PLSC 502 – Fall 2022
Linear Regression II

December 1, 2022

Model Fit

- The closeness of the mapping between model-based values of Y and actual values of Y...
- Can be *in-sample* or *out-of-sample* ($\rightarrow$ "overfitting")
- Is (in part) a function of *model specification* (choice of predictors, functional form, interactions, etc.)
- Related (but not identical) to prediction / predictive ability

R^2 Introduced

Recall that for

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

We have:

$$\begin{aligned}
\text{“TSS”} &= \sum (Y_i - \bar{Y})^2 \\
\text{“MSS”} &= \sum (\hat{Y}_i - \bar{Y})^2 \\
\text{“RSS”} &= \sum (Y_i - \hat{Y})^2 \equiv \sum \hat{u}_i^2
\end{aligned}$$

Then:

$$\begin{aligned}
R^2 &= \frac{\sum (\hat{Y}_i - \bar{Y})^2}{\sum (Y_i - \bar{Y})^2} \\
&= \frac{\text{MSS}}{\text{TSS}} \\
&= 1 - \frac{\text{RSS}}{\text{TSS}} \\
&= 1 - \frac{\sum \hat{u}_i^2}{\sum (Y_i - \bar{Y})^2}
\end{aligned}$$

R^2 Is...

R-squared:

- is "the proportion of variance explained"
- $\in [0, 1]$
  - $R^2 = 1.0 \equiv$ a "perfect (linear) fit" '
  - $R^2 = 0 \equiv$ no (linear) $X - Y$ association

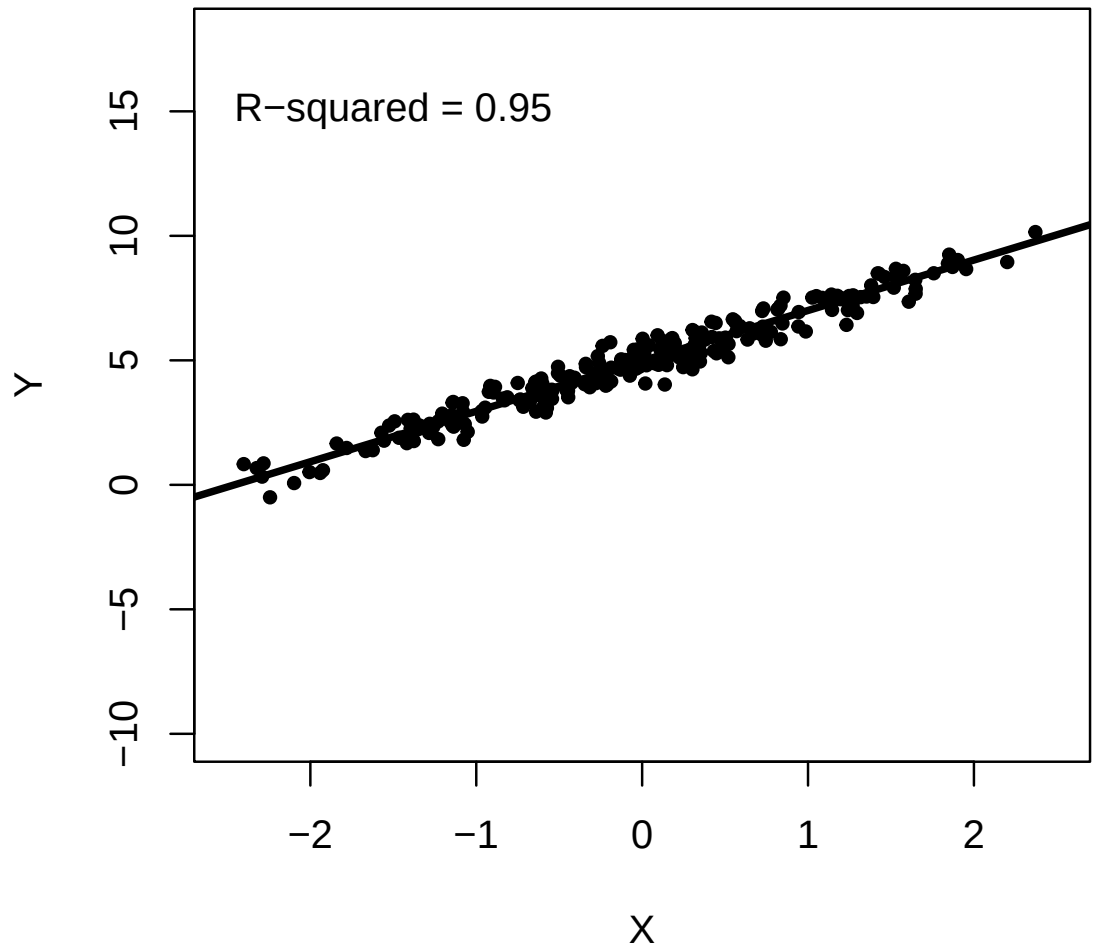
For a single X,

$$\begin{aligned} R^2 &= \hat{\beta}_1^2 \frac{\sum (X_i - \bar{X})^2}{\sum (Y_i - \bar{Y})^2} \\ &= (r_{XY})^2 \end{aligned}$$

A (Simulated) Example

```
seed <- 7222009
set.seed(seed)
> X<-rnorm(250)
> Y1<-5+2*X+rnorm(250,mean=0,sd=sqrt(0.2))
> Y2<-5+2*X+rnorm(250,mean=0,sd=sqrt(20))
> fit<-lm(Y1~X)
> summary(fit)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  4.97712    0.02846  174.86   <2e-16 ***
X            2.02529    0.02785   72.73   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.4491 on 248 degrees of freedom
Multiple R-squared:  0.9552,Adjusted R-squared:  0.955
F-statistic:  5290 on 1 and 248 DF,  p-value: < 2.2e-16
```

