PLSC 502 – Fall 2022
Linear Regression I

November 17, 2022

Recall that a (real-valued) *random variable* Y is:

$$Y_i = \underbrace{\mu}_{\text{"systematic"}} + \underbrace{u_i}_{\text{"stochastic"}}$$

Note that we typically require that:

$$\text{Cov}(\mu, u) = 0.$$

Allow μ to vary *linearly* with some other variable X:

$$\mu_i = \beta_0 + \beta_1 X_i$$

so that we get

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

Goals:

- *Point estimates* of β_0 and β_1 (call them $\hat{\beta}_0$ and $\hat{\beta}_1$)
- Estimates of their *variability* $\rightarrow$ inference

Estimating β_0 and β_1

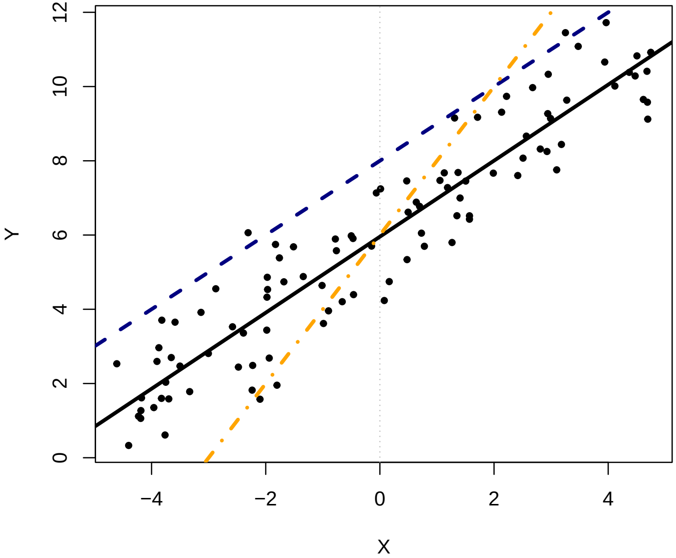
Suppose we have some estimates $\hat{\beta}_0$ and $\hat{\beta}_1$. Then:

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

$\rightarrow$ estimated "residuals":

$$\begin{aligned}\hat{u}_i &= Y_i - \hat{Y}_i \\ &= Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\end{aligned}$$

Intuition



"Loss Function"

Key Idea: Select $\hat{\beta}_0$ and $\hat{\beta}_1$ to make the $\hat{u}_i$s as small as possible.

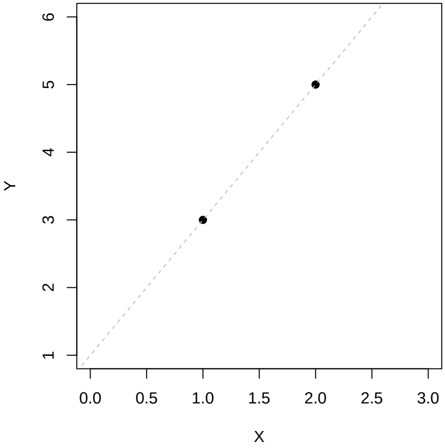
Possibilities:

- Pick $\hat{\beta}_0$ and $\hat{\beta}_1$ so as to minimize $\sum_{i=1}^{N} \hat{u}_i$
- Pick $\hat{\beta}_0$ and $\hat{\beta}_1$ so as to minimize $\sum_{i=1}^{N} |\hat{u}_i|$ ("MAD")
- Pick $\hat{\beta}_0$ and $\hat{\beta}_1$ so as to minimize $\sum_{i=1}^{N} \hat{u}_i^2$ ("least squares")

$\rightarrow$ "ordinary least squares" ("OLS") regression...

