PLSC 503 – Spring 2025
Regression Models for Nominal Outcomes

April 14, 2025

Outcome variable has $J > 2$ *unordered* categories:

$$Y_i \in \{1, 2, ...J\}$$

Write:

$$\Pr(Y_i = j) = P_{ij}$$

Means that:

$$\sum_{j=1}^{J} P_{ij} = 1$$

And set:

$$P_{ij} = \exp(\mathbf{X}_i \boldsymbol{\beta}_j)$$

Motivation, continued

Rescale:

$$\Pr(Y_i = j) \equiv P_{ij} = \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)}$$

Ensures

- $\Pr(Y_i = j) \in (0, 1)$
- $\sum_{j=1}^{J} \Pr(Y_i = j) = 1.0$

Constrain $\boldsymbol{\beta}_1 = \mathbf{0}$; then:

$$\Pr(Y_i = 1) = \frac{1}{1 + \sum_{j=2}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}'_j)}$$

$$\Pr(Y_i = j) = \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}'_j)}{1 + \sum_{j=2}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}'_j)}$$

where $\boldsymbol{\beta}'_j = \boldsymbol{\beta}_j - \boldsymbol{\beta}_1$.

Alternative Motivation: Discrete *Choice*

Utility:

$$U_{ij} = \mu_i + \epsilon_{ij}$$

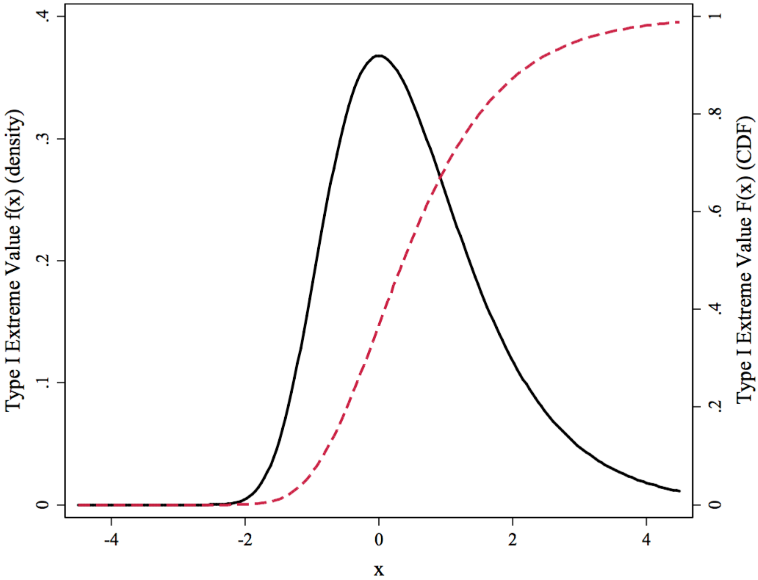
$$\mu_i = \mathbf{X}_i \boldsymbol{\beta}_j$$

$$\begin{aligned}
\Pr(Y_i = j) &= \Pr(U_{ij} > U_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\mu_i + \epsilon_{ij} > \mu_i + \epsilon_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\mathbf{X}_i \boldsymbol{\beta}_j + \epsilon_{ij} > \mathbf{X}_i \boldsymbol{\beta}_\ell + \epsilon_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\epsilon_{ij} - \epsilon_{i\ell} > \mathbf{X}_i \boldsymbol{\beta}_\ell - \mathbf{X}_i \boldsymbol{\beta}_j \,\forall\, \ell \neq j \in J)
\end{aligned}$$

$\epsilon \sim$???

- *Type I Extreme Value*
- Density: $f(\epsilon) = \exp[-\epsilon - \exp(-\epsilon)]$
- CDF: $\int f(\epsilon) \equiv F(\epsilon) = \exp[-\exp(-\epsilon)]$

Type I Extreme Value



The probability of choosing choice j is:

$$
\begin{aligned}
\Pr(Y_i = j) &= \Pr(U_j > U_1, U_j > U_2, \ldots U_j > U_J) \\
&= \int f(\epsilon_j) \left[\int_{-\infty}^{\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_1} f(\epsilon_1) d\epsilon_1 \times \int_{-\infty}^{\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_2} f(\epsilon_2) d\epsilon_2 \times \ldots \right] d\epsilon_j \\
&= \int f(\epsilon_j) \times \exp[-\exp(\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_1)] \times \\
&\quad \exp[-\exp(\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_2)] \times \ldots d\epsilon_j \\
&= \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)}
\end{aligned}
$$

Define:
$$\begin{aligned}\delta_{ij} &= 1 \text{ if } Y_i = j, \\ &= 0 \text{ otherwise.}\end{aligned}$$

Then:

$$\begin{aligned}L_i &= \prod_{j=1}^{J} [\Pr(Y_i = j)]^{\delta_{ij}} \\ &= \prod_{j=1}^{J} \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)}\right]^{\delta_{ij}}\end{aligned}$$

