PLSC 503 – Spring 2021 Bootstrapping and Missing Data

March 10, 2021

The population is to the sample as the sample is to the bootstrap sample.

Practical (Nonparametric) Bootstrapping

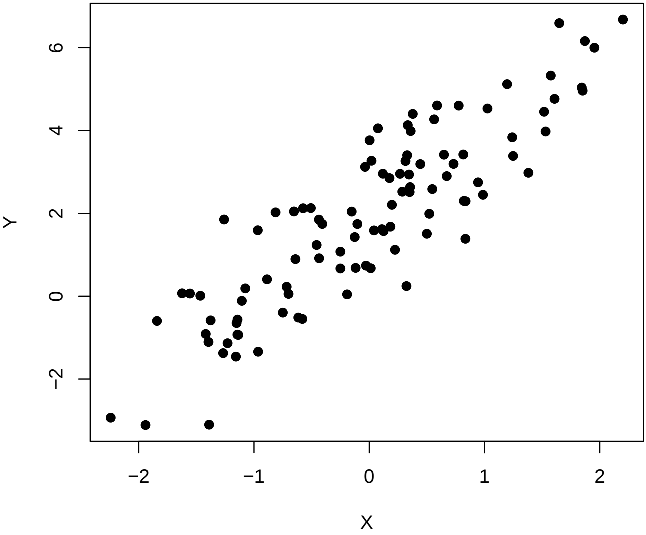
- Draw one bootstrap sample of size N **with replacement** from the original data,
- Estimate the parameter(s) $\underset{k\times 1}{\tilde{\theta}}$,
- Repeat steps 1 and 2 R times, to get $\tilde{\theta}_r$, $r \in \{1, 2, ...R\}$, comprising elements $\tilde{\theta}_{rk}$,
- Examine the empirical characteristics of the resulting distribution(s) of $\tilde{\theta}_{rk}$.

Why Bootstrap?

- **It's intuitive.**
- **It's simple.**
- **It's robust.**

```
N<-100
reps<-999

set.seed(1337)
X<-rnorm(N)
Y<-2+2*X+rnorm(N)
data<-data.frame(Y,X)
fitOLS<-lm(Y~X)
CI<-confint(fitOLS)

B0<-numeric(reps)
B1<-numeric(reps)

for (i in 1:reps) {
  temp<-data[sample(1:N,N,replace=TRUE),]
  temp.lm<-lm(Y~X,data=temp)
  B0[i]<-temp.lm$coefficients[1]
  B1[i]<-temp.lm$coefficients[2]
}

ByHandB0<-median(B0)
ByHandB1<-median(B1)
ByHandCI.B0<-quantile(B0,probs=c(0.025,0.975)) # <-- 95% c.i.s
ByHandCI.B1<-quantile(B1,probs=c(0.025,0.975))
```

Normal Residuals...



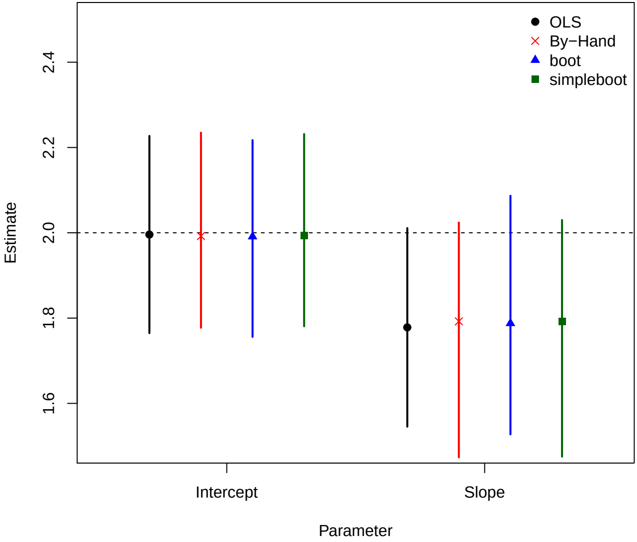
Bootstrapping Via boot

```
library(boot)

Bs<-function(formula, data, indices) { # <- regression function
    dat <- data[indices,]
    fit <- lm(formula, data=dat)
    return(coef(fit))
}

Boot.fit<-boot(data=data, statistic=Bs,
          R=reps, formula=Y~X)

BootB0<-median(Boot.fit$t[,1])
BootB1<-median(Boot.fit$t[,2])
BootCI.B0<-boot.ci(Boot.fit,type="basic",index=1)
BootCI.B1<-boot.ci(Boot.fit,type="basic",index=2)
```

