PLSC 504 - Fall 2022
Introduction to Survival Analysis

October 24, 2022

"Survival" / "Duration" / "Event History" Models

- Models for *time-to-event data*.
- Roots in biostats/epidemiology, plus engineering, sociology, economics.

Characteristics of time-to-event data:

- *Discrete* events (i.e., not continuous),
- Take place over *time*,
- May not (or *never*) experience the event (i.e., possibility of *censoring*).

Survival Data Basics: Terminology

$$
\begin{aligned}
Y_i &= \text{the duration until the event occurs,} \\
Z_i &= \text{the duration until the observation is ``censored''} \\
T_i &= \min\{Y_i, Z_i\}, \\
C_i &= \text{0 if observation } i \text{ is censored, 1 if it is not.}
\end{aligned}
$$

Survival Data Basics: The Density

$$f(t) = \Pr(T_i = t)$$

Issues:

- $T_i = t$ iff $T_i > t-1,\ t-2$, etc.

- $C_i = 0$ (censoring)

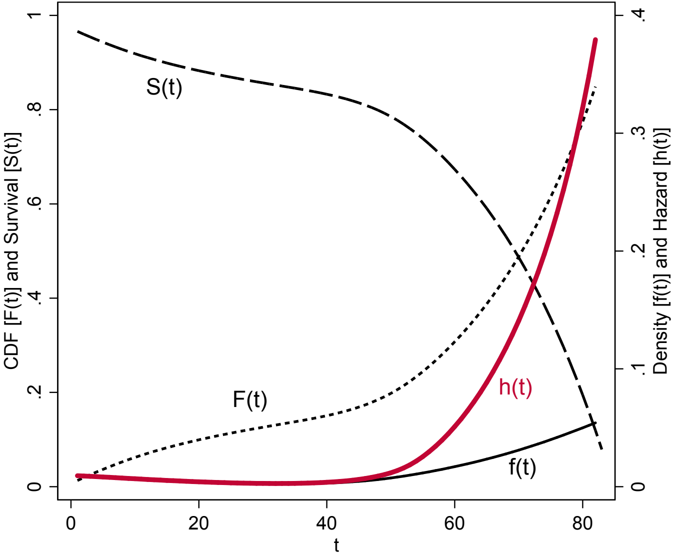
Survival Data Basics: Survivor Function

$$\Pr(T_i \leq t) \equiv F(t) = \int_0^t f(t)\, dt$$

$$\begin{aligned}\Pr(T_i \geq t) \equiv S(t) &= 1 - F(t) \\ &= 1 - \int_0^t f(t)\, dt\end{aligned}$$

$$
\begin{aligned}
\Pr(T_i = t | T_i \geq t) \equiv h(t) &= \frac{f(t)}{S(t)} \\
&= \frac{f(t)}{1 - \int_0^t f(t)\, dt}
\end{aligned}
$$

Example: Human Mortality



$$f(t) = \frac{-\partial S(t)}{\partial t}$$

Implies

$$\begin{aligned} h(t) &= \frac{\frac{-\partial S(t)}{\partial t}}{S(t)} \\ &= \frac{-\partial lnS(t)}{\partial t} \end{aligned}$$

More Useful Things: Integrated Hazard

Define

$$H(t) = \int_0^t h(t)\, dt.$$

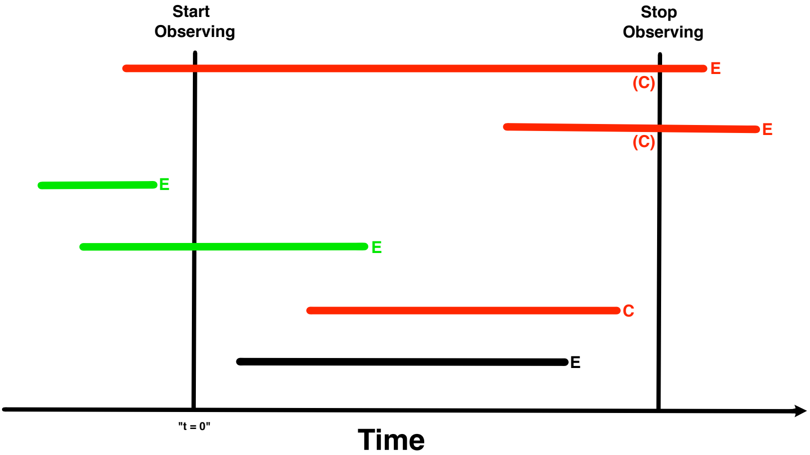
Implies

$$\begin{aligned} H(t) &= \int_0^t \frac{-\partial lnS(t)}{\partial t}\, dt \\ &= -ln[S(t)] \end{aligned}$$

and

$$S(t) = \exp[-H(t)]$$

Censoring and Truncation



- Defined by the researcher
- Conditionally independent of both T_i and $\mathbf{X}_i$
- Doesn't mean that the observation provides no information