So:

$$L = \prod_{i=1}^{N} \prod_{j=1}^{J} \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]^{\delta_{ij}}$$

and (of course):

$$\ln L = \sum_{i=1}^{N} \sum_{j=1}^{J} \delta_{ij} \ln \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]$$

Example: The 1992 U.S. Presidential Election

1992 American National Election Study

Data:

- Y (PresVote) $\in \{\text{Bush}(=1), \text{Clinton}(=2), \text{Perot}(=3)\}$
- $\mathbf{X}$ = political demographic characteristics + "feeling thermometers"

```
> describe(NES92)
```

	vars	n	mean	sd	median	trimmed	mad	min	max	range	skew	kurtosis	se
ID	1	1473	4671.15	1104.02	5113	4681.28	1546.35	3001	6251	3250	-0.11	-1.66	28.77
VotedFor*	2	1473	1.85	0.71	2	1.82	1.48	1	3	2	0.22	-1.03	0.02
PresVote	3	1473	1.85	0.71	2	1.82	1.48	1	3	2	0.22	-1.03	0.02
PartyID	4	1473	3.75	2.11	3	3.69	2.97	1	7	6	0.15	-1.39	0.06
Age	5	1473	45.89	16.67	43	44.85	17.79	18	91	73	0.50	-0.72	0.43
FamIncome	6	1473	15.53	5.76	16	16.10	5.93	1	24	23	-0.78	-0.17	0.15
Female	7	1473	0.51	0.50	1	0.52	0.00	0	1	1	-0.06	-2.00	0.01
White*	8	1473	1.88	0.33	2	1.97	0.00	1	2	1	-2.31	3.36	0.01
FT.Bush	9	1473	51.75	27.26	60	52.99	29.65	0	100	100	-0.30	-0.72	0.71
FT.Clinton	10	1473	55.77	25.08	60	57.35	29.65	0	100	100	-0.45	-0.37	0.65
FT.Perot	11	1473	44.85	26.51	50	44.90	29.65	0	100	100	-0.16	-0.68	0.69

Model:

$$\texttt{PresVote}_i = f(\beta_0 + \beta_1 \times \texttt{PartyID}_i + \beta_2 \times \texttt{Age}_i + \beta_3 \times \texttt{White}_i + \beta_4 \times \texttt{Female}_i)$$

MNL #1, using `vglm` ("Baseline" = Perot)

```
> NES92.mlogit<-vglm(PresVote~PartyID+Age+White+Female,multinomial,data=NES92)
> summary(NES92.mlogit)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial,
    data = NES92)

Coefficients:
              Estimate Std. Error z value  Pr(>|z|)
(Intercept):1 -1.98008    0.52454   -3.77   0.00016 ***
(Intercept):2  3.82657    0.46402    8.25   < 2e-16 ***
PartyID:1      0.50132    0.04870   10.29   < 2e-16 ***
PartyID:2     -0.63429    0.04918  -12.90   < 2e-16 ***
Age:1          0.01556    0.00504    3.09   0.00203 **
Age:2          0.01296    0.00510    2.54   0.01096 *
WhiteWhite:1  -0.87918    0.43605   -2.02   0.04377 *
WhiteWhite:2  -1.86826    0.38611   -4.84 0.0000013 ***
Female:1       0.50928    0.16266    3.13   0.00174 **
Female:2       0.38427    0.16267    2.36   0.01816 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Names of linear predictors: log(mu[,1]/mu[,3]), log(mu[,2]/mu[,3])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
No Hauck-Donner effect found in any of the estimates
Reference group is level  3  of the response
```

MNL #2, using `multinom` ("Baseline" = Perot)

```
> NES92$PresVote2<-factor(NES92$PresVote,
+                         levels = c("3", "1", "2"),
+                         labels = c("Perot", "Bush", "Clinton"))

> NES92.mlogit2<-multinom(PresVote2~PartyID+Age+White+Female,data=NES92)
# weights:  18 (10 variable)
initial  value 1618.255901
iter  10 value 1080.908630
final  value 1053.650588
converged

> summary(NES92.mlogit2)
Call:
multinom(formula = PresVote2 ~ PartyID + Age + White + Female,
    data = NES92)

Coefficients:
        (Intercept) PartyID    Age WhiteWhite Female
Bush          -1.98   0.501 0.0156     -0.879  0.509
Clinton        3.83  -0.634 0.0130     -1.868  0.384

Std. Errors:
        (Intercept) PartyID     Age WhiteWhite Female
Bush          0.525  0.0487 0.00504      0.436  0.163
Clinton       0.464  0.0492 0.00510      0.386  0.163

Residual Deviance: 2107
AIC: 2127
```