Bootstrapping Via `simpleboot`

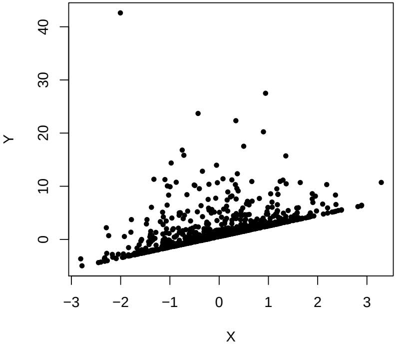
```
library(simpleboot)

Simple<-lm.boot(fitOLS,reps)
SimpleB0<-perc(Simple,.50)[1]
SimpleB1<-perc(Simple,.50)[2]
Simple.CIs<-perc(Simple,p=c(0.025,0.975))
```

Bootstrapping Results

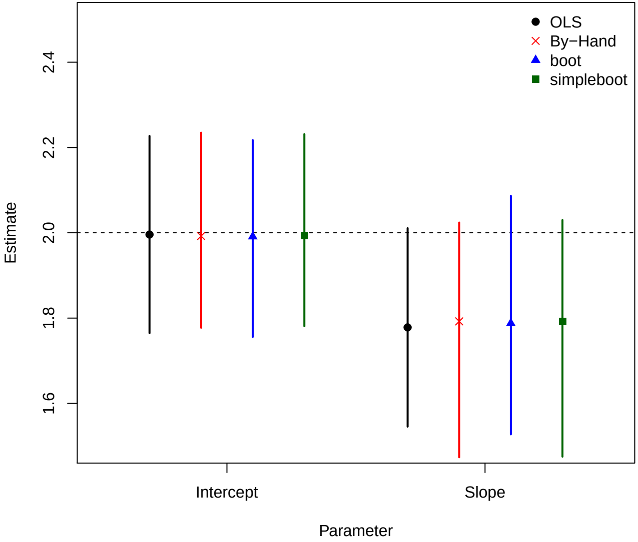


Bootstrapping: Skewed Residuals

```
N<-100
reps<-999

set.seed(1337)
X<-rnorm(N)
ustar<-rchisq(N,2) # <- skewed u.s
Y<-2+2*X+(ustar-mean(ustar))
data<-data.frame(Y,X)
fitOLS<-lm(Y~X)
CI<-confint(fitOLS)
```



Skewed Residuals: Results



R things:

- A simple introduction at StatMethods
- Bootstrap in R (at DataCamp)
- Packages: `boot`, `bootstrap`, `simpleboot`, `car::Boot`, `broom` (tidy), many more

Other Resources:

- Efron's original (1979) paper
- Chernick and Labudde (2011) (a solid R-based bootstrapping book)
- Many other books, etc.

Missing Data

Why are data missing?

- The observation itself does not exist
- Data don't exist for that observation
- Data exist, but are *impossible* to measure
- Data exist, but were not measured

Missing Data, Part II: Flavors

Notation:

$$\mathbf{X}_i \in \{\mathbf{W}_i, \mathbf{Z}_i\}$$

$\mathbf{W}_i$ have some missing values,
$\mathbf{Z}_i$ are "complete"

$$R_{ik} = \begin{cases} 1 & \text{if } W_{ik} \text{ is missing,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_{ik} = \Pr(R_{ik} = 1)$$

Missing Data, Part II: Rubin's Flavors

Missing completely at random ("MCAR"):