Estimating $S(t)$

Assume N observations, *absorbing* events, and no ties. Then define

$$
\begin{aligned}
n_t &= \text{number of observations ``at risk'' for the event at } t \text{, and} \\
d_t &= \text{number of observations which experience the event} \\
&\quad\ \text{at time } t.
\end{aligned}
$$

Then

$$\widehat{S(t_k)} = \prod_{t \leq t_k} \frac{n_t - d_t}{n_t}$$

Variance of $\widehat{S(t)}$

$$\text{Var}[\widehat{S(t_k)}] = \left[\widehat{S(t_k)}\right]^2 \sum_{t \leq t_k} \frac{d_t}{n_t(n_t - d_t)}$$

Note:

- $\text{Var}[\widehat{S(t_k)}]$ is increasing in $S(t)$,
- is also increasing in d_t, but
- is decreasing in n_t.

Estimating $H(t)$

"Nelson-Aalen":

$$\widehat{H(t_k)} = \sum_{t \leq t_k} \frac{d_t}{n_t}$$

...which gives an alternative estimator for the survival function equal to:

$$\begin{aligned}
\widehat{S(t_k)} &= \exp[-\widehat{H(t_k)}] \\
&= \exp\left[-\sum_{t \leq t_k} \frac{d_t}{n_t}\right]
\end{aligned}$$

	Treatment	Placebo	Total
Event	d_{1t}	d_{0t}	d_t
No Event	$n_{1t} - d_{1t}$	$n_{0t} - d_{0t}$	$n_t - d_t$
Total	n_{1t}	n_{0t}	n_t

Log-Rank Test:

$$
\begin{aligned}
Q &= \frac{\left[\sum (d_{1t} - \frac{n_{1t} d_t}{n_t})\right]^2}{\left[\frac{n_{1t} n_{0t} d_t (n_t - d_t)}{n_t^2 (n_t - 1)}\right]} \\
&\sim \chi^2_1
\end{aligned}
$$

Data Structure and Organization: Non-Time-Varying

id	durat	censor	timein	timeout	X
1	4	0	30	34	0.12
2	2	1	12	14	0.19
3	5	1	5	10	0.09
...	...	...	...	...	...
N	10	1	21	31	0.22

Time-Varying Data

id	durat	censor	timein	timeout	X	Z
1	1	0	30	31	0.12	331
1	2	0	31	32	0.12	412
1	3	0	32	33	0.12	405
1	4	0	33	34	0.12	416
2	1	0	12	13	0.19	226
2	2	1	13	14	0.19	296
3	1	0	5	6	0.09	253
3	2	0	6	7	0.09	311
3	3	0	7	8	0.09	327
3	4	0	8	9	0.09	344
3	5	1	9	10	0.09	301
...	...	...	...	...	...	...

survival object (non-time-varying):

```
library(survival)
NonTV<-read.csv(NonTVdata.csv)
NonTV.S<-Surv(NonTV$duration, NonTV$censor)
```

survival object (time-varying):

```
TV<-read.csv(TVdata.csv)
TV.S<-Surv(TV$starttime, TV$endtime, TV$censor)
```

An Example

OECD Cabinet survival [Strom (1985); King et al. (1990)],

$N = 314$ cabinets in 15 countries

Outcome: Duration of cabinet, in months

Covariates (all non-time varying):

- *Fractionalization*
- *Polarization*
- *Formation Attempts*
- **Investiture**
- *Numerical Status*
- *Post-Election*
- *Caretaker*

Also: Indicator for whether the cabinet ended within 12 months of the end of the "constitutional inter-election period" ($\rightarrow$ censored)

