PLSC 503 – Spring 2023
Bootstrapping and Missing Data

March 13, 2023

The population is to the sample as the sample is to the bootstrap sample.

Practical (Nonparametric) Bootstrapping

The General Idea:

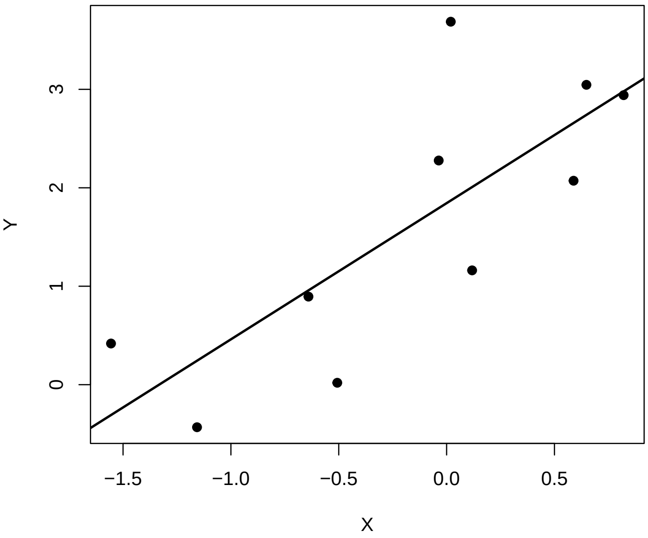
- Draw one bootstrap sample of size N **with replacement** from the original data,
- Estimate the parameter(s) $\underset{k\times 1}{\tilde{\theta}}$,
- Repeat steps 1 and 2 R times, to get $\tilde{\theta}_r$, $r \in \{1, 2, ...R\}$, comprising elements $\tilde{\theta}_{rk}$,
- Examine the empirical characteristics of the resulting distribution(s) of $\tilde{\theta}_{rk}$.

- **It's intuitive.**
- **It's simple.**
- **It's robust.**

Bootstrapping: "By Hand"

```
N<-10 # small sample!
reps<-1001

set.seed(1337)
X<-rnorm(N)
Y<-2+2*X+rnorm(N)
data<-data.frame(Y,X)
fitOLS<-lm(Y~X)
CI<-confint(fitOLS)

B0<-numeric(reps)
B1<-numeric(reps)

for (i in 1:reps) {
  temp<-data[sample(1:N,N,replace=TRUE),]
  temp.lm<-lm(Y~X,data=temp)
  B0[i]<-temp.lm$coefficients[1]
  B1[i]<-temp.lm$coefficients[2]
}

ByHandB0<-median(B0)
ByHandB1<-median(B1)
ByHandCI.B0<-quantile(B0,probs=c(0.025,0.975)) # <-- 95% c.i.s
ByHandCI.B1<-quantile(B1,probs=c(0.025,0.975))
```

Normal Residuals...



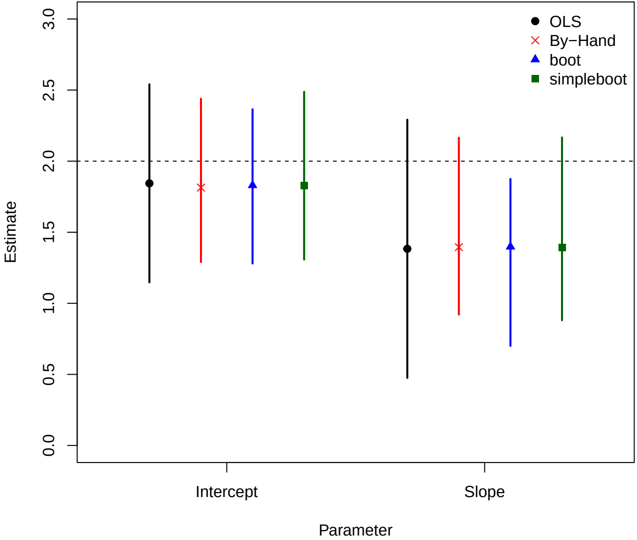
Bootstrapping Via boot

```
library(boot)

Bs<-function(formula, data, indices) { # <- regression function
    dat <- data[indices,]
    fit <- lm(formula, data=dat)
    return(coef(fit))
}

Boot.fit<-boot(data=data, statistic=Bs,
          R=reps, formula=Y~X)

BootB0<-median(Boot.fit$t[,1])
BootB1<-median(Boot.fit$t[,2])
BootCI.B0<-boot.ci(Boot.fit,type="basic",index=1)
BootCI.B1<-boot.ci(Boot.fit,type="basic",index=2)
```