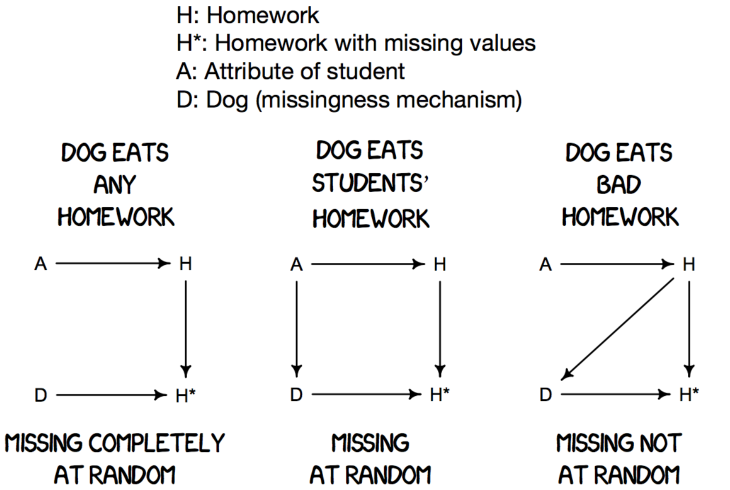
$$\mathbf{R} \perp \{\mathbf{Z}, \mathbf{W}\}$$

Missing at random ("MAR"):

$$\mathbf{R} \perp \mathbf{W} | \mathbf{Z}$$

Anything else is "informatively" (or "non-ignorably," or sometimes "MNAR") missing.

MCAR vs. MAR vs. MNAR, Explained



(Source)

Missing Data: Consequences

In general:

- MCAR:
  - Missing data are a fully random sample of all the data
  - $\rightarrow$ Missingness does not bias $\hat{\theta}$, *but*
  - There is some loss of information (and therefore efficiency)

- MAR
  - Missing data are a nonrandom sample of all the data
  - Ignoring missingness can lead to bias in $\hat{\theta}$, *but*
  - Conditioning on the variable(s) that drive the missingness can eliminate the bias

- Informative Missingness / MNAR
  - Missing data are a nonrandom sample of all the data
  - Ignoring missingness can lead to bias in $\hat{\theta}$
  - In general, conditioning cannot eliminate the bias

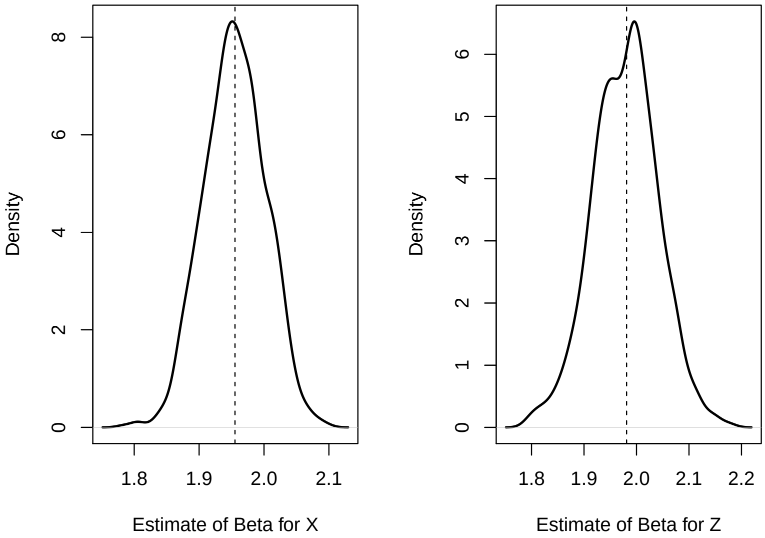
Example, Simulated

```
> set.seed(7222009)
> Npop <- 1000
> X<-runif(Npop,0,10)   # NOTE: X, Z are correlated a bit...
> Z<-(0.3*X)+(0.7*runif(Npop,0,10))
> Y<-0+(2*X)+(2*Z)+rnorm(Npop,mean=0,sd=4)
> DF<-data.frame(X=X,Z=Z,Y=Y)
> fit.pop<-lm(Y~X+Z,DF)
> summary(fit.pop)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.4051     0.3260    1.24     0.21
X             1.9553     0.0466   41.97   <2e-16 ***
Z             1.9812     0.0617   32.09   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.98 on 997 degrees of freedom
Multiple R-squared:  0.823,Adjusted R-squared:  0.823
F-statistic: 2.32e+03 on 2 and 997 DF,  p-value: <2e-16
```