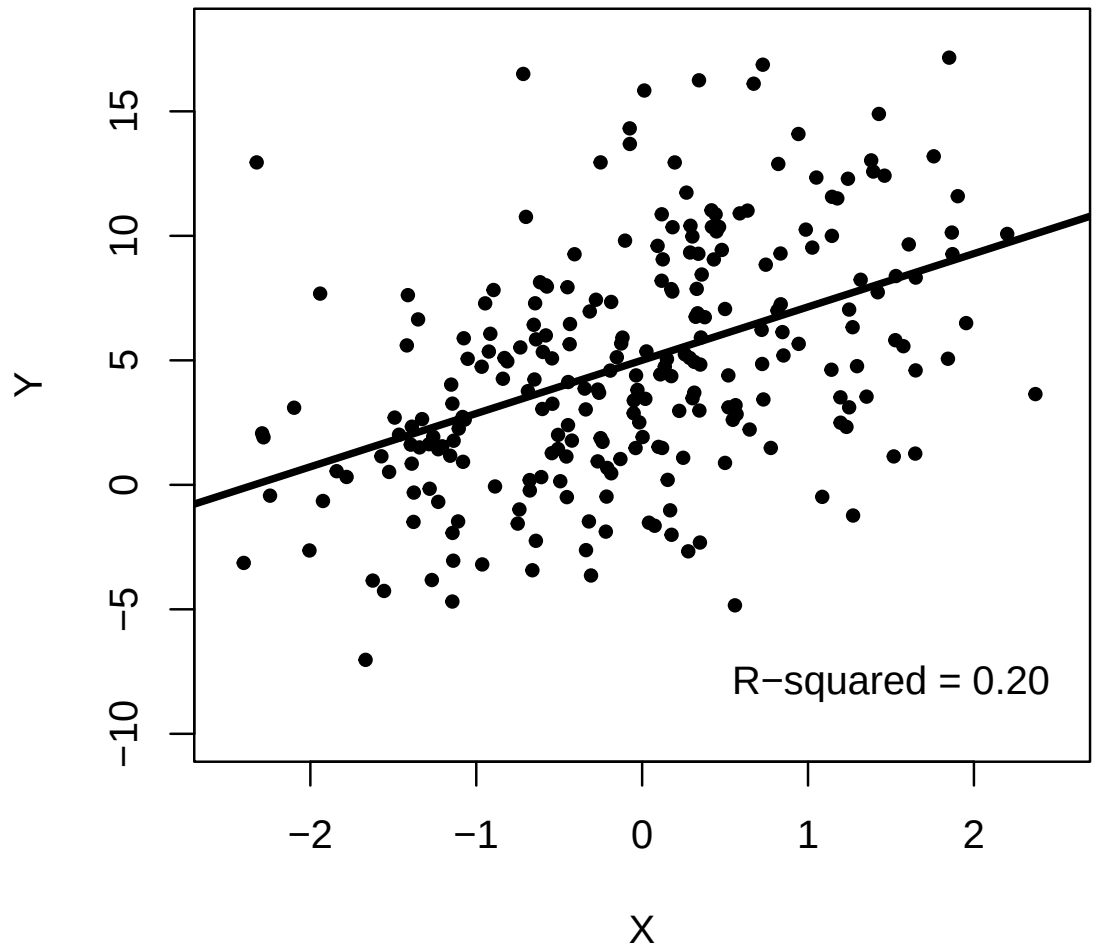
Regression of $Y_i = 5 + 2X_i + u_i$ $(R^2 = 0.95)$



Same Slope/Intercept, Different R^2

```
> fit2<-lm(Y2~X)
> summary(fit2)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   5.0048     0.2757  18.151  < 2e-16 ***
X             2.1402     0.2697   7.934 7.29e-14 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 4.351 on 248 degrees of freedom
Multiple R-squared:  0.2024,Adjusted R-squared:  0.1992
F-statistic: 62.95 on 1 and 248 DF,  p-value: 7.288e-14
```

Regression of $Y_i = 5 + 2X_i + u_i$ $(R^2 = 0.20)$



R^2 is Also an *Estimate*...

Luskin: Population analogue "P^2":

$$P^2 = 1 - \frac{\sigma^2}{\sigma_Y^2}$$

Then $\hat{P}^2 = R^2$ has variance:

$$\widehat{\text{Var}(R^2)} = \frac{4R^2(1-R^2)^2(N-k)^2}{(N^2-1)(N+3)}$$

and standard error:

$$\widehat{\text{s.e.}(R^2)} = \sqrt{\frac{4R^2(1-R^2)^2(N-k)^2}{(N^2-1)(N+3)}}.$$

Adjusted R^2

"Adjusted" R^2 is:

$$R^2_{adj.} = 1 - \frac{(1 - R^2)(N - c)}{(N - k)}$$

where $c = 1$ if there is a constant in the model and $c = 0$ otherwise.

$R^2_{adj.}$ characteristics:

- $R^2_{adj.} \to R^2$ as $N \to \infty$
- $R^2_{adj.}$ can be > 1, or < 0...
- $R^2_{adj.}$ increases with model "fit," but
- The extent of that increase is discounted by a factor proportional to the number of covariates.

Other R^2 / Goodness-Of-Fit Alternatives

- Standard Error of the Estimate:

$$\text{SEE} = \sqrt{\frac{\text{RSS}}{N-k}}$$

- F-statistic (bivariate regression, for $\beta_1 = 0$):

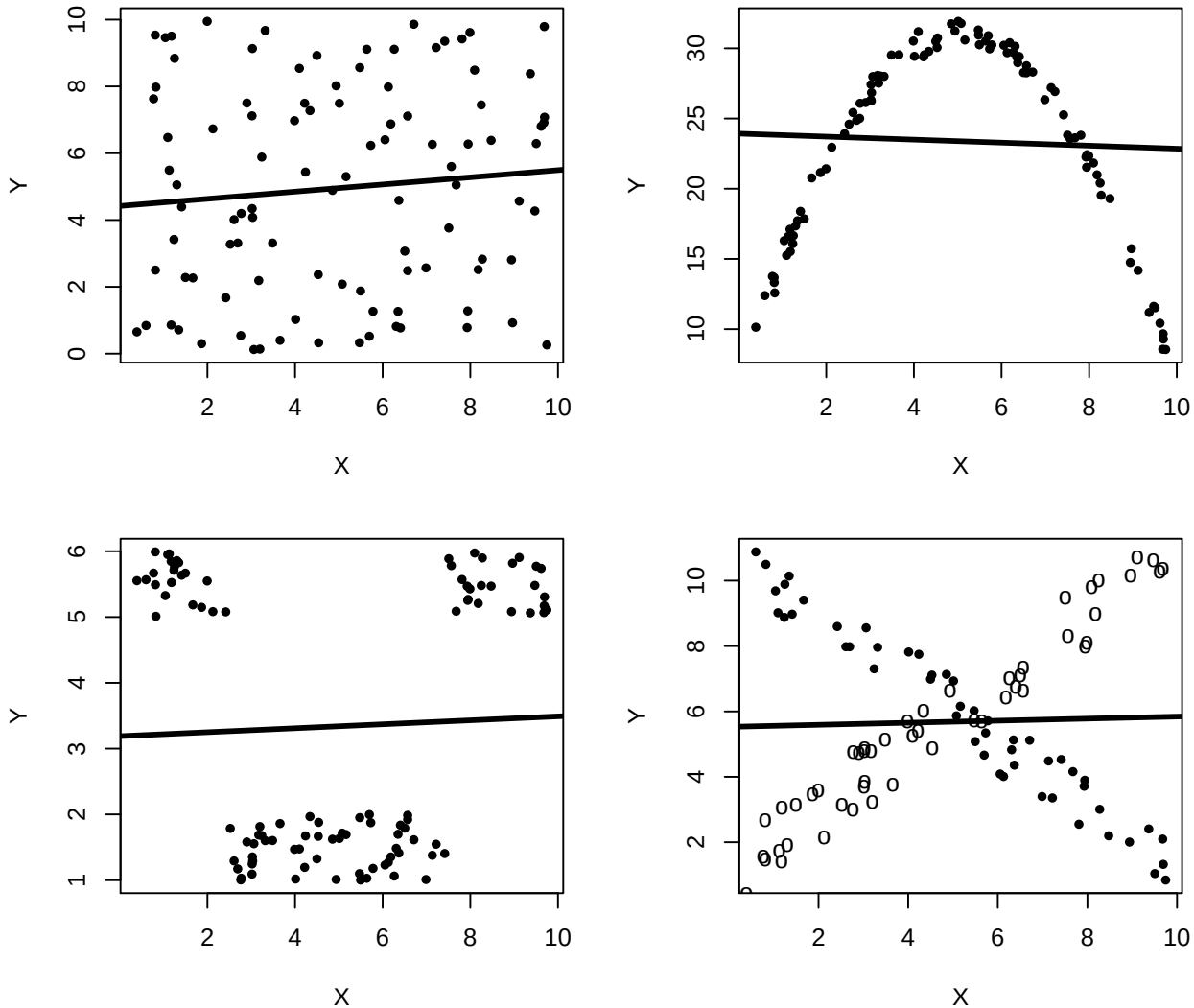
$$\begin{aligned} F &= \frac{\sum(Y_i - \bar{Y})^2 - \sum(Y_i - \hat{Y}_i)^2}{(N-1)-(N-2)} \div \frac{\sum(Y_i - \hat{Y}_i)^2}{(N-2)} \\ &= \frac{\text{“explained” variance}}{\text{“unexplained” variance}} \end{aligned}$$

  which is $\sim F(1, N-2)$.

