PLSC 502 – Fall 2022
Two-Group Comparisons

October 27, 2022

“Student’s” t...

“...the T-Distribution, also known as Student’s t-distribution, gets its name from William Sealy Gosset who first published it in English in 1908 in the scientific journal *Biometrika* using the pseudonym “Student” because his employer preferred staff to use pen names when publishing scientific papers. Gosset worked at the Guinness Brewery in Dublin, Ireland, and was interested in the problems of small samples – for example, the chemical properties of barley with small sample sizes.

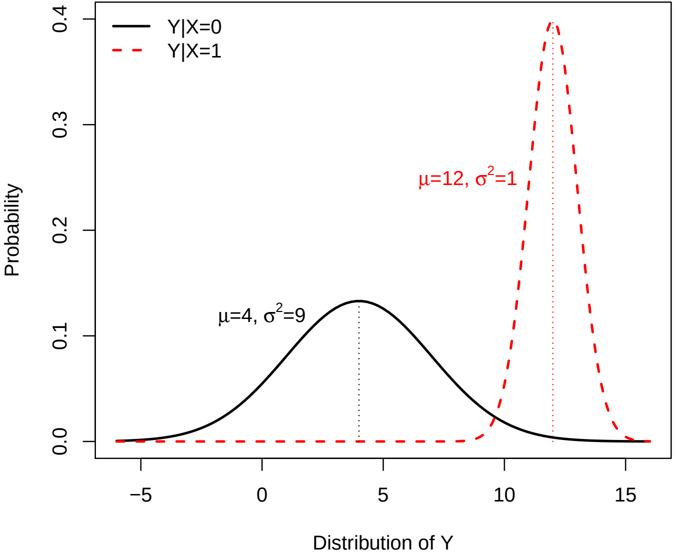
Gosset had been hired due to Claude Guinness’s policy of recruiting the best graduates from Oxford and Cambridge to apply biochemistry and statistics to Guinness’s industrial processes. Gosset devised the t-test as an economical way to monitor the quality of stout. The Student’s t-test work was submitted to and accepted in the journal *Biometrika* and published in 1908.”

- Student’s t-test (Wikipedia)

The Setup

- N observations, $i \in \{1, 2, ...N\}$
- A dichotomous predictor X, so that $X_i \in \{0, 1\}$
- n_0 and n_1 are the number of observations in the data with $X = 0$ and $X = 1$, respectively (so $n_0 + n_1 = N$)
- An continuous (interval/ratio) outcome variable Y, with
  - $Y|X = 0 \sim N(\mu_0, \sigma_0^2)$ and
  - $Y|X = 1 \sim N(\mu_1, \sigma_1^2)$.
- Call
  - $\bar{Y}_0 = \bar{Y}|X = 0$, and
  - $\bar{Y}_1 = \bar{Y}|X = 1$

Example



Y|X=0
Y|X=1
μ=12, σ²=1
μ=4, σ²=9
Probability
0.0
0.1
0.2
0.3
0.4
−5
0
5
10
15
Distribution of Y

Difference of Means

Difference of (sample) means:

$$\bar{Y}_1 - \bar{Y}_0 = \frac{1}{n_1}\sum_{i=1}^{n_1} Y_{1i} - \frac{1}{n_0}\sum_{i=1}^{n_0} Y_{0i}$$

Has:

$$E(\bar{Y}_1 - \bar{Y}_0) = \mu_1 - \mu_0$$

and

$$Var(\bar{Y}_1 - \bar{Y}_0) = \sigma^2_{\mu_1 - \mu_0}.$$

Difference of Means (continued)

We can show that:

$$\sigma^2_{\mu_1-\mu_0} = \frac{\sigma^2_0}{n_0} + \frac{\sigma^2_1}{n_1}$$

In practice we use:

$$s^2_{\bar{Y}_1-\bar{Y}_0} = \frac{s^2_0}{n_0} + \frac{s^2_1}{n_1}$$

The t Statistic

The t-statistic is:

$$
\begin{aligned}
t &= \frac{\bar{Y}_1 - \bar{Y}_0}{s_{\bar{Y}_1 - \bar{Y}_0}} \\
&= \frac{\bar{Y}_1 - \bar{Y}_0}{\sqrt{\frac{s_0^2}{n_0} + \frac{s_1^2}{n_1}}}
\end{aligned}
$$

