PLSC 503 – Spring 2021
Multivariate Regression

February 3, 2021

The Model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

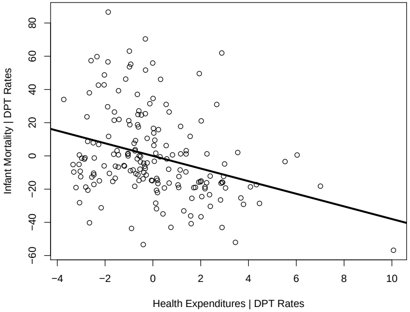
$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_K X_{Ki} + u_i$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{K1} \\ 1 & X_{12} & X_{22} & \cdots & X_{K2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1N} & X_{2N} & \cdots & X_{KN} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}.$$

Diversion: "Added Variable Plots"

- Regress Y on X_1 and save the residuals $\hat{u}_i$,
- Regress X_2 on X_1 and save the residuals (call these $\hat{e}_i$),
- Plot $\hat{u}_i$ (conventionally on the y-axis) vs. $\hat{e}_i$ (conventionally on the x-axis).

Example: Infant Mortality and Health Expenditures Given DPT Immunization Rates



Estimating $\boldsymbol{\beta}$

Residuals:

$$\mathbf{u} = \mathbf{Y} - \mathbf{X}\boldsymbol{\beta}$$

The inner product of **u**:

$$\begin{aligned}
\mathbf{u}'\mathbf{u} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \\
&= u_1^2 + u_2^2 + ... + u_N^2 \\
&= \sum_{i=1}^{N} u_i^2
\end{aligned}$$

Estimating $\boldsymbol{\beta}$

$$
\begin{aligned}
\mathbf{u}'\mathbf{u} &= (\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}) \\
&= \mathbf{Y}'\mathbf{Y} - 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{Y}' + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}
\end{aligned}
$$

Now get:

$$
\frac{\partial \mathbf{u}'\mathbf{u}}{\partial \boldsymbol{\beta}} = -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}
$$

Solve:

$$
\begin{aligned}
-2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\
-\mathbf{X}'\mathbf{Y} + \mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\
\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= \mathbf{X}'\mathbf{Y} \\
(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\
\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}
\end{aligned}
$$

Estimation Issues

"Do not compute the least squares estimates using $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$!"
– *Weisberg (p. 61)*

Most software uses:

$$\mathbf{X} = \mathbf{QR}$$

where $\mathbf{Q}$ is orthogonal ($\mathbf{Q}'\mathbf{Q} = \mathbf{I}$) and $\mathbf{R}$ is upper-triangular.

Why??? See e.g. here, or section 3.19, here

OLS Assumptions

1. Expectation-Zero Disturbances

$$\mathsf{E}(\mathbf{u}) = \mathbf{0}$$

OLS Assumptions

2. Homoscedasticity / No Error Correlation

$$
\begin{aligned}
\mathbf{uu}' &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \\
&= \begin{bmatrix} u_1^2 & u_1 u_2 & \cdots & u_1 u_N \\ u_2 u_1 & u_2^2 & \cdots & u_2 u_N \\ \vdots & \vdots & \ddots & \vdots \\ u_N u_1 & u_N u_2 & \cdots & u_N^2 \end{bmatrix}
\end{aligned}
$$

Expectation must be:

$$
\mathrm{E}(\mathbf{uu}') = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}
$$

OLS Assumptions

<u>3. "Fixed" **X**...</u>

- No *measurement error* in the $\mathbf{X}$s, and
- $\text{Cov}(\mathbf{X}, \mathbf{u}) = \mathbf{0}$.

<u>4. **X** is full column rank.</u>

Means:

- no exact linear relationship among $\mathbf{X}$, and
- $K < N$.

<u>5. Normal Disturbances</u>

$$\mathbf{u} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

OLS: Unbaisedness

Unbiasedness:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

Substitute OLS $\hat{\boldsymbol{\beta}}$:

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\
&= \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}
\end{aligned}$$

and so:

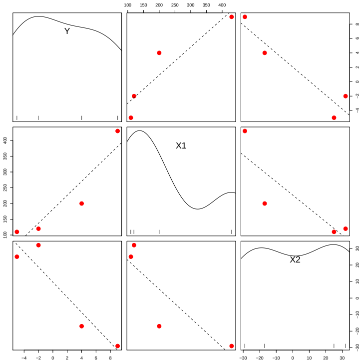
$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}.$$

By $\mathrm{Cov}(\mathbf{X}, \mathbf{u}) = \mathbf{0}$, we have $\mathrm{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$.