- ROC / AUC (later...)

- Graphical methods

Caution: Different Ways to get $R^2 \approx 0$



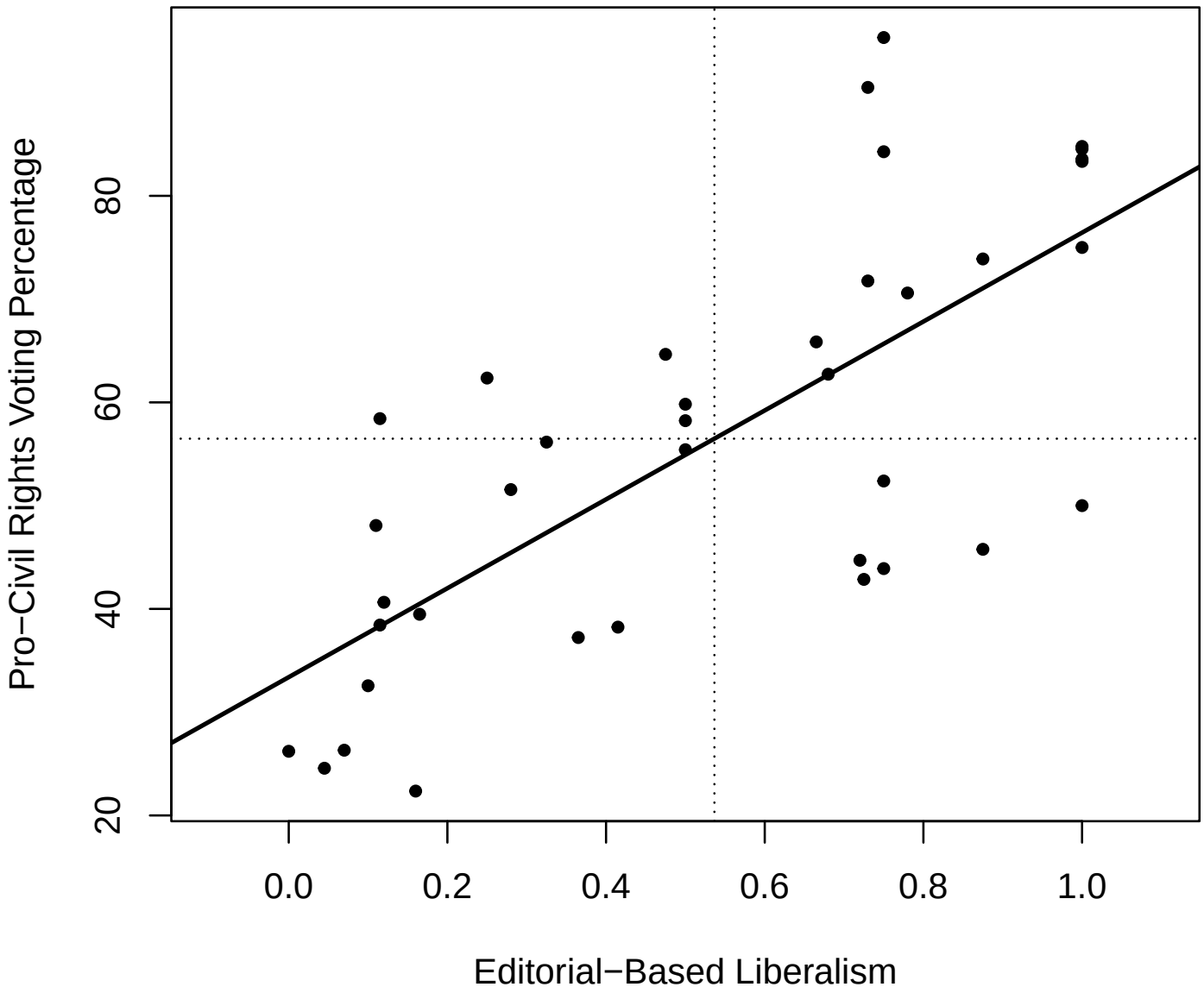
SCOTUS Redux (OT1946-2021)

Data from the Supreme Court Database and the justices' Segal-Cover scores...

- Y is `CivLibs` = liberal voting percentage in civil rights & liberties cases
- X is `IdeologyScore` $\in [0, 1] \rightarrow$ SCOTUS justice liberalism

```
> describe(SCOTUS,skew=FALSE,trim=0)
                     vars  n    mean    sd     min     max range   se
justice                 1 38   97.37 11.32   78.00  116.00 38.00 1.84
justiceName*            2 38   19.50 11.11    1.00   38.00 37.00 1.80
CivLibs                 3 38   56.49 19.94   22.36   95.33 72.97 3.23
Nom.Order*              4 38   19.50 11.11    1.00   38.00 37.00 1.80
Nominee*                5 38   19.50 11.11    1.00   38.00 37.00 1.80
ChiefJustice*           6  4    1.00  0.00    1.00    1.00  0.00 0.00
SenateVote*             7 38   17.05  8.23    1.00   25.00 24.00 1.33
IdeologyScore           8 38    0.54  0.33    0.00    1.00  1.00 0.05
QualificationsScore*    9 38   16.45  7.91    1.00   25.00 24.00 1.28
Nominator (Party)*     10 38    7.03  3.72    1.00   13.00 12.00 0.60
Year                   11 38 1969.74 24.70 1937.00 2018.00 81.00 4.01
```

Scatterplot



lm Results

```
> fit<-lm(CivLibs~IdeologyScore,data=SCOTUS)
> summary(fit)

Call:
lm(formula = CivLibs ~ IdeologyScore, data = SCOTUS)

Residuals:
    Min      1Q  Median      3Q     Max
-26.433 -10.587   2.460   7.858  29.655

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)     33.389      4.354   7.669 4.44e-09 ***
IdeologyScore   43.044      6.917   6.223 3.51e-07 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 14.03 on 36 degrees of freedom
Multiple R-squared:  0.5182,Adjusted R-squared:  0.5048
F-statistic: 38.72 on 1 and 36 DF,  p-value: 3.505e-07
```

