PLSC 503 – Spring 2021
Generalized Linear Models

April 28, 2021

$$f(z|\psi) = \Pr(Z = z|\psi)$$

Exponential if:

$$f(z|\psi) = r(z)s(\psi)\exp[q(z)h(\psi)]$$

provided that $r(z) > 0$ and $s(\psi) > 0$.

$$f(z|\psi) = \exp[\underbrace{\ln r(z) + \ln s(\psi)}_{\text{“additive”}} + \underbrace{q(z)h(\psi)}_{\text{“interactive”}}]$$

Canonical Forms

$$y = q(z)$$

$$\theta = h(\psi)$$

$$f[y|\theta] = \exp[y\theta - b(\theta) + c(y)].$$

- $b(\theta)$ is a "normalizing constant"
- $c(y)$ is a function solely of y
- $y\theta$ is a multiplicative term

A Familiar Family Member: Poisson

$$f(y|\lambda) = \frac{\exp(-\lambda)\lambda^y}{y!}.$$

$$\begin{aligned} f(y|\lambda) &= \exp\left\{\ln\left[\exp(-\lambda)\lambda^y / y!\right]\right\} \\ &= \exp[\underbrace{y\ln(\lambda)}_{y\theta} - \underbrace{\lambda}_{b(\theta)} - \underbrace{\ln(y!)}_{c(y)}] \end{aligned}$$

$$f(y|\theta, \phi) = \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right]$$

Familiar Family Member II: Normal

$$f(y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{(y-\mu)^2}{2\sigma^2}\right]$$

$$\begin{aligned}
f(y|\mu,\sigma^2) &= \exp\left[-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y^2 - 2y\mu + \mu^2)\right] \\
&= \exp\left[-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}y^2 + \frac{1}{2\sigma^2}2y\mu - \frac{1}{2\sigma^2}\mu^2\right] \\
&= \exp\left[\frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{y^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)\right] \\
&= \exp\left\{\frac{y\mu - \frac{\mu^2}{2}}{\sigma^2} + \frac{-1}{2}\left[\frac{y^2}{\sigma^2} + \ln(2\pi\sigma^2)\right]\right\}
\end{aligned}$$

Normal, continued

$$f(y|\mu,\sigma^2) = \exp\left\{ \frac{y\mu - \frac{\mu^2}{2}}{\sigma^2} + \frac{-1}{2}\left[\frac{y^2}{\sigma^2} + \ln(2\pi\sigma^2)\right] \right\}$$

$\theta = \mu$, so:

- $y\theta = y\mu$
- $b(\theta) = \frac{\mu^2}{2}$
- $a(\phi) = \sigma^2$
- $c(y,\phi) = \frac{-1}{2}\left[\frac{y^2}{\sigma^2} + \ln(2\pi\sigma^2)\right]$

Other Family Members

- Binomial ($\supset$ Bernoulli; also Multinomial)
- Exponential
- Gamma
- Logarithmic
- Inverse Gaussian
- Negative Binomial
- others...

$$\begin{aligned}\ln L(\theta,\phi|y) &= \ln f(y|\theta,\phi) \\ &= \ln\left\{\exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right]\right\} \\ &= \frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\end{aligned}$$

$$\frac{\partial \ln L(\theta,\phi|y)}{\partial\theta} \equiv \mathbf{S} = \frac{\partial}{\partial\theta}\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right] = \frac{y - \frac{\partial}{\partial\theta} b(\theta)}{a(\phi)}.$$

Among family members:

- **S** is a sufficient statistic for θ.
- $\mathrm{E}(\mathbf{S}) = 0$.
- $\mathrm{Var}(\mathbf{S}) \equiv \mathcal{I}(\theta) = \mathrm{E}[(\mathbf{S})^2|\theta]$

$$\mathrm{E}(Y) = \frac{\partial}{\partial\theta} b(\theta)$$

and

$$\mathrm{Var}(Y) = a(\phi) \frac{\partial^2}{\partial\theta^2} b(\theta)$$

Example: Poisson Again

$$
\begin{aligned}
\mathsf{E}(Y) &= \frac{\partial}{\partial\theta}\,\mathsf{exp}(\theta) \\
&= \mathsf{exp}(\theta)|_{\theta=\mathsf{ln}(\lambda)} \\
&= \lambda
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Var}(Y) &= 1 \times \frac{\partial^2}{\partial\theta^2}\,\mathsf{exp}(\theta)|_{\theta=\mathsf{ln}(\lambda)} \\
&= \mathsf{exp}[\mathsf{ln}(\lambda)] \\
&= \lambda
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{E}(Y) &= \frac{\partial}{\partial\theta}\left(\frac{\theta^2}{2}\right) \\
&= \theta|_{\theta=\mu} \\
&= \mu
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Var}(Y) &= \sigma^2 \times \frac{\partial^2}{\partial\theta^2}\left(\frac{\theta^2}{2}\right) \\
&= \sigma^2 \times \frac{\partial}{\partial\theta}\theta \\
&= \sigma^2
\end{aligned}
$$

