PLSC 503 – Fall 2024
Central Tendency and Variation

September 23, 2024

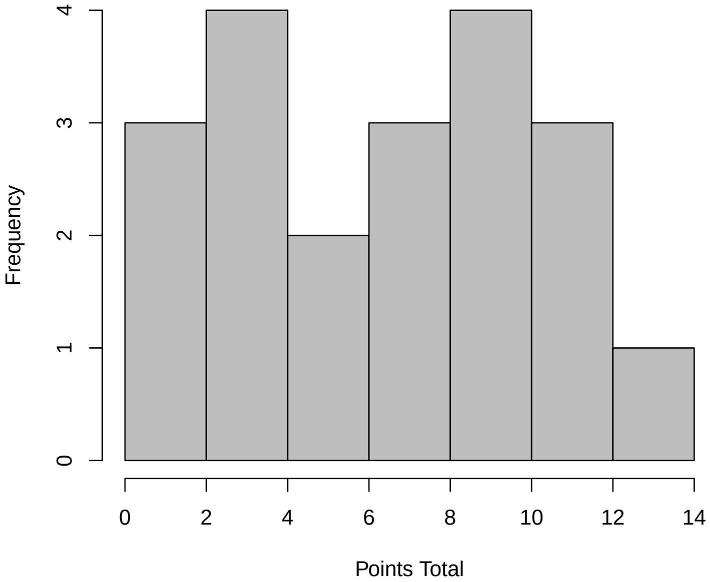
Link Of The Day

Link: https://x.com/RandVegan/status/1837958470890348970

Example: Today's Premier League

Team	GamesPlayed	Won	Drew	Lost	GoalsFor	GoalsAgainst	GoalDifference	Points
Manchester City	5	4	1	0	13	5	8	13
Liverpool	5	4	0	1	10	1	9	12
Aston Villa	5	4	0	1	10	7	3	12
Arsenal	5	3	2	0	8	3	5	11
Chelsea	5	3	1	1	11	5	6	10
Newcastle United	5	3	1	1	7	6	1	10
Brighton and Hove Albion	5	2	3	0	8	4	4	9
Nottingham Forest	5	2	3	0	6	4	2	9
Fulham	5	2	2	1	7	5	2	8
Tottenham Hotspur	5	2	1	2	9	5	4	7
Manchester United	5	2	1	2	5	5	0	7
Brentford	5	2	0	3	7	9	-2	6
Bournemouth	5	1	2	2	5	8	-3	5
West Ham United	5	1	1	3	5	9	-4	4
Leicester City	5	0	3	2	6	8	-2	3
Crystal Palace	5	0	3	2	4	7	-3	3
Ipswich Town	5	0	3	2	3	8	-5	3
Southampton	5	0	1	4	2	9	-7	1
Everton	5	0	1	4	5	14	-9	1
Wolverhampton Wanderers	5	0	1	4	5	14	-9	1

Premier League Points: Histogram



The Arithmetic Mean

The "mean":

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

implies that:

$$\sum_{i=1}^{N} X_i = N\bar{X}$$

and so:

$$\sum_{i=1}^{N}(X_i) - N\bar{X} \; = \; \sum_{i=1}^{N}(X_i - \bar{X}) \; = \; 0.$$

$\bar{X}$ Minimizes Squared Deviations

Find the value of X μ that minimizes the sum of squared deviations...

$$
\begin{aligned}
f(X) &= \sum_{i=1}^{N}(X_i - \mu)^2 \\
&= \sum_{i=1}^{N}(X_i^2 + \mu^2 - 2\mu X_i) \\
\frac{\partial f(X)}{\partial \mu} &= \sum_{i=1}^{N}(2\mu - 2X_i)
\end{aligned}
$$

$\bar{X}$ Minimizes Squared Deviations

Solve:

$$
\begin{aligned}
\sum_{i=1}^{N}(2\mu - 2X_i) &= 0 \\
2N\mu - 2\sum_{i=1}^{N} X_i &= 0 \\
2N\mu &= 2\sum_{i=1}^{N} X_i \\
\mu &= \frac{1}{N}\sum_{i=1}^{N} X_i \equiv \bar{X}
\end{aligned}
$$

Means from Sums of Frequencies

Frequency table:

Points	Frequency f_j
1	3
3	3
4	1
5	1
$\vdots$	$\vdots$
13	1

For J different unique values of X:

$$\bar{X} = \frac{1}{N} \sum_{j=1}^{J} f_j X_j$$

Weighted Means

For "weights" w_i, the *weighted (arithmetic) mean* is:

$$\bar{W} = \frac{\sum_{i=1}^{N} w_i X_i}{\sum_{i=1}^{N} w_i}$$

Things to remember:

- If $w_i = \frac{1}{N} \; \forall \; i$, then $\bar{W} = \bar{X}$
- If $w_i = w \; \forall \; i$, then $\bar{W} = w\bar{X}$
- Weighted means are simpler if $\sum_{i=1}^{N} w_i = 1.0$...
- ... we can normalize any set of weights by $w_i' = \frac{w_i}{\sum_{i=1}^{N} w_i}$.

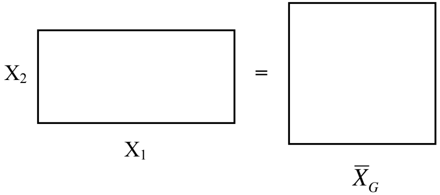
Geometric Mean

The *geometric* mean is:

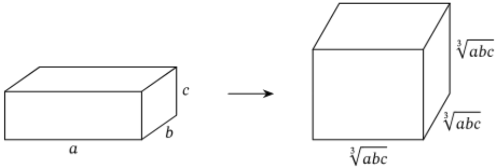
$$\begin{aligned}
\bar{X}_G &= \left(\prod_{i=1}^{N} X_i \right)^{\frac{1}{N}} \\
&= \sqrt[N]{X_1 \cdot X_2 \cdot \ldots \cdot X_N} \\
&= \exp \left[\frac{1}{N} \sum_{i=1}^{N} \ln X_i \right]
\end{aligned}$$

Geometric Mean (graphically)

For example, with $N = 2$:



With $N = 3$:



Geometric Mean (continued)

Note: Geometric means don't like zero or negative values...

- Formally, $\bar{X}_G$ is defined only if $X_i > 0 \; \forall \; i$
- R's `geometric.mean()` defaults to removing them before calculation...
- If *all* values of X are negative, the geometric mean will be `NaN`.

Consider percentage changes:

$$\{ +12\%, +5\%, \text{-}9\%, +2\%, \text{-}10\% \}$$

```
> geometric.mean(c(12,5,-9,2,-10))
[1] 4.932424

Warning message:
In log(x) : NaNs produced

> geometric.mean(c(1.12,1.05,0.91,1.02,0.90))
[1] 0.9964563
```

Harmonic Mean

The harmonic mean is:

$$\bar{X}_H = \frac{N}{\sum_{i=1}^{N} \frac{1}{X_i}} = \frac{1}{\left(\overline{\frac{1}{X}}\right)}$$

Note that:

$$\bar{X}_H \leq \bar{X}_G \leq \bar{X}$$

The Median

The *arithmetic median*:

$$\begin{aligned}\check{X} &= \text{“middle observation” of } X \\ &= \text{50th } \textit{percentile} \text{ of } X.\end{aligned}$$

The median minimizes *absolute* distance:

$$\check{X} = \min\left(\sum_{i=1}^{N} |X_i - c|\right).$$

Some Fun Facts About Medians

1. Median observations need not be unique; more than one observation can be the "median" observation. Likewise...

2. By convention, when N is even, $\check{X}$ is the arithmetic mean of the two "middle observations"...
   - This means for N even, there is *no* observation in the data that is the median observation. Conversely,
   - A *medoid* is the observation in the data that is the most similar / least distant from the others.

3. Medians can be calculated on *discrete* ordinal data (e.g., ranks).

4. Medians can be calculated on *censored* data (e.g., durations).

5. If a distribution has finite variance (see below), the difference between the median and the mean can be no more than one standard deviation.

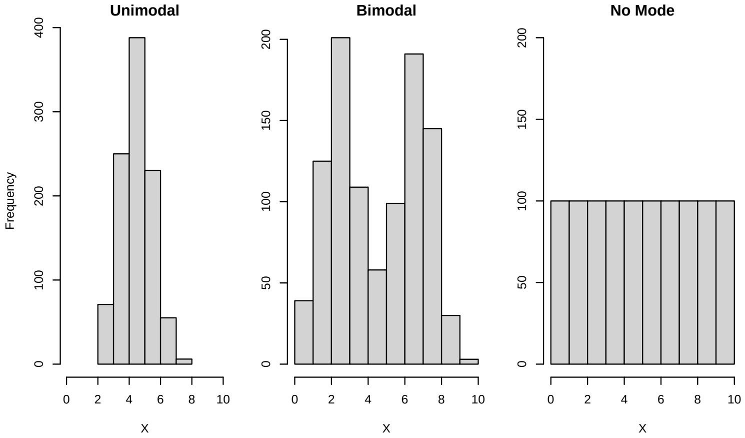
The Mode

The mode of X is "the value of X that appears most frequently in the data."