ANOVA Redux

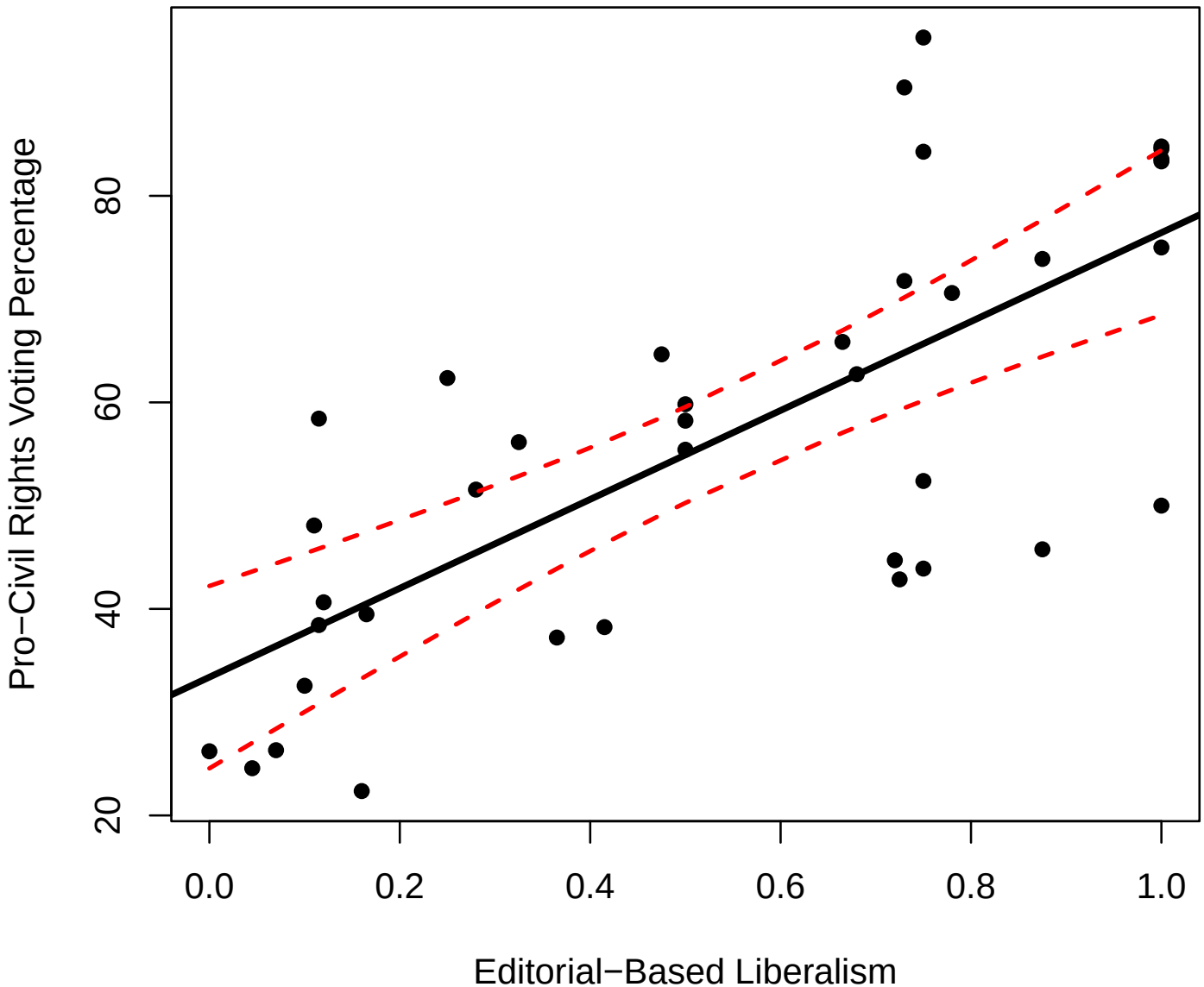
```
> anova(fit)
Analysis of Variance Table

Response: CivLibs
               Df Sum Sq Mean Sq F value     Pr(>F)
IdeologyScore   1   7621    7621    38.7 0.00000035 ***
Residuals      36   7086     197
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
>
> # R-squared:
>
> anova(fit)$`Sum Sq`[1] / (anova(fit)$`Sum Sq`[1] + anova(fit)$`Sum Sq`[2])
[1] 0.5182
>
> # F-statistic:
>
> anova(fit)$`Mean Sq`[1] / anova(fit)$`Mean Sq`[2]
[1] 38.72
```

Stupid Regression Tricks

SCOTUS Regression Redux

```
> fit<-lm(CivLibs~IdeologyScore,data=SCOTUS)
> summary(fit)

Call:
lm(formula = CivLibs ~ IdeologyScore, data = SCOTUS)

Residuals:
    Min      1Q  Median      3Q     Max
-26.433 -10.587   2.460   7.858  29.655

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)     33.389      4.354   7.669 4.44e-09 ***
IdeologyScore   43.044      6.917   6.223 3.51e-07 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 14.03 on 36 degrees of freedom
Multiple R-squared:  0.5182,Adjusted R-squared:  0.5048
F-statistic: 38.72 on 1 and 36 DF,  p-value: 3.505e-07
```