Simulated MCAR

```
> pmis<-0.50
> DF$Ymcar<-rbinom(Npop,1,pmis)
> DF$Ymcar<-ifelse(DF$Ymcar==1,NA,DF$Y)
>
> # Regression w/listwise deletion:
>
> fit.s<-lm(Ymcar~X+Z,DF) # <-- looks fine
> summary(fit.s)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.4442     0.4653    0.95     0.34
X             1.9661     0.0658   29.87   <2e-16 ***
Z             1.9763     0.0862   22.92   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4 on 507 degrees of freedom
  (490 observations deleted due to missingness)
Multiple R-squared:  0.822,Adjusted R-squared:  0.821
F-statistic: 1.17e+03 on 2 and 507 DF,  p-value: <2e-16
```

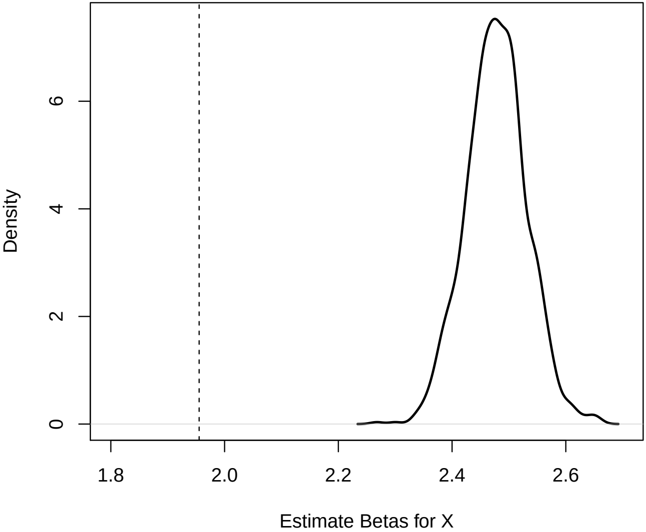
Do That A Bunch Of Times...



Simulated MAR Y

```
> set.seed(7222009)
> DF$Ymar<-rbinom(Npop,1,(DF$Z/10))
> DF$Ymar<-ifelse(DF$Ymar==1,NA,DF$Y)
>
> summary(lm(Ymar~X,DF))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   3.6600     0.3610    10.1   <2e-16 ***
X             2.9923     0.0648    46.2   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.75 on 547 degrees of freedom
  (451 observations deleted due to missingness)
Multiple R-squared:  0.796,Adjusted R-squared:  0.795
F-statistic: 2.13e+03 on 1 and 547 DF,  p-value: <2e-16
```

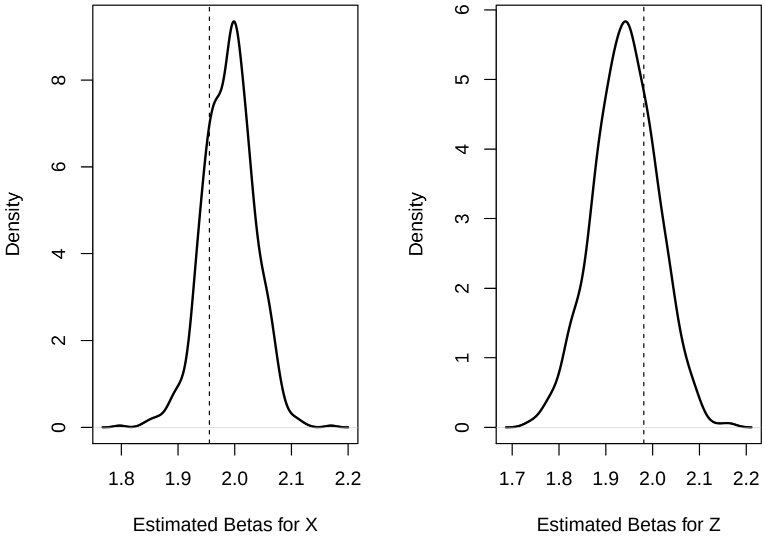
Do That A Bunch Of Times...