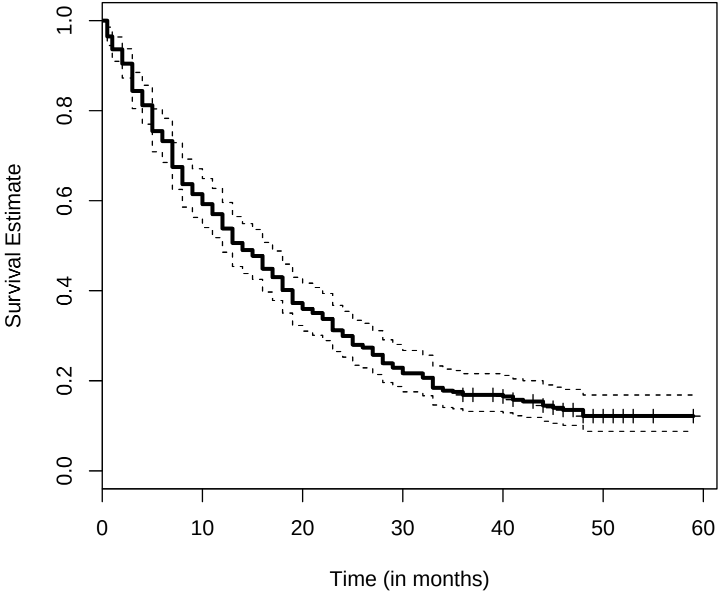
KABL Data

```
> head(KABL)
  id country durat ciep12 fract polar format invest numst2 eltime2 caretk2
1  1       1   0.5      1   656    11      3      1      0       1       0
2  2       1   3.0      1   656    11      2      1      1       0       0
3  3       1   7.0      1   656    11      5      1      1       0       0
4  4       1  20.0      1   656    11      2      1      1       0       0
5  5       1   6.0      1   656    11      3      1      1       0       0
6  6       1   7.0      1   634     6      4      1      1       1       0

> KABL.S<-Surv(KABL$durat,KABL$ciep12)

> KABL.S[1:50,]

 [1]  0.5   3.0   7.0  20.0   6.0   7.0   2.0  17.0  27.0  49.0+
[11]  4.0  29.0  49.0+  6.0  23.0  41.0+ 10.0  12.0   2.0  33.0
[21]  1.0  16.0   2.0   9.0   3.0   5.0   5.0   6.0  45.0+ 23.0
[31] 41.0   7.0  49.0+ 46.0   9.0  51.0+ 10.0  32.0  28.0   3.0
[41] 53.0+ 17.0  59.0+  9.0  52.0+  3.0  23.0  33.0   1.0  30.0
```

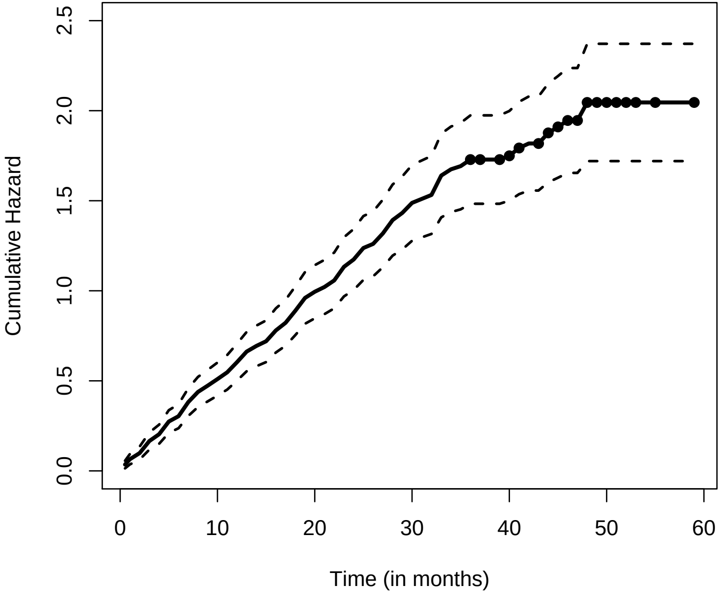
Example `survfit` Object

```
> KABL.fit<-survfit(KABL.S~1)

> str(KABL.fit)

List of 13
 $ n        : int 314
 $ time     : num [1:54] 0.5 1 2 3 4 5 6 7 8 9 ...
 $ n.risk   : num [1:54] 314 303 294 284 265 255 237 230 212 200 ...
 $ n.event  : num [1:54] 11 9 10 19 10 18 7 18 12 7 ...
 $ n.censor : num [1:54] 0 0 0 0 0 0 0 0 0 0 ...
 $ surv     : num [1:54] 0.965 0.936 0.904 0.844 0.812 ...
 $ type     : chr "right"
 $ std.err  : num [1:54] 0.0108 0.0147 0.0183 0.0243 0.0271 ...
 $ upper    : num [1:54] 0.986 0.964 0.938 0.885 0.856 ...
 $ lower    : num [1:54] 0.945 0.91 0.873 0.805 0.77 ...
 $ conf.type: chr "log"
 $ conf.int : num 0.95
 $ call     : language survfit(formula = KABL.S ~ 1)
 - attr(*, "class")= chr "survfit"
```