A Toy Example

$$\mathbf{Y} = \begin{bmatrix} 4 \\ -2 \\ 9 \\ -5 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 200 & -17 \\ 1 & 120 & 32 \\ 1 & 430 & -29 \\ 1 & 110 & 25 \end{bmatrix}$$



Example, continued

So:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 4 & 860 & 11 \\ 860 & 251400 & -9280 \\ 11 & -9280 & 2779 \end{bmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 3.2453 & -0.0132 & -0.05694 \\ -0.0132 & 0.000058 & 0.0002468 \\ -0.0569 & 0.000247 & 0.001409 \end{bmatrix}$$

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} 6 \\ 3880 \\ 518 \end{bmatrix}$$

So:

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= \begin{bmatrix} 3.2453 & -0.0132 & -0.05694 \\ -0.0132 & 0.000058 & 0.0002468 \\ -0.0569 & 0.000247 & 0.001409 \end{bmatrix} \begin{bmatrix} 6 \\ 3880 \\ 518 \end{bmatrix} \\ &= \begin{bmatrix} -2.264 \\ 0.0190 \\ -0.1141 \end{bmatrix} \end{aligned}$$

R Example: Correlation

```
Y<-c(4,-2,9,-5)
X1<-c(200,120,430,110)
X2<-c(-17,32,-29,25)
data<-cbind(Y,X1,X2)
```

```
cor(data)
```

```
         Y      X1      X2
Y   1.0000  0.9285 -0.9425
X1  0.9285  1.0000 -0.8613
X2 -0.9425 -0.8613  1.0000
```

Regression

```
fit<-lm(Y~X1+X2)
summary(fit)

Call:
lm(formula = Y ~ X1 + X2)

Residuals:
     1      2      3      4
 0.531  1.639 -0.201 -1.970

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2.2643     4.7284   -0.48     0.72
X1            0.0190     0.0200    0.95     0.52
X2           -0.1141     0.0985   -1.16     0.45

Residual standard error: 2.62 on 1 degrees of freedom
Multiple R-Squared: 0.941,Adjusted R-squared: 0.823
F-statistic: 7.99 on 2 and 1 DF,  p-value: 0.243
```

Inference, In General

- Pick some $\mathbf{H}_A : \boldsymbol{\beta} = \boldsymbol{\beta}_A$
- Estimate $\hat{\boldsymbol{\beta}}$
- Determine distribution of $\hat{\boldsymbol{\beta}}$ under $\mathbf{H}_A$
- Form a *test statistic* $\hat{\mathbf{S}} = h(\boldsymbol{\beta}, \hat{\boldsymbol{\beta}})$
- Assess $\Pr(\hat{\mathbf{S}}|\mathbf{H}_A)$

The Importance of $\mathbf{V}(\hat{\boldsymbol{\beta}})$

$$\begin{aligned}
\mathbf{V}(\hat{\boldsymbol{\beta}}) &= \mathsf{E}[\hat{\boldsymbol{\beta}} - \mathsf{E}(\hat{\boldsymbol{\beta}})]^2 \\
&= \mathsf{E}\{[\hat{\boldsymbol{\beta}} - \mathsf{E}(\hat{\boldsymbol{\beta}})][\hat{\boldsymbol{\beta}} - \mathsf{E}(\hat{\boldsymbol{\beta}})]'\}
\end{aligned}$$

Rewrite:

$$\begin{aligned}
\mathbf{V}(\hat{\boldsymbol{\beta}}) &= \mathsf{E}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \\
&= \mathsf{E}\{[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}]'\} \\
&= \mathsf{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]
\end{aligned}$$

The Importance of $\mathbf{V}(\hat{\beta})$

Taking expectations:

$$\begin{aligned}
\mathbf{V}(\hat{\beta}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{E}(\mathbf{u}\mathbf{u}')\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

Estimating $\mathbf{V}(\hat{\beta})$

Empirical estimate:

$$\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{N - K}$$

Yields:

$$\widehat{\mathbf{V}(\hat{\beta})} = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$$