First, we have to "reshape" the data:

```
> head(NES92)
    ID VotedFor PresVote PartyID Age FamIncome Female     White FT.Bush FT.Clinton FT.Perot PresVote2
1 3001     Bush        1       6  31        20      0     White      85         30        0      Bush
2 3002     Bush        1       7  89         9      1     White     100          0        0      Bush
3 3003     Bush        1       7  35        17      1     White      85         30       60      Bush
4 3005  Clinton        2       6  27         3      1 Non-White      40         60       60   Clinton
5 3006  Clinton        2       2  54        15      1     White      30         70       50   Clinton
6 3007  Clinton        2       1  45         2      1 Non-White      15         70       50   Clinton

> AltNES92<-dfidx(NES92,varying=9:11,shape="wide",choice="VotedFor")

> head(AltNES92)
~~~~~~~
 first 10 observations out of 4419
~~~~~~~
     ID VotedFor PresVote PartyID Age FamIncome Female     White PresVote2  FT    idx
1  3001     TRUE        1       6  31        20      0     White      Bush  85 1:Bush
2  3001    FALSE        1       6  31        20      0     White      Bush  30 1:nton
3  3001    FALSE        1       6  31        20      0     White      Bush   0 1:erot
4  3002     TRUE        1       7  89         9      1     White      Bush 100 2:Bush
5  3002    FALSE        1       7  89         9      1     White      Bush   0 2:nton
6  3002    FALSE        1       7  89         9      1     White      Bush   0 2:erot
7  3003     TRUE        1       7  35        17      1     White      Bush  85 3:Bush
8  3003    FALSE        1       7  35        17      1     White      Bush  30 3:nton
9  3003    FALSE        1       7  35        17      1     White      Bush  60 3:erot
```

MNL #3, using mlogit (continued)

Now, fit the model:

```
> NES92.mlogit3<-mlogit(VotedFor~0|PartyID+Age+White+Female,data=AltNES92,reflevel="Perot")
> summary(NES92.mlogit3)

Call:
mlogit(formula = VotedFor ~ 0 | PartyID + Age + White + Female,
    data = AltNES92, reflevel = "Perot", method = "nr")

Frequencies of alternatives:choice
  Perot    Bush Clinton
  0.191   0.339   0.469

nr method
5 iterations, 0h:0m:0s
g'(-H)^-1g = 4.94E-08
gradient close to zero

Coefficients :
                    Estimate Std. Error z-value Pr(>|z|)
(Intercept):Bush    -1.98008    0.52454   -3.77  0.00016 ***
(Intercept):Clinton  3.82657    0.46403    8.25  2.2e-16 ***
PartyID:Bush         0.50132    0.04870   10.29  < 2e-16 ***
PartyID:Clinton     -0.63429    0.04918  -12.90  < 2e-16 ***
Age:Bush             0.01556    0.00504    3.09  0.00203 **
Age:Clinton          0.01296    0.00510    2.54  0.01096 *
WhiteWhite:Bush     -0.87918    0.43606   -2.02  0.04378 *
WhiteWhite:Clinton  -1.86826    0.38612   -4.84  1.3e-06 ***
Female:Bush          0.50928    0.16266    3.13  0.00174 **
Female:Clinton       0.38427    0.16267    2.36  0.01816 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log-Likelihood: -1050
McFadden R^2:  0.311
Likelihood ratio test : chisq = 952 (p.value = <2e-16)
```

MNL: 1992 Election ("Baseline" = Bush)

```
> Bush.nes92.mlogit<-vglm(PresVote~PartyID+Age+White+Female,
+                         data=NES92,family=multinomial(refLevel=1))
> summary(Bush.nes92.mlogit)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial(refLevel = 1),
    data = NES92)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  5.80665    0.44301   13.11  < 2e-16 ***
(Intercept):2  1.98008    0.52454    3.77  0.00016 ***
PartyID:1     -1.13561    0.05486  -20.70  < 2e-16 ***
PartyID:2     -0.50132    0.04870  -10.29  < 2e-16 ***
Age:1         -0.00260    0.00514   -0.51  0.61276
Age:2         -0.01556    0.00504   -3.09  0.00203 **
WhiteWhite:1  -0.98908    0.31346   -3.16  0.00160 **
WhiteWhite:2   0.87918    0.43605    2.02  0.04377 *
Female:1      -0.12500    0.16895   -0.74  0.45936
Female:2      -0.50928    0.16266   -3.13  0.00174 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Names of linear predictors: log(mu[,2]/mu[,1]), log(mu[,3]/mu[,1])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
No Hauck-Donner effect found in any of the estimates
Reference group is level  1  of the response
```