Plotting $\widehat{S(t)}$



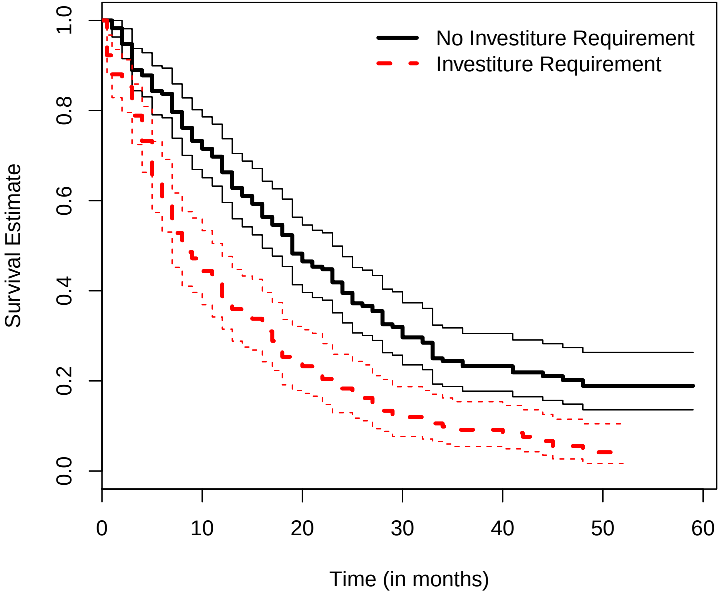
Plotting $\widehat{H(t)}$



Log-rank test:

```
> survdiff(KABL.S~invest,data=KABL,rho=0)

Call:
survdiff(formula = KABL.S ~ invest, data = KABL, rho = 0)

           N Observed Expected (O-E)^2/E (O-E)^2/V
invest=0 172      137    178.7      9.72      30.5
invest=1 142      134     92.3     18.81      30.5

 Chisq= 30.5  on 1 degrees of freedom, p= 3.26e-08
```

Comparing $\widehat{S(t)}$s



Parametric Survival Regression

$$f(t) = \lim_{\Delta t \to 0} \frac{\Pr(t \leq T < t + \Delta t)}{\Delta t}$$

$$\begin{aligned} S(t) &= \Pr(T \geq t) \\ &= 1 - \int_0^t f(t)\, dt \\ &= 1 - F(t) \end{aligned}$$

$$\begin{aligned} h(t) &= \frac{f(t)}{S(t)} \\ &= \lim_{\Delta t \to 0} \frac{\Pr(t \leq T < t + \Delta t | T \geq t)}{\Delta t} \end{aligned}$$

Likelihood

$$L = \prod_{i=1}^{N} \left[f(T_i)\right]^{C_i} \left[S(T_i)\right]^{1-C_i}$$

$$\ln L = \sum_{i=1}^{N} \left\{ C_i \ln\left[f(T_i)\right] + (1 - C_i) \ln\left[S(T_i)\right] \right\}$$

$$\ln L | \mathbf{X}, \beta = \sum_{i=1}^{N} \left\{ C_i \ln\left[f(T_i | \mathbf{X}, \beta)\right] + (1 - C_i) \ln\left[S(T_i | \mathbf{X}, \beta)\right] \right\}$$