Bootstrapping Via `simpleboot`

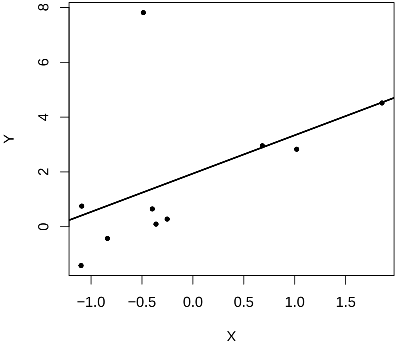
```
library(simpleboot)

Simple<-lm.boot(fitOLS,reps)
SimpleB0<-perc(Simple,.50)[1]
SimpleB1<-perc(Simple,.50)[2]
Simple.CIs<-perc(Simple,p=c(0.025,0.975))
```

Bootstrapping Results

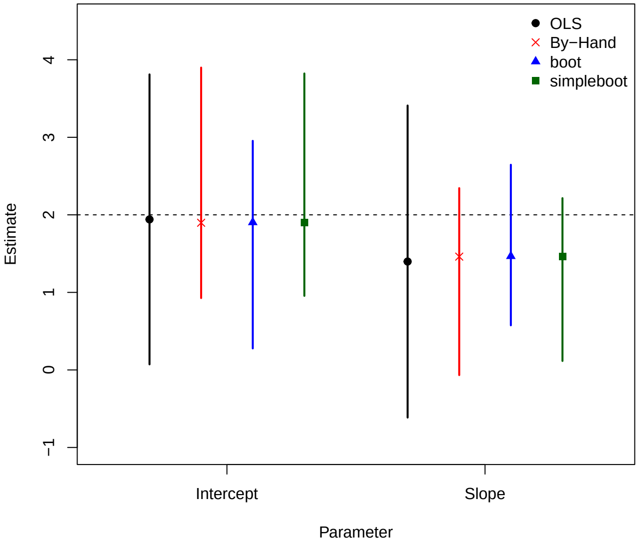


Bootstrapping: Skewed Residuals

```
N<-10
reps<-1001

set.seed(1337)
X<-rnorm(N)
ustar<-rgamma(N,shape=0.2,scale=1)*6 # <- skewed u.s
Y<-2+2*X+(ustar-mean(ustar))
data<-data.frame(Y,X)
fitOLS<-lm(Y~X)
CI<-confint(fitOLS)
```



Skewed Residuals: Results



R things:

- A simple introduction at StatMethods
- Bootstrap in R (at DataCamp)
- Packages: `boot`, `bootstrap`, `simpleboot`, `car::Boot`, `broom` (tidy), many more

Other Resources:

- Efron's original (1979) paper
- Chernick and Labudde (2011) (a solid R-based bootstrapping book)
- Many other books, etc.

Missing Data

Why are data missing?

- The observation itself does not exist
- Data don't exist for that observation
- Data exist, but are *impossible* to measure
- Data exist, but were not measured

Missing Data, Part II: Flavors

Notation:

$$\mathbf{X}_i \in \{\mathbf{W}_i, \mathbf{Z}_i\}$$

$\mathbf{W}_i$ have some missing values,
$\mathbf{Z}_i$ are "complete"

$$R_{ik} = \begin{cases} 1 & \text{if } W_{ik} \text{ is missing,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_{ik} = \Pr(R_{ik} = 1)$$

Missing Data, Part II: Rubin's Flavors

Missing completely at random ("MCAR"):