MNL: 1992 Election ("Baseline" = Clinton)

```
> Clinton.nes92.mlogit<-vglm(PresVote~PartyID+Age+White+Female,
+                         data=NES92,family=multinomial(refLevel=2))
> summary(Clinton.nes92.mlogit)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial(refLevel = 2),
    data = NES92)

Coefficients:
              Estimate Std. Error z value  Pr(>|z|)
(Intercept):1 -5.80665    0.44301  -13.11   < 2e-16 ***
(Intercept):2 -3.82657    0.46402   -8.25   < 2e-16 ***
PartyID:1      1.13561    0.05486   20.70   < 2e-16 ***
PartyID:2      0.63429    0.04918   12.90   < 2e-16 ***
Age:1          0.00260    0.00514    0.51    0.6128
Age:2         -0.01296    0.00510   -2.54    0.0110 *
WhiteWhite:1   0.98908    0.31346    3.16    0.0016 **
WhiteWhite:2   1.86826    0.38611    4.84 0.0000013 ***
Female:1       0.12500    0.16895    0.74    0.4594
Female:2      -0.38427    0.16267   -2.36    0.0182 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Names of linear predictors: log(mu[,1]/mu[,2]), log(mu[,3]/mu[,2])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
Warning: Hauck-Donner effect detected in the following estimate(s):
’(Intercept):2’
Reference group is level  2  of the response
```

`PartyID` Coefficient Estimates and "Baselines"

Note: `PartyID` is 1 (strong Democrat) $\rightarrow$ 7 (strong Republican)

		"Baseline" category		
		Clinton	Perot	Bush
Comparison	Clinton	–	-0.63	-1.14
Category	Perot	0.63	–	-0.50
	Bush	1.14	0.50	–

Consider the choice of Bush vs. Perot:

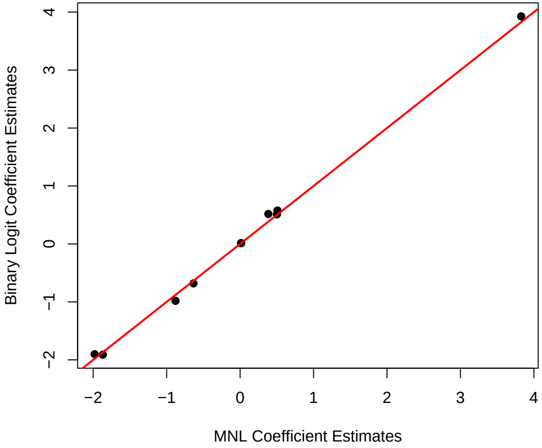
```
> NES92$PickBush<-NA
> NES92$PickBush<-ifelse(NES92$VotedFor=="Bush",1,NES92$PickBush)
> NES92$PickBush<-ifelse(NES92$VotedFor=="Perot",0,NES92$PickBush)
> BushBinary<-glm(PickBush~PartyID+Age+White+Female,data=NES92,family="binomial")
> summary(BushBinary)

Call:
glm(formula = PickBush ~ PartyID + Age + White + Female, family = "binomial",
    data = NES92)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -1.9024     0.5372   -3.54  0.00040 ***
PartyID       0.5106     0.0505   10.12  < 2e-16 ***
Age           0.0143     0.0052    2.75  0.00595 **
WhiteWhite   -0.9817     0.4586   -2.14  0.03230 *
Female        0.5768     0.1683    3.43  0.00061 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1022.50  on 781  degrees of freedom
Residual deviance:  880.28  on 777  degrees of freedom
  (691 observations deleted due to missingness)
AIC: 890.3

Number of Fisher Scoring iterations: 4
```

What about Clinton vs. Perot?:

```
> NES92$PickClinton<-NA
> NES92$PickClinton<-ifelse(NES92$VotedFor=="Clinton",1,NES92$PickClinton)
> NES92$PickClinton<-ifelse(NES92$VotedFor=="Perot",0,NES92$PickClinton)
> ClintonBinary<-glm(PickClinton~PartyID+Age+White+Female,data=NES92,family="binomial")
> summary(ClintonBinary)

Call:
glm(formula = PickClinton ~ PartyID + Age + White + Female, family = "binomial",
    data = NES92)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  3.92614    0.48490    8.10  5.6e-16 ***
PartyID     -0.68125    0.05301  -12.85  < 2e-16 ***
Age          0.01381    0.00537    2.57   0.0101 *
WhiteWhite  -1.91056    0.39879   -4.79  1.7e-06 ***
Female       0.51690    0.17024    3.04   0.0024 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1171.48  on 972  degrees of freedom
Residual deviance:  861.57  on 968  degrees of freedom
  (500 observations deleted due to missingness)
AIC: 871.6

Number of Fisher Scoring iterations: 5
```

MNL and Binary Logit (continued)

Are the $\hat{\beta}$s the same? (A: Yes, basically...)



It is exactly the same as the multinomial logit model. Period.