Linear Model(s)

$$Y_i = \mathbf{X}_i \boldsymbol{\beta} + u_i$$

$$\mathrm{E}(Y_i) \equiv \boldsymbol{\mu}_i = \mathbf{X}_i \boldsymbol{\beta}$$

The "Generalized" Part

$$g(\boldsymbol{\mu}_i) = \mathbf{X}_i\boldsymbol{\beta}.$$

$$\begin{aligned} \boldsymbol{\eta}_i &= \mathbf{X}_i\boldsymbol{\beta} \\ &= g(\boldsymbol{\mu}_i) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\mu}_i &= g^{-1}(\boldsymbol{\eta}_i) \\ &= g^{-1}(\mathbf{X}_i\boldsymbol{\beta}) \end{aligned}$$

Random component $\sim$ EF($\cdot$) with

$$E(Y_i) = \boldsymbol{\mu}_i.$$

Systematic component:

$$g(\boldsymbol{\mu}_i) = \boldsymbol{\eta}_i$$

"Link function":

$$g(\boldsymbol{\mu}_i) = \boldsymbol{\eta}_i$$

or

$$g^{-1}(\boldsymbol{\eta}_i) = \boldsymbol{\mu}_i.$$

$$
\begin{aligned}
\boldsymbol{\theta}_i &= g(\boldsymbol{\mu}_i) \\
&= \boldsymbol{\eta}_i \\
&= \mathbf{X}_i \boldsymbol{\beta}
\end{aligned}
$$

$$g^{-1}(\boldsymbol{\theta}_i) = \boldsymbol{\eta}_i$$

GLM Example: Linear-Normal

$$f(y|\mu, \sigma^2) = \mathcal{N}(\mu, \sigma^2)$$

$$\boldsymbol{\mu}_i = \boldsymbol{\eta}_i$$

$$\begin{aligned} \boldsymbol{\mu}_i \equiv \boldsymbol{\theta}_i &= \boldsymbol{\eta}_i \\ Y_i &\sim \mathcal{N}(\boldsymbol{\mu}_i, \sigma^2) \end{aligned}$$

GLM Example: Binary

$$f(y|\pi) = \pi^y (1-\pi)^{1-y}$$

$$\boldsymbol{\theta}_i = \ln\left(\frac{\boldsymbol{\mu}_i}{1-\boldsymbol{\mu}_i}\right)$$

$$\begin{aligned}
\boldsymbol{\mu}_i &= g^{-1}(\boldsymbol{\theta}_i) \\
&= \frac{\exp(\boldsymbol{\eta}_i)}{1+\exp(\boldsymbol{\eta}_i)} \\
Y_i &\sim \text{Bernoulli}(\boldsymbol{\mu}_i)
\end{aligned}$$

GLM Example: Counts (Independent Events)

$$f(y|\lambda) = \frac{\exp(-\lambda)\lambda^y}{y!}$$

$$\ln(\lambda_i) = \boldsymbol{\eta}_i$$

$$\begin{aligned}
\boldsymbol{\mu}_i &= g^{-1}(\boldsymbol{\theta}_i) \\
&= \exp(\boldsymbol{\eta}_i) \\
Y_i &\sim \text{Poisson}(\lambda_i)
\end{aligned}$$