We can show that:

$$t \sim t(\nu)$$

where

$$\nu \approx \frac{\left(\frac{s_0^2}{n_0} + \frac{s_1^2}{n_1} \right)^2}{\frac{s_0^4}{n_0^2(n_0 - 1)} + \frac{s_1^4}{n_1^2(n_1 - 1)}}$$

Other Uses

Test statistic for $H_0: \ \mu_1 - \mu_0 = k$ is:

$$t = \frac{(\bar{Y}_1 - \bar{Y}_0) - k}{s_{\bar{Y}_1 - \bar{Y}_0}}$$

The $(1 - \alpha) \times 100$ c.i. for $\bar{Y}_1 - \bar{Y}_0$ is:

$$(\bar{Y}_1 - \bar{Y}_0) \pm t_{\alpha/2}(s_{\bar{Y}_1 - \bar{Y}_0}),$$

Differences of Proportions

For a proportion:

$$E(\mu) = \pi$$

and

$$\sigma^2_\mu = \frac{\pi(1-\pi)}{\mathfrak{N}}.$$

So $\hat{\pi} = \bar{Y}$ and:

$$\begin{aligned} s^2 &= \frac{\hat{\pi}(1-\hat{\pi})}{N} \\ &= \frac{\bar{Y}(1-\bar{Y})}{N}, \end{aligned}$$

For two samples:

$$s_0 = \sqrt{\frac{\bar{Y}_0(1-\bar{Y}_0)}{n_0}} \quad \text{and} \quad s_1 = \sqrt{\frac{\bar{Y}_1(1-\bar{Y}_1)}{n_1}}$$

Two-Sample t-test

Key things to remember:

- Assumes $Y \sim \ i.i.d.\ N(\mu, \sigma^2)$
  - · *Independence* (vs. dependence)
  - · *Normality* (vs. skewness)
- Note that if $s_0^2 = s_1^2$, then $\nu = n_0 + n_1 - 2$.
- $\nu = n_0 + n_1 - 2$ is also good if n_0 and $n_1 > 50$ or so

Variances, Independence, & Skewness

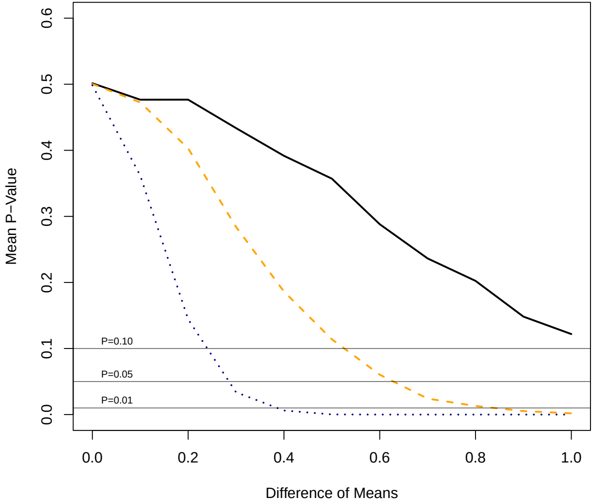
A simulation:

- $Y_0 \sim N(0, 1)$
- $Y_1 \sim N(\mu_1, \sigma_1^2)$
- $\mu_1 \in \{0, 0.1, 0.2, ...1.0\}$
- $\sigma_1^2 \in \{1, 25\}$
- Y_0, $Y_1 \in \{\text{independent}, \text{dependent}\}$
- $N \in \{10, 40, 200\}$ (*per group*)
- $N_{sims} = 1000$

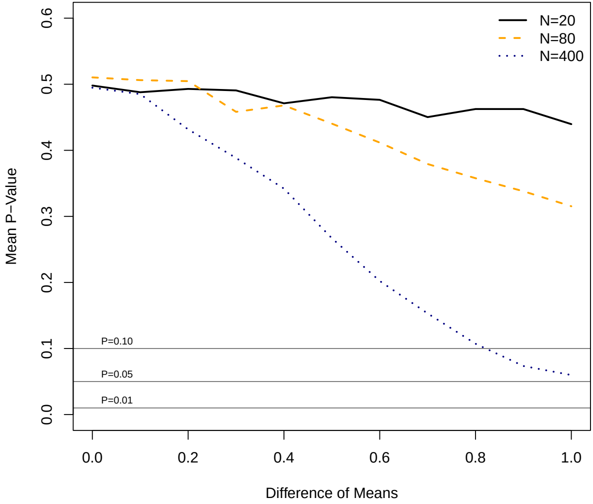
Simulation

```
Nsims <- 1000    # number of sims

N <- c(10,40,200)   # sample sizes
D <- seq(0,1,by=0.1) # differences in means

P1<-as.data.frame(matrix(Nsims,length(N)*length(D)))
set.seed(7222009)

z=1                          # counter...
for(j in 1:length(N)){
  for(k in 1:length(D)){
    for(i in 1:Nsims){
      x<-rnorm(N[j],0,1)            # independent samples,
      y<-rnorm(N[j],0+D[k],1)       # same variance...
      t<-t.test(x,y,var.equal=TRUE) # t-test
      P1[i,z]<-t$p.value            # P-value
    }
    z<-z+1                   # increment counter
  }
}
```