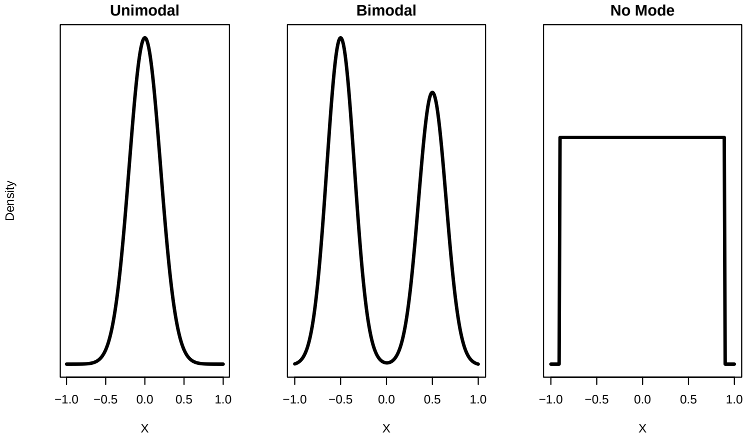
- That works fine for discrete variables...
  - There can be zero, one, two, or more modes,
  - If (say) two values of X have *nearly* the same number of cases, we often refer to that as "bimodal" data.

- For continuous variables:
  - There is often no mode (no two observations have *exactly* the same values of X)
  - Modes are usually defined as any *local maximum* of the probability density function of X

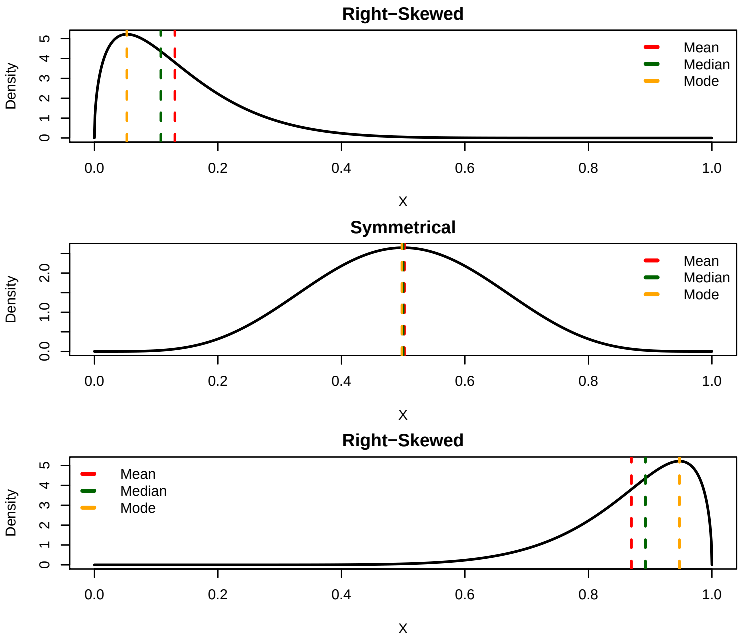
Modes: Discrete X



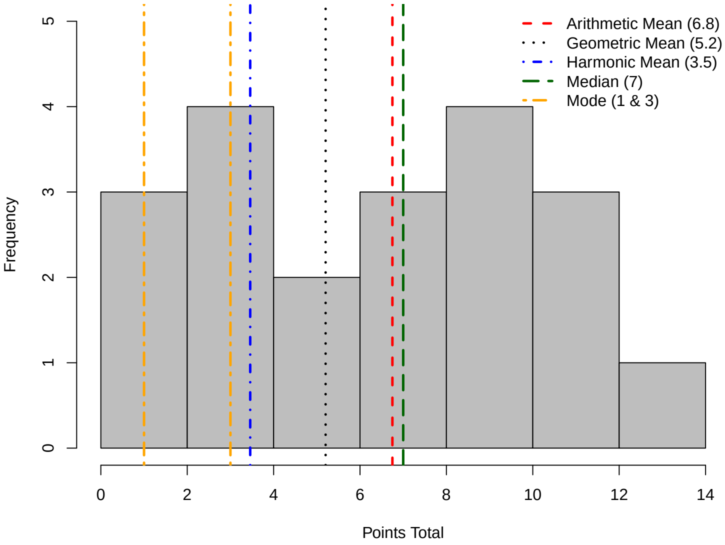
Modes: Continuous X



Means, Medians, Modes, and Skewness



Central Tendencies: Premier League Data



Variation

Range and Percentiles

Range:

