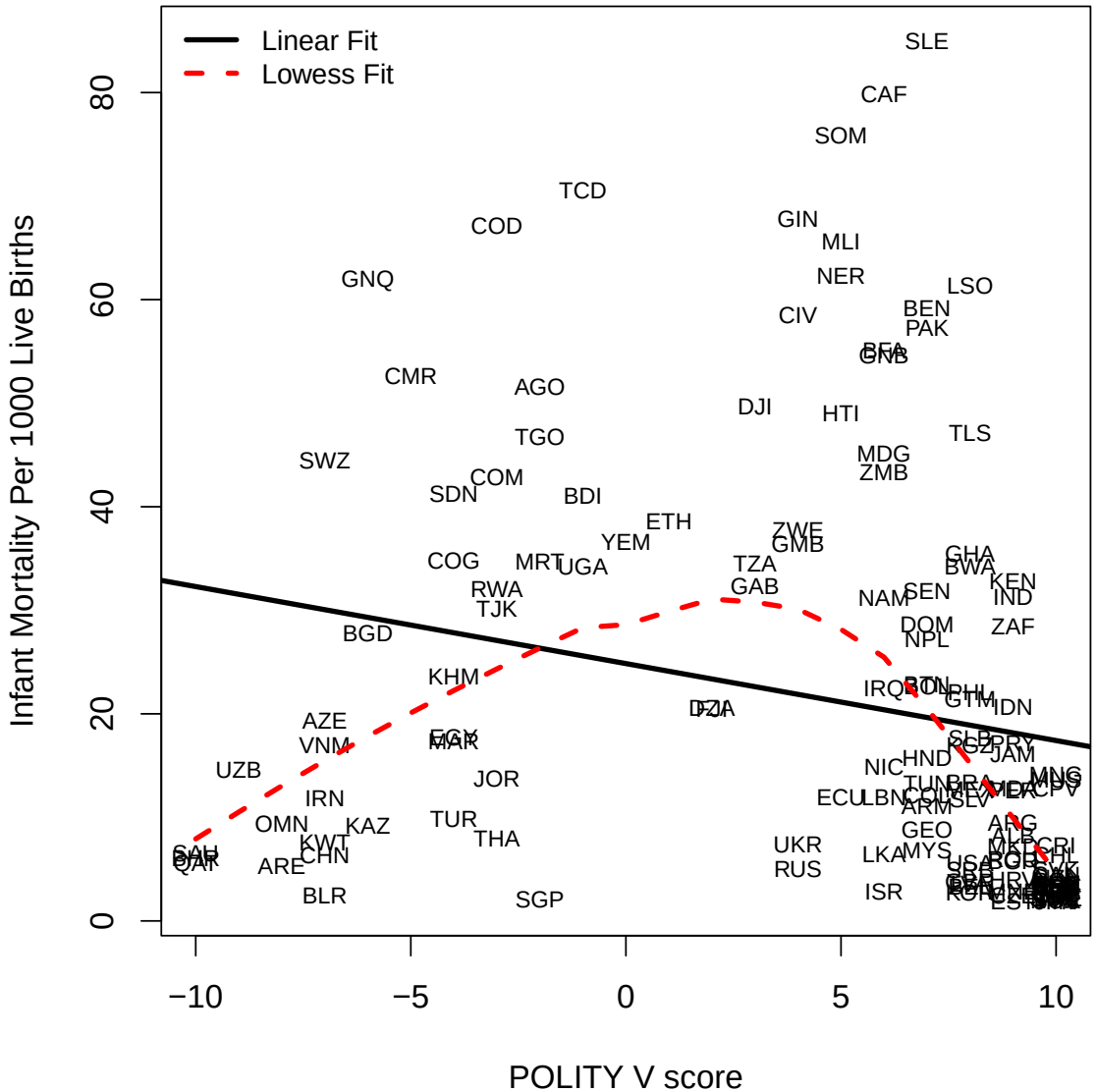
Infant Mortality and Wealth



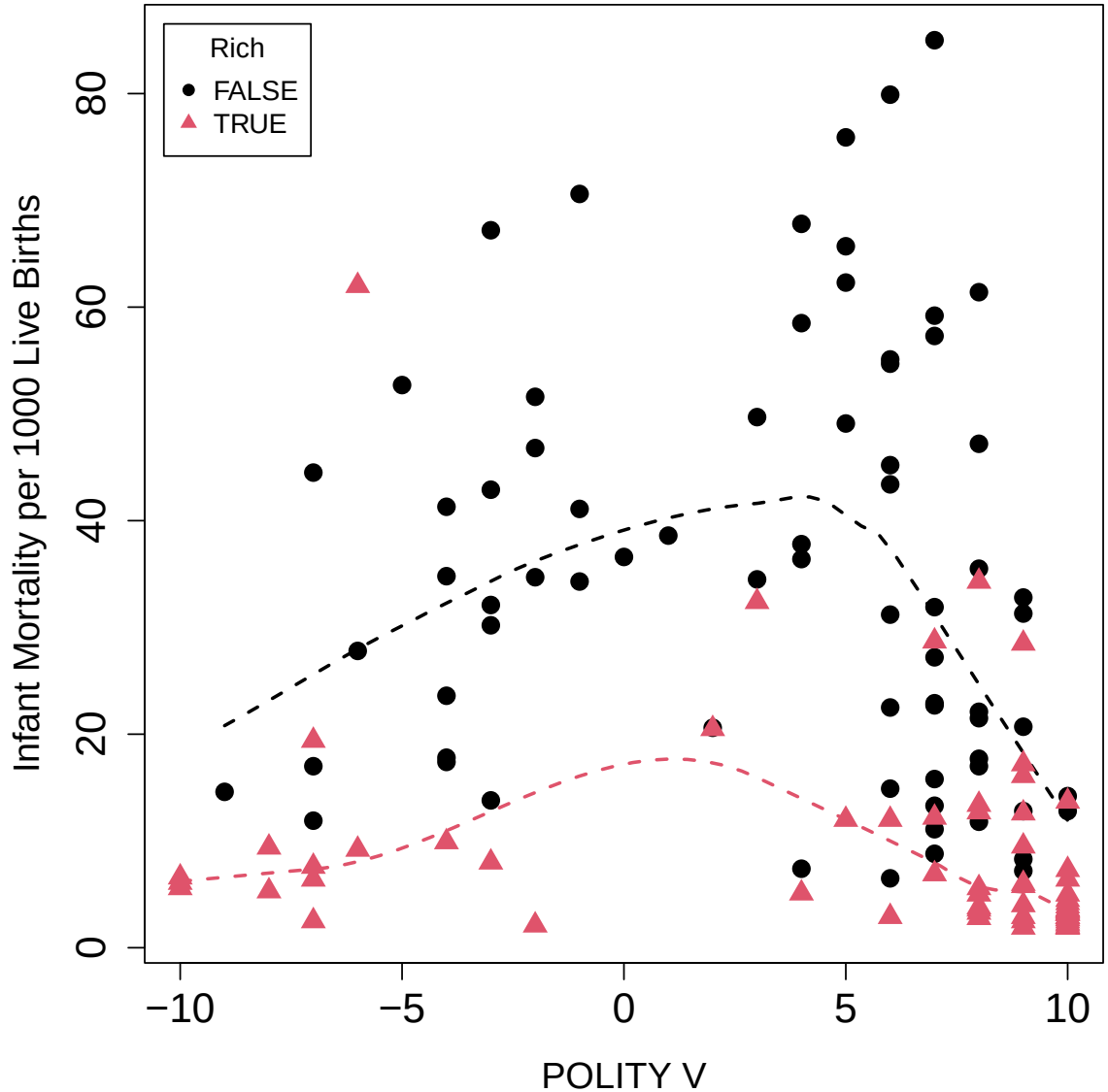
(Logged) Infant Mortality and (Logged) Wealth



Infant Mortality and Democracy



Infant Mortality, (Dichotomized) Wealth, and Democracy



Consider random variable Y:

$$Y_i = \mu + u_i \tag{1}$$

$$\mu_i = \beta_0 + \beta_1 X_i$$

so:

$$Y_i = \beta_0 + \beta_1 X_i + u_i \tag{2}$$

Goals:

- Estimate $\hat{\beta}_0$ and $\hat{\beta}_1$
- Estimate the *variability* $\hat{\beta}_0$ and $\hat{\beta}_1$
- Assess *model fit*

Bivariate OLS - Estimation

For a bivariate regression:

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2} \\ &= \frac{\text{Covariance of } X \text{ and } Y}{\text{Variance of } X} \end{aligned} \tag{3}$$

and

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \tag{4}$$

Assume (for now):

$$u_i \sim \text{i.i.d. } N(0, \sigma^2)$$

meaning:

$$\text{Var}(Y|X, \beta) = \sigma^2$$

so:

$$\begin{aligned}
\text{Var}(\hat{\beta}_1) &= \text{Var}\left[\frac{\sum_{i=1}^{N}(X_i - \bar{X})Y_i}{\sum_{i=1}^{N}(X_i - \bar{X})^2}\right] \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2 \,\text{Var}(Y) \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2 \,\sigma^2 \\
&= \frac{\sigma^2}{\sum(X_i - \bar{X})^2}.
\end{aligned}$$

$\text{Var}(\hat{\beta}_0)$ and $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)$

Similarly:

$$\text{Var}(\hat{\beta}_0) = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2$$

and :

$$\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2$$

Important Things

Note that:

- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto \sigma^2$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -\sum(X_i - \bar{X})$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -N$
- $\text{sign}[\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)] = -\text{sign}(\bar{X})$

If $u_i \sim N(0, \sigma^2)$, then:

$$\hat{\beta}_0 \sim N[\beta_0, \text{Var}(\hat{\beta}_0)]$$

and

$$\hat{\beta}_1 \sim N[\beta_1, \text{Var}(\hat{\beta}_1)]$$

Means:

$$\begin{aligned} z_{\hat{\beta}_1} &= \frac{(\hat{\beta}_1 - \beta_1)}{\sqrt{\text{Var}(\hat{\beta}_1)}} \\ &= \frac{(\hat{\beta}_1 - \beta_1)}{\text{s.e.}(\hat{\beta}_1)} \\ &= \sim N(0, 1) \end{aligned}$$

A Small Problem...

$\sigma^2 =$???

Solution: use

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{N-k}$$

Gives:

$$\widehat{\text{Var}(\hat{\beta}_1)} = \frac{\hat{\sigma}^2}{\sum (X_i - \bar{X})^2},$$

and

$$\widehat{\text{Var}(\hat{\beta}_0)} = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \hat{\sigma}^2$$

So:

$$\begin{aligned}
\widehat{\text{s.e.}(\hat{\beta}_1)} &= \sqrt{\widehat{\text{Var}(\hat{\beta}_1)}} \\
&= \sqrt{\frac{\hat{\sigma}^2}{\sum(X_i - \bar{X})^2}} \\
&= \frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}
\end{aligned}$$

which implies:

$$\begin{aligned}
t_{\hat{\beta}_1} \equiv \frac{(\hat{\beta}_1 - \beta_1)}{\widehat{\text{s.e.}(\hat{\beta}_1)}} &= \frac{(\hat{\beta}_1 - \beta_1)}{\frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}} \\
&= \frac{(\hat{\beta}_1 - \beta_1)\sqrt{\sum(X_i - \bar{X})^2}}{\hat{\sigma}} \\
&\sim t_{N-k}
\end{aligned}$$

Predictions and Variance

Point prediction:

$$\hat{Y}_k = \hat{\beta}_0 + \hat{\beta}_1 X_k$$

$\hat{Y}_k$ is unbiased:

$$\begin{aligned} \mathsf{E}(\hat{Y}_k) &= \mathsf{E}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\ &= \mathsf{E}(\hat{\beta}_0) + X_k \mathsf{E}(\hat{\beta}_1) \\ &= \beta_0 + \beta_1 X_k \\ &= \mathsf{E}(Y_k) \end{aligned}$$

Variability:

$$\begin{aligned} \mathsf{Var}(\hat{Y}_k) &= \mathsf{Var}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\ &= \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2 + \left[\frac{\sigma^2}{\sum (X_i - \bar{X})^2} \right] X_k^2 + 2 \left[\frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2 \right] X_k \\ &= \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right] \end{aligned}$$

Variability of Predictions

Prediction variation:

$$\text{Var}(\hat{Y}_k) = \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]$$

means that $\text{Var}(\hat{Y}_k)$:

- Decreases in N
- Decreases in $\text{Var}(X)$
- Increases in $|X - \bar{X}|$

Standard error of the prediction:

$$\widehat{\text{s.e.}(\hat{Y}_k)} = \sqrt{\sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]}$$

$\rightarrow$ (e.g.) confidence intervals:

$$95\%\,\text{c.i.}(\hat{Y}_k) = \hat{Y}_k \pm [1.96 \times \widehat{\text{s.e.}(\hat{Y}_k)}]$$

Variation in Y

We can decompose variation in Y:

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(\hat{Y} + \hat{u}) \\ &= \mathrm{Var}(\hat{Y}) + \mathrm{Var}(\hat{u}) + 2\,\mathrm{Cov}(\hat{Y}, \hat{u}) \\ &= \mathrm{Var}(\hat{Y}) + \mathrm{Var}(\hat{u}) \end{aligned}$$

TSS	$=$	**MSS**	$+$	**RSS**
("Total")		("Estimated," or "Model")		("Residual")

Model Fit: R^2

"R-squared":

$$
\begin{aligned}
R^2 &= \frac{\text{MSS}}{\text{TSS}} \\
&= \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} \\
&= 1 - \frac{\text{RSS}}{\text{TSS}} \\
&= 1 - \frac{\sum \hat{u}_i^2}{\sum(Y_i - \bar{Y})^2}
\end{aligned}
$$

R^2 Is...

R-squared:

- is "the proportion of variance explained"
- $\in [0, 1]$
  - $R^2 = 1.0 \equiv$ a "perfect (linear) fit"
  - $R^2 = 0 \equiv$ no (linear) $X - Y$ association

For a single X,

$$\begin{aligned} R^2 &= \hat{\beta}_1^2 \frac{\sum (X_i - \bar{X})^2}{\sum (Y_i - \bar{Y})^2} \\ &= r_{XY}^2 \end{aligned}$$

Adjusted R^2

"Adjusted" R^2:

$$R^2_{adj.} = 1 - \frac{(1 - R^2)(N - c)}{(N - k)}$$

where $c = 1$ if there is a constant in the model and $c = 0$ otherwise.

$R^2_{adj.}$:

- $R^2_{adj.} \to R^2$ as $N \to \infty$
- $R^2_{adj.}$ can be > 1, or < 0...
- $R^2_{adj.}$ increases with model "fit," but
- The extent of that increase is discounted by a factor proportional to the number of covariates.