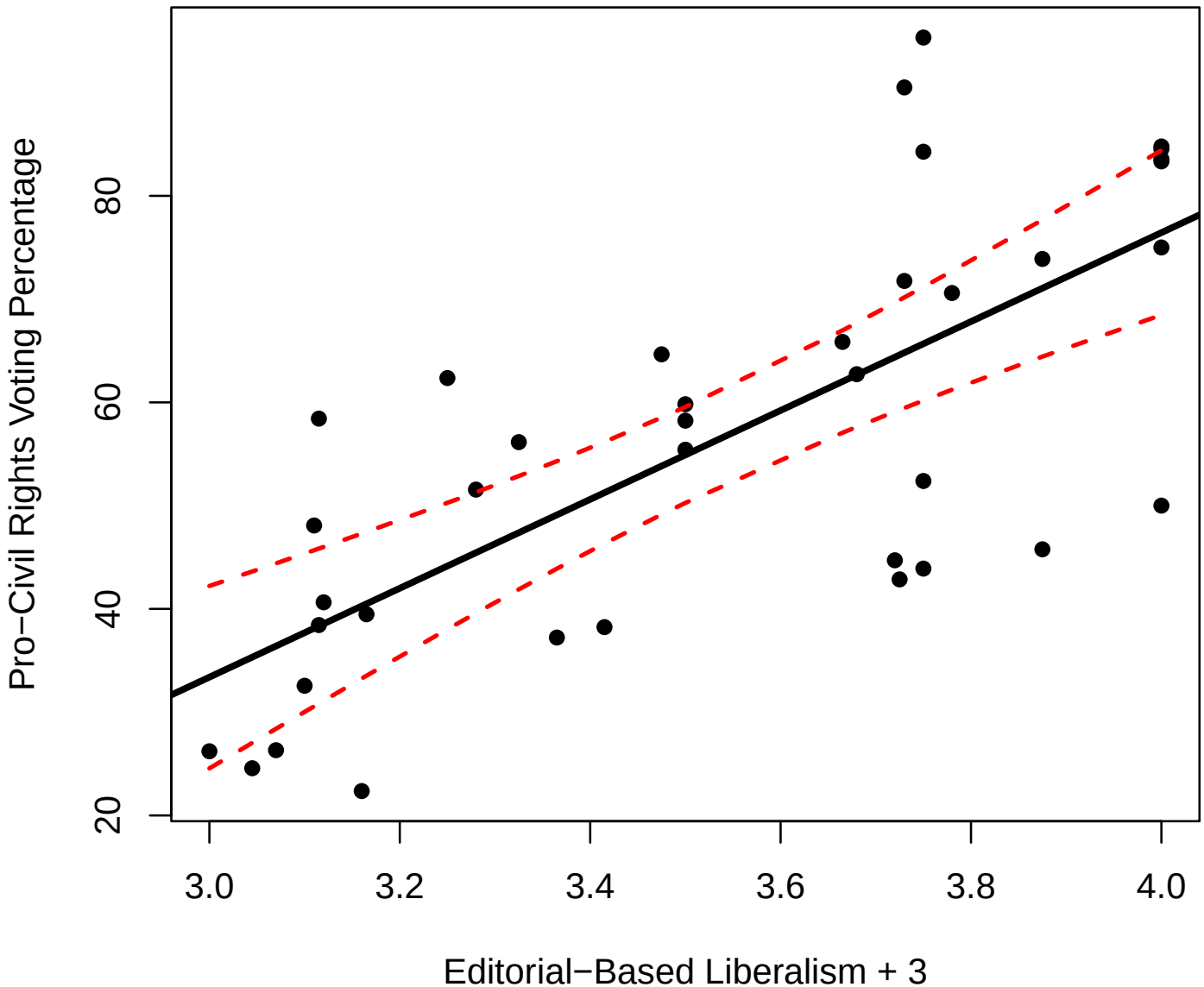
SCOTUSplot


Pro−Civil Rights Voting Percentage
20
40
60
80
0.0
0.2
0.4
0.6
0.8
1.0
Editorial−Based Liberalism

Add Three to IdeologyScore

```
> SCOTUS$IdeoPlus3 <- SCOTUS$IdeologyScore + 3
>
> fit2<-lm(CivLibs~IdeoPlus3,data=SCOTUS)
> summary(fit2)

Call:
lm(formula = CivLibs ~ IdeoPlus3, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-26.43 -10.59   2.46   7.86  29.66

Coefficients:
            Estimate Std. Error t value   Pr(>|t|)
(Intercept)   -95.74      24.57   -3.90    0.00041 ***
IdeoPlus3      43.04       6.92    6.22 0.00000035 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 14 on 36 degrees of freedom
Multiple R-squared:  0.518,Adjusted R-squared:  0.505
F-statistic: 38.7 on 1 and 36 DF,  p-value: 0.000000351
```

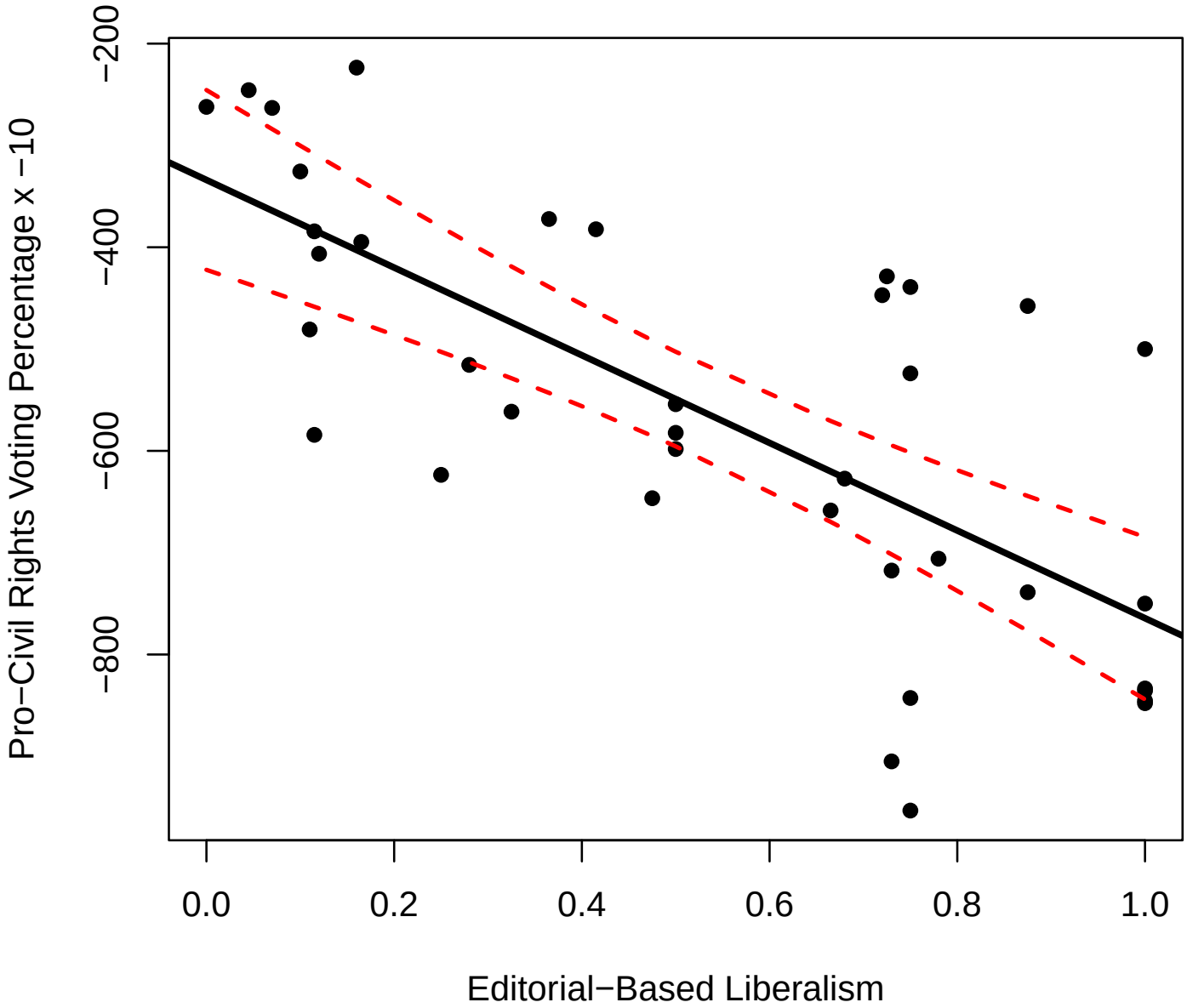
SCOTUSplot With Rescaled X



Multiply CivLibs Times -10

```
> SCOTUS$CivLibNeg10 <- -10 * SCOTUS$CivLibs
>
> fit3<-lm(CivLibNeg10~IdeologyScore,data=SCOTUS)
> summary(fit3)

Call:
lm(formula = CivLibNeg10 ~ IdeologyScore, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-296.6  -78.6  -24.6  105.9  264.3

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)     -333.9       43.5   -7.67  4.4e-09 ***
IdeologyScore   -430.4       69.2   -6.22  3.5e-07 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 140 on 36 degrees of freedom
Multiple R-squared:  0.518,Adjusted R-squared:  0.505
F-statistic: 38.7 on 1 and 36 DF,  p-value: 3.51e-07
```