Single Coefficient Hypothesis Tests

We know that:

$$\hat{\boldsymbol{\beta}} \sim \mathcal{N}[\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1}].$$

In practice, using $\hat{\sigma}^2$ means

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \sim t_{N-K}$$

<u>Procedure</u>:

- Choose a value of β_k that you want to test (say, $\beta_k = 0$),
- Calculate the t-statistic for the coefficient associated with X_k, which is:

$$\hat{t}_k = \frac{\hat{\beta}_k - \beta_k}{\sqrt{\widehat{\mathbf{V}(\hat{\beta}_k)}}}$$

- Compare $\hat{t}_k$ to a t distribution with $N - K$ degrees of freedom.

Multivariate Hypothesis Testing

E.g.: $\quad \mathrm{H}_0 : \beta_1 = \beta_2 = ... = \beta_K = 0$

or: $\quad \mathrm{H}_0 : \beta_3 = \beta_6 = 0$

Generally: *Linear restrictions*:

$$\underset{q \times k}{\mathbf{R}} \underset{k \times 1}{\boldsymbol{\beta}} = \underset{q \times 1}{\mathbf{r}}$$

E.g.:

$$\beta_2 = -2 \iff (0\ 1\ 0) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = -2$$

The F-test

Recall:

$$\textbf{TSS} \quad = \quad \textbf{MSS} \quad + \quad \textbf{RSS}$$

Consider:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_{Ui}$$

and the restriction:

$$\mathrm{H}_a : \beta_2 = \beta_4 = 0.$$

Restricted model:

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_{1i} + 0X_{2i} + \beta_3 X_{3i} + 0X_{4i} + u_i \\ &= \beta_0 + \beta_1 X_{1i} + \beta_3 X_{3i} + u_{Ri} \end{aligned}$$

F-tests: Sums of Squared Residuals

"Unrestricted":

$$\text{RSS}_U \equiv \hat{\mathbf{u}}_U' \hat{\mathbf{u}}_U = \sum_{i=1}^{N} \hat{u}_{Ui}^2$$

"Restricted":

$$\text{RSS}_R \equiv \hat{\mathbf{u}}_R' \hat{\mathbf{u}}_R = \sum_{i=1}^{N} \hat{u}_{Ri}^2$$

The F-test

F-statistic:

$$\mathbf{F} = \frac{(\mathrm{RSS}_R - \mathrm{RSS}_U)/q}{\mathrm{RSS}_U/(N-K)}$$

$$= \frac{(R_U^2 - R_R^2)/q}{(1-R_U^2)/N-K}$$

Testing:

$$\mathbf{F} \sim F_{q,N-K}$$

F-Test: Example

Consider:

$$\begin{aligned} \mathrm{H}_b: \quad & \beta_1 + \beta_4 = 1 \\ & \beta_1 = 1 - \beta_4 \end{aligned}$$

Implies:

$$\begin{aligned} Y_i &= \beta_0 + (1 - \beta_4)X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_{R'i} \\ &= \beta_0 + X_{1i} - \beta_4 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_{R'i} \\ &= \beta_0 + X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 (X_{4i} - X_{1i}) + u_{R'i} \end{aligned}$$

implying restricted model:

$$Y_i - X_{1i} = \beta_0 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 (X_{4i} - X_{1i}) + u_{R'i}$$

Confidence Regions

$$F = \frac{(\hat{\boldsymbol{\beta}}_q - \boldsymbol{\beta}_q^H)'\hat{\mathbf{V}}_q^{-1}(\hat{\boldsymbol{\beta}}_q - \boldsymbol{\beta}_q^H)}{q\hat{\sigma}^2}$$

Implies:

$$\Pr\left[\frac{(\hat{\boldsymbol{\beta}}_q - \boldsymbol{\beta}_q^H)'\hat{\mathbf{V}}_q^{-1}(\hat{\boldsymbol{\beta}}_q - \boldsymbol{\beta}_q^H)}{q\hat{\sigma}^2} \leq F_{q,N-K}\right] = 1 - \alpha. \qquad (1)$$

$\rightarrow$ "confidence region" of all points satisfying:

$$(\hat{\boldsymbol{\beta}}_q - \boldsymbol{\beta}_q^H)'\hat{\mathbf{V}}_q^{-1}(\hat{\boldsymbol{\beta}}_q - \boldsymbol{\beta}_q^H) \leq q\hat{\sigma}^2 F_{q,N-K}.$$