R^2 Alternatives

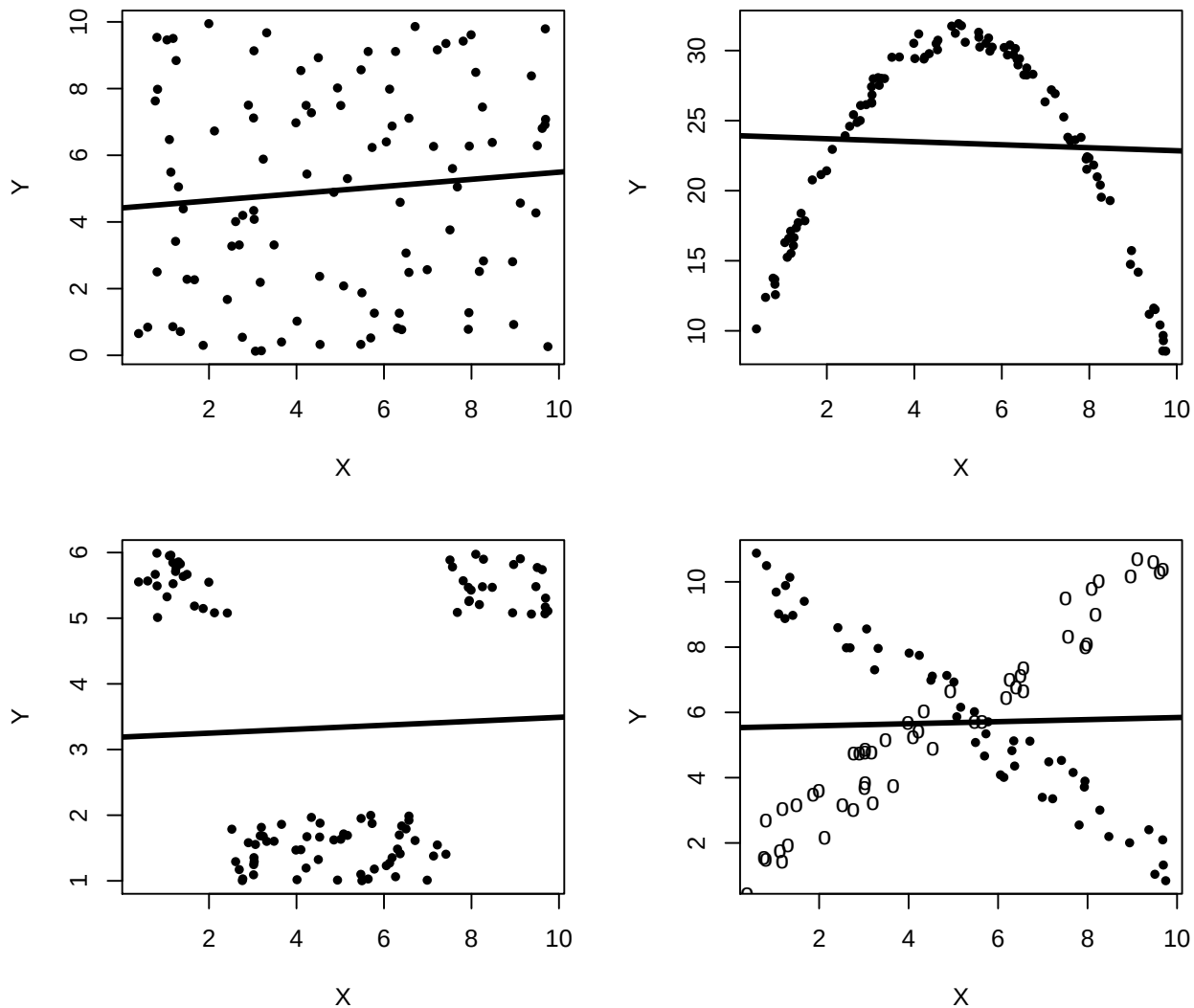
Alternative single measures of model fit include:

- The Standard Error of the Estimate:

$$\text{SEE} = \sqrt{\frac{\text{RSS}}{N-k}}$$

- F-tests
- ROC / AUC
- Graphical methods

Caution: Different Ways to get $R^2 = 0$



Linear Regression: K Predictors

Now consider:

$$\underset{N\times 1}{\mathbf{Y}} = \underset{N\times K}{\mathbf{X}}\,\underset{K\times 1}{\boldsymbol{\beta}} + \underset{N\times 1}{\mathbf{u}}$$

equivalently:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_K X_{Ki} + u_i$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{K1} \\ 1 & X_{12} & X_{22} & \cdots & X_{K2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1N} & X_{2N} & \cdots & X_{KN} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}.$$

Residuals:

$$\mathbf{u} = \mathbf{Y} - \mathbf{X}\boldsymbol{\beta}$$

The inner product of $\mathbf{u}$:

$$\begin{aligned}
\mathbf{u}'\mathbf{u} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \\
&= u_1^2 + u_2^2 + ... + u_N^2 \\
&= \sum_{i=1}^{N} u_i^2
\end{aligned}$$

Estimating β

Sum of squared residuals:

$$\begin{aligned}\mathbf{u}'\mathbf{u} &= (\mathbf{Y}-\mathbf{X}\beta)'(\mathbf{Y}-\mathbf{X}\beta) \\ &= \mathbf{Y}'\mathbf{Y} - 2\beta'\mathbf{X}'\mathbf{Y}' + \beta'\mathbf{X}'\mathbf{X}\beta\end{aligned}$$

Now get:

$$\frac{\partial \mathbf{u}'\mathbf{u}}{\partial \beta} = -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\beta$$

Solve:

$$\begin{aligned}-2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\beta &= 0 \\ -\mathbf{X}'\mathbf{Y} + \mathbf{X}'\mathbf{X}\beta &= 0 \\ \mathbf{X}'\mathbf{X}\beta &= \mathbf{X}'\mathbf{Y} \\ (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\ \beta &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}\end{aligned}$$

The Importance of $\mathbf{V}(\hat{\boldsymbol{\beta}})$

Variance-covariance matrix:

$$\begin{aligned}\mathbf{V}(\hat{\boldsymbol{\beta}}) &= \mathrm{E}[\hat{\boldsymbol{\beta}} - \mathrm{E}(\hat{\boldsymbol{\beta}})]^2 \\ &= \mathrm{E}\{[\hat{\boldsymbol{\beta}} - \mathrm{E}(\hat{\boldsymbol{\beta}})][\hat{\boldsymbol{\beta}} - \mathrm{E}(\hat{\boldsymbol{\beta}})]'\}\end{aligned}$$

Rewrite:

$$\begin{aligned}\mathbf{V}(\hat{\boldsymbol{\beta}}) &= \mathrm{E}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \\ &= \mathrm{E}\{[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}]'\} \\ &= \mathrm{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\end{aligned}$$

The Importance of $\mathbf{V}(\hat{\boldsymbol{\beta}})$

Taking expectations:

$$\begin{aligned}
\mathbf{V}(\hat{\boldsymbol{\beta}}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{E}(\mathbf{u}\mathbf{u}')\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

Estimating $\mathbf{V}(\hat{\boldsymbol{\beta}})$

Empirical estimate:

$$\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{N-K}$$

Yields:

$$\widehat{\mathbf{V}(\hat{\boldsymbol{\beta}})} = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$$

OLS Assumptions

1. Zero Expectation Disturbances

$$\mathrm{E}(\mathbf{u}) = \mathbf{0}$$

2. Homoscedasticity / No Error Correlation

$$\mathrm{E}(\mathbf{u}\mathbf{u}') = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

OLS Assumptions (continued)

3. "Fixed" $\mathbf{X}$...

- No *measurement error* in the $\mathbf{X}$s, and
- $\text{Cov}(\mathbf{X}, \mathbf{u}) = \mathbf{0}$.

4. $\mathbf{X}$ is of full column rank.

Means:

- no exact linear relationship among $\mathbf{X}$, and
- $K < N$.

5. Normal Disturbances

$$\mathbf{u} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

OLS: Properties

Under these assumptions, the OLS estimate of $\hat{\beta}$ is:

- **Unbiased**
- **Fully Efficient**

(i.e., **"BLUE"**)

Multivariate Regression, Conceptually

Begin with:

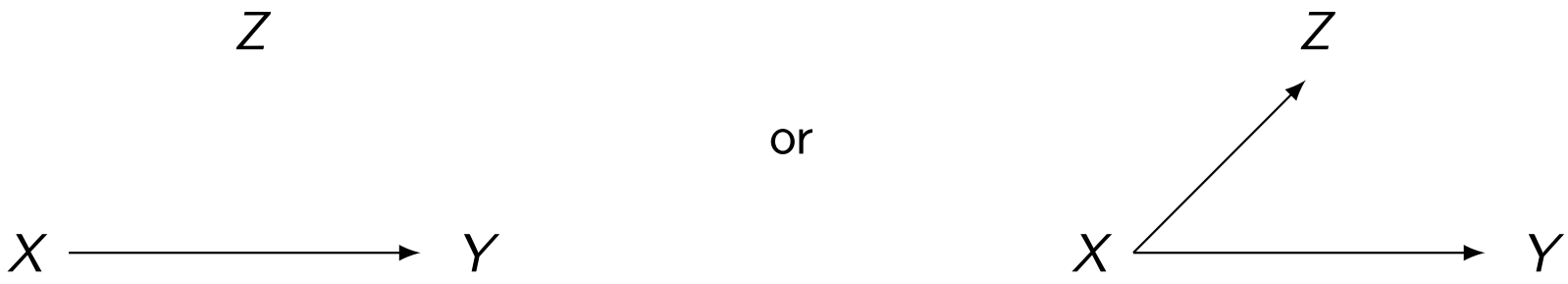
- An *outcome* Y
- A *predictor* X
- Another variable Z

We are mainly interested in $\text{Cov}(Y, X|Z)$...

- Possibly *but not necessarily* the causal relationship (cf. Berk 2010)
- Key question is *model specification*...

The Easiest Case

Things are easiest if:



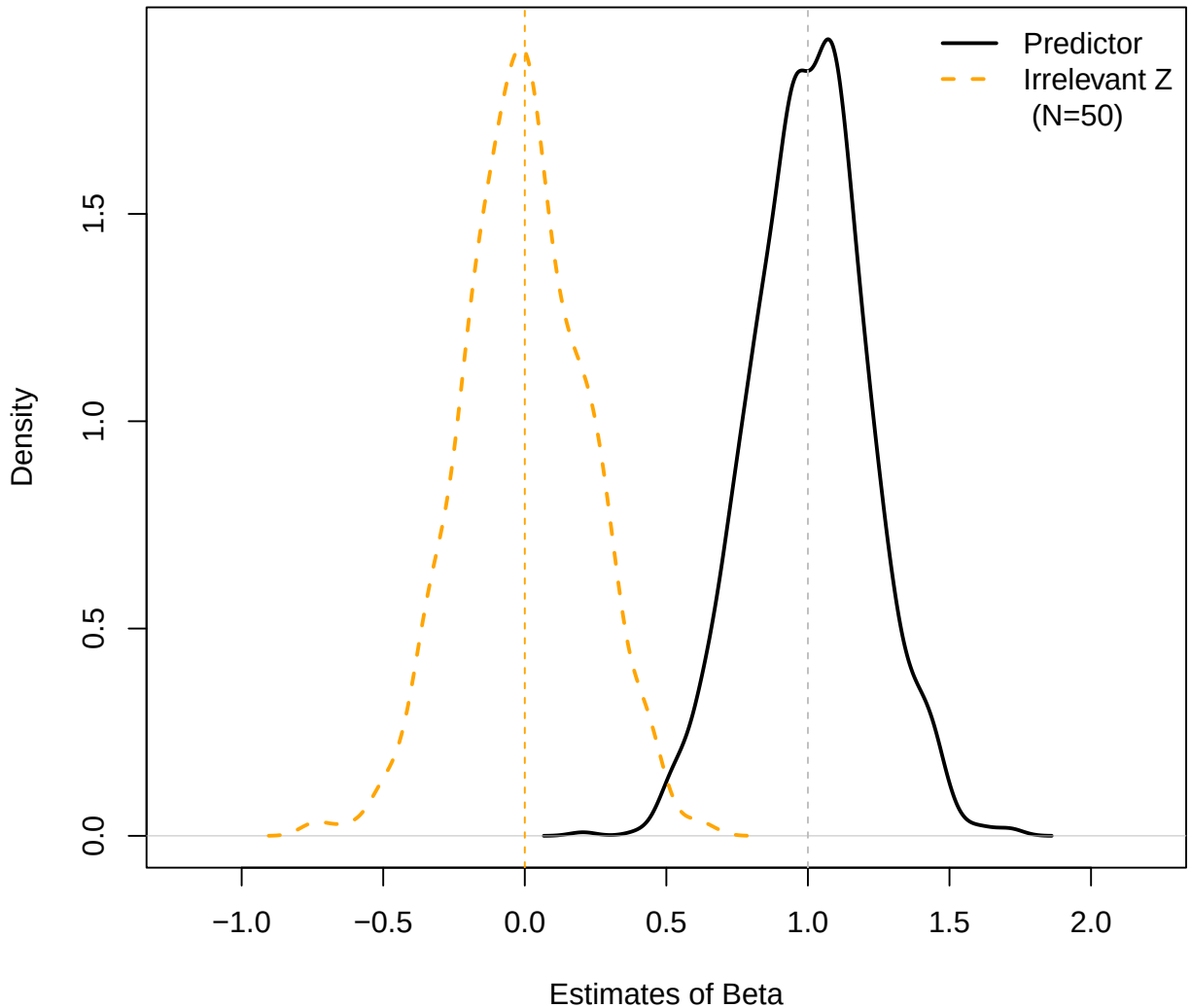
Implies that:

- Z is unimportant to understanding $\text{Cov}(X, Y)$
- $\rightarrow$ Z is *ignorable*