The Simplest Regression In Human History

```
> d
  x y
1 1 3
2 2 5
```



World's Simplest Regression

Recall:

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i.$$

So, for $i = 1$

$$\hat{Y}_1 = \hat{\beta}_0 + \hat{\beta}_1(1)$$

and for $i = 2$

$$\hat{Y}_2 = \hat{\beta}_0 + \hat{\beta}_1(2)$$

Means:

$$\begin{aligned}
\hat{u}_i &= Y_i - \hat{Y}_i \\
&= 3 - [\hat{\beta}_0 + \hat{\beta}_1(1)] \text{ for } i = 1, \text{and} \\
&= 5 - [\hat{\beta}_0 + \hat{\beta}_1(2)] \text{ for } i = 2
\end{aligned}$$

Sum of Squared Residuals

$$
\begin{aligned}
\hat{S} &= u_1^2 + u_1^2 \\
&= [3 - \hat{\beta}_0 - \hat{\beta}_1(1)]^2 + [5 - \hat{\beta}_0 - \hat{\beta}_1(2)]^2 \\
&= (9 + \hat{\beta}_0^2 + \hat{\beta}_1^2 - 6\hat{\beta}_0 - 6\hat{\beta}_1 + 2\hat{\beta}_0\hat{\beta}_1) + \\
&\qquad (25 + \hat{\beta}_0^2 + 4\hat{\beta}_1^2 - 10\hat{\beta}_0 - 20\hat{\beta}_1 + 4\hat{\beta}_0\hat{\beta}_1) \\
&= 2\hat{\beta}_0^2 + 5\hat{\beta}_1^2 - 16\hat{\beta}_0 - 26\hat{\beta}_1 + 6\hat{\beta}_0\hat{\beta}_1 + 34
\end{aligned}
$$

Choose $\hat{\beta}_0$ and $\hat{\beta}_1$ that minimize this...

Minimizing...

For:

$$\hat{S} = 2\hat{\beta}_0^2 + 5\hat{\beta}_1^2 - 16\hat{\beta}_0 - 26\hat{\beta}_1 + 6\hat{\beta}_0\hat{\beta}_1 + 34$$

We have:

$$\begin{aligned}
\frac{\partial \hat{S}}{\partial \hat{\beta}_0} &= 4\hat{\beta}_0 + 6\hat{\beta}_1 - 16 \\
\frac{\partial \hat{S}}{\partial \hat{\beta}_1} &= 6\hat{\beta}_0 + 10\hat{\beta}_1 - 26
\end{aligned}$$

So for $\hat{\beta}_1$:

$$\begin{aligned} 4\hat{\beta}_0 + 6\hat{\beta}_1 - 16 = 0 \quad &\Rightarrow \quad 2\hat{\beta}_0 = -3\hat{\beta}_1 + 8 \\ &\Rightarrow \quad \hat{\beta}_0 = -3/2\hat{\beta}_1 + 4 \end{aligned}$$

$$\begin{aligned} 6\hat{\beta}_0 + 10\hat{\beta}_1 - 26 = 0 \quad &\Rightarrow \quad 5\hat{\beta}_1 - 3(-3/2\hat{\beta}_1 + 4) - 13 = 0 \\ &\Rightarrow \quad 5\hat{\beta}_1 - 9/2\hat{\beta}_1 + 12 - 13 = 0 \\ &\Rightarrow \quad \frac{1}{2}\hat{\beta}_1 - 1 = 0 \\ &\Rightarrow \quad \boldsymbol{\hat{\beta}_1 = 2} \end{aligned}$$

And for $\hat{\beta}_0$:

$$\begin{aligned} 4\hat{\beta}_0 + 6(2) - 16 = 0 \quad &\Rightarrow \quad 4\hat{\beta}_0 = 4 \\ &\Rightarrow \quad \boldsymbol{\hat{\beta}_0 = 1} \end{aligned}$$