SCOTUSplot With Rescaled Y



Linear Transformations

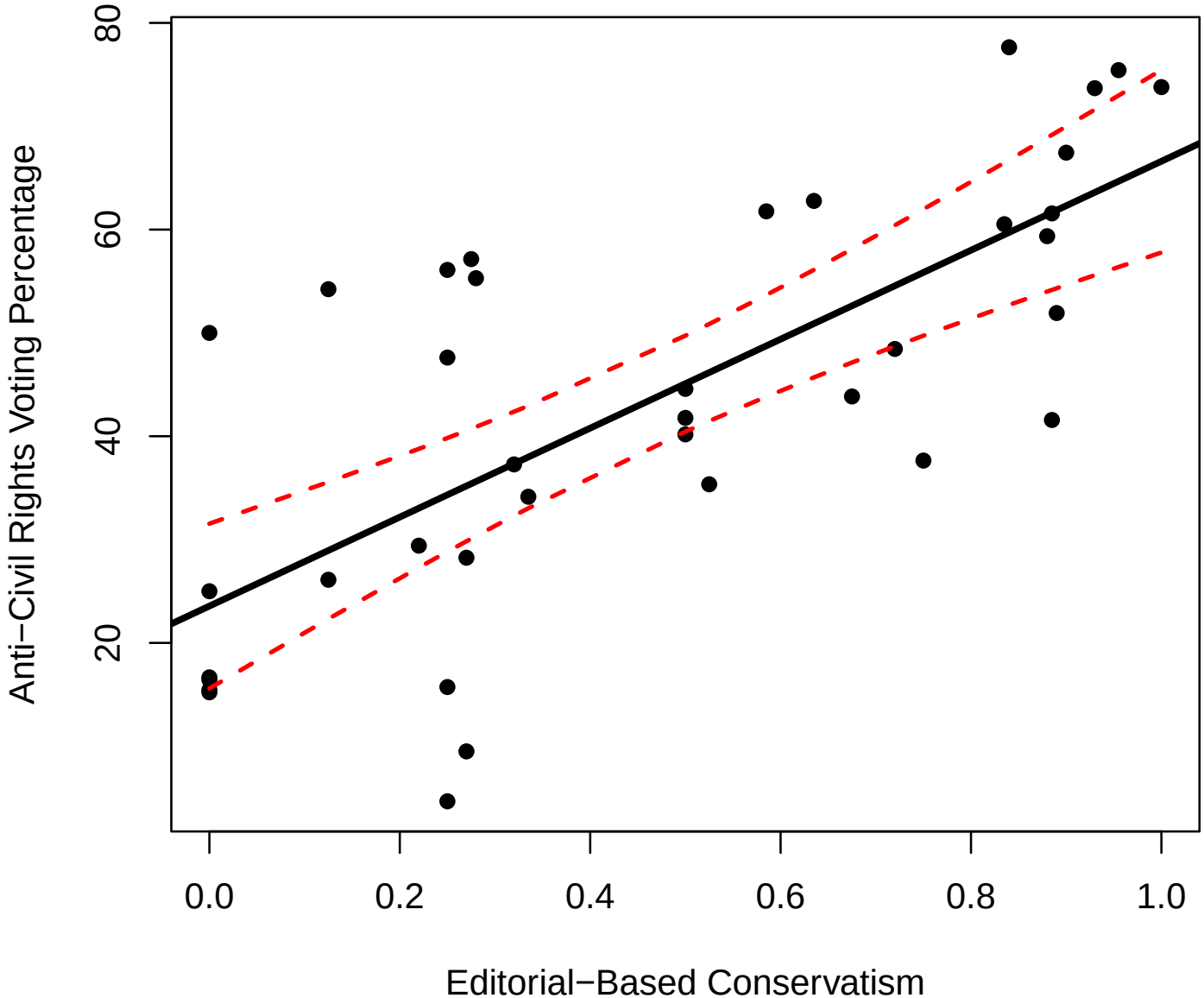
- Adding (subtracting) a positive constant to X shifts the X-axis to the left (right).
- Adding (subtracting) a positive constant to Y shifts the Y-axis downwards (upwards).
- Multiplying X (Y) times a positive constant greater than 1.0 stretches the X (Y) axis.
- Multiplying X (Y) times a positive constant less than 1.0 shrinks the X (Y) axis.
- Multiplying X (Y) times a negative constant inverts the X (Y) axis, and stretches / shrinks it as above.

Linear transformations do not alter the model in a statistically / substantively important way.

Application: Reversing The Scales

```
> SCOTUS$CivLibCons <- 100 - SCOTUS$CivLibs
> SCOTUS$IdeolCons <- 1 - SCOTUS$IdeologyScore
>
> fit4<-lm(CivLibCons~IdeolCons,data=SCOTUS)
> summary(fit4)

Call:
lm(formula = CivLibCons ~ IdeolCons, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-29.66  -7.86  -2.46  10.59  26.43

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    23.57       3.93    5.99  7.1e-07 ***
IdeolCons      43.04       6.92    6.22  3.5e-07 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 14 on 36 degrees of freedom
Multiple R-squared:  0.518,Adjusted R-squared:  0.505
F-statistic: 38.7 on 1 and 36 DF,  p-value: 3.51e-07
```

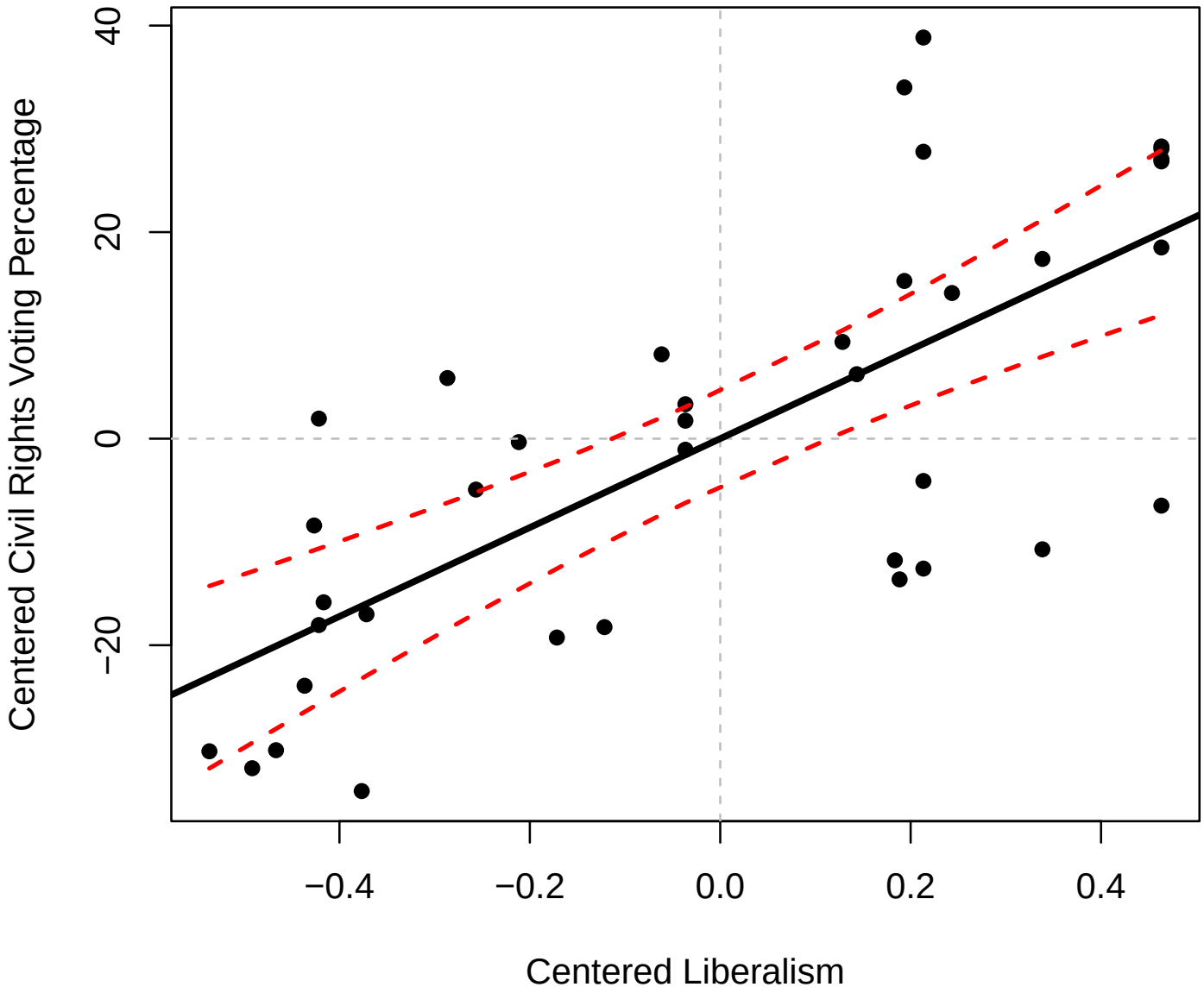
Plot of Civil Liberties Conservatism vs. Ideological Conservatism