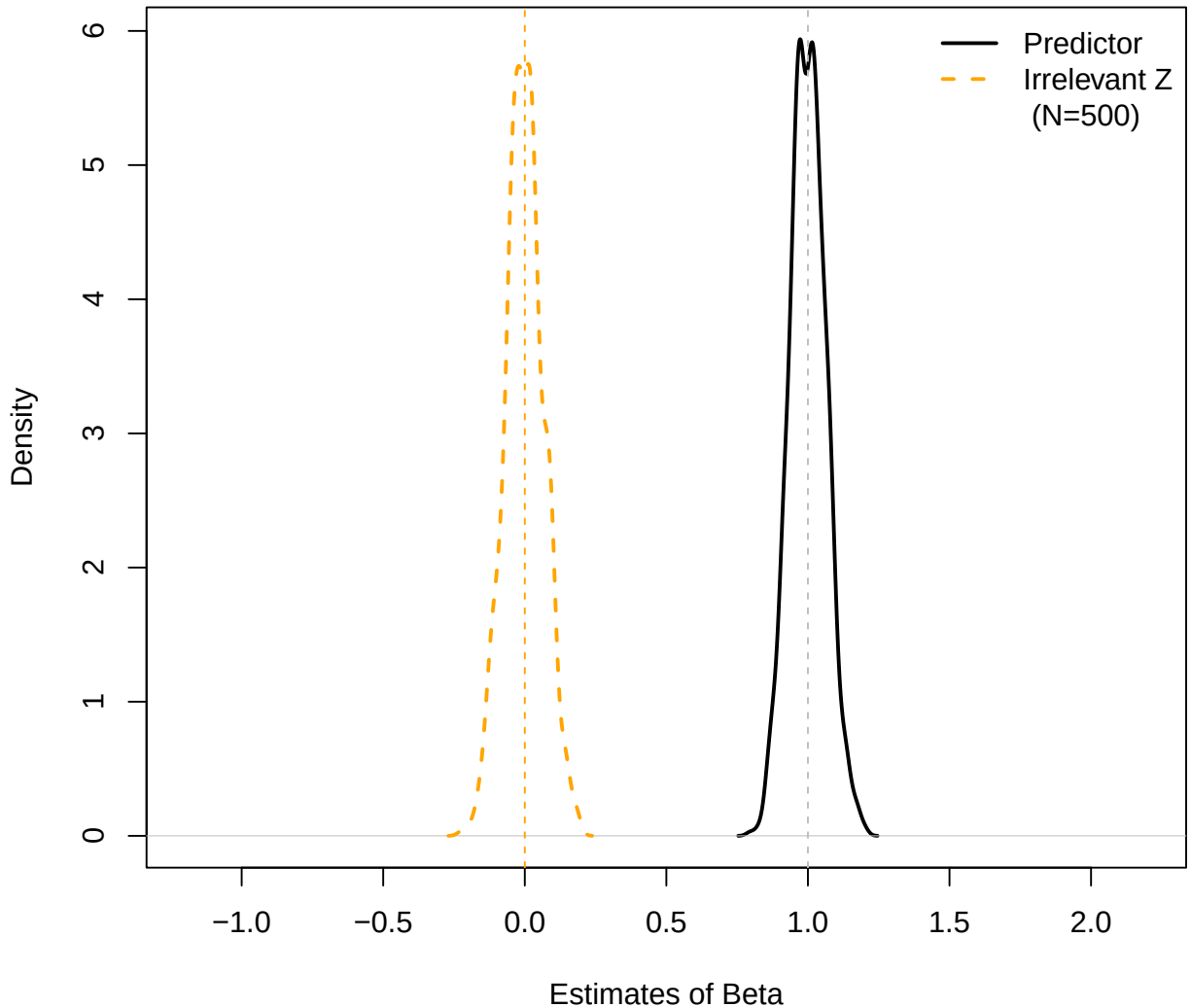
Simulations For Everyone!

```
> N<-50
> set.seed(7222009)
> X<-rnorm(N)         # Predictor
> Z<-(X+rnorm(N))/1.5 # Z "caused by" X
> Y<-X+rnorm(N)       # Outcome Y (unrelated to Z)

> print(summary(lm(Y~X)))

Coefficients:
            Estimate Std. Error t value   Pr(>|t|)
(Intercept)    0.028      0.144    0.19       0.85
X              0.978      0.162    6.05 0.00000021 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1 on 48 degrees of freedom
Multiple R-squared:  0.432, Adjusted R-squared:  0.421
F-statistic: 36.6 on 1 and 48 DF,  p-value: 0.000000212


> print(summary(lm(Y~X+Z)))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.0335     0.1460    0.23     0.82
X             0.9322     0.2161    4.31 0.000082 ***
Z             0.0659     0.2019    0.33     0.75
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1 on 47 degrees of freedom
Multiple R-squared:  0.434, Adjusted R-squared:  0.41
F-statistic:   18 on 2 and 47 DF,  p-value: 0.00000157
```

Do That 999 More Times

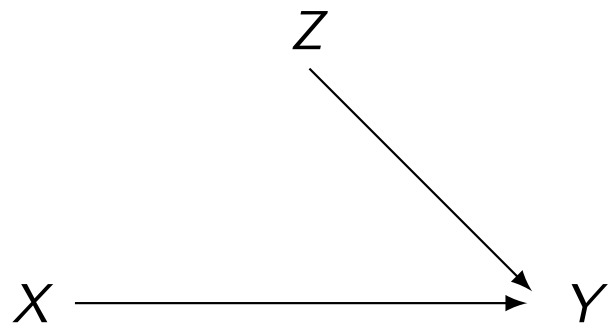


Same, But With $N = 500$



Slightly More Challenging

Suppose instead that:



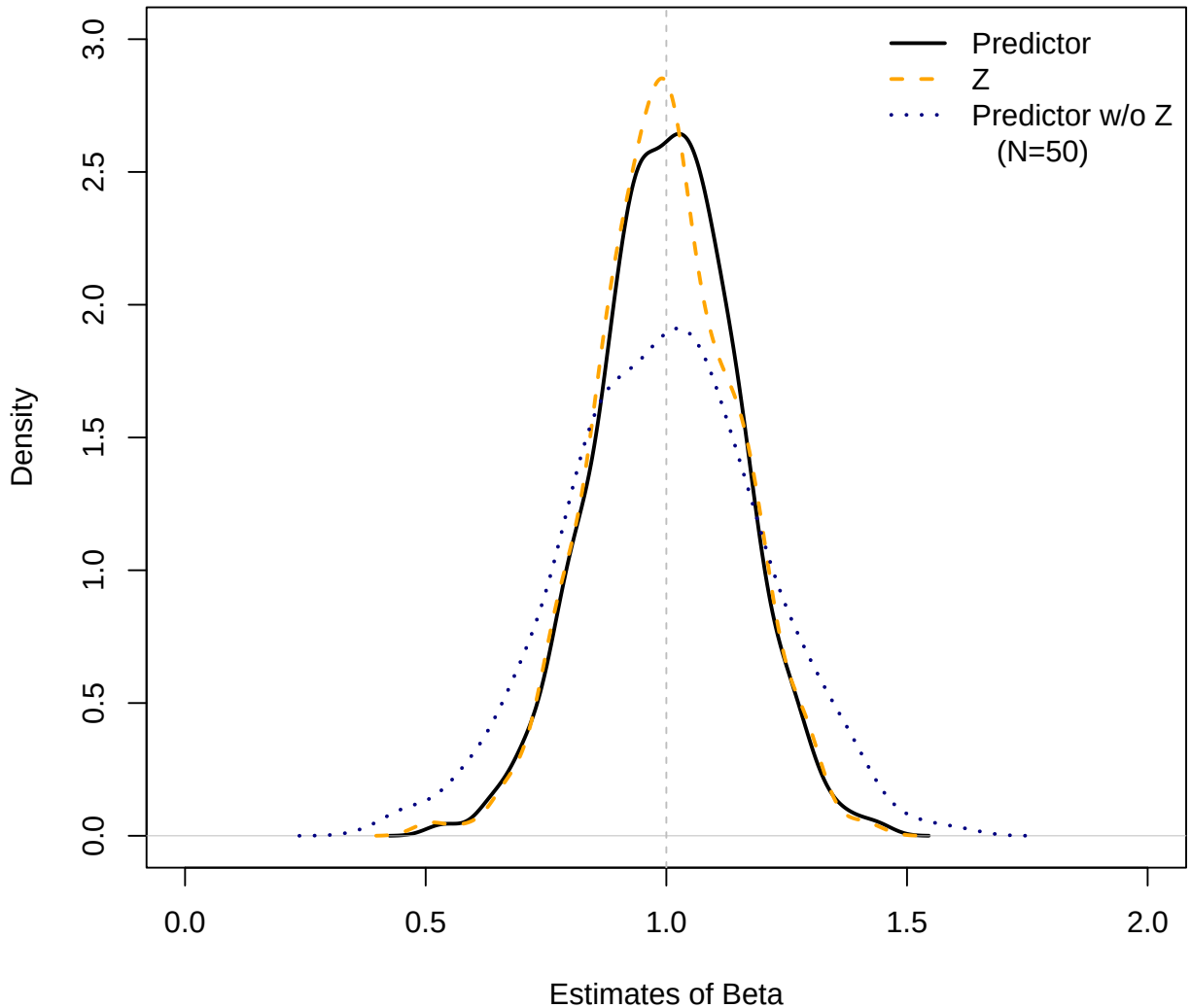
This means that:

- Z is important to / influential on understanding Y, *but*
- Z is unrelated to X...

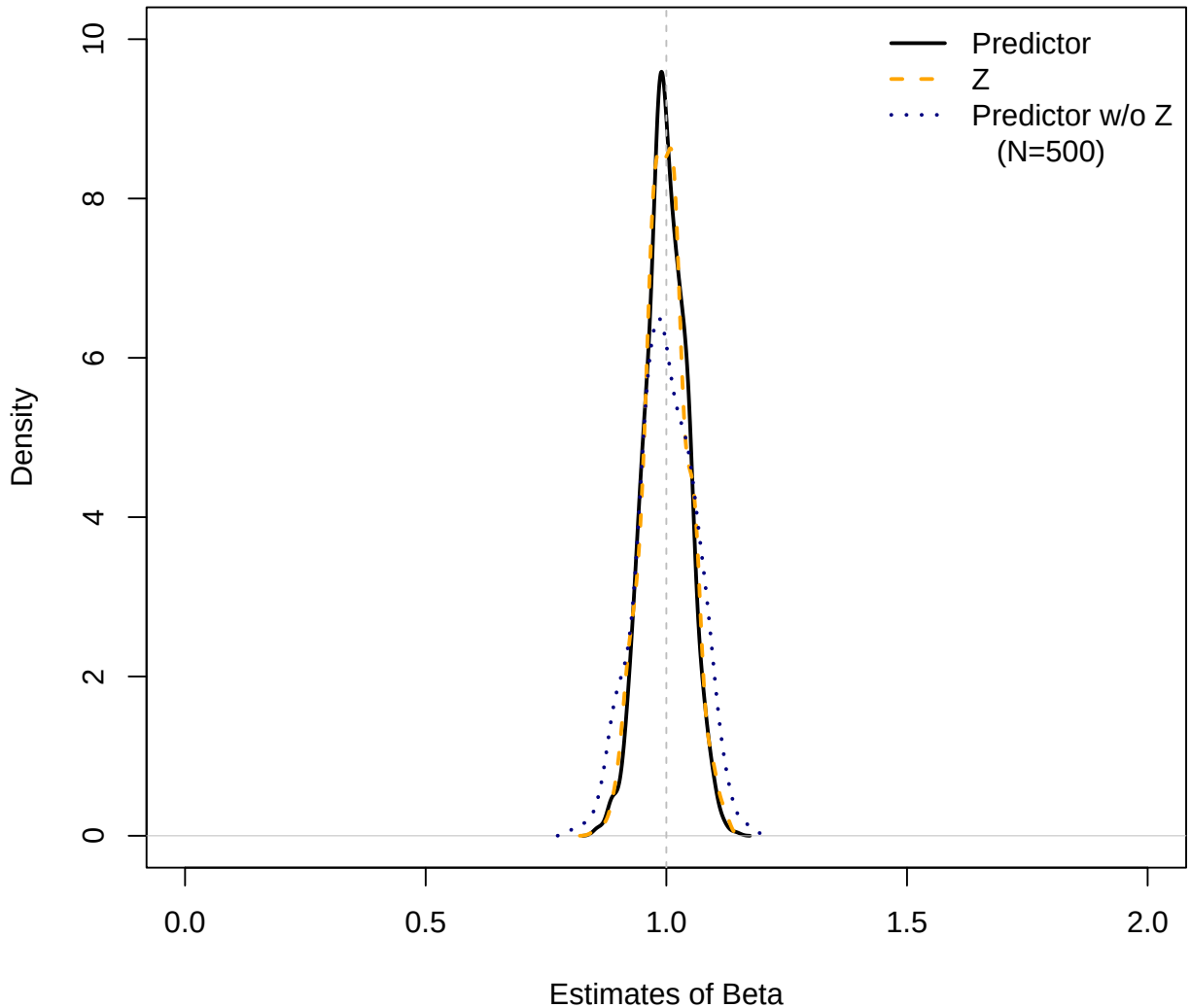
One Regression

```
> N<-50
> set.seed(7222009)
> X<-rnorm(N)         # Predictor
> Z<-rnorm(N)         # Z orthogonal to X
> Y<-X+Z+rnorm(N)     # Outcome Y


> print(summary(lm(Y~X)))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.0956     0.2167   -0.44  0.66119
X             1.0301     0.2439    4.22  0.00011 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.5 on 48 degrees of freedom
Multiple R-squared:  0.271, Adjusted R-squared:  0.256
F-statistic: 17.8 on 1 and 48 DF,  p-value: 0.000107


> print(summary(lm(Y~X+Z)))

Coefficients:
            Estimate Std. Error t value      Pr(>|t|)
(Intercept)   0.0335     0.1460    0.23          0.82
X             0.9761     0.1634    5.97 0.00000029670 ***
Z             1.0439     0.1346    7.75 0.00000000059 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1 on 47 degrees of freedom
Multiple R-squared:  0.68, Adjusted R-squared:  0.666
F-statistic:   50 on 2 and 47 DF,  p-value: 0.00000000000233
```