So:

$$Y_i = 1 + 2X_i + u_i$$

Note that, in this (N=2) case:

$$\begin{aligned}\hat{\beta}_1 &= (5-3)/(2-1) \\ &= 2, \text{ and}\end{aligned}$$

$$\begin{aligned}\hat{\beta}_0 &= -2(2) + 5 \\ &= 1\end{aligned}$$

Least Squares with > 2 Observations

$$
\begin{aligned}
\hat{S} &= \sum_{i=1}^{N}(Y_i - \hat{Y}_i)^2 \\
&= \sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \\
&= \sum_{i=1}^{N}(Y_i^2 - 2Y_i\hat{\beta}_0 - 2Y_i\hat{\beta}_1 X_i + \hat{\beta}_0^2 + 2\hat{\beta}_0\hat{\beta}_1 X_i + \hat{\beta}_1^2 X_i^2)
\end{aligned}
$$

Then:

$$\begin{aligned}
\frac{\partial \hat{S}}{\partial \hat{\beta}_0} &= \sum_{i=1}^{N}(-2Y_i + 2\hat{\beta}_0 + 2\hat{\beta}_1 X_i) \\
&= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) \\
&= -2\sum_{i=1}^{N}\hat{u}_i
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \hat{S}}{\partial \hat{\beta}_1} &= \sum_{i=1}^{N}(-2Y_i X_i + 2\hat{\beta}_0 X_i + 2\hat{\beta}_1 X_i^2) \\
&= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)X_i \\
&= -2\sum_{i=1}^{N}\hat{u}_i X_i
\end{aligned}$$

Next, set:

$$-2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$$

and

$$-2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)X_i = 0$$

... and solve...

Least Squares "Normal Equations"

(Algebra happens...):

$$\sum_{i=1}^{N} Y_i = N\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{N} X_i$$

and

$$\sum_{i=1}^{N} Y_i X_i = \hat{\beta}_0 \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2$$

Least Squares: Solutions!

(More algebra...):

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2}$$

and

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X}$$

<u>The intuition</u>:

$$\hat{\beta}_1 = \frac{\text{Covariance of } X \text{ and } Y}{\text{Variance of } X}$$

Parsing Variation in Y: ANOVA

Note that the "total" variation in Y around its mean $\bar{Y}$ is:

$$\text{SS}_{Total} = \sum_{i=1}^{N}(Y_i - \bar{Y})^2$$

which comprises:

$$\begin{aligned}\text{SS}_{Residual} &= \sum_{i=1}^{N}(\hat{u}_i)^2 \\ &= \sum_{i=1}^{N}(Y_i - \hat{Y})^2\end{aligned}$$

and:

$$\text{SS}_{Model} = \sum_{i=1}^{N}(\hat{Y}_i - \bar{Y})^2$$

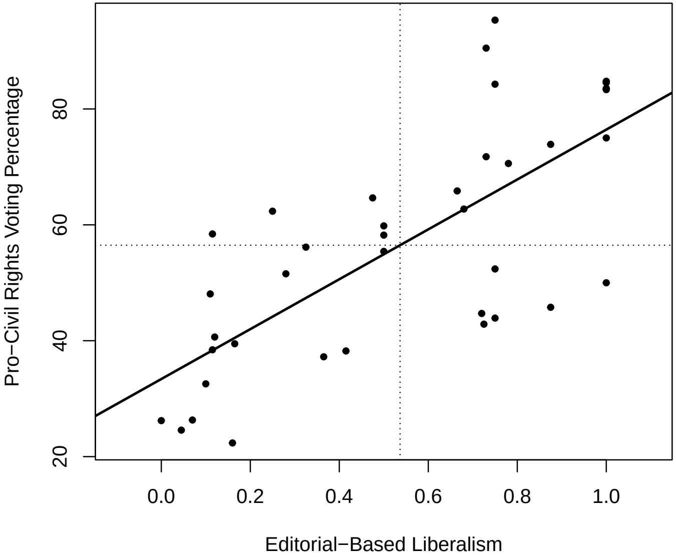
SCOTUS Data, OT1946-2021

Data from the Supreme Court Database and the justices' Segal-Cover scores...