The Exponential Model

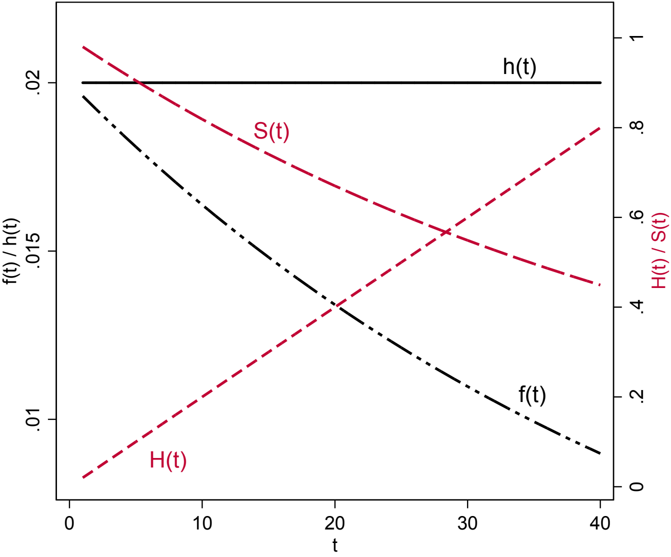
$$h(t) = \lambda$$

$$\begin{aligned} H(t) &= \int_0^t h(t)\,dt \\ &= \lambda t \end{aligned}$$

$$\begin{aligned} S(t) &= \exp[-H(t)] \\ &= \exp(-\lambda t) \end{aligned}$$

$$\begin{aligned} f(t) &= h(t)S(t) \\ &= \lambda \exp(-\lambda t) \end{aligned}$$

The Exponential Model, Illustrated



Exponential Model (continued)

Covariates:

$$\lambda_i = \exp(\mathbf{X}_i \beta).$$

$$S_i(t) = \exp(-e^{\mathbf{X}_i \beta} t).$$

Log-likelihood:

$$\begin{aligned}
\ln L &= \sum_{i=1}^{N} \left\{ C_i \ln \left[\exp(\mathbf{X}_i \beta) \exp(-e^{\mathbf{X}_i \beta} t) \right] + \right. \\
&\quad \left. (1 - C_i) \ln \left[\exp(-e^{\mathbf{X}_i \beta} t) \right] \right\} \\
&= \sum_{i=1}^{N} \left\{ C_i \left[(\mathbf{X}_i \beta)(-e^{\mathbf{X}_i \beta} t) \right] + (1 - C_i)(-e^{\mathbf{X}_i \beta} t) \right\}
\end{aligned}$$

Interpretation: Hazard Ratios

$$\text{HR}_k = \frac{\widehat{h(t)|X_k} = 1}{\widehat{h(t)|X_k} = 0}$$

$$h_i(t) = \exp(\beta_0)\exp(\mathbf{X}_i\beta)$$

$$\begin{aligned}
\text{HR}_k &= \frac{\widehat{h(t)|X_k} = 1}{\widehat{h(t)|X_k} = 0} \\
&= \frac{\exp(\hat{\beta}_0 + X_1\hat{\beta}_1 + ... + \hat{\beta}_k(1) + ...)}{\exp(\hat{\beta}_0 + X_1\hat{\beta}_1 + ... + \hat{\beta}_k(0) + ...)} \\
&= \frac{\exp(\hat{\beta}_k \times 1)}{\exp(\hat{\beta}_k \times 0)} \\
&= \exp(\hat{\beta}_k)
\end{aligned}$$

$$\begin{aligned}
\mathrm{HR}_k &= \frac{\hat{h}(t)|X_k + \delta}{\hat{h}(t)|X_k} \\
&= \exp(\delta\,\hat{\beta}_k)
\end{aligned}$$

$$\mathrm{HR}_{\frac{i}{j}} = \frac{\exp(\mathbf{X}_i\hat{\beta})}{\exp(\mathbf{X}_j\hat{\beta})}$$

Interpretation: Survival Rates

Predicted survival is:

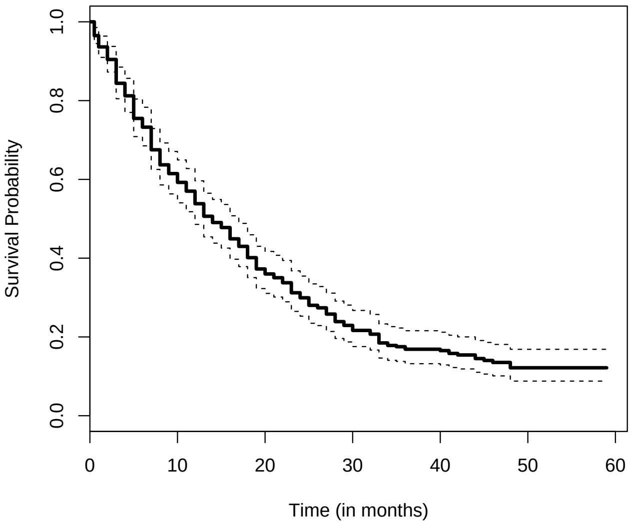
$$\widehat{S(t)} = \exp(-e^{\mathbf{X}_i\hat{\beta}}t).$$