More MAR: Add Z...

```
> summary(lm(Ymar~X+Z,DF))

Call:
lm(formula = Ymar ~ X + Z, data = DF)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.2529     0.4367    0.58     0.56
X             2.0200     0.0663   30.49   <2e-16 ***
Z             1.9499     0.0979   19.91   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.02 on 499 degrees of freedom
  (498 observations deleted due to missingness)
Multiple R-squared:  0.801,Adjusted R-squared:   0.8
F-statistic: 1e+03 on 2 and 499 DF,  p-value: <2e-16
```

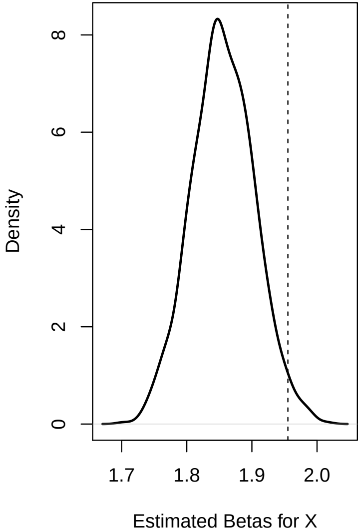
Do That A Bunch Of Times...

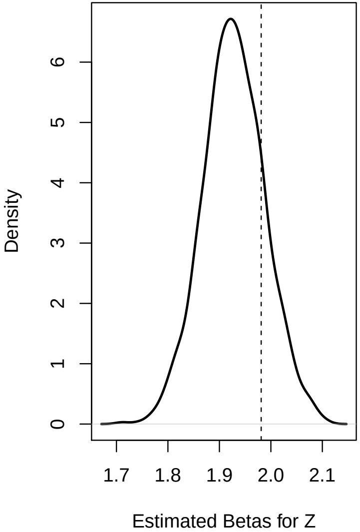


Informative Missingness / “MNAR”

```
> set.seed(7222009)
> DF$Yim<-rbinom(Npop,1,rescale(DF$Z-(4*DF$Y)))
> DF$Yim<-ifelse(DF$Yim==1,NA,DF$Y)
>
> summary(lm(Yim~X+Z,DF))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.0518     0.5463    3.76  0.00019 ***
X             1.8420     0.0671   27.45  < 2e-16 ***
Z             1.9171     0.0859   22.32  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.85 on 465 degrees of freedom
  (532 observations deleted due to missingness)
Multiple R-squared:  0.797,Adjusted R-squared:  0.796
F-statistic:  911 on 2 and 465 DF,  p-value: <2e-16
```

Do That A Bunch Of Times...





How Much Missing Data Is A Problem?

"It is often supposed that there exists something like a critical missing rate up to which missing values are not too dangerous. The belief in such a global missing rate is rather stupid."

– *Vach (1994, 113)*

What to Do About Missing Data?

- Listwise Deletion...
- Mean Substitution / Imputation
- "Nearest Neighbor" methods
- "Hot Deck" Imputation
- Multiple Imputation
- Model-Based Solutions

MAR Data

For MAR data:

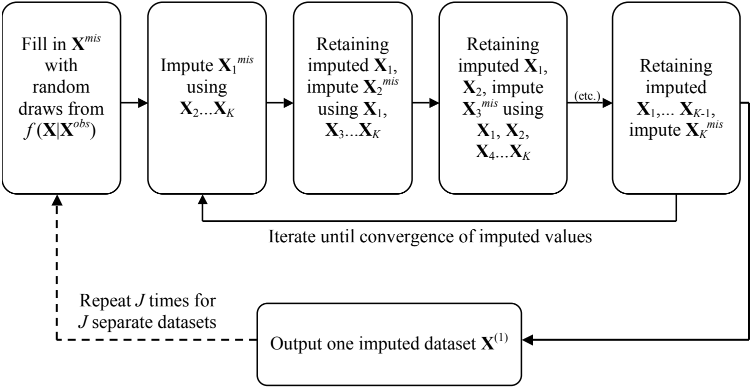
$$\mathbf{R} \perp \mathbf{W} | \mathbf{Z}$$

so **W** and **Z** factorize independently.

Sources of variation we need to consider:

1. Prediction
2. Predictive variation
3. Parameter variation / uncertainty

MAR: Multiple Imputation



Multiple Imputation (continued)

Original Data **X** With Missing Data

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$\bullet$	X_{22}	X_{32}	$\bullet$	...	X_{K2}
3	X_{13}	X_{23}	$\bullet$	X_{43}	...	X_{K3}
4	X_{14}	$\bullet$	X_{34}	X_{44}	...	X_{K4}
5	$\bullet$	X_{25}	X_{35}	$\bullet$	...	$\bullet$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Iteration One:

Step One: "Fill In" Missing Values of $\mathbf{X}$

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	R_{12}	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	R_{33}	X_{43}	...	X_{K3}
4	X_{14}	R_{24}	X_{34}	X_{44}	...	X_{K4}
5	R_{15}	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Step Two: Use $\{X_2, X_3, ...X_K\}$ To Impute X_1^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	R_{33}	X_{43}	...	X_{K3}
4	X_{14}	R_{24}	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Step Three: Use The Imputed X_1, Along With $\{X_3, X_4, ... X_K\}$ To Impute X_2^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	R_{33}	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Step Four: Use The Imputed X_1 and X_2, Along With $\{X_4, ...X_K\}$ To Impute X_3^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

(etc.)