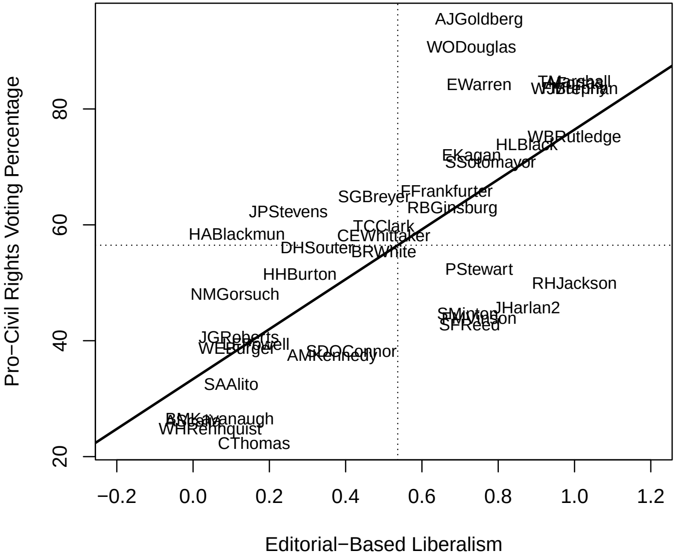
- Y is `CivLibs` = liberal voting percentage in civil rights & liberties cases
- X is `IdeologyScore` $\in [0, 1] \rightarrow$ SCOTUS justice liberalism

```
> describe(SCOTUS,skew=FALSE,trim=0)
                     vars  n    mean    sd     min     max range   se
justice                 1 38   97.37 11.32   78.00  116.00 38.00 1.84
justiceName*            2 38   19.50 11.11    1.00   38.00 37.00 1.80
CivLibs                 3 38   56.49 19.94   22.36   95.33 72.97 3.23
Nom.Order*              4 38   19.50 11.11    1.00   38.00 37.00 1.80
Nominee*                5 38   19.50 11.11    1.00   38.00 37.00 1.80
ChiefJustice*           6  4    1.00  0.00    1.00    1.00  0.00 0.00
SenateVote*             7 38   17.05  8.23    1.00   25.00 24.00 1.33
IdeologyScore           8 38    0.54  0.33    0.00    1.00  1.00 0.05
QualificationsScore*    9 38   16.45  7.91    1.00   25.00 24.00 1.28
Nominator (Party)*     10 38    7.03  3.72    1.00   13.00 12.00 0.60
Year                   11 38 1969.74 24.70 1937.00 2018.00 81.00 4.01
```

Le Scatterplot



Le Labeled Scatterplot



Estimating $\hat{\beta}$

```
> # Betas:

> Beta1 <- with(SCOTUS, (sum((IdeologyScore - mean(IdeologyScore)) *
+                        (CivLibs - mean(CivLibs))) /
+                        sum((IdeologyScore - mean(IdeologyScore))^2)))
> Beta1
[1] 43.04


> Beta0 <- with(SCOTUS, mean(CivLibs) - (Beta1 * mean(IdeologyScore)))
> Beta0
[1] 33.39
```

$\hat{Y}$, $\hat{u}$, etc.