Common GLM Flavors

Distribution	Range of Y	Link(s) $g(\cdot)$	Inverse Link $g^{-1}(\cdot)$
Normal	$(-\infty, \infty)$	Identity: $\boldsymbol{\theta} = \boldsymbol{\mu}$ (Canonical)	$\boldsymbol{\theta}$
Binomial	$\{0,...n\}$	Logit: $\boldsymbol{\theta} = \ln\left(\frac{\boldsymbol{\mu}}{1-\boldsymbol{\mu}}\right)$ (Canonical)	$\frac{\exp(\boldsymbol{\theta})}{1+\exp(\boldsymbol{\theta})}$
		Probit: $\boldsymbol{\theta} = \Phi^{-1}(\boldsymbol{\mu})$	$\Phi(\boldsymbol{\theta})$
		C-Log-Log: $\boldsymbol{\theta} = \ln[-\ln(1-\boldsymbol{\mu})]$	$1 - \exp[-\exp(\boldsymbol{\theta})]$
Bernoulli	$\{0,1\}$	(same as Binomial)	(same as Binomial)
Multinomial	$\{0,...J\}$	(same as Binomial)	(same as Binomial)
Poisson	$[0, \infty]$ (integers)	Log: $\boldsymbol{\theta} = \ln(\boldsymbol{\mu})$ (Canonical)	$\exp(\boldsymbol{\theta})$
Gamma	$(0, \infty)$	Reciprocal: $\boldsymbol{\theta} = -\frac{1}{\boldsymbol{\mu}}$ (Canonical)	$-\frac{1}{\boldsymbol{\theta}}$

Note: The Bernoulli is a special case of the Binomial with $n = 1$. The multinomial is the J-outcome variant of the Binomial, and is also related to the Poisson (see, e.g., Agresti 2002).

GLMs: How-To

- Pick $f(Y)$
- Pick $g(\cdot)$
- Specify $\mathbf{X}$
- Estimate

- MLE
- IRLS ($\approx$ MLE):

$$\hat{\boldsymbol{\beta}}^{(t+1)} = [\mathbf{X}'\mathbf{W}^{(t)}\mathbf{X}]^{-1}\mathbf{X}'\mathbf{W}^{(t)}\mathbf{z}^{(t)}$$

with

$$\underset{N\times N}{\mathbf{W}^{(t)}} = \mathrm{diag}\left[\frac{\left(\partial\mu_i^{(t)}/\partial\eta_i^{(t)}\right)^2}{\mathrm{Var}(Y_i)}\right]$$

and

$$\mathbf{z}^{(t)} \quad = \quad \boldsymbol{\eta}^{(t)} + (Y - \boldsymbol{\mu}^{(t)})\left(\frac{\partial\boldsymbol{\eta}^{(t)}}{\partial\boldsymbol{\mu}}\right).$$

IRLS, Intuitively

At iteration t:

1. Calculate $\mathbf{z}^{(t)}$, $\mathbf{W}^{(t)}$
2. Regress $\mathbf{z}^{(t)}$ on $\mathbf{X}$, using $\mathbf{W}^{(t)}$ as weights, to obtain $\hat{\boldsymbol{\beta}}^{(t+1)}$
3. Generate $\boldsymbol{\eta}^{(t+1)} = \mathbf{X}\hat{\boldsymbol{\beta}}^{(t+1)}$
4. Generate $\boldsymbol{\mu}^{(t+1)} = g^{-1}(\boldsymbol{\eta}^{(t+1)})$
5. Use $\boldsymbol{\eta}^{(t+1)}$ and $\boldsymbol{\mu}^{(t+1)}$ to calculate $\mathbf{z}^{(t+1)}$ and $\mathbf{W}^{(t+1)}$
6. Repeat until convergence.

"Response" Residuals:

$$\begin{aligned}\hat{u}_i &= Y_i - \hat{\mu}_i \\ &= Y_i - g^{-1}(\mathbf{X}_i\hat{\beta})\end{aligned}$$

"Pearson" Residuals:

$$\hat{P}_i = \frac{\hat{u}_i}{[\text{Var}(\hat{u}_i)]^{1/2}}$$

"Deviance":

$$
\begin{aligned}
\hat{d}_i &= -2[\ln L_i(\hat{\theta}) - \ln L_i(\theta_S)] \\
&= 2\left\{\left[\frac{Y_i\theta_S - b(\theta_S)}{a(\phi)} + c(Y_i,\phi)\right] - \left[\frac{Y_i\hat{\theta} - b(\hat{\theta})}{a(\phi)} + c(Y_i,\phi)\right]\right\} \\
&= 2\left[\frac{Y_i(\theta_S - \hat{\theta}) - b(\theta_S) + b(\hat{\theta})}{a(\phi)}\right]
\end{aligned}
$$

"Deviance" Residuals:

$$
\hat{r}_{Di} = \left(\frac{\hat{u}_i}{|\hat{u}_i|}\right)\sqrt{\hat{d}_i^2}
$$