So:

- Increases in hazards $\rightarrow$ decreases in survival times.
- *Proportional* increases in hazards *rightarrow proportional* decreases in survival times.
- Specifically, a one-unit increase in X implies a proportional change in the predicted survival time of:

$$1 - \exp(-\hat{\beta})$$

Cabinet Durations: Kaplan-Meier



```
> KABL.S<-Surv(KABL$durat,KABL$ciep12)
> xvars<-c("fract","polar","format","invest","numst2","eltime2","caretk2")
> MODEL<-as.formula(paste(paste("KABL.S ~ ", paste(xvars,collapse="+"))))
> KABL.exp<-phreg(MODEL,data=KABL,dist="weibull",shape=1)

> KABL.exp
Call:
phreg(formula = MODEL, data = KABL, dist = "weibull", shape = 1)

Covariate          W.mean      Coef Exp(Coef)  se(Coef)    Wald p
fract             692.734     0.001     1.001     0.001     0.198
polar              10.521     0.016     1.016     0.006     0.008
format              1.690     0.091     1.095     0.046     0.046
invest              0.332     0.369     1.447     0.139     0.008
numst2              0.713    -0.515     0.598     0.129     0.000
eltime2             0.665    -0.723     0.485     0.135     0.000
caretk2             0.009     1.300     3.671     0.260     0.000

log(scale)                    3.725               0.631     0.000

 Shape is fixed at  1

Events                     271
Total time at risk         5789.5
Max. log. likelihood       -1025.6
LR test statistic          150.21
Degrees of freedom         7
Overall p-value            0
```

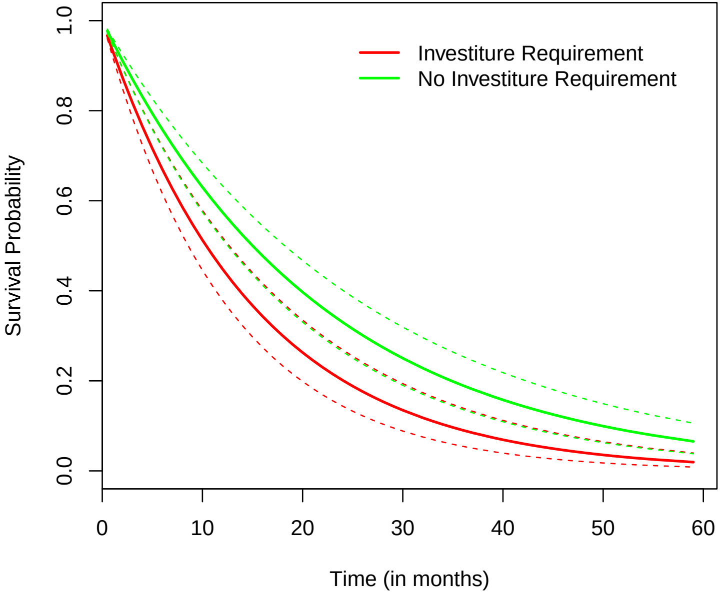
Hazard Ratios: Interpretation

- On average, an investiture requirement *increases* the *hazard* of cabinet failure by $100 \times (1.447 - 1) = 44.7$ percent.

- On average, an investiture requirement *decreases* the predicted *survival* time by

$$\begin{aligned} 100 \times [1 - \exp(-0.369)] &= 100 \times (1 - 0.691) \\ &= 30.1 \text{ percent.} \end{aligned}$$

Comparing Predicted Survival



The Weibull Model

Hazard is:

$$h(t) = \lambda p(\lambda t)^{p-1}$$

Survival function:

$$\begin{aligned} S(t) &= \exp\left[-\int_0^t \lambda p(\lambda t)^{p-1}\, dt\right] \\ &= \exp(-\lambda t)^p \end{aligned}$$