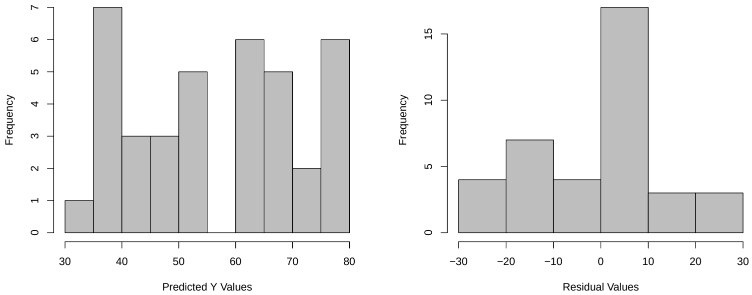
```
> SCOTUS$Yhats <- with(SCOTUS, Beta0 + Beta1*IdeologyScore)
> SCOTUS$Uhats <- with(SCOTUS, CivLibs - Yhats)

> # Y itself:
> describe(SCOTUS$CivLibs)
   vars  n  mean    sd median trimmed   mad   min   max range skew kurtosis   se
X1    1 38 56.49 19.94  55.78   56.33 23.07 22.36 95.33 72.97 0.15    -1.03 3.23

> # Predicted Ys:
> describe(SCOTUS$Yhats)
   vars  n  mean    sd median trimmed   mad   min   max range  skew kurtosis   se
X1    1 38 56.49 14.35  58.46   56.63 18.67 33.39 76.43 43.04 -0.11    -1.46 2.33

> # Residuals:
> describe(SCOTUS$Uhats)
   vars  n mean    sd median trimmed   mad    min   max range  skew kurtosis   se
X1    1 38    0 13.84   2.46   -0.06 11.18 -26.43 29.66 56.09 -0.08    -0.59 2.24
```

$\hat{Y}$ and $\hat{u}$ Plots



What's a "typical" residual?

Note that because

$$\sum_{i=1}^{N} \hat{u}_i = 0$$

it's also true that:

$$\begin{aligned} \bar{\hat{u}} &= \frac{\sum_{i=1}^{N} \hat{u}_i}{N} \\ &= 0 \end{aligned}$$

Consider instead:

$$\text{“Residual Standard Error” (RSE)} = \sqrt{\left(\frac{\sum_{i=1}^{N} \hat{u}_i^2}{N-2}\right)}$$

Sums of Squares, RSE, etc.

```
> # Sums of squares:
>
> TotalYVar <- with(SCOTUS, sum((CivLibs - mean(CivLibs))^2))
> TotalYVar
[1] 14707

> TotalUVar <- with(SCOTUS, sum((Uhats)^2))
> TotalUVar
[1] 7086

> TotalModelVar <- with(SCOTUS, sum((Yhats - mean(CivLibs))^2))
> TotalModelVar
[1] 7621

> RSE <- with(SCOTUS, sqrt(TotalUVar / (nrow(SCOTUS)-2)))
> RSE
[1] 14.03
```