Choice-Specific Covariates: Data Structure

```
> head(AltNES92)

      ID VotedFor PresVote PartyID   Age FamIncome Female White     PresVote2    FT idx$id1 $id2
 * <dbl> <lgl>       <dbl>   <dbl> <dbl>     <dbl>  <dbl> <chr>     <fct>     <dbl>   <int> <fct>
 1  3001 TRUE            1       6    31        20      0 White     Bush         85       1 Bush
 2  3001 FALSE           1       6    31        20      0 White     Bush         30       1 Clinton
 3  3001 FALSE           1       6    31        20      0 White     Bush          0       1 Perot
 4  3002 TRUE            1       7    89         9      1 White     Bush        100       2 Bush
 5  3002 FALSE           1       7    89         9      1 White     Bush          0       2 Clinton
 6  3002 FALSE           1       7    89         9      1 White     Bush          0       2 Perot
 7  3003 TRUE            1       7    35        17      1 White     Bush         85       3 Bush
 8  3003 FALSE           1       7    35        17      1 White     Bush         30       3 Clinton
 9  3003 FALSE           1       7    35        17      1 White     Bush         60       3 Perot
10  3005 FALSE           2       6    27         3      1 Non-White Clinton      40       4 Bush
# 4,409 more rows
# Use 'print(n = ...)' to see more rows
```

Note that:

$$\Pr(Y_{ij} = 1) = \frac{\exp(\mathbf{Z}_{ij}\gamma)}{\sum_{j=1}^{J} \exp(\mathbf{Z}_{ij}\gamma)}$$

Combinations: $\mathbf{X}_i\beta$ and $\mathbf{Z}_{ij}\gamma$:

- "Fixed effects" (choice-specific intercepts), plus
- Observation-specific $\mathbf{X}$s, plus
- Interactions...

CL in R (Feeling Thermometers only)

```
> NES92.clogit<-mlogit(VotedFor~FT,data=AltNES92,reflevel="Perot")
> summary(NES92.clogit)

Call:
mlogit(formula = VotedFor ~ FT, data = AltNES92, reflevel = "Perot",
    method = "nr")

Frequencies of alternatives:choice
  Perot    Bush Clinton
  0.191   0.339   0.469

nr method
6 iterations, 0h:0m:0s
g'(-H)^-1g = 0.00219
successive function values within tolerance limits

Coefficients :
                    Estimate Std. Error z-value   Pr(>|z|)
(Intercept):Bush     0.03307    0.10039    0.33       0.74
(Intercept):Clinton  0.45841    0.09253    4.95 0.00000073 ***
FT                   0.07512    0.00314   23.89    < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log-Likelihood: -801
McFadden R^2:  0.476
Likelihood ratio test : chisq = 1460 (p.value = <2e-16)
```

```
> NES92.clogit2<-mlogit(VotedFor~FT|PartyID+Age+White+Female,data=AltNES92,reflevel="Perot")
> summary(NES92.clogit2)

Frequencies of alternatives:choice
  Perot    Bush Clinton
  0.191   0.339   0.469

nr method
6 iterations, 0h:0m:0s
g'(-H)^-1g = 3.06E-08
gradient close to zero

Coefficients :
                    Estimate Std. Error z-value Pr(>|z|)
(Intercept):Bush    -0.39416    0.58730   -0.67  0.50214
(Intercept):Clinton  2.69235    0.50403    5.34  9.2e-08 ***
FT                   0.06231    0.00325   19.17  < 2e-16 ***
PartyID:Bush         0.22298    0.05842    3.82  0.00014 ***
PartyID:Clinton     -0.40620    0.05798   -7.01  2.4e-12 ***
Age:Bush             0.00868    0.00612    1.42  0.15639
Age:Clinton          0.00839    0.00598    1.40  0.16040
WhiteWhite:Bush     -1.31961    0.47725   -2.77  0.00569 **
WhiteWhite:Clinton  -1.57156    0.41404   -3.80  0.00015 ***
Female:Bush          0.39271    0.19995    1.96  0.04953 *
Female:Clinton       0.28585    0.19474    1.47  0.14213
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -723
McFadden R^2:  0.527
Likelihood ratio test : chisq = 1610 (p.value = <2e-16)
```

Interpretation: Baseline MNL Results

```
> NES.MNL<-vglm(PresVote~PartyID+Age+White+Female,data=NES92,
+               multinomial(refLevel=1)) # Bush is comparison category
> summaryvglm(NES.MNL)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial(refLevel = 1),
    data = NES92)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  5.80665    0.44301   13.11  < 2e-16 ***
(Intercept):2  1.98008    0.52454    3.77  0.00016 ***
PartyID:1     -1.13561    0.05486  -20.70  < 2e-16 ***
PartyID:2     -0.50132    0.04870  -10.29  < 2e-16 ***
Age:1         -0.00260    0.00514   -0.51  0.61276
Age:2         -0.01556    0.00504   -3.09  0.00203 **
WhiteWhite:1  -0.98908    0.31346   -3.16  0.00160 **
WhiteWhite:2   0.87918    0.43605    2.02  0.04377 *
Female:1      -0.12500    0.16895   -0.74  0.45936
Female:2      -0.50928    0.16266   -3.13  0.00174 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Names of linear predictors: log(mu[,2]/mu[,1]), log(mu[,3]/mu[,1])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
No Hauck-Donner effect found in any of the estimates
Reference group is level  1  of the response
```