$$\text{Range}(X) = \max(X) - \min(X)$$

The k**th percentile** is the value of the variable below which k percent of the observations fall.

- 50th percentile $= \check{X}$
- 0th percentile $=$ minimum(X)
- 100th percentile $=$ maximum(X)

More Percentiles

- *Quartiles* $= \{$25th, 50th, 75th percentiles$\}$

- *Interquartile Range* (IQR):

$$\text{IQR}(X) = \text{75th percentile}(X) - \text{25th percentile}(X)$$

- *Deciles* $= \{$10th, 20th, 30th, etc. percentiles$\}$

"Mean Deviation"

$$\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X}).$$

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X}) &= \frac{1}{N}\left[\left(\sum_{i=1}^{N} X_i\right) - N\bar{X}\right] \\
&= \frac{1}{N}\left[\sum_{i=1}^{N} X_i - N\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)\right] \\
&= \frac{1}{N}\left(\sum_{i=1}^{N} X_i - \sum_{i=1}^{N} X_i\right) = \frac{1}{N}(0) \\
&= 0
\end{aligned}$$

Squared Deviation

Mean squared deviation:

$$\text{MSD} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2$$

Also known as *mean squared error* ("MSE") in regression models...

Note that MSD is "average squared difference from the mean" $\rightarrow$ expressed in "squared" units of X...

A more useful quantity is "root mean squared deviation":

$$\text{RMSD} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2}$$

An Important Fact

Consider $N = 1$:

Team	Points
Tottenham Hotspur	7

This gives:

$$\bar{X} = \frac{14}{1} = 7 \quad \text{and} \quad RMSD = \sqrt{\frac{(7-7)^2}{1}} = 0$$

For $N = 2$:

Team	Points
Tottenham Hotspur	7
Everton	1

we get:

$$\bar{X} = \frac{7+1}{2} = 4 \quad \text{and} \quad RMSD = \sqrt{\frac{(7-4)^2 + (1-4)^2}{2}} = 3$$

You cannot learn about more characteristics of data than you have observations.

Variance:

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2$$

Standard deviation:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2}$$

"Geometric" Standard Deviation:

$$\sigma_G = \exp\left[\sqrt{\frac{\sum_{i=1}^{N}(\ln X_i - \ln \bar{X}_G)^2}{N}}\right]$$

PL Points Data

```
> summary(PL$Points)

   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   1.00    3.00    7.00    6.75   10.00   13.00

> var(PL$Points)

[1] 15.7

> sd(PL$Points)

[1] 3.96
```

Standardizing Variables

Sometimes useful to put variables on a common scale... ("z-scores")...

Typically:

$$Z_i = \frac{X_i - \bar{X}}{\sigma}$$

A standardized variable Z has:

- A mean of zero, and
- A standard deviation (and therefore variance) of 1.0

Standardizing Example

```
> library(psych)

> PLSmall<-PL[,4:10]
> describe(PLSmall,trim=0,skew=FALSE)
               vars  n mean   sd median min max range   se
Won               1 20 1.75 1.45    2.0   0   4     4 0.32
Drew              2 20 1.50 1.05    1.0   0   3     3 0.24
Lost              3 20 1.75 1.33    2.0   0   4     4 0.30
GoalsFor          4 20 6.80 2.78    6.5   2  13    11 0.62
GoalsAgainst      5 20 6.80 3.27    6.5   1  14    13 0.73
GoalDifference    6 20 0.00 5.30    0.5  -9   9    18 1.19
Points            7 20 6.75 3.96    7.0   1  13    12 0.89

> PL.Z<-scale(PLSmall)
> describe(PL.Z,trim=0,skew=FALSE)
               vars  n mean sd median   min  max range   se
Won               1 20    0  1   0.17 -1.21 1.56  2.77 0.22
Drew              2 20    0  1  -0.48 -1.43 1.43  2.85 0.22
Lost              3 20    0  1   0.19 -1.31 1.69  3.00 0.22
GoalsFor          4 20    0  1  -0.11 -1.72 2.23  3.95 0.22
GoalsAgainst      5 20    0  1  -0.09 -1.77 2.20  3.98 0.22
GoalDifference    6 20    0  1   0.09 -1.70 1.70  3.40 0.22
Points            7 20    0  1   0.06 -1.45 1.58  3.03 0.22
```