Estimating $\hat{\beta}$ via `lm`

```
> fit<-lm(CivLibs~IdeologyScore,data=SCOTUS)
> summary(fit)

Call:
lm(formula = CivLibs ~ IdeologyScore, data = SCOTUS)

Residuals:
    Min      1Q  Median      3Q     Max
-26.433 -10.587   2.460   7.858  29.655

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)     33.389      4.354   7.669 4.44e-09 ***
IdeologyScore   43.044      6.917   6.223 3.51e-07 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 14.03 on 36 degrees of freedom
Multiple R-squared:  0.5182,Adjusted R-squared:  0.5048
F-statistic: 38.72 on 1 and 36 DF,  p-value: 3.505e-07
```

```
> anova(fit)

Analysis of Variance Table

Response: CivLibs
              Df Sum Sq Mean Sq F value    Pr(>F)
IdeologyScore  1 7621.4  7621.4  38.723 3.505e-07 ***
Residuals     36 7085.5   196.8
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Inference

Setup

For the linear model:

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

Note that we can rewrite the formula for $\hat{\beta}_1$:

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^N (X_i - \bar{X})^2} \\
&= \frac{\sum_{i=1}^N (X_i - \bar{X})Y_i - \sum_{i=1}^N (X_i - \bar{X})\bar{Y}}{\sum_{i=1}^N (X_i - \bar{X})^2} \\
&= \frac{\sum_{i=1}^N (X_i - \bar{X})Y_i - \bar{Y}\sum_{i=1}^N (X_i - \bar{X})}{\sum_{i=1}^N (X_i - \bar{X})^2} \\
&= \frac{\sum_{i=1}^N (X_i - \bar{X})Y_i - \bar{Y}(0)}{\sum_{i=1}^N (X_i - \bar{X})^2} \\
&= \frac{\sum (X_i - \bar{X})Y_i}{\sum (X_i - \bar{X})^2}
\end{aligned}$$

Variation in $\hat{\beta}_0$ and $\hat{\beta}_1$

$\hat{\beta}_0$ and $\hat{\beta}_1$ are *random variables*...

- Q: Where does their variation come from?
- A: From the *stochastic* variation in Y...
- ...that is, from u.

Next question: What does the random variation in Y "look like"?

Getting To Var($\hat{\beta}_1$)

An assumption:

$$u_i \sim \text{i.i.d. } N(0, \sigma^2)$$

Implies:

$$\text{Var}(Y|X, \beta) = \sigma^2$$

so:

$$\begin{aligned}
\text{Var}(\hat{\beta}_1) &= \text{Var}\left[\frac{\sum_{i=1}^{N}(X_i - \bar{X})Y_i}{\sum_{i=1}^{N}(X_i - \bar{X})^2}\right] \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2 \,\text{Var}(Y) \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2 \,\sigma^2 \\
&= \frac{\sigma^2}{\sum(X_i - \bar{X})^2}.
\end{aligned}$$

$\text{Var}(\hat{\beta}_0)$ and $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)$

Similarly:

$$\text{Var}(\hat{\beta}_0) = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2$$

and :

$$\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2$$

Note that:

- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto \sigma^2$

  $\hookrightarrow$ $\text{Var}(\hat{\beta}\text{s})$ increases as Y gets "noisier"...

- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -\sum(X_i - \bar{X})$

  $\hookrightarrow$ $\text{Var}(\hat{\beta}\text{s})$ decreases with greater variation in X...

- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -N$

  $\hookrightarrow$ $\text{Var}(\hat{\beta}\text{s})$ decreases as N gets larger...

- $\text{sign}[\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)] = -\text{sign}(\bar{X})$

  $\hookrightarrow$ The sign of the covariance between $\hat{\beta}_0$ and $\hat{\beta}_1$ is the opposite of the sign of the mean of X

"Given the assumptions of the classical linear regression model, the least squares estimators are the minimum variance estimators among the class of unbiased linear estimators. (They are BLUE)."

Gauss-Markov, continued

Imagine:

$$\hat{\beta}_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

Rewrite:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})Y_i}{\sum_{i=1}^{N}(X_i - \bar{X})^2}.$$

k are "weights":

$$\hat{\beta}_1 = \sum k_i Y_i$$

with $k_i = \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2}$.

Alternative (non-LS) estimator:

$$\tilde{\beta}_1 = \sum w_i Y_i$$

Unbiasedness requires:

$$\begin{aligned} \mathsf{E}(\tilde{\beta}_1) &= \sum w_i \mathsf{E}(Y_i) \\ &= \sum w_i(\beta_0 + \beta_1 X_i) \\ &= \beta_0 \sum w_i + \beta_1 \sum w_i X_i \end{aligned}$$

Variance:

$$
\begin{aligned}
\mathrm{Var}(\tilde{\beta}_1) &= \mathrm{Var}\left(\sum w_i Y_i\right) \\
&= \sigma^2 \sum w_i^2 \\