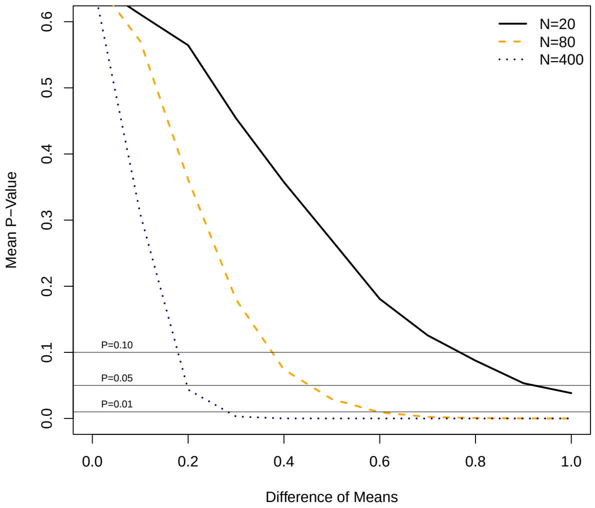
Equal Variances, Independent Samples



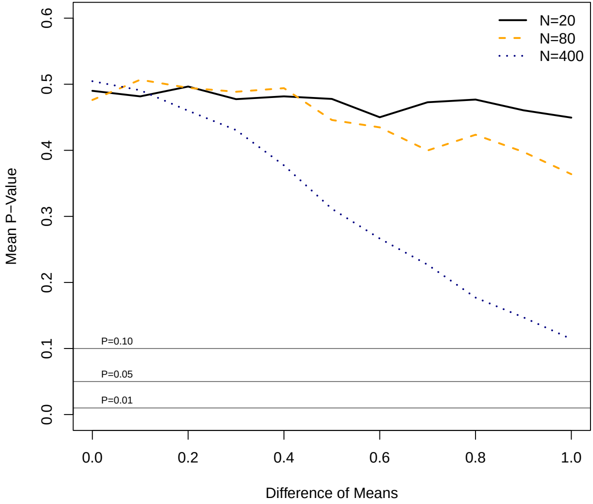
Different Variances, Independent Samples



Equal Variances, Dependent Samples



High Skewness ($Y \sim$ Exponential with $\lambda = 4$)



t Mnemonics

Rough Values of t You'll Want To Get To Know

Absolute Value of t	One-Tailed P-Value*	Two-Tailed P-Value
≈ 1.3	0.10	0.20
≈ 1.65	0.05	0.10
≈ 2	0.025	0.05
≈ 2.4	0.01	0.02
≈ 2.6	0.005	0.01
> 3	< 0.001	< 0.002

Note: Assumes d.f. $= \infty$. * indicates that the directionality of the difference in means is "correct."

Example: Federal District Court Judges

The Biographical Directory of Article III Federal Judges contains "the biographies of judges presidentially appointed to serve during good behavior since 1789 on the U.S. district courts, U.S. courts of appeals, Supreme Court of the United States, and U.S. Court of International Trade, as well as the former U.S. circuit courts, Court of Claims, U.S. Customs Court, and U.S. Court of Customs and Patent Appeals."