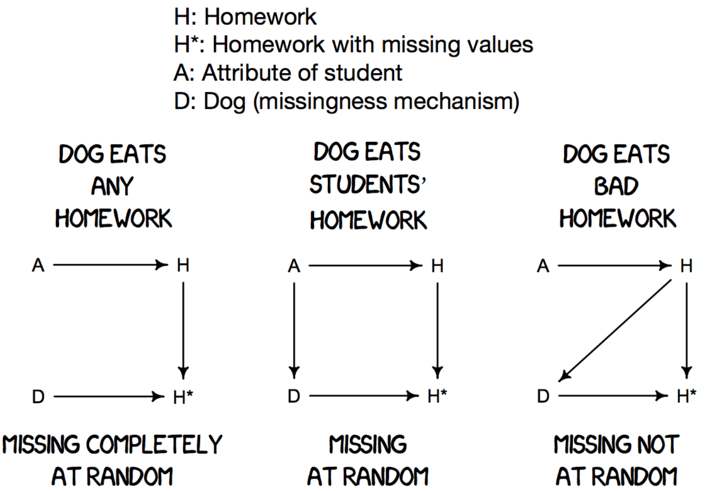
$$\mathbf{R} \perp \{\mathbf{Z}, \mathbf{W}\}$$

Missing at random ("MAR"):

$$\mathbf{R} \perp \mathbf{W} | \mathbf{Z}$$

Anything else is "informatively" (or "non-ignorably," or sometimes "MNAR") missing.

MCAR vs. MAR vs. MNAR, Explained



(Source)

Missing Data: Consequences

In general:

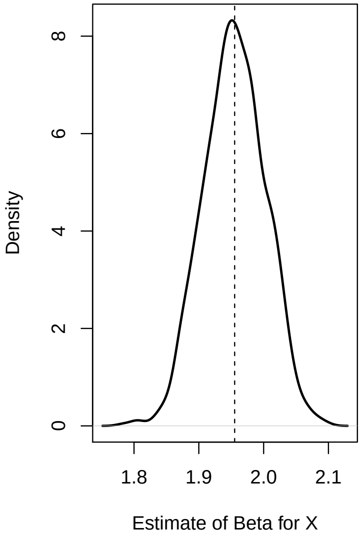
- MCAR:
  - Missing data are a fully random sample of all the data
  - $\rightarrow$ Missingness does not bias $\hat{\theta}$, *but*
  - There is some loss of information (and therefore efficiency)
- MAR
  - Missing data are a nonrandom sample of all the data
  - Ignoring missingness can lead to bias in $\hat{\theta}$, *but*
  - Conditioning on the variable(s) that drive the missingness can eliminate the bias
- Informative Missingness / MNAR
  - Missing data are a nonrandom sample of all the data
  - Ignoring missingness can lead to bias in $\hat{\theta}$
  - In general, conditioning cannot eliminate the bias

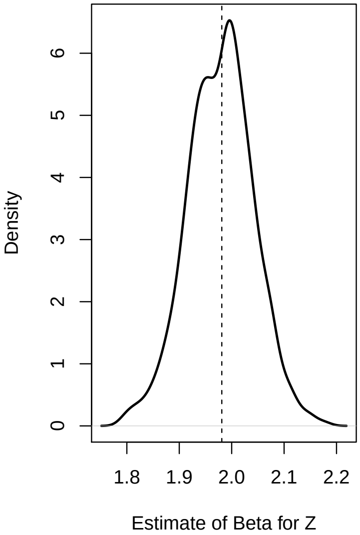
```
> set.seed(7222009)
> Npop <- 1000
> X<-runif(Npop,0,10)   # NOTE: X, Z are correlated a bit...
> Z<-(0.3*X)+(0.7*runif(Npop,0,10))
> Y<-0+(2*X)+(2*Z)+rnorm(Npop,mean=0,sd=4)
> DF<-data.frame(X=X,Z=Z,Y=Y)
> fit.pop<-lm(Y~X+Z,DF)
> summary(fit.pop)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.4051     0.3260    1.24     0.21
X             1.9553     0.0466   41.97   <2e-16 ***
Z             1.9812     0.0617   32.09   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.98 on 997 degrees of freedom
Multiple R-squared:  0.823,Adjusted R-squared:  0.823
F-statistic: 2.32e+03 on 2 and 997 DF,  p-value: <2e-16
```

Simulated MCAR

```
> pmis<-0.50
> DF$Ymcar<-rbinom(Npop,1,pmis)
> DF$Ymcar<-ifelse(DF$Ymcar==1,NA,DF$Y)
>
> # Regression w/listwise deletion:
>
> fit.s<-lm(Ymcar~X+Z,DF) # <-- looks fine
> summary(fit.s)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.4442     0.4653    0.95     0.34
X             1.9661     0.0658   29.87   <2e-16 ***
Z             1.9763     0.0862   22.92   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4 on 507 degrees of freedom
  (490 observations deleted due to missingness)
Multiple R-squared:  0.822,Adjusted R-squared:  0.821
F-statistic: 1.17e+03 on 2 and 507 DF,  p-value: <2e-16
```

Do That A Bunch Of Times...