&= \sigma^2 \sum \left[w_i - \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} + \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right]^2 \\
&= \sigma^2 \sum \left[w_i - \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right]^2 + \sigma^2 \left[\frac{1}{\sum (X_i - \bar{X})^2} \right]
\end{aligned}
$$

Gauss-Markov (continued)

Because $\sigma^2 \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]$ is a constant, $\min[\text{Var}(\tilde{\beta}_1)]$ minimizes

$$\sum \left[w_i - \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2}\right]^2.$$

Minimized at:

$$w_i = \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2}.$$

implying:

$$\begin{aligned} \text{Var}(\tilde{\beta}_1) &= \frac{\sigma^2}{\sum(X_i - \bar{X})^2} \\ &= \text{Var}(\hat{\beta}_1) \end{aligned}$$

For the Gauss-Markov theorem to hold, it must be the case that:

1. $\mathrm{E}(u) = 0$
2. $\mathrm{Cov}(X,u) = 0$

3a. $\mathrm{Var}(u) = \sigma^2 \; \forall \; i$

3b. $\mathrm{Cov}(u_i, u_j) = 0$

4. $\mathrm{Rank}(\mathbf{X}) = k$
5. $u \sim$ i.i.d. $N(0, \sigma^2)$

(...don't sweat these too much for now...)

BLUE, BUE, and Linearity

BLUE vs. BUE:

- OLS has been BLUE since about 1821 (see, e.g., Plackett 1949).
- Hansen (2022): OLS is "BUE" – most efficient among *all* unbiased estimators, linear or otherwise...
- Challenged by others; resolved by Portnoy (2022): Any unbiased estimator *must* be linear (so "BLUE" = "BUE").
- A pretty good nontechnical discussion of all this by Paul Allison is here.

If $u_i \sim N(0, \sigma^2)$, then:

$$\hat{\beta}_0 \sim N[\beta_0, \text{Var}(\hat{\beta}_0)]$$

and

$$\hat{\beta}_1 \sim N[\beta_1, \text{Var}(\hat{\beta}_1)]$$

Means:

$$\begin{aligned} z_{\hat{\beta}_1} &= \frac{(\hat{\beta}_1 - \beta_1)}{\sqrt{\text{Var}(\hat{\beta}_1)}} \\ &= \frac{(\hat{\beta}_1 - \beta_1)}{\text{s.e.}(\hat{\beta}_1)} \\ &= \sim N(0, 1) \end{aligned}$$

A Small Problem...

$\sigma^2 =$???

Solution: use

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{N - k}$$

Gives:

$$\widehat{\text{Var}(\hat{\beta}_1)} = \frac{\hat{\sigma}^2}{\sum (X_i - \bar{X})^2},$$

and

$$\widehat{\text{Var}(\hat{\beta}_0)} = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \hat{\sigma}^2$$

Inference (continued)

$$\begin{aligned}\widehat{\text{s.e.}(\hat{\beta}_1)} &= \sqrt{\widehat{\text{Var}(\hat{\beta}_1)}} \\ &= \sqrt{\frac{\hat{\sigma}^2}{\sum(X_i - \bar{X})^2}} \\ &= \frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}\end{aligned}$$

implies:

$$\begin{aligned}t_{\hat{\beta}_1} \equiv \frac{(\hat{\beta}_1 - \beta_1)}{\widehat{\text{s.e.}(\hat{\beta}_1)}} &= \frac{(\hat{\beta}_1 - \beta_1)}{\frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}} \\ &= \frac{(\hat{\beta}_1 - \beta_1)\sqrt{\sum(X_i - \bar{X})^2}}{\hat{\sigma}} \\ &\sim t_{N-k}\end{aligned}$$

Point prediction:

$$\hat{Y}_k = \hat{\beta}_0 + \hat{\beta}_1 X_k$$

Y_k is unbiased:

$$\begin{aligned} \mathrm{E}(\hat{Y}_k) &= \mathrm{E}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\ &= \mathrm{E}(\hat{\beta}_0) + X_k \mathrm{E}(\hat{\beta}_1) \\ &= \beta_0 + \beta_1 X_k \\ &= \mathrm{E}(Y_k) \end{aligned}$$

Variability:

$$\begin{aligned} \mathrm{Var}(\hat{Y}_k) &= \mathrm{Var}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\ &= \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2 + \left[\frac{\sigma^2}{\sum (X_i - \bar{X})^2} \right] X_k^2 + 2 \left[\frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2 \right] X_k \\ &= \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right] \end{aligned}$$

Variability of Predictions

$$\mathrm{Var}(\hat{Y}_k) = \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]$$

means that $\mathrm{Var}(\hat{Y}_k)$:

- Decreases in N
- Decreases in $\mathrm{Var}(X)$
- Increases in $|X - \bar{X}|$

Standard error of the prediction:

$$\widehat{\text{s.e.}(\hat{Y}_k)} = \sqrt{\sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right]}$$

$\rightarrow$ (e.g.) confidence intervals:

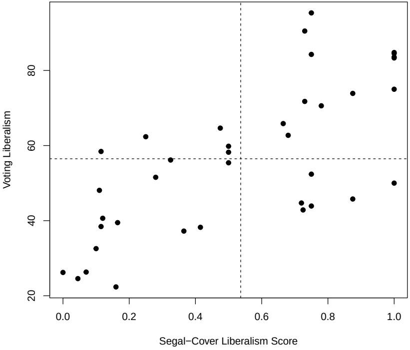
$$95\%\,\text{c.i.}(\hat{Y}_k) = \hat{Y}_k \pm [1.96 \times \widehat{\text{s.e.}(\hat{Y}_k)}]$$

Example: SCOTUS Liberalism