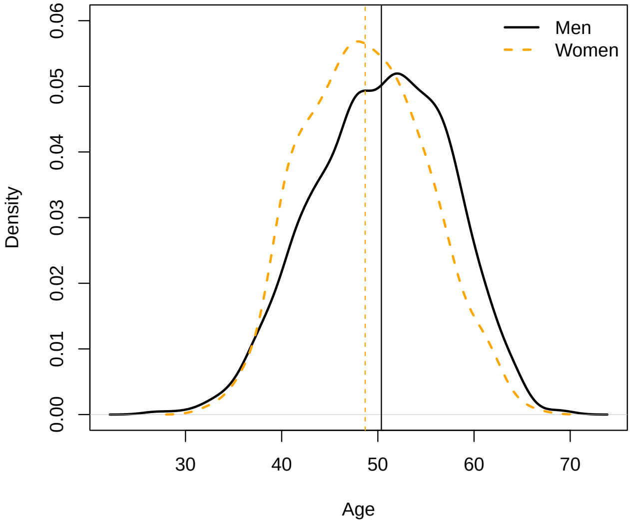
Here: Federal district court judges:

- First appointments *only*
- $N = 3142$ (as of yesterday)
- Variables of interest:
  - `AppAge`: The age at which each judge was appointed
  - `Gender`: The sex (male or female) of the appointee

Federal District Court Judges

```
> describe(Js$AppAge)
   vars    n  mean   sd median trimmed  mad min max range  skew kurtosis   se
X1    1 3117 50.13 6.88     50   50.23 7.41  26  70    44 -0.15     -0.4 0.12

> table(Js$Gender)

Female   Male
   428   2714

> tapply(Js$AppAge,Js$Gender,describe) # Appointment age by gender

$Female
   vars   n  mean   sd median trimmed  mad min max range skew kurtosis  se
X1    1 427 48.68 6.23     48   48.57 7.41  33  66    33 0.13    -0.54 0.3

$Male
   vars    n  mean   sd median trimmed  mad min max range skew kurtosis   se
X1    1 2690 50.36 6.95     51   50.51 7.41  26  70    44 -0.2    -0.37 0.13
```

D. Ct. Judge Appointment Age by Gender



$$\bar{Y}_{Male} - \bar{Y}_{Female} = 1.69$$

and

$$\begin{aligned}
s^2_{\bar{Y}_{Male} - \bar{Y}_{Female}} &= \frac{s^2_{Male}}{n_{Male}} + \frac{s^2_{Female}}{n_{Female}} \\
&= \frac{48.33}{2690} + \frac{38.76}{427} \\
&= 0.018 + 0.091 \\
&= 0.109
\end{aligned}$$

42 and:

$$\begin{aligned}
s_{\bar{Y}_{Male} - \bar{Y}_{Female}} &= \sqrt{0.109} \\
&= 0.33.
\end{aligned}$$

Then:

$$\begin{aligned}
t &= \frac{1.69 - 0}{0.33} \\
&= \mathbf{5.12}
\end{aligned}$$

t-test (via `t.test`)

```
> T1<-t.test(AppAge~Gender,data=Js)
> T1

Welch Two Sample t-test

data:  AppAge by Gender
t = -5.1, df = 608, p-value = 4e-07
alternative hypothesis: true difference in means between group Female
    and group Male is not equal to 0
95 percent confidence interval:
 -2.333 -1.038
sample estimates:
mean in group Female   mean in group Male
               48.68                50.36
```

“Reverse” the Difference

```
> Js$Female<-ifelse(Js$Gender=="Female",1,0)

> Ta<-t.test(AppAge~Female,data=Js)
> Ta

Welch Two Sample t-test

data:  AppAge by Female
t = 5.1, df = 608, p-value = 4e-07
alternative hypothesis: true difference in means between group 0
    and group 1 is not equal to 0
95 percent confidence interval:
 1.038 2.333
sample estimates:
mean in group 0 mean in group 1
          50.36           48.68
```

$$H_0: \overline{\text{AppAge}}_{Male} > \overline{\text{AppAge}}_{Female}$$

```
> Tg<-t.test(AppAge~Female,data=Js,alternative="greater")
> Tg

Welch Two Sample t-test

data:  AppAge by Female
t = 5.1, df = 608, p-value = 2e-07
alternative hypothesis: true difference in means between group 0
    and group 1 is greater than 0
95 percent confidence interval:
 1.142   Inf
sample estimates:
mean in group 0 mean in group 1
          50.36           48.68
```

$H_0: \overline{\text{AppAge}}_{Male} < \overline{\text{AppAge}}_{Female}$

```
> Tl<-t.test(AppAge~Female,data=Js,alternative="less")
> Tl

Welch Two Sample t-test

data:  AppAge by Female
t = 5.1, df = 608, p-value = 1
alternative hypothesis: true difference in means between group 0
    and group 1 is less than 0
95 percent confidence interval:
  -Inf 2.229
sample estimates:
mean in group 0 mean in group 1
          50.36           48.68
```