Many Regressions

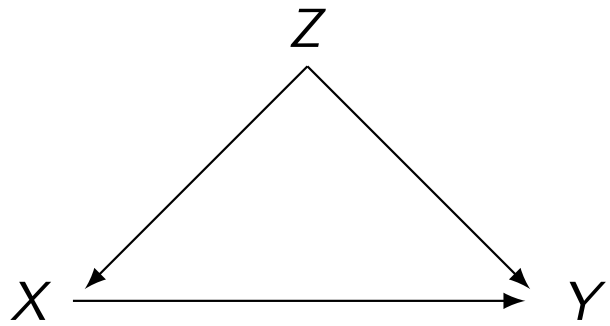


Same, But With $N = 500$



Confounding

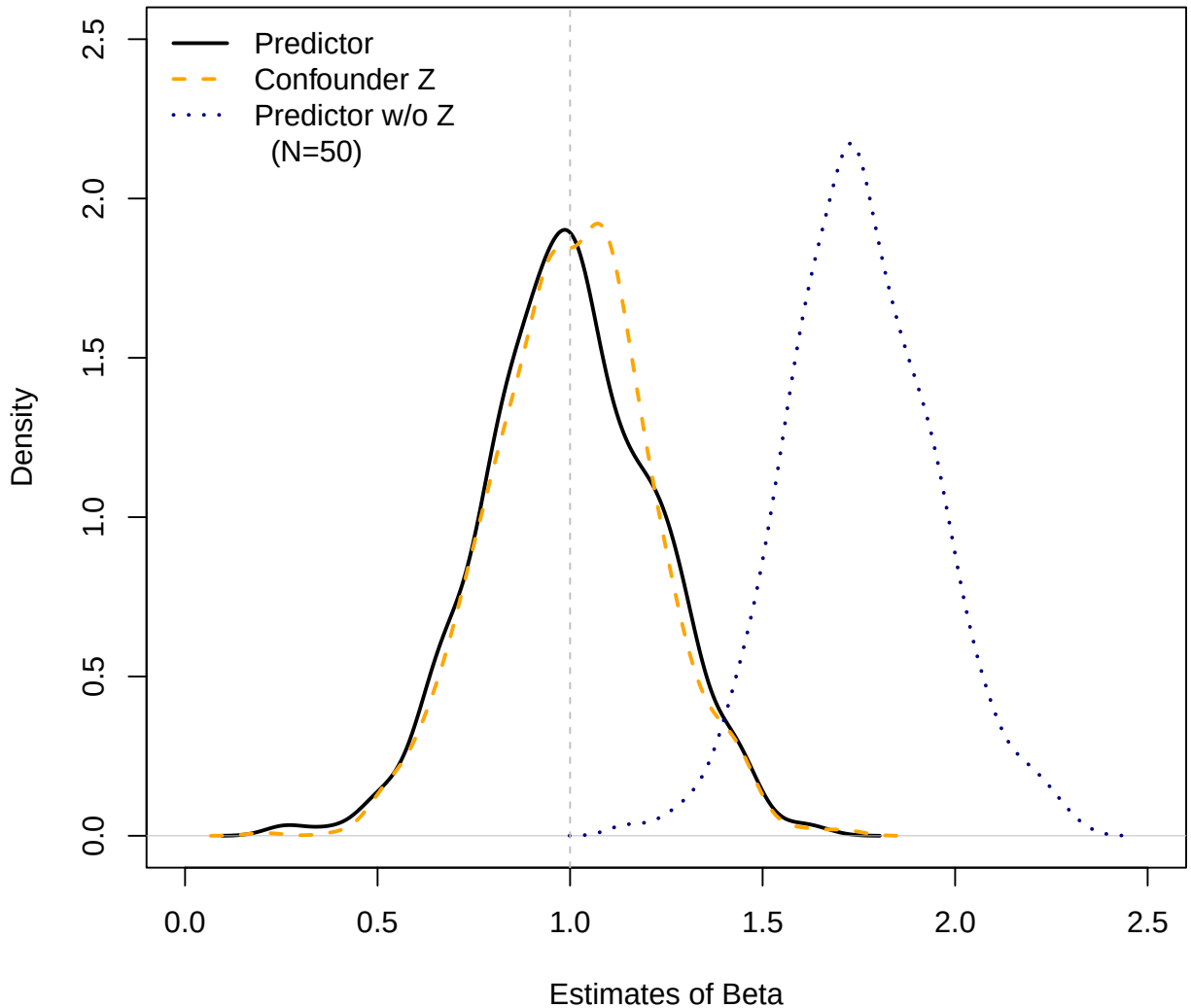
The classic example is:



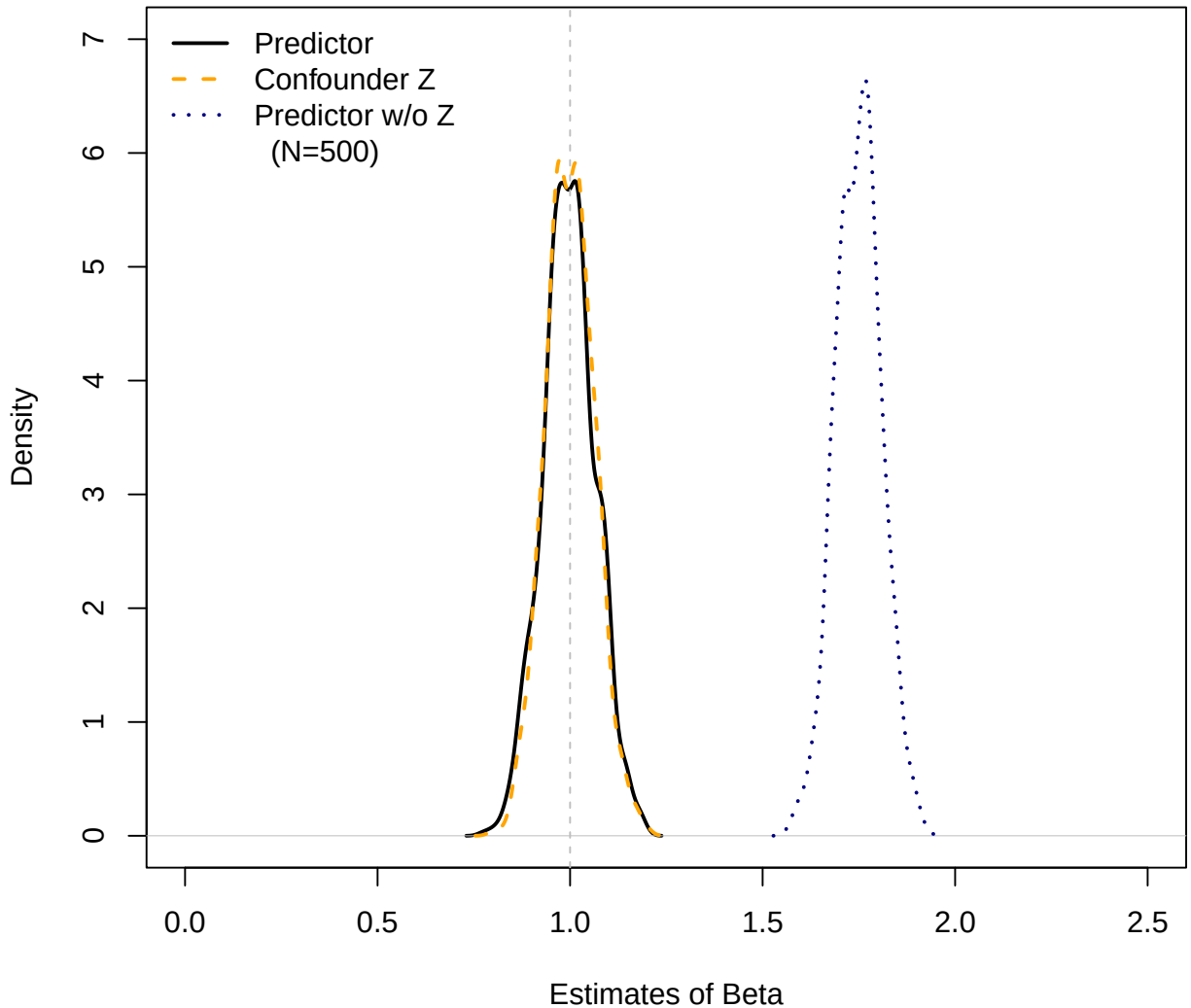
This means that:

- Z is important to / influential on both X and Y
- The marginal association $\text{Cov}(X, Y|Z)$ (obviously) depends on Z...
- More specifically, $\text{Cov}(X, Y|Z) \neq \text{Cov}(X, Y)$

So Much Confounding

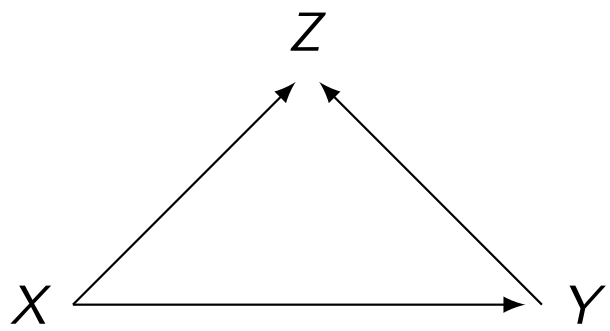


Same, But With $N = 500$



"Collider Bias"

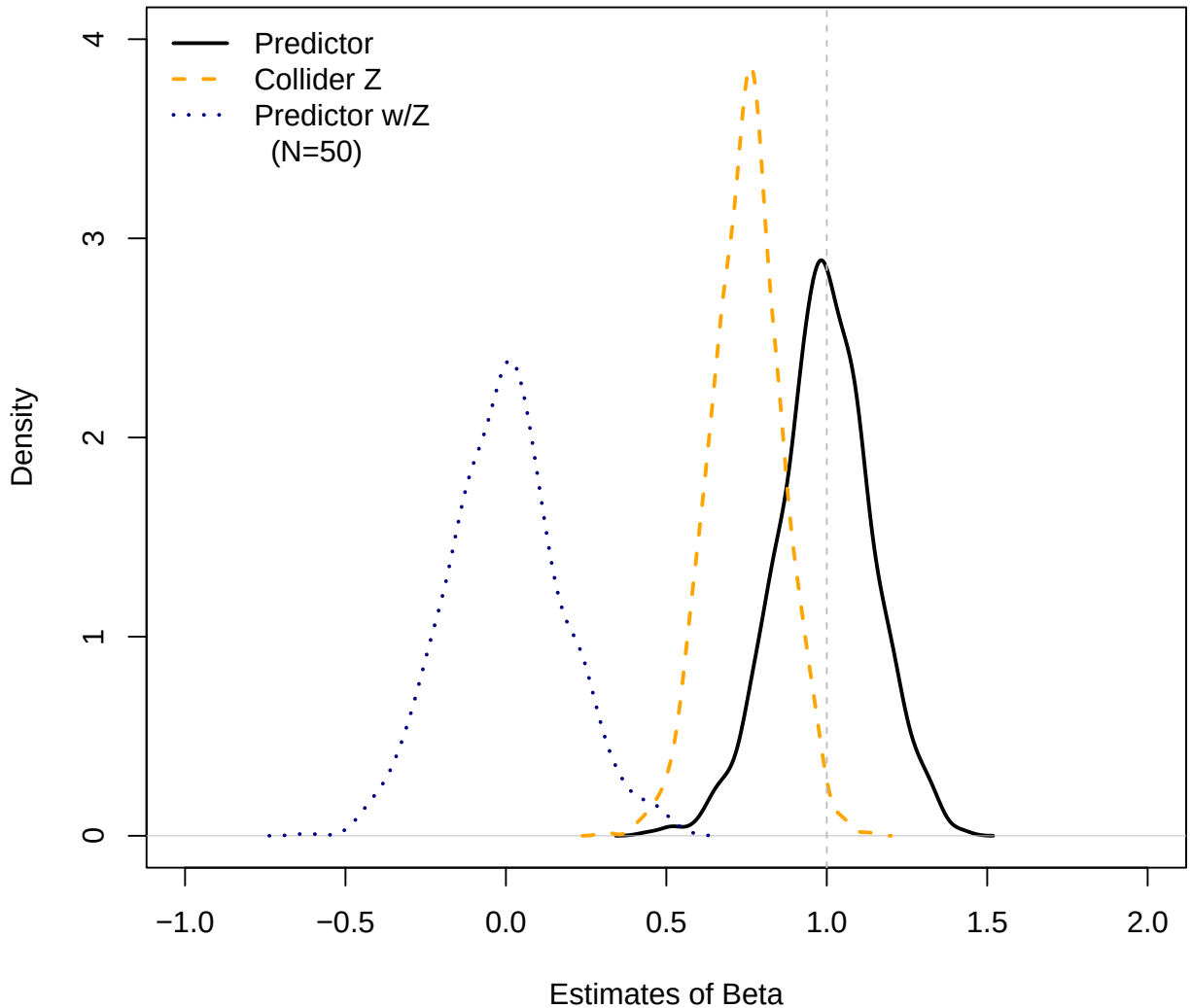
Z is a "collider":



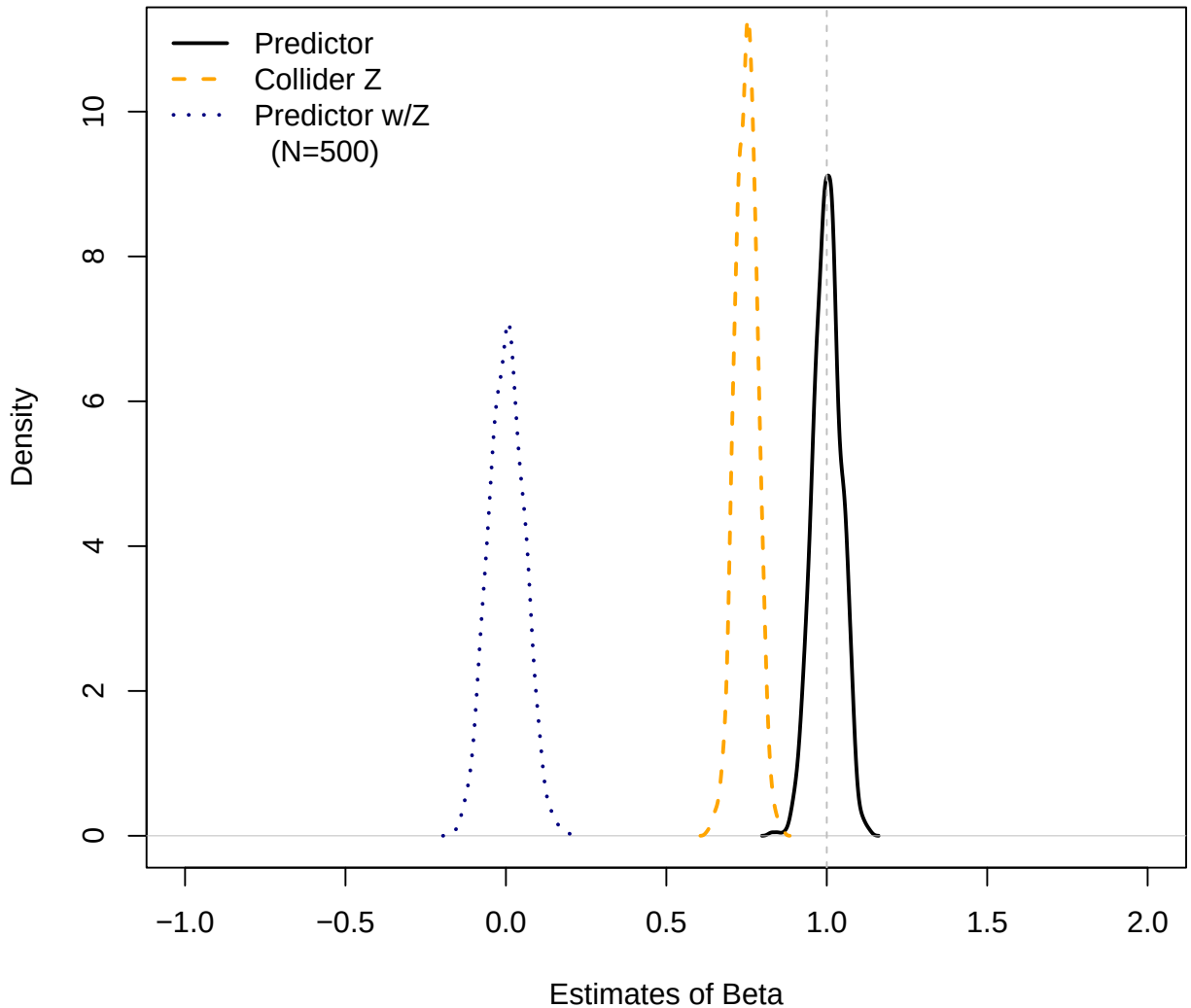
This means that:

- Z is influenced *by* both X and Y
- Once again, $\text{Cov}(X, Y|Z) \neq \text{Cov}(X, Y)$
- Sometimes referred to as Berkson's paradox