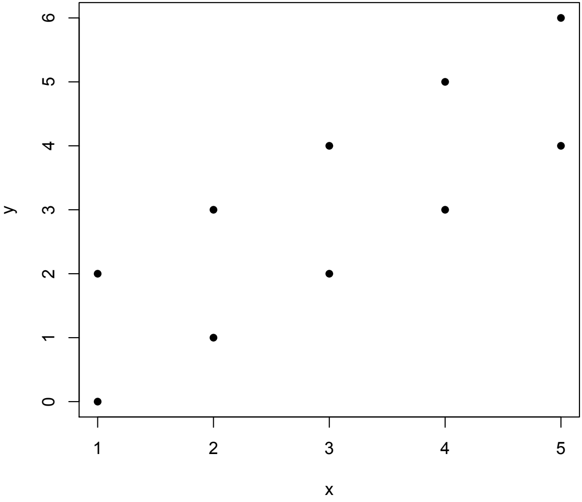
Toy Example: Linear-Normal

$$
\begin{aligned}
X &= \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5\} \\
Y &= \{0, 2, 1, 3, 2, 4, 3, 5, 4, 6\}
\end{aligned}
$$

$$
\begin{aligned}
Y_i &= 0 + 1X_i + u_i \\
\hat{u}_i^2 &= 1 \,\forall\, i
\end{aligned}
$$

$$
\begin{aligned}
\text{"TSS"} \equiv \sum (Y_i - \bar{Y})^2 &= 30 \\
\text{"RSS"} \equiv \sum \hat{u}_i^2 &= 10 \\
\text{"MSS" / "ESS"} &= 20
\end{aligned}
$$

Toy Example: Plot



Toy Example: OLS

```
> linmod<-lm(y~x)
> summary(linmod)

Call:
lm(formula = y ~ x)

Residuals:
       Min         1Q     Median         3Q        Max
-1.000e+00 -1.000e+00  1.110e-16  1.000e+00  1.000e+00

Coefficients:
              Estimate Std. Error   t value Pr(>|t|)
(Intercept) -5.617e-16  8.292e-01 -6.77e-16  1.00000
x            1.000e+00  2.500e-01         4  0.00395 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.118 on 8 degrees of freedom
Multiple R-squared: 0.6667,Adjusted R-squared: 0.625
F-statistic:    16 on 1 and 8 DF,  p-value: 0.00395
```

Toy Example: Linear-Normal GLM

```
> linglm<-glm(y~x,family="gaussian")
> summary(linglm)

Deviance Residuals:
       Min          1Q      Median          3Q         Max
-1.000e+00  -1.000e+00  -5.551e-17   1.000e+00   1.000e+00

Coefficients:
              Estimate Std. Error   t value Pr(>|t|)
(Intercept) -5.617e-16  8.292e-01 -6.77e-16  1.00000
x            1.000e+00  2.500e-01         4  0.00395 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for gaussian family taken to be 1.25)

    Null deviance: 30  on 9  degrees of freedom
Residual deviance: 10  on 8  degrees of freedom
AIC: 34.379

Number of Fisher Scoring iterations: 2
```

Better GLM Example: Political Knowledge

- 2008 NES political knowledge
- Identify Speaker of the House, VP, British PM, and Chief Justice
- Y_i = number of correct answers (out of four)

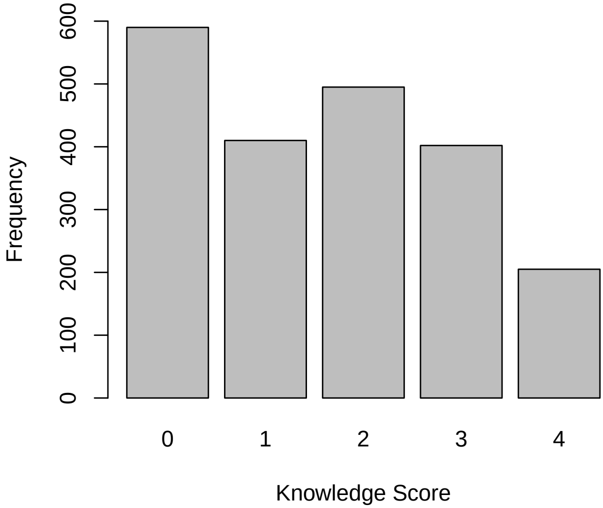
$$f(Y_i, p_i) = \binom{4}{Y_i} p_i^{Y_i}(1 - p_i)^{4-Y_i}$$

$$\begin{aligned} Y &\sim \text{Binomial}(4, p) \\ \text{E}(Y_i) &= \frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1 + \exp(\mathbf{X}_i\boldsymbol{\beta})} \end{aligned}$$