Global $\ln LR$ statistic Q tests:

$$\hat{\boldsymbol{\beta}} = \mathbf{0} \, \forall \, j, k$$

$$Q \sim \chi^2_{(J-1)(k-1)}$$

Test H: No Effect of Age

Is the effect of Age across the three candidates equal to zero?

```
> library(aod)
> wald.test(b=c(t(coef(NES.MNL))),Sigma=vcov(NES.MNL),Terms=c(5,6))

Wald test:
----------

Chi-squared test:
X2 = 11.0, df = 2, P(> X2) = 0.0042
```

Test H: No Difference – Clinton vs. Bush

Are the estimated coefficients for Clinton (vs. Bush) jointly equal to zero?

```
> wald.test(b=c(t(coef(NES.MNL))),Sigma=vcov(NES.MNL),Terms=c(1,3,5,7,9))

Wald test:
----------

Chi-squared test:
X2 = 444.6, df = 5, P(> X2) = 0.0
```

Interpretation: Marginal Effects

$$\frac{\partial \Pr(Y_i = j)}{\partial X_k} = \Pr(Y_i = j|\mathbf{X}) \left[\hat{\boldsymbol{\beta}}_{jk} - \sum_{j=1}^{J} \hat{\boldsymbol{\beta}}_{jk} \times \Pr(Y_i = j|\mathbf{X}) \right]$$

Depends on:

- $\Pr\widehat{(Y_i = j)}$
- $\hat{\beta}_{jk}$
- $\sum_{j=1}^{J} \hat{\boldsymbol{\beta}}_{jk}$

Available for `-multinom-` (in the `-nnet-` package) via the `-margins-` package...

```
> MNL.alt<-multinom(PresVote2~PartyID+Age+White+Female,data=NES92,Hess=TRUE)
# weights:  18 (10 variable)
initial  value 1618.255901
iter  10 value 1080.908630
final  value 1053.650588
converged

> summary(marginal_effects(MNL.alt))
  dydx_PartyID         dydx_Age            dydx_Female         dydx_WhiteWhite
 Min.   :-0.0908   Min.   :-0.00362   Min.   :-0.1158   Min.   :0.0416
 1st Qu.:-0.0439   1st Qu.:-0.00282   1st Qu.:-0.0892   1st Qu.:0.0943
 Median : 0.0185   Median :-0.00215   Median :-0.0674   Median :0.1352
 Mean   : 0.0083   Mean   :-0.00207   Mean   :-0.0648   Mean   :0.1435
 3rd Qu.: 0.0618   3rd Qu.:-0.00138   3rd Qu.:-0.0420   3rd Qu.:0.1848
 Max.   : 0.1070   Max.   :-0.00011   Max.   :-0.0034   Max.   :0.2926
```

Odds ("Relative Risk") Ratios

MNL has:

$$\ln\left[\frac{\Pr(Y_i = j|\mathbf{X})}{\Pr(Y_i = j'|\mathbf{X})}\right] = \mathbf{X}(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}_{j'})$$

Setting $\hat{\boldsymbol{\beta}}_{j'} = \mathbf{0}$:

$$\ln\left[\frac{\Pr(Y_i = j|\mathbf{X})}{\Pr(Y_i = j'|\mathbf{X})}\right] = \mathbf{X}\hat{\boldsymbol{\beta}}_j$$

One-Unit Change in X_k:

$$RRR_{jk} = \exp(\boldsymbol{\beta}_{jk})$$

δ-Unit Change in X_k:

$$RRR_{jk} = \exp(\boldsymbol{\beta}_{jk} \times \delta)$$

Odds ("Relative Risk") Ratios

```
> mnl.or <- function(model) {
    coeffs <- c(t(coef(NES.MNL)))
    lci <- exp(coeffs - 1.96 * diag(vcov(NES.MNL))^0.5)
    or <- exp(coeffs)
    uci <- exp(coeffs + 1.96* diag(vcov(NES.MNL))^0.5)
    lreg.or <- cbind(lci, or, uci)
    lreg.or
  }

> mnl.or(NES.MNL)
                    lci      or     uci
(Intercept):1 139.540 332.504 792.309
(Intercept):2   2.591   7.243  20.250
PartyID:1       0.288   0.321   0.358
PartyID:2       0.551   0.606   0.666
Age:1           0.987   0.997   1.008
Age:2           0.975   0.985   0.994
WhiteWhite:1    0.201   0.372   0.688
WhiteWhite:2    1.025   2.409   5.662
Female:1        0.634   0.882   1.229
Female:2        0.437   0.601   0.827
```

Odds Ratios: Interpretation

Odds ratio interpretations:

- A one unit increase in `partyid` corresponds to:
  - A decrease in the odds of a Clinton vote, versus a vote for Bush, of $\exp(-1.136) = 0.321$ (or about 68 percent), and
  - A decrease in the odds of a Perot vote, versus a vote for Bush, of $\exp(-0.501) = 0.606$ (or about 40 percent).
  - These are *large* decreases in the odds – not surprisingly, more Republican voters are *much* more likely to vote for Bush than for Perot or Clinton.
- Similarly, `female` voters are:
  - No more or less likely to vote for Clinton vs. Bush (OR=0.88), but
  - Roughly 40 percent less likely to have voted for Perot (OR=0.60).

$$\begin{aligned}\Pr(\widehat{\texttt{presvote}_i} = \text{Bush}) &= \frac{\exp(\mathbf{X}_i\hat{\boldsymbol{\beta}}_{\text{Bush}})}{\sum_{j=1}^{J}\exp(\mathbf{X}_i\hat{\boldsymbol{\beta}}_j)} \\ &= \frac{1}{1+\sum_{j=2}^{J}\exp(\mathbf{X}_i\hat{\boldsymbol{\beta}}_j)}\end{aligned}$$

In-Sample Predicted Outcomes

Generate predicted vote choices:

```
> NES92$Predictions<-" "
> NES92$Predictions<-ifelse(fitted.values(NES.MNL)[,1]>fitted.values(NES.MNL)[,2]
+                   & fitted.values(NES.MNL)[,1]>fitted.values(NES.MNL)[,3],
+                   paste("Bush"),NES92$Predictions) # Bush
> NES92$Predictions<-ifelse(fitted.values(NES.MNL)[,2]>fitted.values(NES.MNL)[,1]
+                   & fitted.values(NES.MNL)[,2]>fitted.values(NES.MNL)[,3],
+                   paste("Clinton"),NES92$Predictions) # Clinton
> NES92$Predictions<-ifelse(fitted.values(NES.MNL)[,3]>fitted.values(NES.MNL)[,1]
+                   & fitted.values(NES.MNL)[,3]>fitted.values(NES.MNL)[,2],
+                   paste("Perot"),NES92$Predictions) # Perot)

> # "Confusion Table":
>
> table(NES92$VotedFor,NES92$Predictions)

        Bush Clinton Perot
Bush     415      77     8
Clinton   56     619    16
Perot    135     133    14
```

$\rightarrow$ PRE

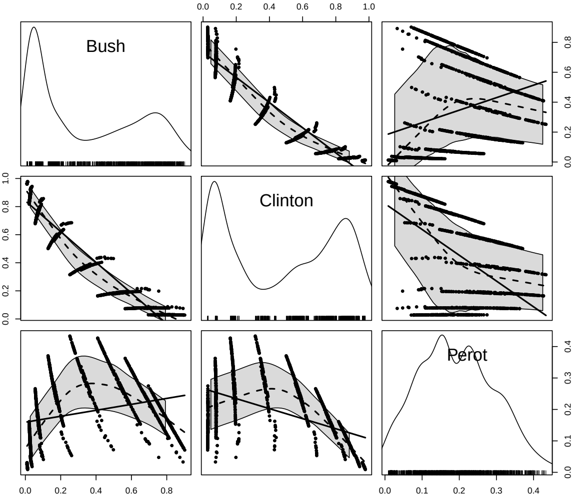
Model fit:

- "Null" Model: $\left(\frac{691}{1473}\right) = 46.9\%$ correct.
- Estimated model: $\frac{(415+619+14)}{1473} = \frac{1048}{1473} = 71.2\%$ correct.
- PRE $= \frac{1048-691}{1473-691} = \frac{357}{782} = 45.7\%$.
- Correct predictions: 90% Clinton, 83% Bush, 5% Perot.

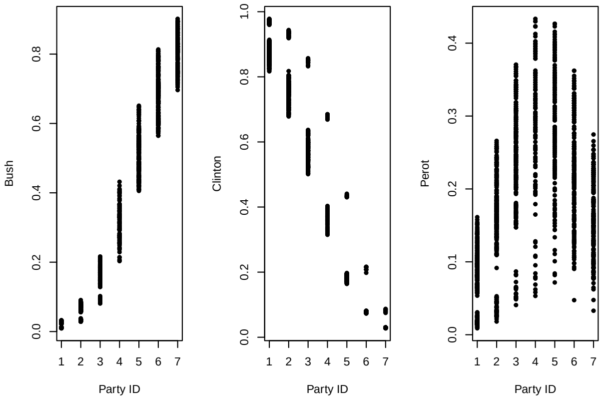
In-Sample Predicted Probabilities

```
> hats<-as.data.frame(fitted.values(NES.MNL))
> names(hats)[3]<-"Perot" # nice names...
> names(hats)[2]<-"Clinton"
> names(hats)[1]<-"Bush"
> attach(hats)

> library(car)
> scatterplot.matrix(~Bush+Clinton+Perot,
  diagonal="histogram",col=c("black","grey"))
```