So Much Colliding

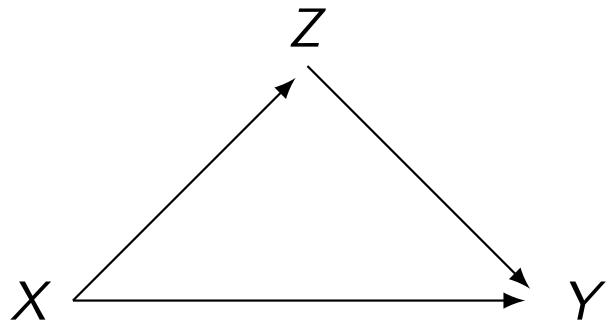


Same, But With $N = 500$



Mediators

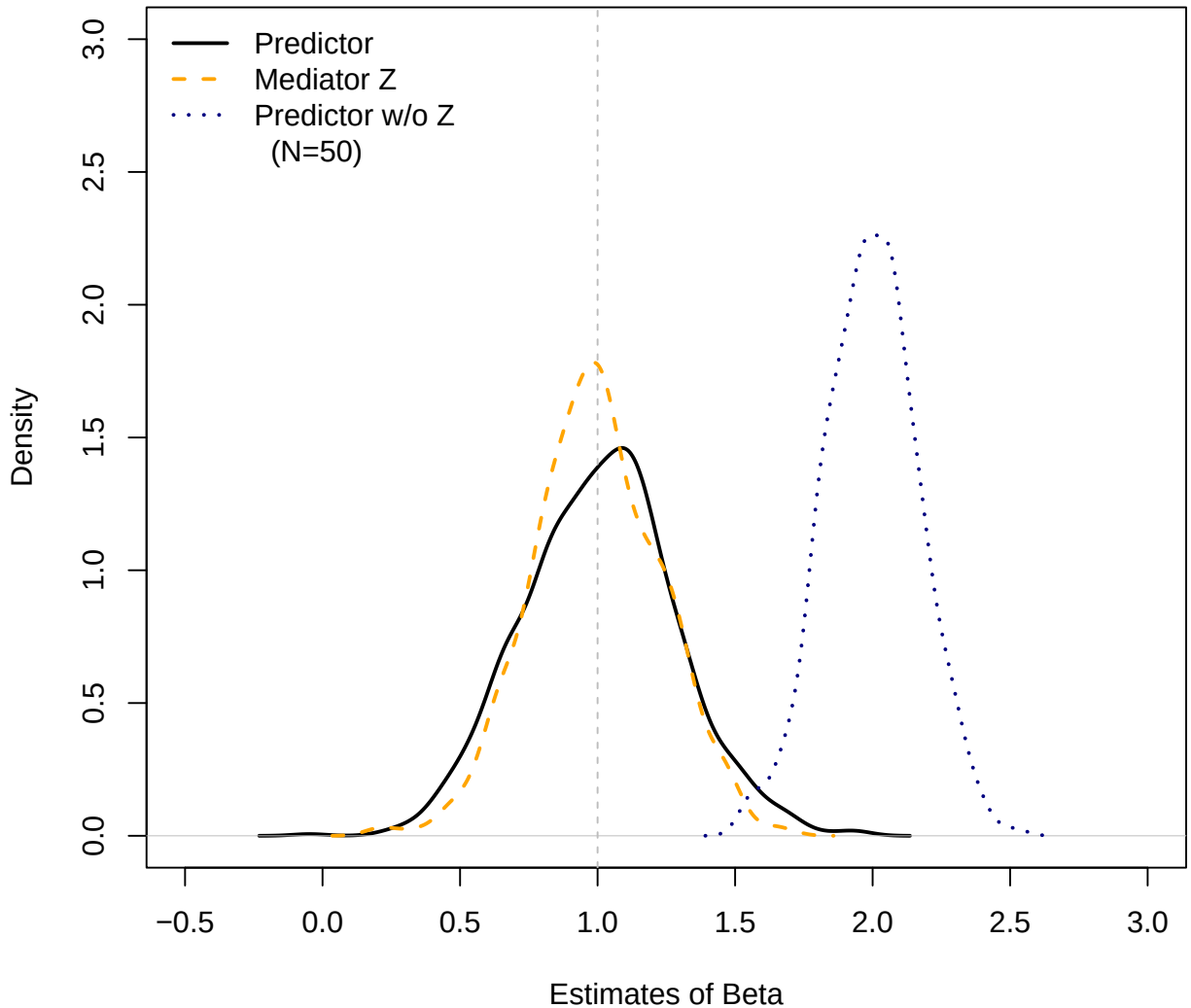
Z is a "mediator":



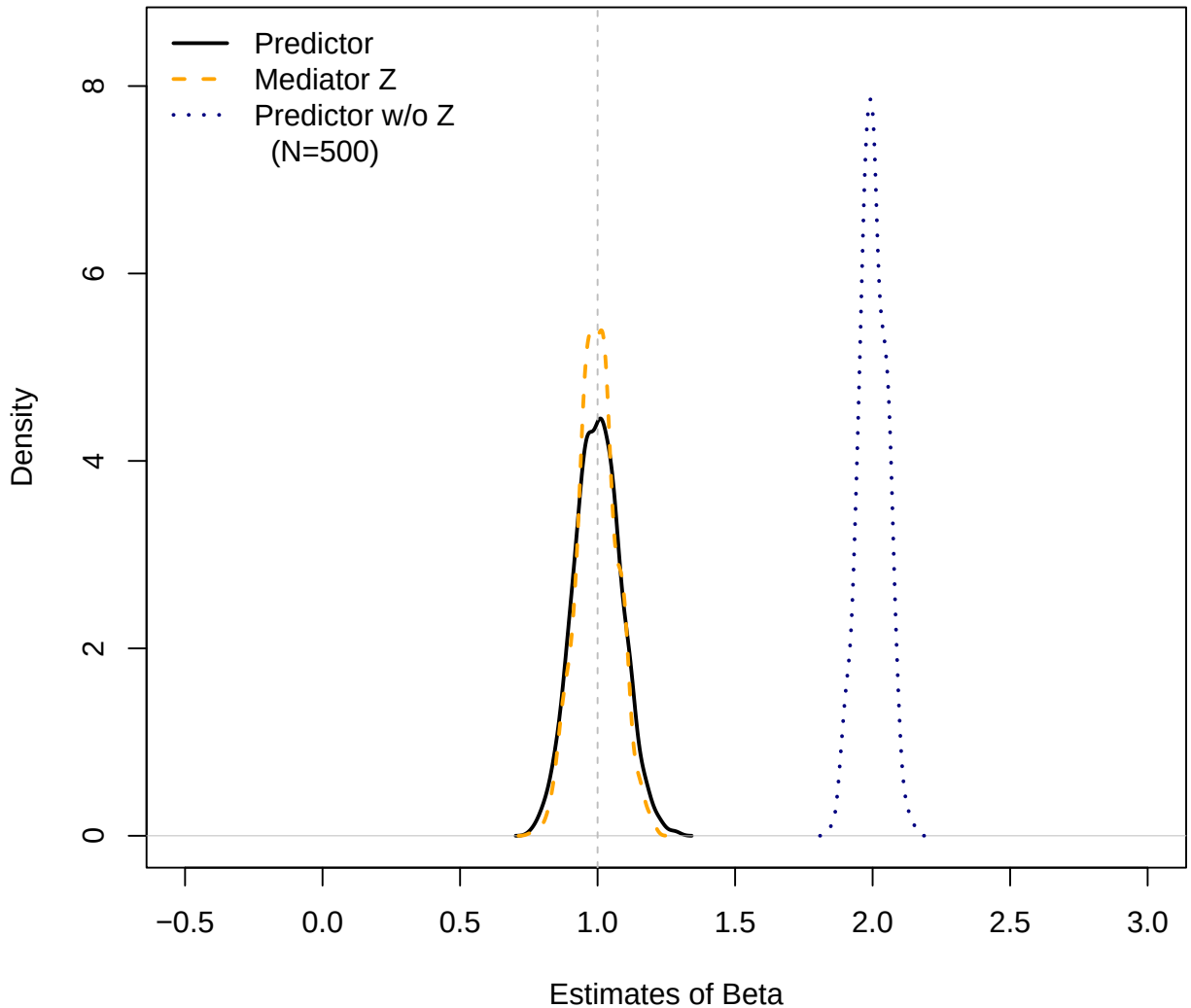
This means that:

- Z is influenced by X; Y by X and Z
- Once again, $\text{Cov}(X, Y|Z) \neq \text{Cov}(X, Y)$
- Think of $\text{Cov}(X, Y)$ = "total effect" and $\text{Cov}(X, Y|Z)$ = "direct effect"

Mediation Illustrated



Same, But With $N = 500$



Lessons...

Some takeaways... *In a linear model*:

- Variables that are irrelevant (to Y), are irrelevant...
- Variables that are relevant to Y but unrelated to X need not be modeled
- Confounders require that we condition on them, or else there's bias
- Colliders require that we *do not* condition on them, or else there's bias
- Mediators may or may not be good to condition on...

Mostly: **Model specification is hard.**

Example Data: The WDI

Data are drawn from the World Development Indicators:

- Cross-national country-level time series data
- $N = 143$ countries (due to missing data); here we'll focus on one recent year (2018)
- Available as file `WDI-2018-Day-One-24.csv` in the `Data` folder at the course Github repo.

Variables (among others):

- `ISO3` - The country's International Standards Organization (ISO) three-letter identification code (e.g., `CHE` for Switzerland).
- `Country` - The name of the country
- `Fertility` - Fertility rate (mean births per woman).
- `Child Mortality` - Average deaths of children under 5 per 1000 live births.
- `Infant Mortality` - Average deaths of infants per 1000 live births.
- `LifeExpectancy` - Life Expectancy at birth (years).
- `DPTPercent` - Percent of children aged 12-24 months w/DPT immunization.
- `GDPPerCapita` - GDP per capita (constant 2010 $US).
- `FDIIn` - Inward Foreign Direct Investment (FDI) (percent of GDP).
- `NaturalResourceRents` - Total natural resource rents (percent of GDP).
- `UrbanPopulation` - Urban Population (percent of total).
- `GovtExpenditures` - Government Expenditures (percent of GDP).
- `PaidParentalLeave` - Paid Parental Leave (0 = No, 1 = Yes).
- `DemocScore` - POLITY V democracy score (0-10).
- `AutocScore` - POLITY V autocracy score (0-10).
- `POLITY` - POLITY V total score (`DemocScore` - `AutocScore`).

Summary Statistics

```
> describe(IR2018)
```

	vars	n	mean	sd	median	trimmed	mad	min	max	range	skew	kurtosis	se
ISO3*	1	143	72.00	41.42	72.00	72.00	53.37	1.00	143.00	142.00	0.00	-1.23	3.46
Country*	2	143	72.00	41.42	72.00	72.00	53.37	1.00	143.00	142.00	0.00	-1.23	3.46
Fertility	3	143	2.71	1.39	2.15	2.52	1.02	0.98	7.02	6.05	1.09	0.31	0.12
ChildMortality	4	143	28.96	30.52	15.40	23.57	17.20	2.40	123.20	120.80	1.35	0.99	2.55
InfantMortality	5	143	21.55	20.47	13.40	18.45	14.97	1.90	85.00	83.10	1.10	0.27	1.71
LifeExpectancy	6	143	72.37	7.82	73.85	72.82	8.91	52.83	84.21	31.39	-0.45	-0.82	0.65
DPTPercent	7	143	88.90	11.73	93.00	91.19	5.93	42.00	99.00	57.00	-2.06	4.53	0.98
GDPPerCapita	8	143	14093.79	19826.50	5233.28	9823.82	6185.18	274.13	106376.78	106102.65	2.08	4.29	1657.97
FDIIn	9	143	1.37	12.07	2.38	2.45	2.11	-117.37	29.21	146.58	-7.05	64.58	1.01
NaturalResourceRents	10	143	6.73	10.55	2.26	4.34	2.98	0.00	62.77	62.77	2.55	7.40	0.88
UrbanPopulation	11	143	60.40	21.62	61.58	61.13	26.32	13.03	100.00	86.97	-0.24	-0.88	1.81
GovtExpenditures	12	143	15.72	6.45	15.48	15.29	5.72	3.60	56.31	52.71	2.00	10.03	0.54
PaidParentalLeave	13	143	0.29	0.46	0.00	0.24	0.00	0.00	1.00	1.00	0.90	-1.20	0.04
DemocScore	14	143	6.01	3.75	7.00	6.26	4.45	0.00	10.00	10.00	-0.58	-1.23	0.31
AutocScore	15	143	1.55	2.55	0.00	0.97	0.00	0.00	10.00	10.00	1.74	2.05	0.21
POLITY	16	143	4.45	6.01	7.00	5.27	4.45	-10.00	10.00	20.00	-0.97	-0.46	0.50
Rich*	17	143	1.50	0.50	1.00	1.50	0.00	1.00	2.00	1.00	0.01	-2.01	0.04
IMDPTres	18	143	0.00	14.75	-3.75	-1.39	10.15	-38.23	68.41	106.64	1.16	2.43	1.23
IMDPThat	19	143	21.55	14.19	16.59	18.78	7.17	9.33	78.29	68.96	2.06	4.53	1.19