Multivariate Prediction

Prediction:

$$\hat{Y}_j = \mathbf{X}_j \hat{\boldsymbol{\beta}}$$

Variance:

$$\widehat{\mathbf{V}(\hat{Y}_j)} = \hat{\sigma}^2[1 + \mathbf{X}_j(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_j']$$

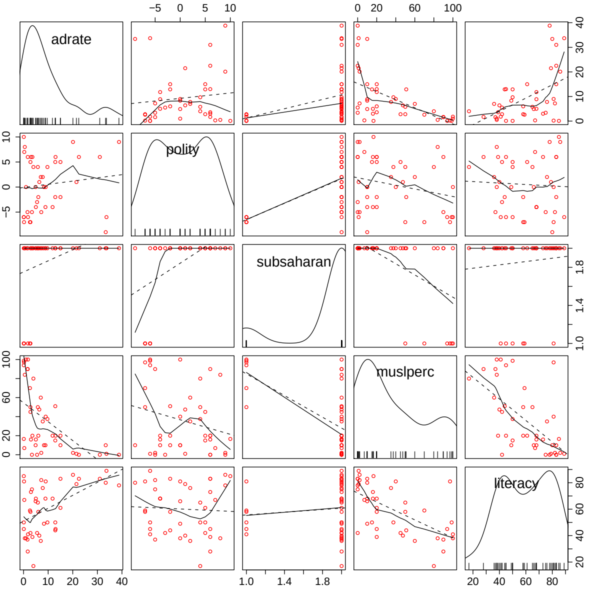
Standard error:

$$\widehat{\text{s.e.}(\hat{Y}_j)} = \sqrt{\hat{\sigma}^2[1 + \mathbf{X}_j(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_j']}$$

Example: Africa Data

```
> library(RCurl)
> temp<-getURL("https://raw.githubusercontent.com/PrisonRodeo/PLSC503-2021-git/master/Data/africa2001.csv'
> Data<-read.csv(text=temp, header=TRUE)
> Data<-with(Data, data.frame(adrate,polity,
+           subsaharan=as.numeric(subsaharan),muslperc,literacy))

> summary(Data)
     adrate           polity          subsaharan      muslperc         literacy
 Min.   : 0.100   Min.   :-9.0000   Min.   :1.00   Min.   :  0.00   Min.   :17.00
 1st Qu.: 2.700   1st Qu.:-4.5000   1st Qu.:2.00   1st Qu.: 10.00   1st Qu.:43.00
 Median : 6.000   Median : 0.0000   Median :2.00   Median : 20.00   Median :61.00
 Mean   : 9.365   Mean   : 0.5116   Mean   :1.86   Mean   : 35.96   Mean   :60.07
 3rd Qu.:12.900   3rd Qu.: 5.5000   3rd Qu.:2.00   3rd Qu.: 55.50   3rd Qu.:78.50
 Max.   :38.800   Max.   :10.0000   Max.   :2.00   Max.   :100.00   Max.   :89.00

> cor(Data)
               adrate      polity  subsaharan   muslperc    literacy
adrate      1.0000000  0.11794182  0.33129420 -0.5709233  0.51489444
polity      0.1179418  1.00000000  0.52819844 -0.2391715 -0.05079354
subsaharan  0.3312942  0.52819844  1.00000000 -0.5772513  0.09472968
muslperc   -0.5709233 -0.23917151 -0.57725134  1.0000000 -0.61960385
literacy    0.5148944 -0.05079354  0.09472968 -0.6196039  1.00000000
```

Africa Data



A Regression

```
> model<-lm(adrate~polity+subsaharan+muslperc+literacy,data=Data)
> summary(model)

Call:
lm(formula = adrate ~ polity + subsaharan + muslperc + literacy,
    data = Data)

Residuals:
     Min       1Q   Median       3Q      Max
-15.4681  -4.3947  -0.5251   3.4246  22.9358

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -4.39843   14.94744  -0.294   0.7702
polity      -0.01390    0.27969  -0.050   0.9606
subsaharan   3.72969    5.43093   0.687   0.4964
muslperc    -0.08689    0.06282  -1.383   0.1747
literacy     0.16575    0.09433   1.757   0.0869 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 8.264 on 38 degrees of freedom
Multiple R-squared:  0.3771,Adjusted R-squared:  0.3115
F-statistic: 5.751 on 4 and 38 DF,  p-value: 0.001013
```