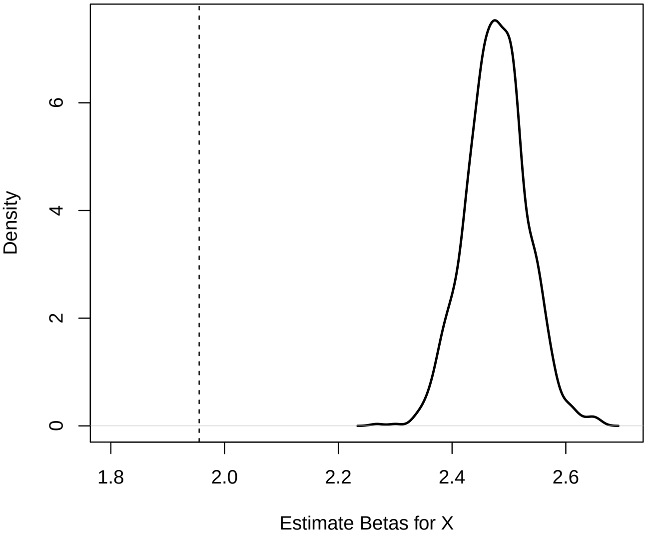




Simulated MAR Y

```
> set.seed(7222009)
> DF$Ymar<-rbinom(Npop,1,(DF$Z/10))
> DF$Ymar<-ifelse(DF$Ymar==1,NA,DF$Y)
>
> summary(lm(Ymar~X,DF))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   3.6600     0.3610    10.1   <2e-16 ***
X             2.9923     0.0648    46.2   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.75 on 547 degrees of freedom
  (451 observations deleted due to missingness)
Multiple R-squared:  0.796,Adjusted R-squared:  0.795
F-statistic: 2.13e+03 on 1 and 547 DF,  p-value: <2e-16
```

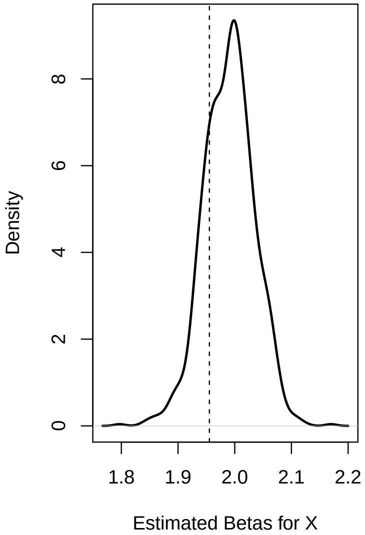
Do That A Bunch Of Times...

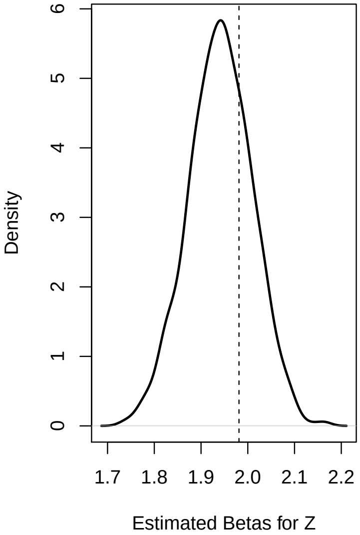


More MAR: Add Z...

```
> summary(lm(Ymar~X+Z,DF))

Call:
lm(formula = Ymar ~ X + Z, data = DF)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.2529     0.4367    0.58     0.56
X             2.0200     0.0663   30.49   <2e-16 ***
Z             1.9499     0.0979   19.91   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.02 on 499 degrees of freedom
  (498 observations deleted due to missingness)
Multiple R-squared:  0.801,Adjusted R-squared:   0.8
F-statistic: 1e+03 on 2 and 499 DF,  p-value: <2e-16
```

Do That A Bunch Of Times...