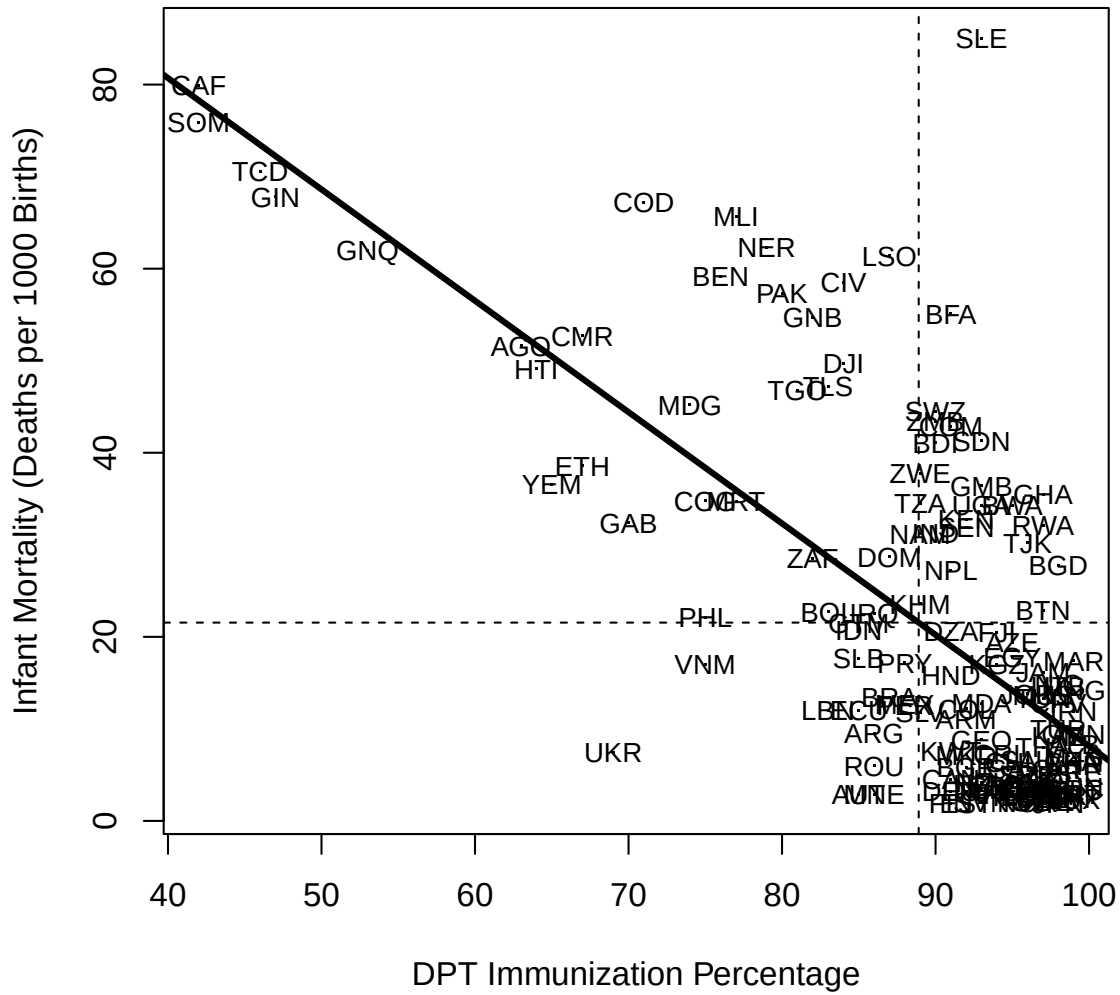
Bivariate OLS Regression

```
> IMDPT<-lm(InfantMortality~DPTPercent,data=IR2018)
> summary.lm(IMDPT)

Call:
lm(formula = InfantMortality ~ DPTPercent, data = IR2018)

Residuals:
   Min     1Q Median     3Q    Max
-38.23  -9.64  -3.75   5.18  68.41

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  129.104      9.491    13.6   <2e-16 ***
DPTPercent    -1.210      0.106   -11.4   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 14.8 on 141 degrees of freedom
Multiple R-squared:  0.481, Adjusted R-squared:  0.477
F-statistic:  131 on 1 and 141 DF,  p-value: <2e-16
```

Analysis of Variance

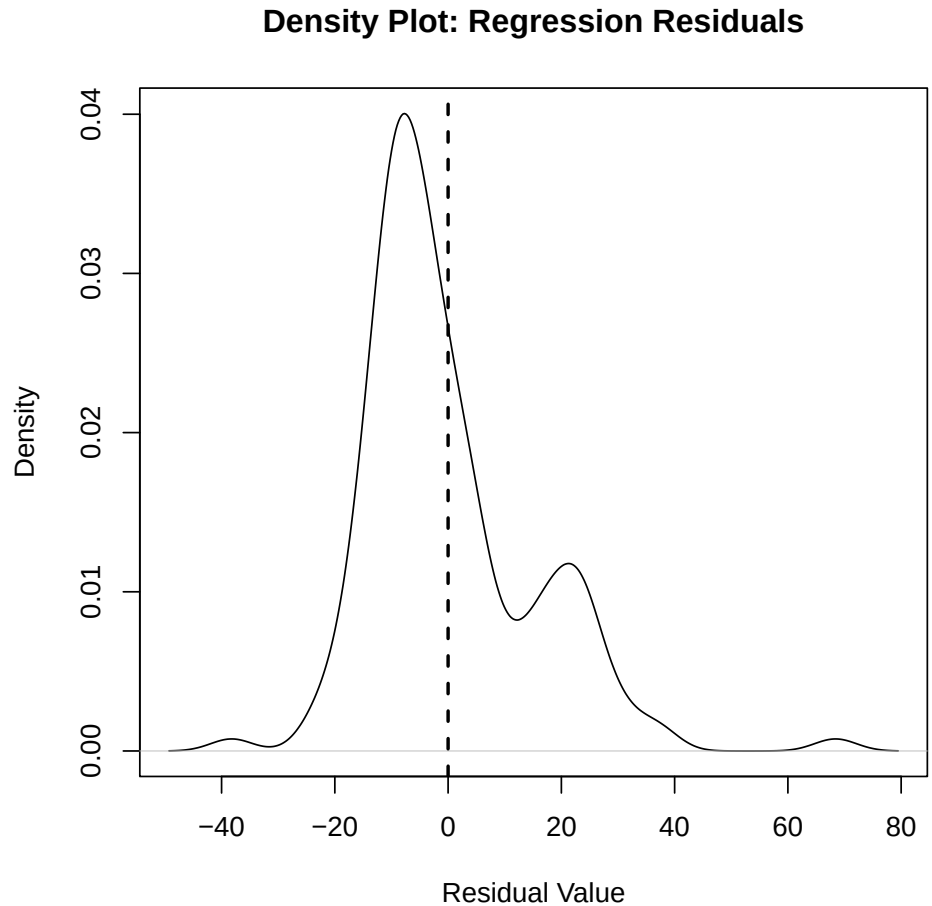
```
> anova(IMDPT)

Analysis of Variance Table

Response: InfantMortality
             Df Sum Sq Mean Sq F value Pr(>F)
DPTPercent    1  28607   28607     131 <2e-16 ***
Residuals   141  30874     219
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Regression of Infant Mortality on DPT Immunization Rates



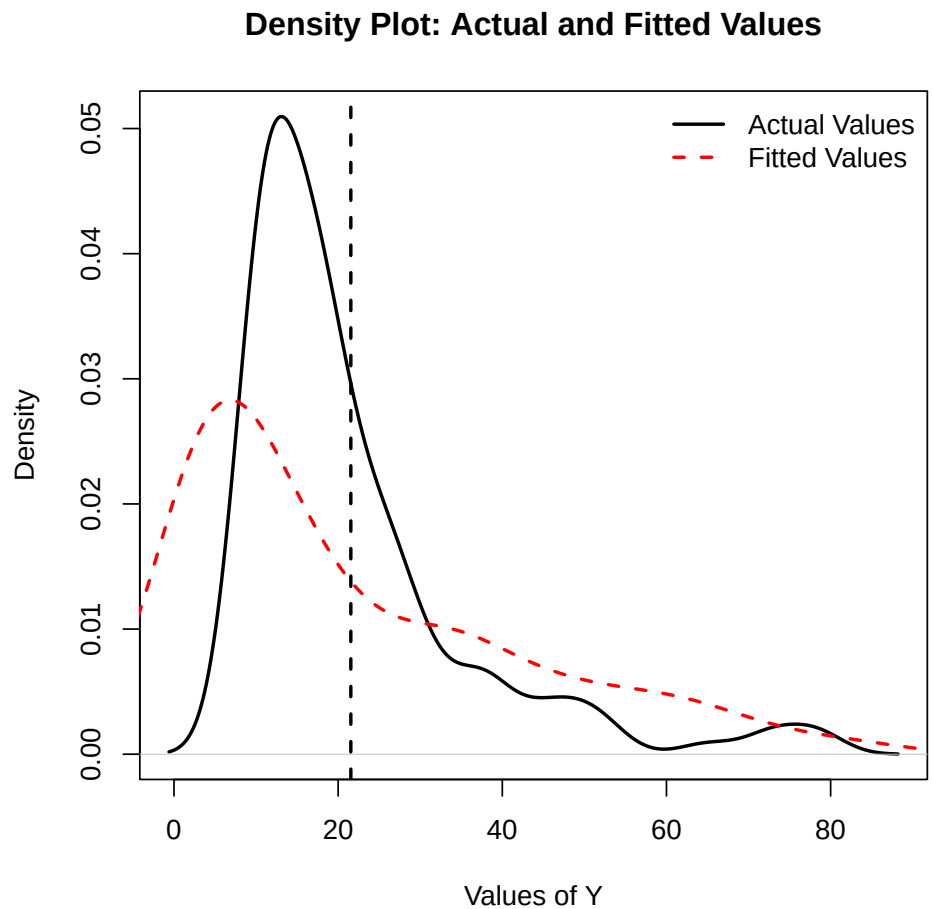
Fitted Values, Residuals, etc.

```
> # Residuals (u):
> IR2018$IMDPTres <- with(IR2018, residuals(IMDPT))
> describe(IR2018$IMDPTres)
   vars   n mean   sd median trimmed  mad   min  max range skew kurtosis   se
X1    1 143    0 14.8  -3.75   -1.39 10.2 -38.2 68.4   107 1.16     2.43 1.23
```

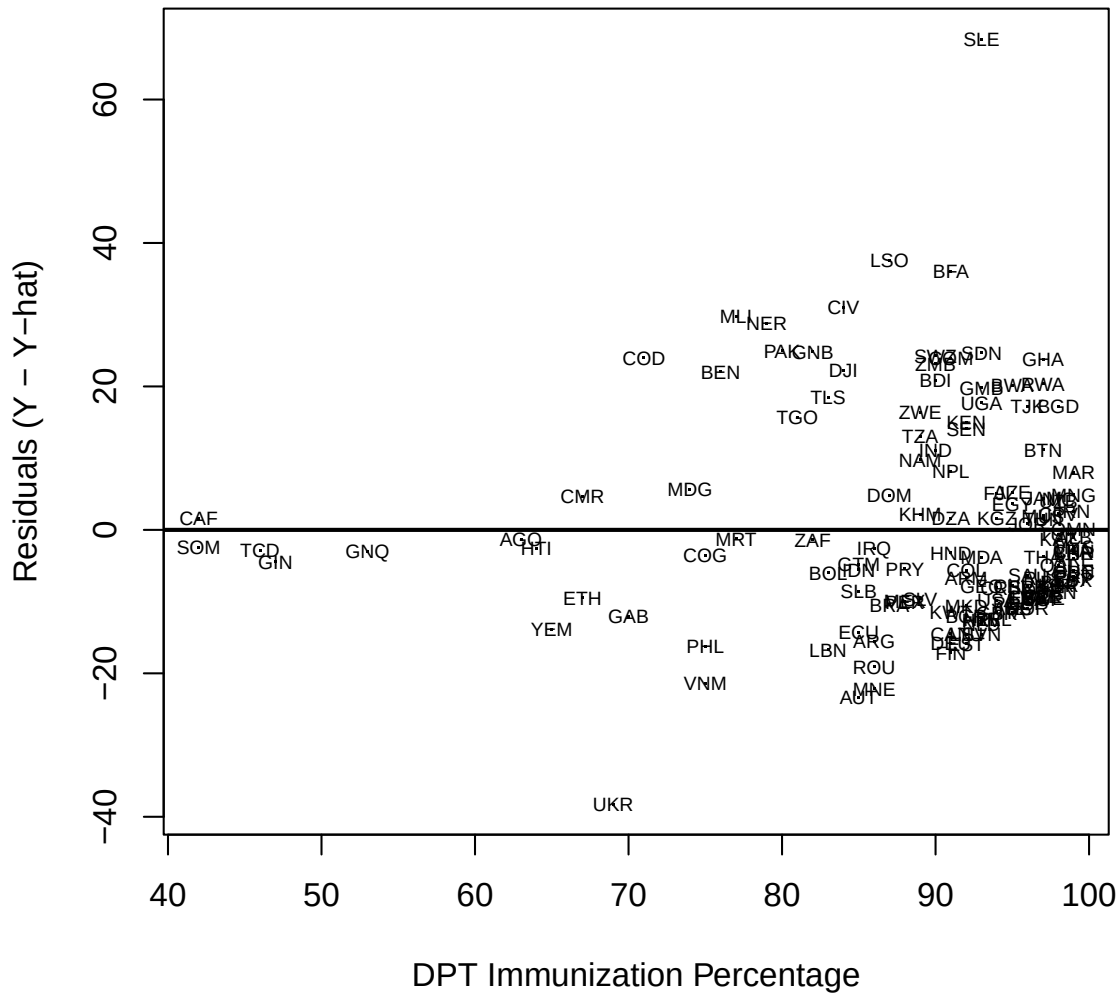


Fitted Values

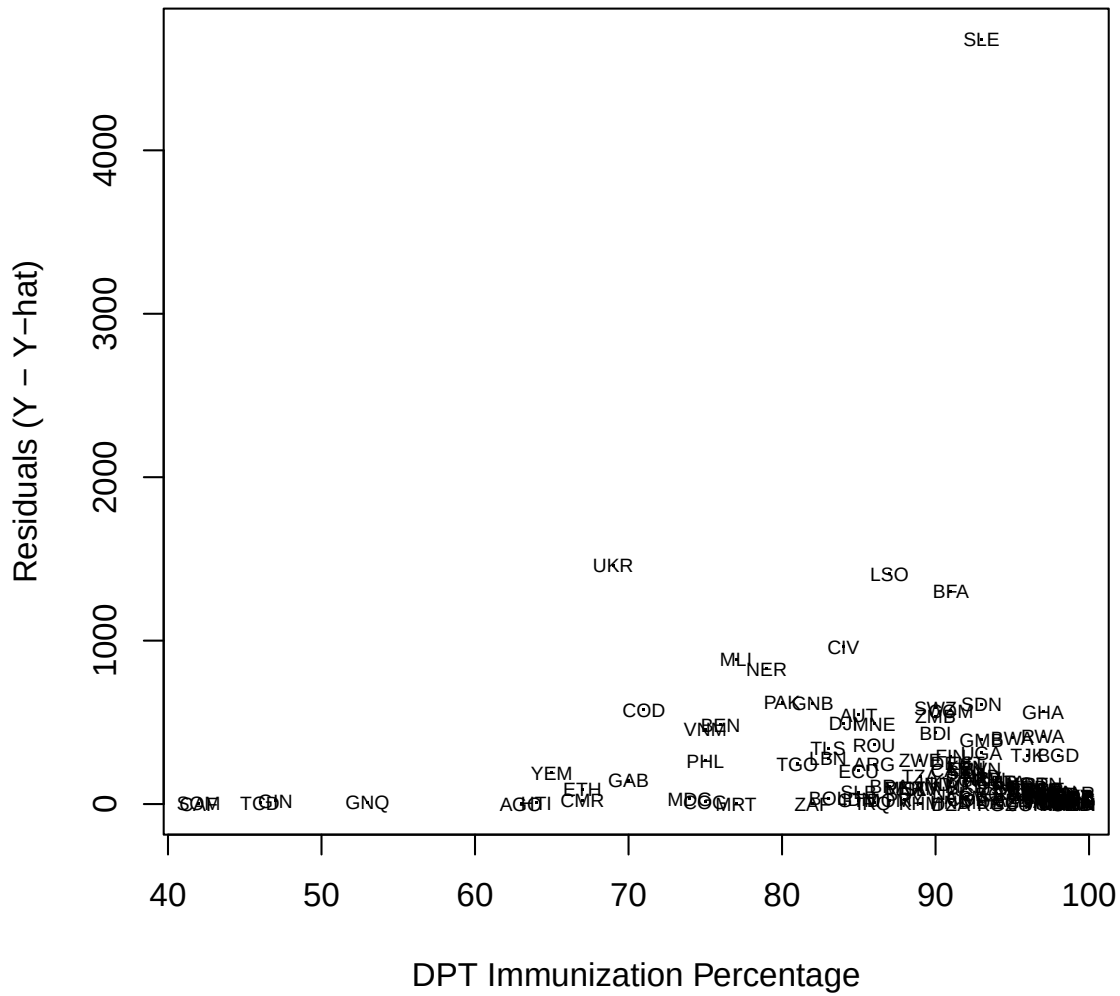
```
> # Fitted Values:
> IR2018$IMDPThat<-fitted.values(IMDPT)
> describe(IR2018$IMDPThat)
   vars   n mean   sd median trimmed  mad  min  max range skew kurtosis   se
X1    1 143 21.6 14.2   16.6    18.8 7.17 9.33 78.3    69 2.06     4.53 1.19
```



Regression Residuals ($\hat{u}$) vs. DPT Percentage



Squared Residuals vs. DPT Percentage



Var($\hat{\beta}$):