Absolute Deviations and MAD

Median Absolute Deviation ("MAD"):

$$\text{MAD} = \text{median}[|X_i - \check{X}|]$$

Mean Absolute Deviation:

$$\text{Mean Absolute Deviation} = \frac{1}{N} \sum_{i=1}^{N} |X_i - \bar{X}|$$

Moments

Moments are functions of distributions that characterize their shape...

For a random variable X, the kth *raw moment* is:

$$m_k = \begin{cases} \sum f(X)\,\Pr(X) \text{ if } X \text{ is discrete} \\ \int f(X)\,\Pr(X) dX \text{ if } X \text{ is continuous.} \end{cases}$$

The kth *central moment* is:

$$M_k = \begin{cases} \mathrm{E}[(X-\mu)^k] \text{ for discrete } X \\ \int_{-\infty}^{+\infty} (X-\mu)^k f(X)\,dX \text{ for continuous } X \end{cases}$$

A distribution for X can be completely characterized by its non-zero moments...

Why Might We Care?

The first (raw) moment of a variable is the mean:[†]

$$\mu = \mathsf{E}(X)$$

The second (central) moment of a variable is its variance:

$$\sigma^2 = \mathsf{E}[(X - \mu)^2]$$

[†] The first central moment is zero (why?)...

Skewness

Third central moment:

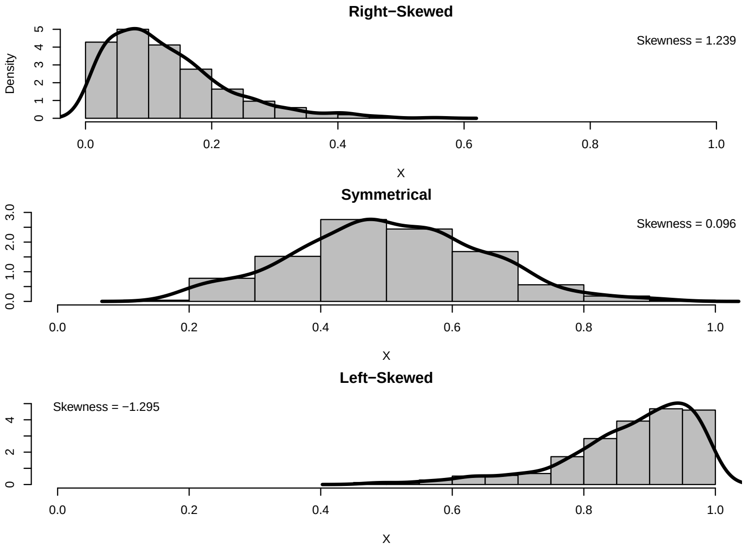
$$M_3 = \mathrm{E}[(X - \mu)^3]$$

More typically, we use the third *standardized moment* (usually called *skewness*):

$$\begin{aligned}\mu_3 &= \frac{M_3^2}{\sigma^3} \\ &= \frac{\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^3}{\left[\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2\right]^{3/2}}\end{aligned}$$

- Skewness $= 0 \rightarrow$ symmetrical
- Skewness $> 0 \rightarrow$ "positive" (tail to the right)
- Skewness $< 0 \rightarrow$ "negative" (tail to the left)

Skewness Illustrated



Symmetry

If a distribution is *symmetrical*, then:

- $\mu_3 = 0$
- $\check{X} = (Q_{25} + Q_{75})/2$,
- $\text{MAD} = \frac{\text{IQR}}{2}$

Note that:

- Both discrete and continuous variables can be symmetrical or asymmetrical;
- Every distribution with *no mode* is symmetrical, but
- Unimodal, bimodal, etc. distributions can be symmetrical or asymmetrical.