Application: "Centering" Variables

```
> SCOTUS$CivLibCentered <- SCOTUS$CivLibs - mean(SCOTUS$CivLibs)
> SCOTUS$IdeolCentered <- SCOTUS$IdeologyScore - mean(SCOTUS$IdeologyScore)
>
> fit5<-lm(CivLibCentered~IdeolCentered,data=SCOTUS)
> summary(fit5)

Call:
lm(formula = CivLibCentered ~ IdeolCentered, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-26.43 -10.59   2.46   7.86  29.66

Coefficients:
                          Estimate          Std. Error t value   Pr(>|t|)
(Intercept)    0.0000000000000025  2.2758451083205631    0.00          1
IdeolCentered 43.0436722235377758  6.9171283031872104    6.22 0.00000035 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 14 on 36 degrees of freedom
Multiple R-squared:  0.518,Adjusted R-squared:  0.505
F-statistic: 38.7 on 1 and 36 DF,  p-value: 0.000000351
```

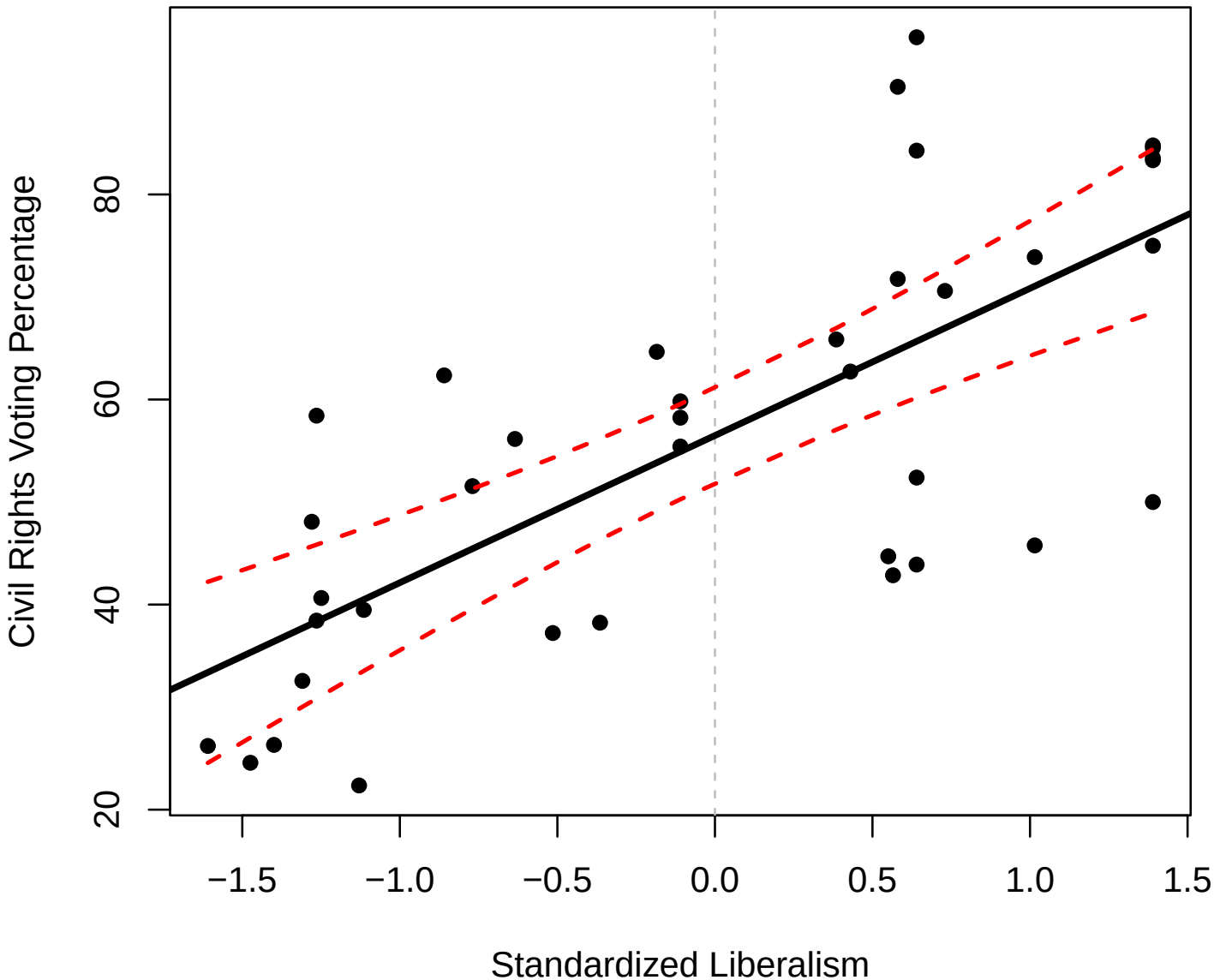
"Regression Through The Origin"



Application: "Standardizing" a Variable

```
> SCOTUS$IdeolStd <- scale(SCOTUS$IdeologyScore)
>
> fit6<-lm(CivLibs~IdeolStd,data=SCOTUS)
> summary(fit6)

Call:
lm(formula = CivLibs ~ IdeolStd, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-26.43 -10.59   2.46   7.86  29.66

Coefficients:
            Estimate Std. Error t value             Pr(>|t|)
(Intercept)    56.49       2.28   24.82 < 0.0000000000000002 ***
IdeolStd       14.35       2.31    6.22            0.00000035 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 14 on 36 degrees of freedom
Multiple R-squared:  0.518,Adjusted R-squared:  0.505
F-statistic: 38.7 on 1 and 36 DF,  p-value: 0.000000351
```

OLS with Standardized X



Rescaling for Interpretability

```
> fit7<-lm(CivLibs~Year,data=SCOTUS)
> summary(fit7)

Call:
lm(formula = CivLibs ~ Year, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-31.25 -15.02  -2.38  14.75  37.45

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  635.783    246.825    2.58    0.014 *
Year          -0.294      0.125   -2.35    0.025 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 18.8 on 36 degrees of freedom
Multiple R-squared:  0.133,Adjusted R-squared:  0.109
F-statistic: 5.51 on 1 and 36 DF,  p-value: 0.0245
```