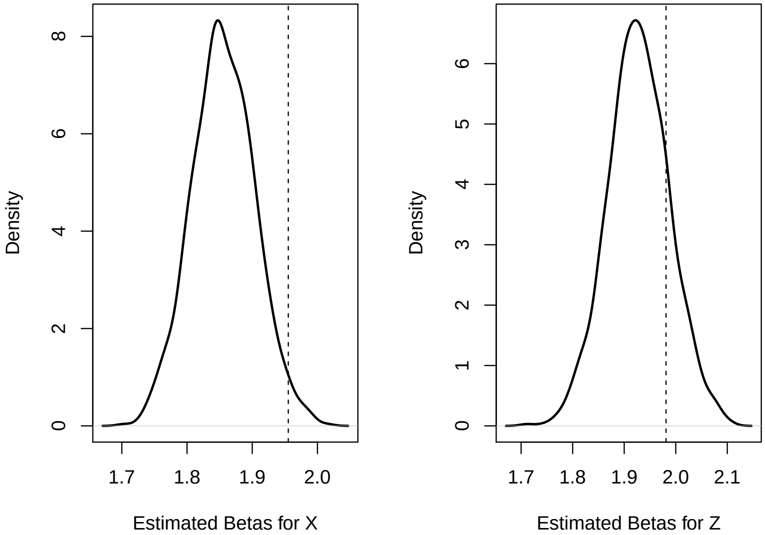




Informative Missingness / "MNAR"

```
> set.seed(7222009)
> DF$Yim<-rbinom(Npop,1,rescale(DF$Z-(4*DF$Y)))
> DF$Yim<-ifelse(DF$Yim==1,NA,DF$Y)
>
> summary(lm(Yim~X+Z,DF))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.0518     0.5463    3.76  0.00019 ***
X             1.8420     0.0671   27.45  < 2e-16 ***
Z             1.9171     0.0859   22.32  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.85 on 465 degrees of freedom
  (532 observations deleted due to missingness)
Multiple R-squared:  0.797,Adjusted R-squared:  0.796
F-statistic:  911 on 2 and 465 DF,  p-value: <2e-16
```

Do That A Bunch Of Times...



A Real-Data Examples: 2020 ANES

Model is:

$$\begin{aligned}
\text{Biden Thermometer}_i &= \beta_0 + \beta_1 \text{R's Conservatism}_i + \\
&= \beta_2 \text{R Labor Union}_i + \beta_3 \text{Female}_i + \\
&= \beta_4 \text{Latino}_i + \beta_5 \text{Age / 10}_i + \\
&= \beta_6 \text{Education}_i + u_i
\end{aligned}$$

Data: ANES 2016-2020 Panel data, 2020 pre-election survey ($N = 2839$).

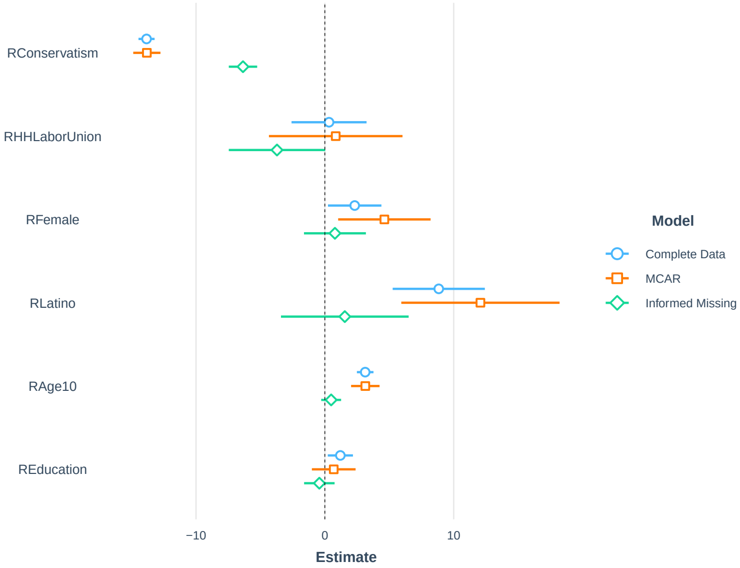
Three models:

- All data ($N = 2291$)
- 67% MCAR (via simulation) ($N = 709$)
- (MNAR) Data *only* on individuals who stated that they "strongly approved" of how then-President Trump was doing his job ($N = 743$)