Kurtosis

Fourth moment:

$$M_4 = \mathrm{E}[(X-\mu)^4]$$

More typically, *kurtosis* ("excess kurtosis"):

$$\begin{aligned}
\mu_4 &= \frac{M_4}{\sigma^4} - 3 \\
&= \frac{\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^4}{\left[\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2\right]^2} - 3
\end{aligned}$$

Note that:

$$\frac{M_4}{\sigma^4} \geq \left(\frac{M_3}{\sigma^3}\right)^2 + 1$$

Kurtosis Intuition

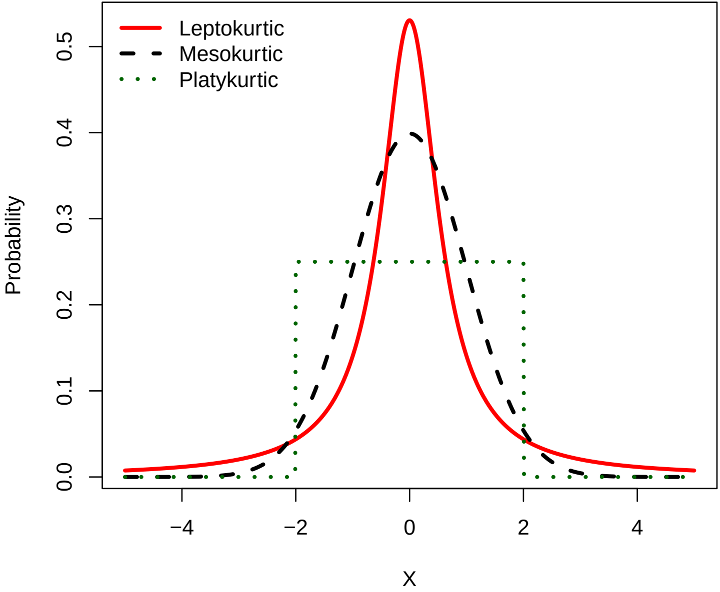
Kurtosis is "the average of the standardized X raised to the fourth power (minus three)."

- Values of standardized variables within one σ of $\bar{X}$ have $|X| \leq 1$
- Taking X^4 when $|X| \leq 1$ gives values very close to 0
- $\rightarrow$ only those values on the "tail" of the distribution contribute significantly to kurtosis

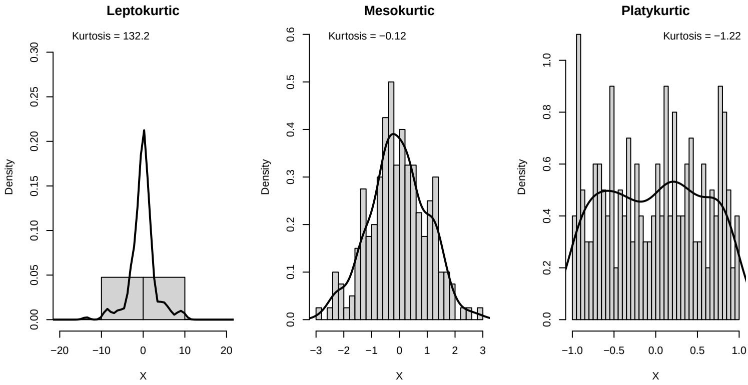
Kurtosis:

- "Fat-tailed" $=$ *leptokurtic*: μ_4 is positive.
- "Medium-tailed" $=$ *mesokurtic*: μ_4 is close to zero.
- "Thin-tailed" $=$ *platykurtic*: μ_4 is negative.

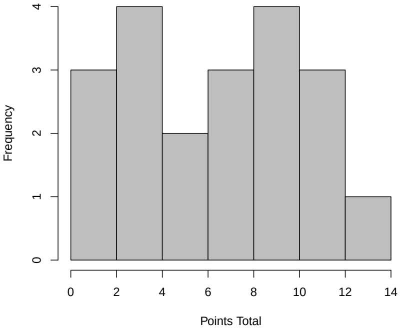
Kurtosis Illustrated



Kurtosis Examples



PL Points Data




```
> library(moments)

> skewness(PL$Points)
[1] -0.0447

> kurtosis(PL$Points)-3
[1] -1.29
```

Binary Variables

For a Bernoulli (binary) variable D:

- $\text{mode}(D) = \check{D}$ (why?)
- The mean of D is:

$$\begin{aligned}\bar{D} &= \frac{1}{N}\sum D_i \\ &= \pi \quad [\equiv \Pr(D=1)]\end{aligned}$$

- The variance is:

$$\sigma_D^2 = \pi \times (1-\pi)$$

- and so the standard deviation is:

$$\sigma_D = \sqrt{\pi \times (1-\pi)}$$

Implies:

- $\sigma_D > \sigma_D^2$
- $\max(\sigma_D^2) \leftrightarrow \pi = 0.5$

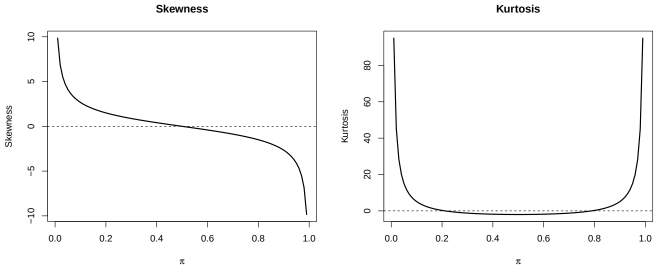
Binary Variables (continued)

For a binary variable, skewness is:

$$\mu_3 = \frac{1 - 2\pi}{\sqrt{\pi(1 - \pi)}}$$

and the (excess) kurtosis is:

$$\mu_4 = \frac{1 - 6\pi(1 - \pi)}{\pi(1 - \pi)}$$



Getting Summary Statistics

Good: `summary`

```
> summary(PLSmall)
      Won            Drew           Lost         GoalsFor      GoalsAgainst  
 Min.   :0.00   Min.   :0.00   Min.   :0.00   Min.   : 2.00   Min.   : 1.00  
 1st Qu.:0.00   1st Qu.:1.00   1st Qu.:1.00   1st Qu.: 5.00   1st Qu.: 5.00  
 Median :2.00   Median :1.00   Median :2.00   Median : 6.50   Median : 6.50  
 Mean   :1.75   Mean   :1.50   Mean   :1.75   Mean   : 6.80   Mean   : 6.80  
 3rd Qu.:3.00   3rd Qu.:2.25   3rd Qu.:2.25   3rd Qu.: 8.25   3rd Qu.: 8.25  
 Max.   :4.00   Max.   :3.00   Max.   :4.00   Max.   :13.00   Max.   :14.00  
 GoalDifference      Points     
 Min.   :-9.00   Min.   : 1.00  
 1st Qu.:-3.25   1st Qu.: 3.00  
 Median : 0.50   Median : 7.00  
 Mean   : 0.00   Mean   : 6.75  
 3rd Qu.: 4.00   3rd Qu.:10.00  
 Max.   : 9.00   Max.   :13.00  
```

Better: `describe` (in `psych`)

```
> describe(PLSmall)
               vars  n mean   sd median trimmed  mad min max range  skew kurtosis   se
Won               1 20 1.75 1.45    2.0    1.69 1.48   0   4     4  0.12    -1.39 0.32
Drew              2 20 1.50 1.05    1.0    1.50 1.48   0   3     3  0.26    -1.31 0.24
Lost              3 20 1.75 1.33    2.0    1.69 1.48   0   4     4  0.31    -1.08 0.30
GoalsFor          4 20 6.80 2.78    6.5    6.69 2.22   2  13    11  0.40    -0.60 0.62
GoalsAgainst      5 20 6.80 3.27    6.5    6.50 2.22   1  14    13  0.66     0.05 0.73
GoalDifference    6 20 0.00 5.30    0.5    0.06 5.19  -9   9    18 -0.10    -1.11 1.19
Points            7 20 6.75 3.96    7.0    6.75 5.19   1  13    12 -0.04    -1.46 0.89
```

Reporting Summary Statistics

```
> stargazer(PLSmall,title="Summary Statistics")
```

Table: Summary Statistics

Statistic	N	Mean	St. Dev.	Min	Max
Won	20	1.750	1.450	0	4
Drew	20	1.500	1.050	0	3
Lost	20	1.750	1.330	0	4
Goals For	20	6.800	2.780	2	13
Goals Against	20	6.800	3.270	1	14
Goal Difference	20	0.000	5.300	-9	9
Points	20	6.750	3.960	1	13

R Things

Packages / commands for summary statistics:

- `summary` – basic summaries by variable
- `describe` (in `psych`) – flexible summary statistics
- `describe` (in `Hmisc`) – more information than you probably want
- `stat.desc` (in `pastecs`) – like `psych::describe`, but sideways

Packages / commands for making pretty tables:

- `stargazer`
- `tinytable`
- `gt`
- `kable` / `kableExtra`
- `flextable`
- `huxtable`