```
> vcov(IMDPT)

             (Intercept) DPTPercent
(Intercept)       90.078    -0.9960
DPTPercent        -0.996     0.0112
```

95 percent c.i.s:

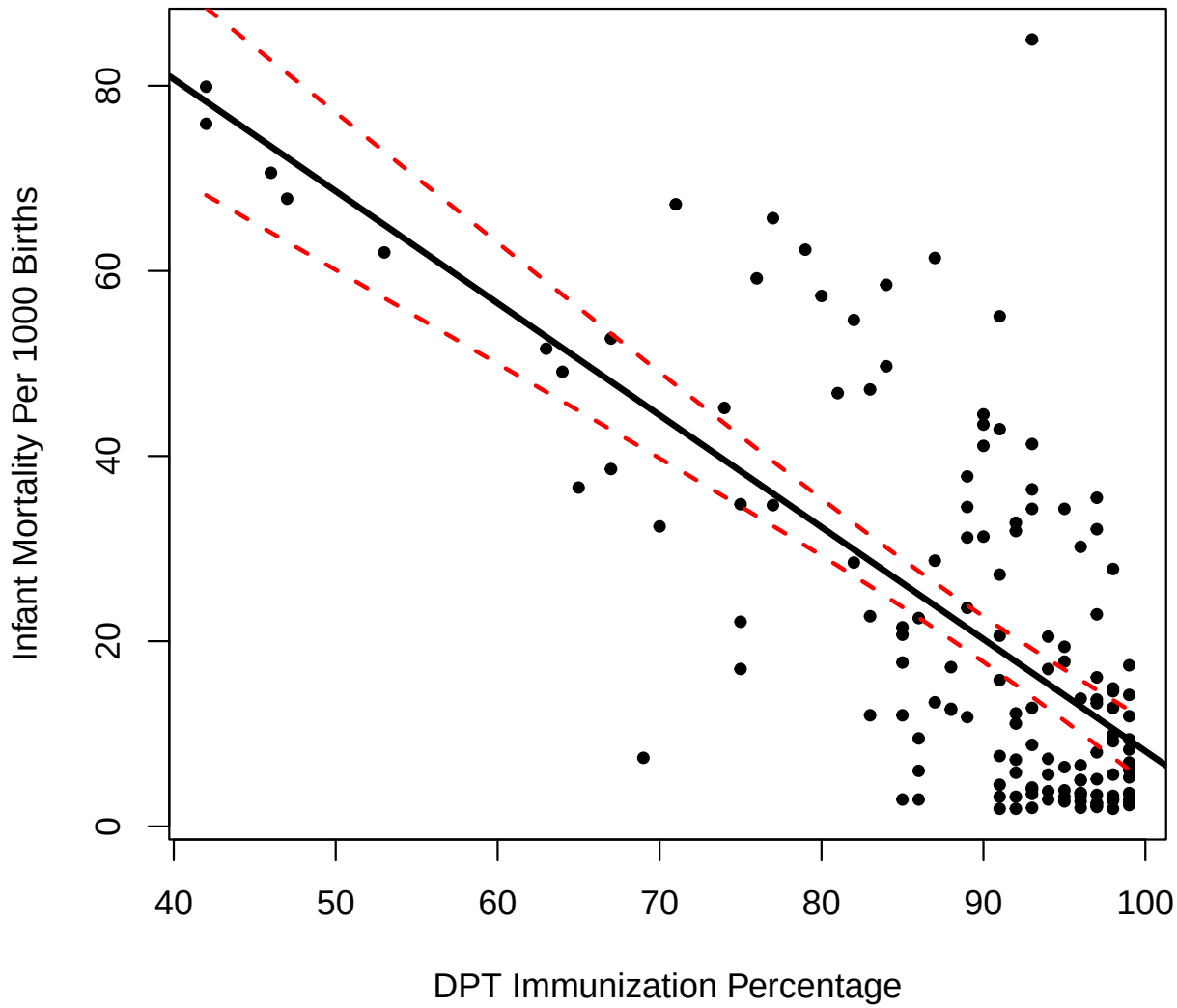
```
> confint(IMDPT)

              2.5 % 97.5 %
(Intercept) 110.34    148
DPTPercent   -1.42     -1
```

Predictions

```
> SEs<-predict(IMDPT,interval="confidence")
> SEs
      fit   lwr  upr
3   52.88 46.94 58.8
4    9.33  6.10 12.6
6    9.33  6.10 12.6
.
.
<rows omitted>
.
.
212 50.46 44.90 56.0
213 29.90 27.06 32.7
214 20.22 17.76 22.7
215 21.43 18.98 23.9
```

A Plot, With CIs

Scatterplot of Infant Mortality and DPT Immunizations, along with Least-Squares Line and 95% Prediction Confidence Intervals



Multivariate Model: Child Mortality

```
> model<-lm(InfantMortality~DPTPercent+GDPPerCapita+FDIIn+
+           NaturalResourceRents+UrbanPopulation+GovtExpenditures+
+           POLITY+PaidParentalLeave,data=IR2018)
> summary(model)

Call:
lm(formula = InfantMortality ~ DPTPercent + GDPPerCapita + FDIIn +
    NaturalResourceRents + UrbanPopulation + GovtExpenditures +
    POLITY + PaidParentalLeave, data = IR2018)

Residuals:
   Min     1Q Median     3Q    Max
-30.77  -5.97  -1.01   5.25  57.79

Coefficients:
                        Estimate  Std. Error t value Pr(>|t|)
(Intercept)          120.1847454   8.2458035   14.58  < 2e-16 ***
DPTPercent            -0.8508569   0.0936331   -9.09  1.2e-15 ***
GDPPerCapita          -0.0000915   0.0000717   -1.28   0.2044
FDIIn                 -0.1000156   0.0897276   -1.11   0.2670
NaturalResourceRents   0.1521474   0.1142324    1.33   0.1852
UrbanPopulation       -0.3295500   0.0571374   -5.77  5.3e-08 ***
GovtExpenditures       0.0036648   0.1727382    0.02   0.9831
POLITY                -0.0928692   0.1947269   -0.48   0.6342
PaidParentalLeave     -7.9426051   2.4968797   -3.18   0.0018 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 11.5 on 134 degrees of freedom
Multiple R-squared:  0.703, Adjusted R-squared:  0.685
F-statistic: 39.6 on 8 and 134 DF,  p-value: <2e-16
```

Variance-Covariance Matrix of $\hat{\beta}$

```
> vcov(model)
                    (Intercept)   DPTPercent   GDPPerCapita       FDIIn NaturalResourceRents
(Intercept)           67.993276 -0.693611256  0.00012395662  0.00773960         -0.234304691
DPTPercent            -0.693611  0.008767158 -0.00000058798 -0.00019383          0.002875263
GDPPerCapita           0.000124 -0.000000588  0.00000000515  0.00000264          0.000000642
FDIIn                  0.007740 -0.000193834  0.00000264192  0.00805104         -0.000201076
NaturalResourceRents  -0.234305  0.002875263  0.00000064215 -0.00020108          0.013049034
UrbanPopulation       -0.084284 -0.000733489 -0.00000204452 -0.00069983         -0.000774746
GovtExpenditures      -0.014527 -0.003418874 -0.00000080316  0.00036455         -0.007817267
POLITY                -0.276336  0.002795967 -0.00000030251  0.00079257          0.010641537
PaidParentalLeave      3.270567 -0.037486336 -0.00004999336 -0.01489057          0.014976307

                    UrbanPopulation GovtExpenditures       POLITY PaidParentalLeave
(Intercept)              -0.08428441     -0.014527322 -0.276336134          3.27057
DPTPercent               -0.00073349     -0.003418874  0.002795967         -0.03749
GDPPerCapita             -0.00000204     -0.000000803 -0.000000303         -0.00005
FDIIn                    -0.00069983      0.000364547  0.000792569         -0.01489
NaturalResourceRents     -0.00077475     -0.007817267  0.010641537          0.01498
UrbanPopulation           0.00326469     -0.000559546 -0.000337250         -0.00822
GovtExpenditures         -0.00055955      0.029838489 -0.010635268         -0.02024
POLITY                   -0.00033725     -0.010635268  0.037918563         -0.07426
PaidParentalLeave        -0.00822062     -0.020241192 -0.074256519          6.23441
```

Inference: Tests...

Test $H_0 : \beta_{\texttt{FDIIn}} = \beta_{\texttt{POLITY}} = 0$:

```
> library(lmtest)
> modelsmall<-lm(InfantMortality~DPTPercent+GDPPerCapita+NaturalResourceRents+
+                       UrbanPopulation+GovtExpenditures+PaidParentalLeave,
+                       data=IR2018)

> waldtest(model,modelsmall)  # from -lmtest- package

Wald test

Model 1: InfantMortality ~ DPTPercent + GDPPerCapita + FDIIn + NaturalResourceRents +
    UrbanPopulation + GovtExpenditures + POLITY + PaidParentalLeave
Model 2: InfantMortality ~ DPTPercent + GDPPerCapita + NaturalResourceRents +
    UrbanPopulation + GovtExpenditures + PaidParentalLeave
  Res.Df Df    F Pr(>F)
1    134
2    136 -2 0.71   0.49
```

More tests...

Test $H_0 : \beta_{\texttt{NaturalResourceRents}} = 1.0$:

```
> library(car)
> linearHypothesis(model,"NaturalResourceRents=1") # from -car-
Linear hypothesis test

Hypothesis:
NaturalResourceRents = 1

Model 1: restricted model
Model 2: InfantMortality ~ DPTPercent + GDPPerCapita + FDIIn + NaturalResourceRents +
    UrbanPopulation + GovtExpenditures + POLITY + PaidParentalLeave

  Res.Df   RSS Df Sum of Sq    F  Pr(>F)
1    135 24946
2    134 17678  1      7268 55.1 1.2e-11 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

More tests...

Test $H_0 : \beta_{\texttt{NaturalResourceRents}} = -\beta_{\texttt{UrbanPopulation}}$:

```
> linearHypothesis(model,"NaturalResourceRents = -UrbanPopulation")

Linear hypothesis test

Hypothesis:
NaturalResourceRents  + UrbanPopulation = 0

Model 1: restricted model
Model 2: InfantMortality ~ DPTPercent + GDPPerCapita + FDIIn + NaturalResourceRents +
    UrbanPopulation + GovtExpenditures + POLITY + PaidParentalLeave

  Res.Df   RSS Df Sum of Sq    F Pr(>F)
1    135 17960
2    134 17678  1       281 2.13   0.15
```

Reporting: Making Tables

R

- LaTeX: `texreg`, `xtable`, `stargazer`, and `modelsummary` packages
- MS Word: generally cut-and-paste (see, e.g., here: https://sejdemyr.github.io/r-tutorials/basics/tables-in-r/); also `KableExtra`
- A pretty good summary of many others is here: https://rfortherestofus.com/2019/11/how-to-make-beautiful-tables-in-r/.

Stata

- `estout` and `esttab` commands are standard
- Others: `outreg2`, `tabout`, `orth_out`, etc. (a summary is here: https://lukestein.github.io/stata-latex-workflows/)
- MS Word: `putdocx`

Reporting (continued)

The output:

```
> summary(model)

Call:
lm(formula = InfantMortality ~ DPTPercent + GDPPerCapita + FDIIn +
    NaturalResourceRents + UrbanPopulation + GovtExpenditures +
    POLITY + PaidParentalLeave, data = IR2018)

Residuals:
   Min     1Q Median     3Q    Max
-30.77  -5.97  -1.01   5.25  57.79

Coefficients:
                        Estimate  Std. Error t value Pr(>|t|)
(Intercept)          120.1847454   8.2458035   14.58  < 2e-16 ***
DPTPercent            -0.8508569   0.0936331   -9.09  1.2e-15 ***
GDPPerCapita          -0.0000915   0.0000717   -1.28   0.2044
FDIIn                 -0.1000156   0.0897276   -1.11   0.2670
NaturalResourceRents   0.1521474   0.1142324    1.33   0.1852
UrbanPopulation       -0.3295500   0.0571374   -5.77  5.3e-08 ***
GovtExpenditures       0.0036648   0.1727382    0.02   0.9831
POLITY                -0.0928692   0.1947269   -0.48   0.6342
PaidParentalLeave     -7.9426051   2.4968797   -3.18   0.0018 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 11.5 on 134 degrees of freedom
Multiple R-squared:  0.703, Adjusted R-squared:  0.685
F-statistic: 39.6 on 8 and 134 DF,  p-value: <2e-16
```

Reporting (continued)

The table:

Table 1: OLS Regression Model of Child Mortality Rates, 2018

	Model I
(Constant)	120.00^{***}
	(8.25)
DPT Immunization Rate	-0.85^{***}
	(0.09)
GDP Per Capita	-0.0001
	(0.0001)
Inward FDI	-0.10
	(0.09)
Natural Resource Rents (Pct. GDP)	0.15
	(0.11)
Urban Population (Pct.)	-0.33^{***}
	(0.06)
Government Expenditures (Pct. GDP)	0.004
	(0.17)
POLITY V (Democracy) Score	-0.09
	(0.19)
Paid Parental Leave	-7.94^{***}
	(2.50)
Observations	143
R^2	0.70
Adjusted R^2	0.69
Residual Std. Error	11.50 (df = 134)
F Statistic	39.60^{***} (df = 8; 134)

Note: Cell entries are coefficient estimates; numbers in parentheses are estimated standard errors. $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$. *See text for details.*

Multiple Models (`stargazer` defaults)

OLS Regression Models of Infant Mortality Rates, 2018

	Univariate	Full	Reduced
(Constant)	129.00***	120.00***	120.00***
	(9.49)	(8.25)	(8.10)
DPT Immunization Rate	−1.21***	−0.85***	−0.85***
	(0.11)	(0.09)	(0.09)
GDP Per Capita		−0.0001	−0.0001
		(0.0001)	(0.0001)
Inward FDI		−0.10	
		(0.09)	
Natural Resource Rents (Pct. GDP)		0.15	0.17*
		(0.11)	(0.10)
Urban Population (Pct.)		−0.33***	−0.34***
		(0.06)	(0.06)
Government Expenditures (Pct. GDP)		0.004	−0.02
		(0.17)	(0.16)
POLITY V (Democracy) Score		−0.09	
		(0.19)	
Paid Parental Leave		−7.94***	−8.29***
		(2.50)	(2.46)
Observations	143	143	143
R^2	0.48	0.70	0.70
Adjusted R^2	0.48	0.69	0.69
Residual Std. Error	14.80 (df = 141)	11.50 (df = 134)	11.50 (df = 136)
F Statistic	131.00*** (df = 1; 141)	39.60*** (df = 8; 134)	52.80*** (df = 6; 136)

Note: *p<0.1; **p<0.05; ***p<0.01

Some Guidelines ("Rules"?)