Biden Thermometer Models

	Dependent variable:		
	Biden Thermometer Rating		
	All Data	MCAR	MNAR
R's Conservatism	−13.841***	−13.817***	−6.354***
	(0.319)	(0.538)	(0.561)
R Labor Union	0.325	0.844	−3.714*
	(1.485)	(2.640)	(1.906)
Female	2.317**	4.621**	0.783
	(1.058)	(1.828)	(1.224)
Latino	8.842***	12.077***	1.550
	(1.824)	(3.129)	(2.524)
Age / 10	3.131***	3.142***	0.490
	(0.328)	(0.563)	(0.394)
Education	1.204**	0.690	−0.427
	(0.498)	(0.864)	(0.603)
Constant	83.222***	83.938***	47.563***
	(3.039)	(5.198)	(4.165)
Observations	2,291	709	743
R^2	0.478	0.512	0.159
Adjusted R^2	0.477	0.508	0.152
Residual Std. Error	25.104 (df = 2284)	24.066 (df = 702)	16.616 (df = 736)
Note:			*p<0.1; **p<0.05; ***p<0.01

Biden Thermometer Models (II)



How Much Missing Data Is A Problem?

"It is often supposed that there exists something like a critical missing rate up to which missing values are not too dangerous. The belief in such a global missing rate is rather stupid."

– *Vach (1994, 113)*

What to Do About Missing Data?

- Listwise Deletion...
- Mean Substitution / Imputation
- "Nearest Neighbor" methods
- "Hot Deck" Imputation
- **Multiple Imputation**
- **Model-Based Solutions**

MAR Data

For MAR data:

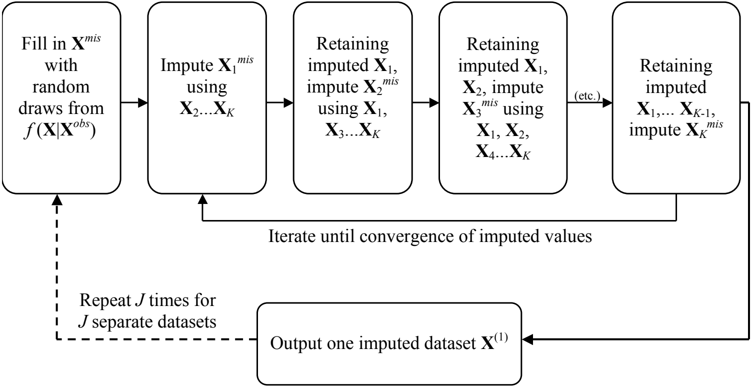
$$\mathbf{R} \perp \mathbf{W} | \mathbf{Z}$$

so **W** and **Z** factorize independently.

Sources of variation we need to consider:

1. Prediction
2. Predictive variation
3. Parameter variation / uncertainty

MAR: Multiple Imputation



Multiple Imputation (continued)

Original Data **X** With Missing Data

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	•	X_{22}	X_{32}	•	...	X_{K2}
3	X_{13}	X_{23}	•	X_{43}	...	X_{K3}
4	X_{14}	•	X_{34}	X_{44}	...	X_{K4}
5	•	X_{25}	X_{35}	•	...	•
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Iteration One:

Step One: "Fill In" Missing Values of $\mathbf{X}$

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	R_{12}	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	R_{33}	X_{43}	...	X_{K3}
4	X_{14}	R_{24}	X_{34}	X_{44}	...	X_{K4}
5	R_{15}	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Step Two: Use $\{X_2, X_3, ...X_K\}$ To Impute X_1^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	R_{33}	X_{43}	...	X_{K3}
4	X_{14}	R_{24}	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation (continued)

Step Three: Use The Imputed X_1, Along With $\{X_3, X_4, ...X_K\}$ To Impute X_2^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	R_{33}	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Step Four: Use The Imputed X_1 and X_2, Along With $\{X_4, ...X_K\}$ To Impute X_3^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	R_{42}	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	R_{45}	...	R_{K5}
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

(etc.)