H_0 : One Year Age Difference

```
> T1<-t.test(AppAge~Female,data=Js,mu=1)
> T1

Welch Two Sample t-test

data:  AppAge by Female
t = 2.1, df = 608, p-value = 0.04
alternative hypothesis: true difference in means between group 0
    and group 1 is not equal to 1
95 percent confidence interval:
 1.038 2.333
sample estimates:
mean in group 0 mean in group 1
          50.36           48.68
```

Forcing Equal Variances

```
> Te<-t.test(AppAge~Female,data=Js,var.equal=TRUE)
> Te

Two Sample t-test

data:  AppAge by Female
t = 4.7, df = 3115, p-value = 2e-06
alternative hypothesis: true difference in means between group 0
    and group 1 is not equal to 0
95 percent confidence interval:
 0.9851 2.3859
sample estimates:
mean in group 0 mean in group 1
          50.36           48.68
```

Power

Power!

Four interrelated components:

- Sample size (N)
- "Effect size" (d)
- Significance level (P):
  - Pr(finding an effect that is not there) / Pr("false positive")
  - Also written as α
- **Power** ($\mathfrak{P}$):
  - Pr(finding an effect that *is* there) / Pr("true positive")
  - Sometimes written $1 - \beta$

Given any three of these, we can determine the fourth.

What's An "Effect Size"?

The size of an effect – e.g., the difference between $\bar{Y}_0$ and $\bar{Y}_1$ – *depends on the "scale" of Y*.

Solution? Cohen's d:

$$d = \frac{\mu_1 - \mu_0}{\sigma}$$

- The *standardized* difference between two means...
- σ is the *pooled standard deviation*:

$$\sigma = \sqrt{\frac{(n_0 - 1)s_0^2 + (n_1 - 1)s_1^2}{n_0 + n_1 - 2}}$$

where $s^2 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{n-1}$ denotes the variance of Y in each group $\{0 \text{ or } 1\}$.

What's a big value for d?

d	Effect Size
0.01	Teeeeny
0.20	Small
0.50	Medium
1.00	Large
2.00	Huuuuge

Example: The t-test

For a given effect size d and sample size N, the t-statistic for testing the hypothesis $d = 0$ (that is, $\mu_0 = \mu_1$) against the alternative hypothesis $d > 0$ (equivalently, $\mu_0 < \mu_1$) is:

$$t = \frac{(\bar{Y}_1 - \bar{Y}_0) - 0}{s_{\bar{Y}_1 - \bar{Y}_0}}.$$

At $P = 0.05$, we reject $d = 0$ if

$$t > 1.64$$

and if N is large, then $t \to N(0, 1)$, and so we can use a z-statistic instead.

Example (continued)

Now suppose $d > 0$ (and N is still large).

The power $\mathfrak{P}(d)$ of t to detect this fact at $P = 0.05$ is:

$$
\begin{aligned}
\mathfrak{P}(d) &= \Pr(t > 1.64 | d) \\
&= \Pr\left[\frac{(\bar{Y}_1 - \bar{Y}_0) - d + d}{s_{\bar{Y}_1 - \bar{Y}_0}} > 1.64\right] \\
&= \Pr\left[\frac{(\bar{Y}_1 - \bar{Y}_0) - d}{s_{\bar{Y}_1 - \bar{Y}_0}} > \left(1.64 - \frac{d}{s_{\bar{Y}_1 - \bar{Y}_0}}\right)\right] \\
&= 1 - \Pr\left[\frac{(\bar{Y}_1 - \bar{Y}_0) - d}{s_{\bar{Y}_1 - \bar{Y}_0}} < \left(1.64 - \frac{d}{s_{\bar{Y}_1 - \bar{Y}_0}}\right)\right] \\
&\approx 1 - \Phi\left(1.64 - \frac{d}{s_{\bar{Y}_1 - \bar{Y}_0}}\right)
\end{aligned}
$$

What's the Point?