Tables:

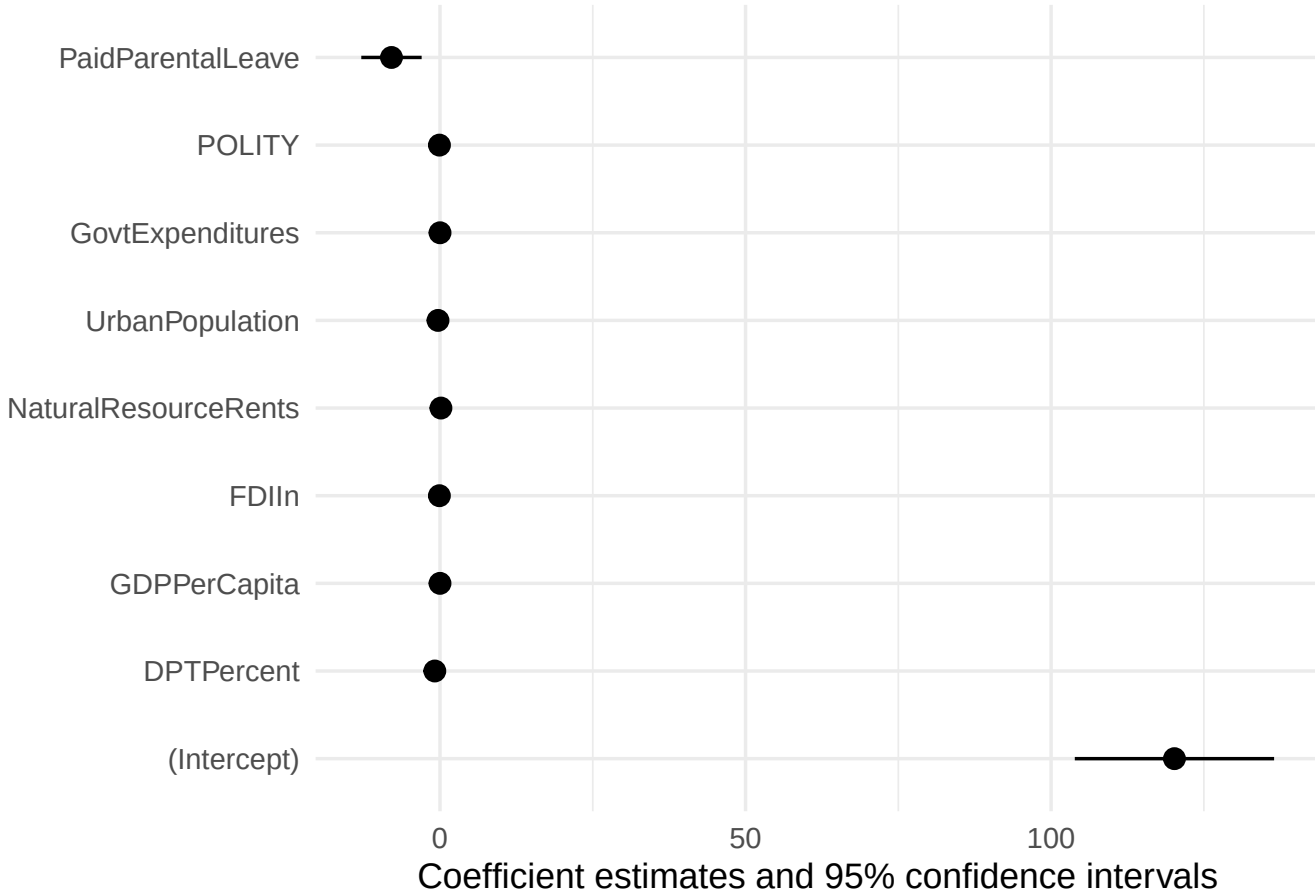
- *Use column headings descriptively.*
- *Use multiple rows / columns rather than multiple tables.*
- *Learn about significant digits, and don't report more than 4-5 of them (at the most).*
- *Use a figure to replace a table when you can.*
- *Be aware of norms about *s.*

Figures:

- *Report the scale of axes, and label them.*
- *Use as much "space" as you need, but no more.*
- *Use color sparingly.*

Plotting Regression Estimates

Ladderplot of OLS Results (using `modelplot` defaults)



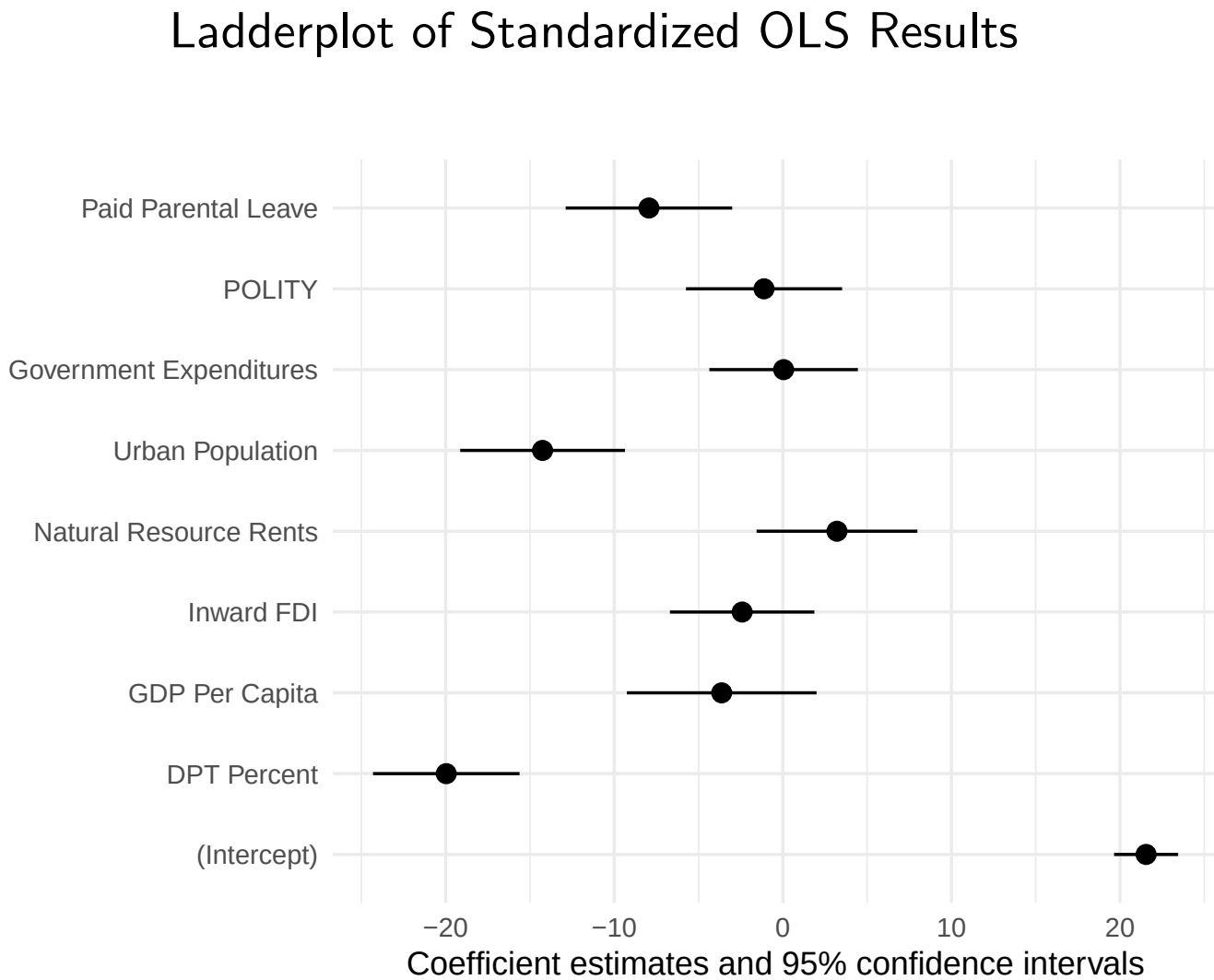
Rescaling Covariates

A la Gelman (2008):

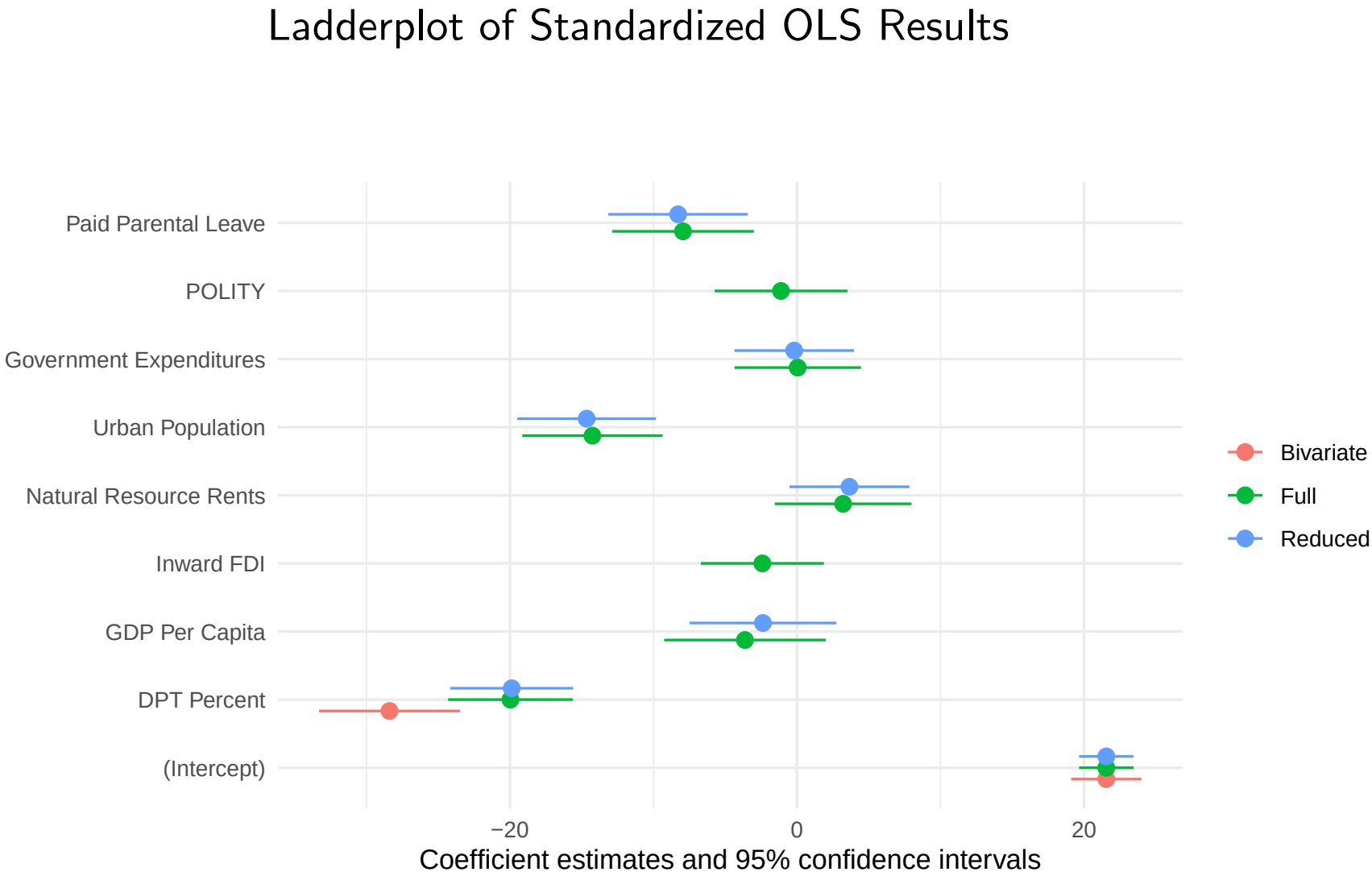
- Continuous = divide by two standard deviations
- Binary = mean 0, difference of 1 between the two categories

```
> modelS<-standardize(model)
> summary(modelS)

Call:
lm(formula = InfantMortality ~ z.DPTPercent + z.GDPPerCapita +
    z.FDIIn + z.NaturalResourceRents + z.UrbanPopulation + z.GovtExpenditures +
    z.POLITY + c.PaidParentalLeave, data = IR2018)

Coefficients:
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)            21.5469     0.9605   22.43  < 2e-16 ***
z.DPTPercent          -19.9644     2.1970   -9.09  1.2e-15 ***
z.GDPPerCapita         -3.6274     2.8445   -1.28   0.2044
z.FDIIn                -2.4140     2.1657   -1.11   0.2670
z.NaturalResourceRents  3.2110     2.4108    1.33   0.1852
z.UrbanPopulation      -14.2529     2.4712   -5.77  5.3e-08 ***
z.GovtExpenditures       0.0472     2.2269    0.02   0.9831
z.POLITY                -1.1163     2.3407   -0.48   0.6342
c.PaidParentalLeave     -7.9426     2.4969   -3.18   0.0018 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 11.5 on 134 degrees of freedom
Multiple R-squared:  0.703, Adjusted R-squared:  0.685
F-statistic: 39.6 on 8 and 134 DF,  p-value: <2e-16
```

A Better Regression Plot



An Even Better Regression Plot



Some Meta-Rules

- *Be aware of the norms in your discipline / field, and follow them.*
- *If it has uncertainty, you should show it.*
- *Ask for advice.*
- *When in doubt, more information is (probably) better.*

Supplement: R Things

Inside the Source Window

This:

```
> table(df$X)
```

... means "Type the phrase 'table(df$X)' on the command line," or – equivalently – "Type the phrase 'table(df$X)' into your Source code, and then run it."

Inside the Source Window

More often, you'll see:

```
with(df, plot(Y~X,pch=19,col="red")) # draw a scatterplot
abline(h=0,lty=2) # add a horizontal line at zero
abline(v=0,lty=2) # add a vertical line at zero
text(df$X,df$Y,labels=df$names,pos=1) # add labels
```

... which means "Put this block of text into your Source code, and then run it."

Note:

- R / RStudio ignores line breaks
- Anything to the right of a "#" is a comment

Very basic R examples...

(see `GSERM-June-2024-R-Intro.R` in the github repo)

Help For Learning R(Studio)

In rough order of preference:

- Quick-R (`http://www.statmethods.net/`)
- The "Level-Zero" R Tutorial (doesn't integrate RStudio, but is otherwise very good)
- Statistics with R
- The Do It Yourself Introduction to R
- Also be sure to consult the Regression for Publishing "Useful R Resources" guide (on GitHub).