```
> with(SCOTUS, describe(CivLibs))
   vars  n  mean    sd median trimmed   mad   min   max range skew kurtosis   se
X1    1 38 56.49 19.94  55.78   56.33 23.07 22.36 95.33 72.97 0.15    -1.03 3.23

> with(SCOTUS, describe(IdeologyScore))
   vars  n mean   sd median trimmed  mad min max range  skew kurtosis   se
X1    1 38 0.54 0.33   0.58    0.54 0.43   0   1     1 -0.11    -1.46 0.05
```

Scatterplot of SCOTUS Voting and Liberalism Scores



Example, Continued

```
> SCLib<-lm(CivLibs~IdeologyScore,data=SCOTUS)
> summary(SCLib)   # regression

Call:
lm(formula = CivLibs ~ IdeologyScore, data = SCOTUS)

Residuals:
   Min     1Q Median     3Q    Max
-26.43 -10.59   2.46   7.86  29.66

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)      33.39       4.35    7.67  4.4e-09 ***
IdeologyScore    43.04       6.92    6.22  3.5e-07 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 14 on 36 degrees of freedom
Multiple R-squared:  0.518,Adjusted R-squared:  0.505
F-statistic: 38.7 on 1 and 36 DF,  p-value: 3.51e-07
```

```
> anova(SCLib)      # ANOVA

Analysis of Variance Table

Response: CivLibs
              Df Sum Sq Mean Sq F value  Pr(>F)
IdeologyScore  1   7621    7621    38.7 3.5e-07 ***
Residuals     36   7086     197
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Var($\hat{\beta}$):

```
> vcov(SCLib)

              (Intercept) IdeologyScore
(Intercept)         18.96        -25.67
IdeologyScore      -25.67         47.85
```

95 percent c.i.s:

```
> confint(SCLib)
              2.5 % 97.5 %
(Intercept)   24.56  42.22
IdeologyScore 29.02  57.07
```

99 percent c.i.s:

```
> confint(SCLib,level=0.99)
              0.5 % 99.5 %
(Intercept)   21.55  45.23
IdeologyScore 24.23  61.85
```

Predictions

```
> SEs<-predict(SCLib,interval="confidence")
> SEs
     fit   lwr   upr
1  71.05 64.43 77.67
2  64.60 59.28 69.91
3  62.01 57.06 66.97
.
.
<rows omitted>
.
.
36 64.81 59.46 70.17
37 38.12 30.57 45.68
38 36.40 28.39 44.41
```

A Plot, With CIs

Scatterplot of SCOTUS Voting and Ideology Scores, along with Least-Squares Line
and 95% Prediction Confidence Intervals