Rescaling for Interpretability (continued)

```
> SCOTUS$Year1900<-SCOTUS$Year-1900
> fit8<-lm(CivLibs~Year1900,data=SCOTUS)
> summary(fit8)

Call:
lm(formula = CivLibs ~ Year1900, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-31.25 -15.02  -2.38  14.75  37.45

Coefficients:
            Estimate Std. Error t value      Pr(>|t|)
(Intercept)   76.995      9.256    8.32 0.00000000067 ***
Year1900      -0.294      0.125   -2.35         0.025 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 18.8 on 36 degrees of freedom
Multiple R-squared:  0.133,Adjusted R-squared:  0.109
F-statistic: 5.51 on 1 and 36 DF,  p-value: 0.0245
```

Binary $X \equiv t$-test

```
> SCOTUS$Chief<-ifelse(is.na(SCOTUS$ChiefJustice),0,1)
> fit9<-lm(CivLibs~Chief,data=SCOTUS)
> summary(fit9)

Call:
lm(formula = CivLibs ~ Chief, data = SCOTUS)

Coefficients:
            Estimate Std. Error t value            Pr(>|t|)
(Intercept)    57.04       3.45   16.51 <0.0000000000000002 ***
Chief          -5.22      10.65   -0.49                0.63
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 20.1 on 36 degrees of freedom
Multiple R-squared:  0.00664,Adjusted R-squared:  -0.021
F-statistic: 0.241 on 1 and 36 DF,  p-value: 0.627


> t.test(CivLibs~Chief,data=SCOTUS,var.equal=TRUE)

Two Sample t-test

data:  CivLibs by Chief
t = 0.49, df = 36, p-value = 0.6
alternative hypothesis: true difference in means between group 0 and group 1 is not equal to 0
95 percent confidence interval:
 -16.37  26.82
sample estimates:
mean in group 0 mean in group 1
          57.04           51.81
```

The results:

```
> summary(fit)

Call:
lm(formula = CivLibs ~ IdeologyScore, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-26.43 -10.59   2.46   7.86  29.66

Coefficients:
              Estimate Std. Error t value     Pr(>|t|)
(Intercept)      33.39       4.35    7.67 0.0000000044 ***
IdeologyScore    43.04       6.92    6.22 0.0000003505 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 14 on 36 degrees of freedom
Multiple R-squared:  0.518,Adjusted R-squared:  0.505
F-statistic: 38.7 on 1 and 36 DF,  p-value: 0.000000351
```

The table:

Table: OLS Regression Model of SCOTUS Voting

Variables	Model I
(Constant)	33.39
	(4.35)
Ideological Liberalism	43.04*
	(6.92)
Adjusted R^2	0.50

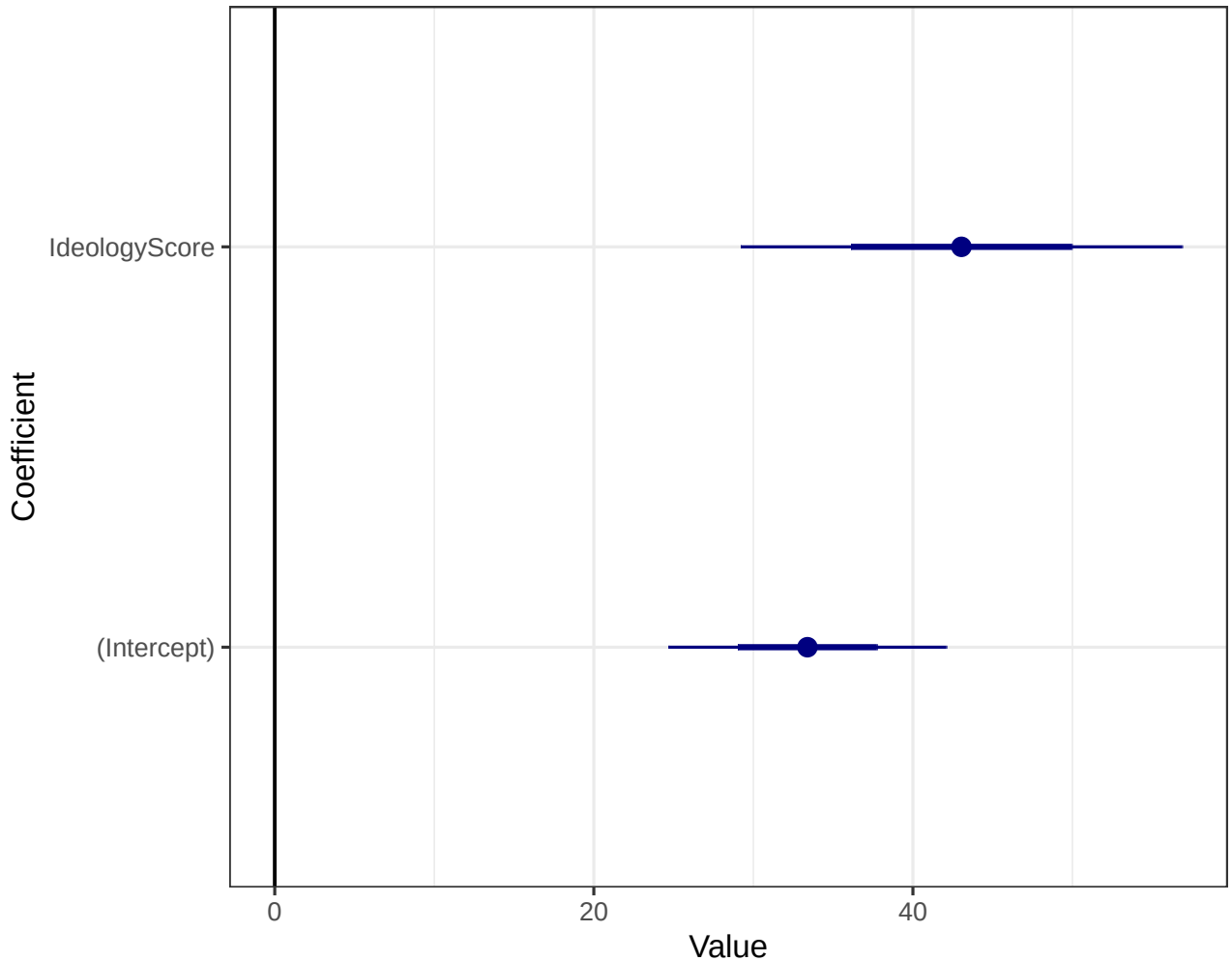
Note: $N = 43$. Cell entries are coefficient estimates; numbers in parentheses are estimated standard errors. Asterisks indicate $p < .05$ (one-tailed). See text for details.

Another Table (using default-y `stargazer`)

Table: OLS Regression Model of SCOTUS Voting

	Model I
(Constant)	33.39^{***}
	(4.35)
Ideological Liberalism	43.04^{***}
	(6.92)
Observations	38
R^2	0.52
Adjusted R^2	0.50
Residual Std. Error	14.03 (df = 36)
F Statistic	38.72^{***} (df = 1; 36)
Note:	*p<0.1; **p<0.05; ***p<0.01

Default-y Ladderplot `-fitplot-`



Some Guidelines ("Rules"?)

Tables:

- *Use column headings descriptively.*
- *Use multiple rows / columns rather than multiple tables.*
- *Learn about significant digits, and don't report more than 4-5 of them.*
- *Use a figure to replace a table when you can.*
- *Be aware of norms about *s.*

Figures:

- *Report the scale of axes, and label them.*
- *Use as much "space" as you need, but no more.*
- *Use color sparingly.*