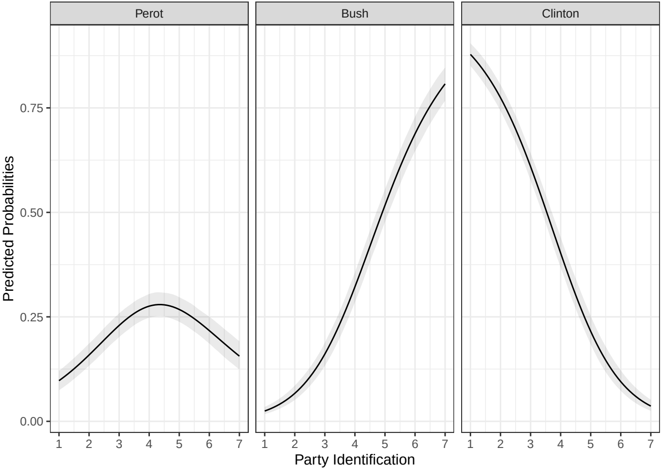
In-Sample $\widehat{\text{Prs}}$



In-Sample $\widehat{\text{Pr}}$s vs. `partyid`

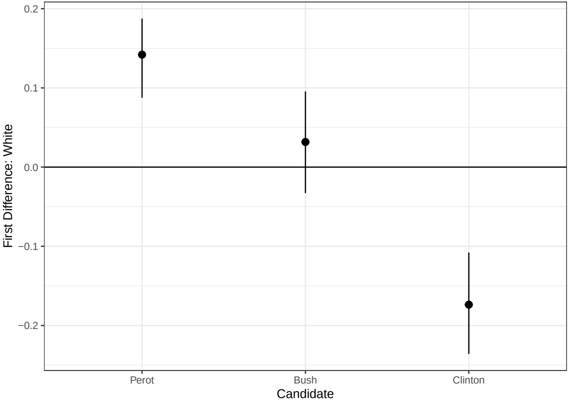


Out-Of-Sample Predictions (using `MNLpred`)



OOS First Differences (using MNLpred)

First differences in probabilities associated with White:

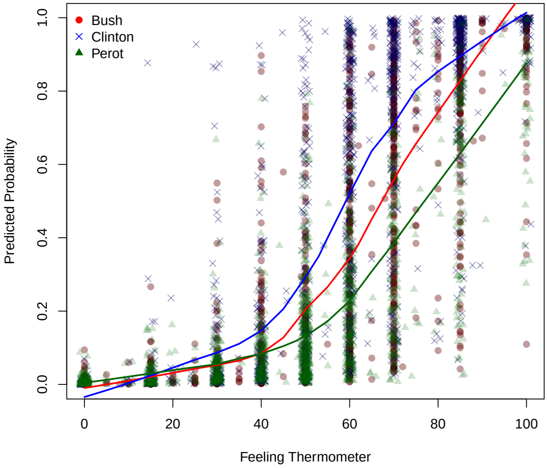


Conditional Logit: Example

```
> summary(NES92.clogit2)

Call:
mlogit(formula = VotedFor ~ FT | PartyID + Age + White + Female,
    data = AltNES92, reflevel = "Perot", method = "nr")

Frequencies of alternatives:choice
  Perot    Bush Clinton
  0.191   0.339   0.469

nr method
6 iterations, 0h:0m:0s
g'(-H)^-1g = 3.06E-08
gradient close to zero

Coefficients :
                    Estimate Std. Error z-value Pr(>|z|)
(Intercept):Bush    -0.39416    0.58730   -0.67  0.50214
(Intercept):Clinton  2.69235    0.50403    5.34  9.2e-08 ***
FT                   0.06231    0.00325   19.17  < 2e-16 ***
PartyID:Bush         0.22298    0.05842    3.82  0.00014 ***
PartyID:Clinton     -0.40620    0.05798   -7.01  2.4e-12 ***
Age:Bush             0.00868    0.00612    1.42  0.15639
Age:Clinton          0.00839    0.00598    1.40  0.16040
WhiteWhite:Bush     -1.31961    0.47725   -2.77  0.00569 **
WhiteWhite:Clinton  -1.57156    0.41404   -3.80  0.00015 ***
Female:Bush          0.39271    0.19995    1.96  0.04953 *
Female:Clinton       0.28585    0.19474    1.47  0.14213
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -723
McFadden R^2:  0.527
Likelihood ratio test : chisq = 1610 (p.value = <2e-16)
```

Conditional Logit: In-Sample Predicted Probabilities

```
> CLhats<-predict(NES92.clogit2,AltNES92)
```



Other Topics (possibly for PLSC 504)

- "Independence of Irrelevant Alternatives"
- $\rightarrow$ Multinomial Probit
- $\rightarrow$ Heteroscedastic Extreme Value model
- "Mixed" Logit
- Nested Logit