Variance-Covariance Matrix of $\hat{\beta}$

```
> options(digits=4)
> vcov(model)

            (Intercept)    polity subsaharan  muslperc  literacy
(Intercept)    223.4259  1.088030   -72.2628 -0.771309 -1.002421
polity           1.0880  0.078229    -0.6642 -0.000293  0.001968
subsaharan     -72.2628 -0.664212    29.4950  0.206067  0.171765
muslperc        -0.7713 -0.000293     0.2061  0.003946  0.004098
literacy        -1.0024  0.001968     0.1718  0.004098  0.008898
```

Tests...

Test $H_0 : \beta_{\texttt{polity}} = \beta_{\texttt{subsaharan}} = 0$:

```
> library(lmtest)
> modelsmall<-lm(adrate~muslperc+literacy,data=Data)
> waldtest(model,modelsmall)

Wald test

Model 1: adrate ~ polity + subsaharan + muslperc + literacy
Model 2: adrate ~ muslperc + literacy
  Res.Df Df    F Pr(>F)
1     38
2     40 -2 0.27   0.76
```

More tests...

Test $H_0 : \beta_{\texttt{muslperc}} = 0.1$:

```
> library(car)
> linearHypothesis(model,"muslperc=0.1")

Linear hypothesis test

Hypothesis:
muslperc = 0.1

Model 1: restricted model
Model 2: adrate ~ polity + subsaharan + muslperc + literacy

  Res.Df  RSS Df Sum of Sq    F Pr(>F)
1     39 3200
2     38 2595  1       605 8.85 0.0051 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

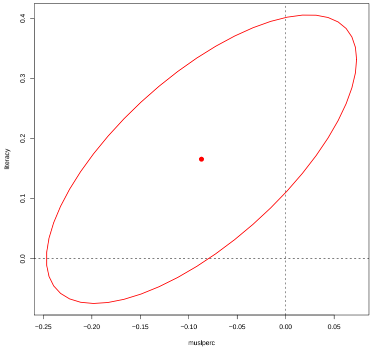
More tests...

Test $H_0 : \beta_{\texttt{literacy}} = \beta_{\texttt{muslperc}}$:

```
> linearHypothesis(model,"literacy=muslperc")

Linear hypothesis test

Hypothesis:
- muslperc  + literacy = 0

Model 1: restricted model
Model 2: adrate ~ polity + subsaharan + muslperc + literacy

  Res.Df  RSS Df Sum of Sq    F  Pr(>F)
1     39 3534
2     38 2595  1       938 13.7 0.00067 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Confidence Regions / Ellipses

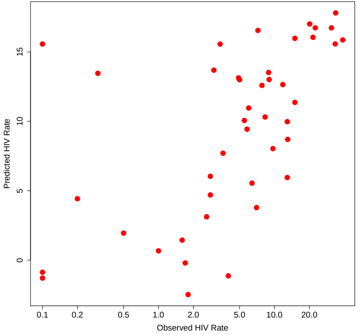
```
> confidenceEllipse(model=model,which.coef=c(4,5),
                    xlab="Muslim Percentage",ylab="Literacy")
> abline(h=0,v=0,lty=2)
```



Predicted Values

```
> hats<-fitted(model)
> # Or, alternatively:
> fitted<-predict(model,se.fit=TRUE, interval=c("confidence"))
> scatterplot(model$fitted~adrate,log="x",smooth=FALSE,boxplots=FALSE,
  reg.line=FALSE,xlab="Observed HIV Rate",ylab="Predicted HIV Rate",
  pch=16,cex=2)
```

Predicted and Actual HIV/AIDS Rates (X-Axis Logged)



An Even More Useful Plot

```
> library(plotrix)
> plotCI(Data$adrate,model$fitted,uiw=(1.96*(fitted$se.fit)),
        log="x",xlab="Observed HIV Rate",ylab="Predicted HIV Rate")
> lines(lowess(Data$adrate,Data$adrate),lwd=2)
```

Predicted and Actual HIV/AIDS Rates, with 95% C.I.s