So:

$$\mathfrak{P}(d) \approx 1 - \Phi\left(1.64 - \frac{d}{s_{\bar{Y}_1 - \bar{Y}_0}}\right)$$

- Power increases as d gets larger...
- For a given value of d, bigger $N \rightarrow$ higher power (via $s_{\bar{Y}_1 - \bar{Y}_0}$)...
- For very small values of d, power will be low
  - The minimum value of $\mathfrak{P}(d)$ as $d \rightarrow 0$ is P
  - For very small values of d, the difference between $d = 0$ and $d > 0$ is usually unimportant

Hypothetical Example

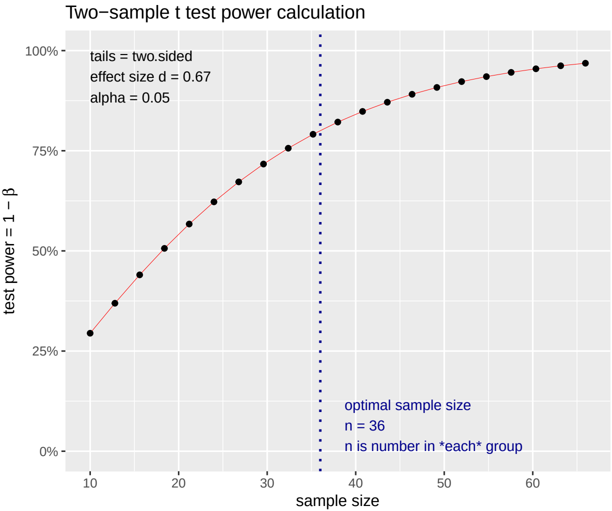
Consider a survey with a standard 101-point "feeling thermometer" (FT) for President Biden. You want to be able to detect the presence of (at the minimum) a 20-point difference in that 101-point scale (say, between Democrats and Republicans) with 80 percent power [$\mathfrak{P} = 0.80$] at $P = 0.05$ (two-tailed). **How big does your sample N need to be?**

Suppose:

- $\sigma_{FT} = 30$, which means
- $d = \frac{20}{30} = 0.67...$

```
> pwr.t.test(d=0.67,sig.level=0.05,power=0.80)

     Two-sample t test power calculation

              n = 35.96
              d = 0.67
      sig.level = 0.05
          power = 0.8
    alternative = two.sided

NOTE: n is number in *each* group
```

Sample Size Plot



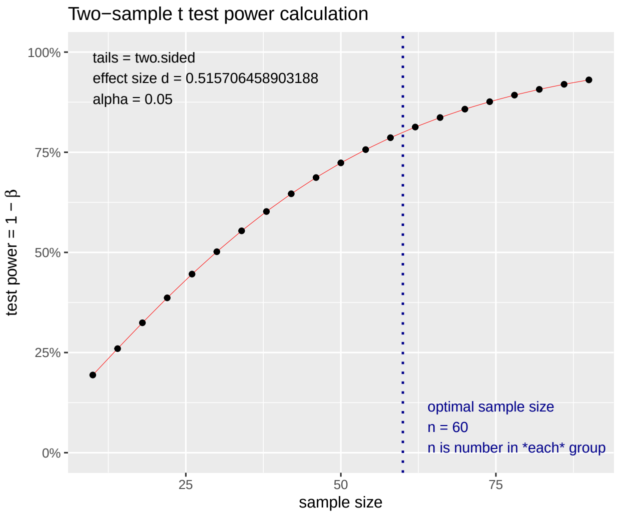
Another Example

I have a small a survey with $N = 120$ (total). Given that same 101-point "feeling thermometer," what is the largest difference d I can detect with $\mathfrak{P} = 0.80$ and $P = 0.05$ (two-tailed)?

```
> pwr.t.test(n=60,sig.level=0.05,power=0.80)

     Two-sample t test power calculation

              n = 60
              d = 0.5157
      sig.level = 0.05
          power = 0.8
    alternative = two.sided

NOTE: n is number in *each* group
```

Note that here, if $\sigma_{FT} = 30$, the actual size of the difference we can detect with $\mathfrak{P} = 0.80$ and $P = 0.05$ is $(0.5157 \times 30) \approx 15.5$ units on the "raw" feeling thermometer scale.

Effect Size Plot



Conducting Power Analyses

How?

- Lots of power calculators on the internet...
- In R, the `pwr` package:
  - Power calculations for t-tests + many others
  - Can specify tailedness, other options
  - Semi-nice plots

Practical considerations:

- Prospective, and largely geared towards experiments (where N is controlled)
- Requires knowledge of d, which we often don't have...
- We (in political science) don't do this enough; BUT
- Retrospective / post-hoc power analyses are bad