Multiple Imputation (continued)

Step $K+1$: Use The Imputed $X_1, X_2, ...X_{K-1}$ To Impute X_K^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(1)}$	X_{22}	X_{32}	$I_{42}^{(1)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(1)}$	X_{25}	X_{35}	$I_{45}^{(1)}$	...	$I_{K5}^{(1)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Iteration Two:
Step One: Use The Imputed $X_2, X_3, ...X_K$ To Impute X_1^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(2)}$	X_{22}	X_{32}	$I_{42}^{(1)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(1)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(2)}$	X_{25}	X_{35}	$I_{45}^{(1)}$	...	$I_{K5}^{(1)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Step Two: Use The Imputed $X_1, X_3, ...X_K$ To Impute X_2^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(2)}$	X_{22}	X_{32}	$I_{42}^{(1)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(1)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(2)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(2)}$	X_{25}	X_{35}	$I_{45}^{(1)}$	...	$I_{K5}^{(1)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

(etc.)

Multiple Imputation (continued)

Step K: Use The Imputed $X_1, X_2, ...X_{K-1}$ To Impute X_K^{mis}

i	X_1	X_2	X_3	X_4	...	X_K
1	X_{11}	X_{21}	X_{31}	X_{41}	...	X_{K1}
2	$I_{12}^{(2)}$	X_{22}	X_{32}	$I_{42}^{(2)}$	...	X_{K2}
3	X_{13}	X_{23}	$I_{33}^{(2)}$	X_{43}	...	X_{K3}
4	X_{14}	$I_{24}^{(2)}$	X_{34}	X_{44}	...	X_{K4}
5	$I_{15}^{(2)}$	X_{25}	X_{35}	$I_{45}^{(2)}$	...	$I_{K5}^{(2)}$
6	X_{16}	X_{26}	X_{36}	X_{46}	...	X_{K6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	X_{1N}	X_{2N}	X_{3N}	X_{4N}	...	X_{KN}

Multiple Imputation: Summary

Basically:

- Repeat this process for $J \approx 10$ iterations until convergence of the $I_{ki}^{(j)}$s.
- Output the resulting imputed data $\mathbf{X}^{(1)}$.
- Repeat this entire process M times to create M imputed datasets $\{\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots \mathbf{X}^{(M)}\}$.
- Rule of thumb: $M \geq$ the percentage of cases in your data with missingness.
- Estimate models and conduct inference using multiple analysis and model averaging (see e.g. Schafer 1997, Ch. 4).

MNAR Data

For MNAR data:

$$\Pr(\mathbf{R}) = f(\mathbf{W}, \mathbf{Z})$$

i.e., missingness is *nonignorable*.

Common causes / situations:

- Omitted variables ($\rightarrow$ can't condition on all elements of $\mathbf{Z}$)
- Differential response due to unmeasured factors
- Truncation / censoring

MNAR and Model-Based Solutions

For MNAR data, we must model the joint distribution $\Pr(\mathbf{X}, \mathbf{R})$...

Approaches:

- *Selection* model: $\Pr(\mathbf{X}, \mathbf{R}) = \Pr(\mathbf{X}) \Pr(\mathbf{R}|\mathbf{X})$
    - E.g., Heckman (1976, 1979, etc.)
    - Specifies a (usually, regression) model for $\Pr(\mathbf{R} \,|X)$

- *Pattern-Mixture* model: $$\begin{aligned} \Pr(\mathbf{X}, \mathbf{R}) &= \Pr(\mathbf{X}|\mathbf{R}) \Pr(\mathbf{X}) \\ &= \Pr(\mathbf{X}|\mathbf{R}=0) \Pr(\mathbf{R}=0) + \\ &\quad \Pr(\mathbf{X}|\mathbf{R}=1) \Pr(\mathbf{R}=1) \end{aligned}$$
    - E.g., Glynn, Laird, and Rubin (1986)
    - Mixture-type model across "responders" and "non-responders"

- Others... [see, e.g., Little and Rubin (2002)]

Missing Data Resources, R and Otherwise

- The Missing Data CRAN Task View
- Packages:
  - `Amelia`
  - `mi`, `mice`, and `miceFast`
  - `miceMNAR` (MNAR imputation using a Heckman-style selection model)
  - `naniar` (tidy-cult, but enables cool visualizations)
  - `VIM` (joint visualization and imputation of missing data; also used to have a GUI)
  - Many others...
- van Buuren's Flexible Imputation of Missing Data e-book