Multiple Imputation (continued)

Step $K+1$: Use The Imputed $X_1, X_2, \ldots X_{K-1}$ To Impute X_K^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	$I_{42}^{(1)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	$I_{45}^{(1)}$	...	$I_{K5}^{(1)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Iteration Two:

Step One: Use The Imputed $X_2, X_3, ...X_K$ To Impute X_1^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(2)}$	X_{22}	X_{32}	$I_{42}^{(1)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(2)}$	X_{25}	X_{35}	$I_{45}^{(1)}$	...	$I_{K5}^{(1)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Step Two: Use The Imputed $X_1, X_3, ...X_K$ To Impute X_2^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(2)}$	X_{22}	X_{32}	$I_{42}^{(1)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(2)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(2)}$	X_{25}	X_{35}	$I_{45}^{(1)}$	...	$I_{K5}^{(1)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

(etc.)

Multiple Imputation (continued)

Step K: Use The Imputed $X_1, X_2, ...X_{K-1}$ To Impute X_K^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(2)}$	X_{22}	X_{32}	$I_{42}^{(2)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(2)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(2)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(2)}$	X_{25}	X_{35}	$I_{45}^{(2)}$	...	$I_{K5}^{(2)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation: Summary

- Repeat this process for $J \approx 10$ iterations until convergence of the $I_{ki}^{(j)}$s.
- Output the resulting imputed data $\mathbf{X}^{(1)}$.
- Repeat this entire process M times to create M imputed datasets $\{\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots \mathbf{X}^{(M)}\}$.
- Rule of thumb: $M \geq$ the percentage of cases in your data with missingness.
- Estimate models and conduct inference using multiple analysis and model averaging (see e.g. Schafer 1997, Ch. 4).

MNAR Data

For MNAR data:

$$\Pr(\mathbf{R}) = f(\mathbf{W}, \mathbf{Z})$$

i.e., missingness is *nonignorable*.

Common causes / situations:

- Omitted variables ($\rightarrow$ can't condition on all elements of $\mathbf{Z}$)
- Differential response due to unmeasured factors
- Truncation / censoring

MNAR and Model-Based Solutions

For MNAR data, we must model the joint distribution $\Pr(\mathbf{X}, \mathbf{R})$...

Approaches:

- *Selection* model: $\Pr(\mathbf{X}, \mathbf{R}) = \Pr(\mathbf{X}) \Pr(\mathbf{R}|\mathbf{X})$
  - E.g., Heckman (1976, 1979, etc.)
  - Specifies a (usually, regression) model for $\Pr(\mathbf{R}\ |X)$

- *Pattern-Mixture* model: $$\begin{aligned}\Pr(\mathbf{X}, \mathbf{R}) &= \Pr(\mathbf{X}|\mathbf{R}) \Pr(\mathbf{X}) \\ &= \Pr(\mathbf{X}|\mathbf{R}=0) \Pr(\mathbf{R}=0) + \\ &\quad \Pr(\mathbf{X}|\mathbf{R}=1) \Pr(\mathbf{R}=1)\end{aligned}$$
  - E.g., Glynn, Laird, and Rubin (1986)
  - Mixture-type model across "responders" and "non-responders"

- Others... [see, e.g., Little and Rubin (2002)]

Missing Data Resources, R and Otherwise

- The Missing Data CRAN Task View
- Packages:
  - `Amelia`
  - `mi`, `mice`, and `miceFast`
  - `miceMNAR` (MNAR imputation using a Heckman-style selection model)
  - `naniar` (tidy-cult, but enables cool visualizations)
  - `VIM` (joint visualization and imputation of missing data; also used to have a GUI)
  - Many others...
- van Buuren's Flexible Imputation of Missing Data e-book