GLM Example Data (2008 NES)

```
> summary(NES08[,4:16])
   knowledge                                  sex                                   race
 Min.   :0.00   1. Male respondent selected  : 999   1. White                   :1442
 1st Qu.:1.00   2. Female respondent selected:1324   2. Black/African-American: 583
 Median :2.00                                        4. Other race              : 262
 Mean   :2.37                                        5. White and another race:  16
 3rd Qu.:4.00                                        6. Black and another race:   6
 Max.   :4.00                                        (Other)                    :   2
 NA's   :221                                         NA's                       :  12
      age          female          white                     oftenvote    conservative
 Min.   :17   Min.   :0.00   Min.   :0.0000   Seldom          :621   Min.   :1.00
 1st Qu.:33   1st Qu.:0.00   1st Qu.:0.0000   Part of the Time:287   1st Qu.:3.00
 Median :46   Median :1.00   Median :1.0000   Nearly Always   :612   Median :4.00
 Mean   :47   Mean   :0.57   Mean   :0.6207   Always          :788   Mean   :4.14
 3rd Qu.:59   3rd Qu.:1.00   3rd Qu.:1.0000   NA's            : 15   3rd Qu.:5.00
 Max.   :90   Max.   :1.00   Max.   :1.0000                          Max.   :7.00
 NA's   :22                                                          NA's   :697
            prayfreq    heterosexual        married          yrsofschool        income
 Never           :235   Min.   :0.0000   Min.   :0.0000   Min.   : 0.00   Min.   : 1.00
 Once/week       :321   1st Qu.:1.0000   1st Qu.:0.0000   1st Qu.:12.00   1st Qu.: 5.00
 Few times a week:416   Median :1.0000   Median :0.0000   Median :13.00   Median :11.00
 Daily           :525   Mean   :0.9591   Mean   :0.4224   Mean   :13.08   Mean   :10.52
 Several/Day     :806   3rd Qu.:1.0000   3rd Qu.:1.0000   3rd Qu.:15.00   3rd Qu.:15.00
 NA's            : 20   Max.   :1.0000   Max.   :1.0000   Max.   :17.00   Max.   :25.00
                        NA's   :49       NA's   :15       NA's   :11      NA's   :151
```

Political Knowledge (2008 NES)



```
> nes08.binom<-glm(cbind(knowledge,4-knowledge)~age+female+white+oftenvote+conservative
+prayfreq+heterosexual+married+yrsofschool+income,data=nes2008,family=binomial)
> summary(nes08.binom)

Deviance Residuals:
     Min        1Q    Median        3Q       Max
-3.59683  -1.01716   0.03124   1.34899   2.85336

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)   2.097696   0.248976   8.425  < 2e-16 ***
age          -0.010789   0.001910  -5.650 1.60e-08 ***
female        0.213865   0.059534   3.592 0.000328 ***
white        -0.154109   0.064613  -2.385 0.017073 *
oftenvote    -0.097272   0.027511  -3.536 0.000407 ***
conservative  0.019421   0.019317   1.005 0.314704
prayfreq      0.048818   0.022248   2.194 0.028216 *
heterosexual  0.070894   0.138471   0.512 0.608665
married      -0.166501   0.058363  -2.853 0.004333 **
yrsofschool  -0.090790   0.013116  -6.922 4.45e-12 ***
income       -0.009015   0.005259  -1.714 0.086492 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 3181.4  on 1359  degrees of freedom
Residual deviance: 2952.9  on 1349  degrees of freedom
  (963 observations deleted due to missingness)
AIC: 4563.1
Number of Fisher Scoring iterations: 4
```

Other Topics:

- Generalizations for Overdispersion (binomial)
- Diagnostics (leverage, etc.)
- Joint Mean-Dispersion Models

Extensions:

- Bias-reduced models (a la Firth 1993)
- "Generalized additive models" (GAMs)
- "Generalized estimating equations" (GEEs)
- "Vector" GLMs (Yee and Wild 1996; Yee and Hastie 2003)

GLMs: References

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Dobson, Annette J., and and Adrian G. Barnett. 2008. *An Introduction to Generalized Linear Models*, 3rd Ed. London: Chapman & Hall.

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Dunn, Peter K., and Gordon K. Smyth. 2018. *Generalized Linear Models With Examples in R*. New York: Springer.

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