Density:

$$f(t) = \lambda p(\lambda t)^{p-1} \times \exp(-\lambda t)^p$$

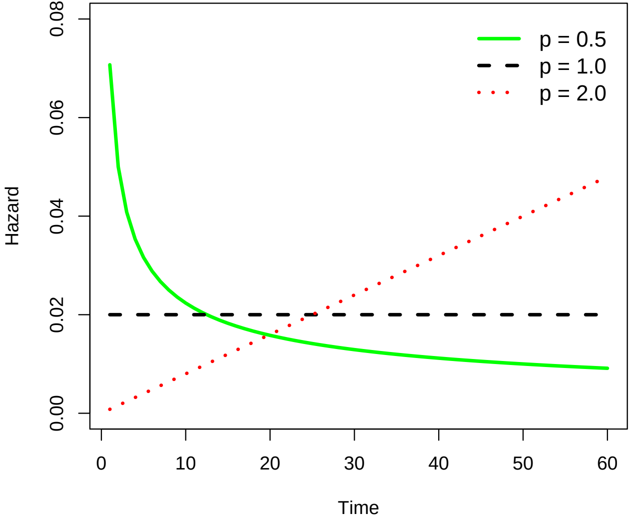
Covariates:

$$\lambda_i = \exp(\mathbf{X}_i \beta)$$

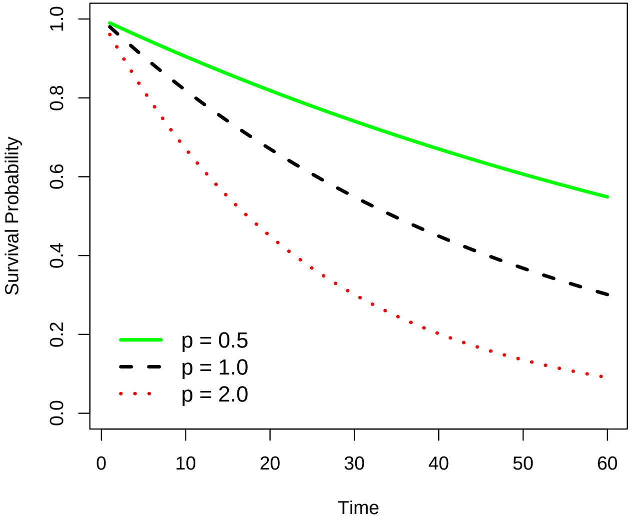
The Importance of p

- $p = 1 \rightarrow$ exponential model
- $p > 1 \rightarrow$ rising hazards
- $0 < p < 1 \rightarrow$ declining hazards

Weibull Hazards Illustrated



Weibull Survival



Weibull Example

```
> KABL.weib<-phreg(MODEL,data=KABL,dist="weibull")
> KABL.weib

Call:
phreg(formula = MODEL, data = KABL, dist = "weibull")

Covariate           W.mean      Coef Exp(Coef)  se(Coef)    Wald p
fract              692.734     0.001     1.001     0.001     0.133
polar               10.521     0.020     1.020     0.006     0.001
format               1.690     0.113     1.119     0.046     0.014
invest               0.332     0.429     1.535     0.139     0.002
numst2               0.713    -0.602     0.548     0.131     0.000
eltime2              0.665    -0.862     0.422     0.138     0.000
caretk2              0.009     1.710     5.530     0.276     0.000

log(scale)                     3.696               0.492     0.000
log(shape)                     0.261               0.050     0.000

Events                    271
Total time at risk        5789.5
Max. log. likelihood      -1013.5
LR test statistic         174.23
Degrees of freedom        7
Overall p-value           0
```

Weibull Example (continued)

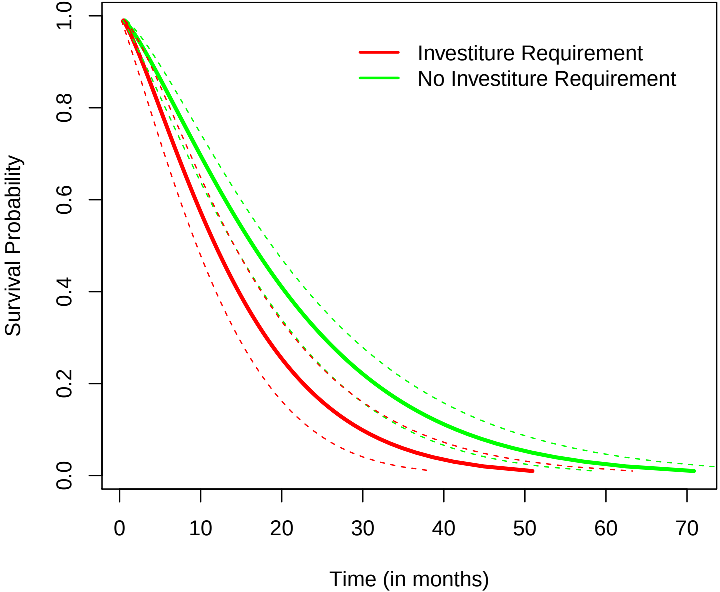
Interpretation:

- Hazard ratios are interpreted as in the exponential model, i.e., $100 \times \exp(\hat{\beta})$ is the percentage change in the hazard associated with a one-unit change in X

- So (e.g.) on average, an investiture requirement *increases* the *hazard* of cabinet failure by $100 \times (1.535 - 1) = 53.5$ percent

- $\widehat{\log(p)} = 0.261$, so $\hat{p} = \exp(0.261) = 1.30$, which implies that the estimated hazard of cabinet failure is *increasing* over time

Comparing Predicted Survival Curves



- Gompertz
- Lognormal / log-Logistic
- Rayleigh (Weibull w/$p = 2$)
- Logistic
- t
- Gamma + Generalized Gamma

R Packages: Parametric Survival Models

A (probably partial) list:

- `survreg` (in `survival`)
- `eha` package
- `rms` package
- `flexsurv` package
- `polspline` package
- `SurvRegCensCov` package (Weibull models)

Notes on *parametric* models with time-varying covariate data:

- Stata handles time-varying data with aplomb.
- R (generally) does not.
  - `survreg` (in the `survival` package) will not estimate models with time-varying data (it will not take a survival object of the form `Surv(start,stop,censor)`).
  - `psm` (in the `rms` package) will also not accept time-varying data.
  - `aftreg` and `phreg` (part of the eha package) will accept time-varying data. `phreg` accepts survival objects of the form `Surv(start,stop,censor)`. `aftreg` does as well, and notes in its documentation that "(I)f there are [sic] more than one spell per individual, it is essential to keep spells together by the `id` argument. This allows for time-varying covariates." In practice, this functions somewhat inconsistently.
- Recommendations: If you want to use R to fit parametric survival models with time-varying covariate data, stick with proportional hazards formulations, and use `phreg`. Also, Weibull models tend to be easier to fit than exponentials in this framework.

Cox's Proportional Hazards Model

Cox (1972)

Basic idea:

$$h_i(t) = h_0(t)\exp(\mathbf{X}_i\beta)$$

Note:

- $h_0(t) \equiv h(t|\mathbf{X} = 0)$
- Changes in $\mathbf{X}$ shift $h(t)$ *proportionally*

Cox (1972) (continued)

$$
\begin{aligned}
\text{HR} &= \frac{h_0(t)\exp(X_1\hat{\beta})}{h_0(t)\exp(X_0\hat{\beta})} \\
&= \exp[(1-0)\hat{\beta}] \\
&= \exp(\hat{\beta})
\end{aligned}
$$

Also, because

$$S(t) = \exp[-H(t)]$$

then

$$\begin{aligned} S(t) &= \exp\left[-\int_0^t h(t)\,dt\right] \\ &= \exp\left[-\exp(\mathbf{X}_i\beta)\int_0^t h_0(t)\,dt\right] \\ &= \left[\exp\left(-\int_0^t h_0(t)\,dt\right)\right]^{\exp(\mathbf{X}_i\beta)} \\ &= [S_0(t)]^{\exp(\mathbf{X}_i\beta)} \end{aligned}$$

Assume N_C distinct event times t_j, with no "ties."

Then:

$$\begin{aligned}
&\Pr(\text{Individual } k \text{ experienced the event at } t_j \mid \textit{One} \text{ observation experienced the event at } t_j) \\
&\quad = \frac{\Pr(\text{At-risk observation } k \text{ experiences the event of interest at } t_j)}{\Pr(\text{One at-risk observation experiences the event of interest at } t_j)} \\
&\quad = \frac{h_k(t_j)}{\sum_{\ell \in R_j} h_\ell(t_j)}
\end{aligned}$$

Partial Likelihood (continued)

$$
\begin{aligned}
L_i &= \frac{h_0(t_j)\exp(\mathbf{X}_i\beta)}{\sum_{\ell\in R_j} h_0(t_j)\exp(\mathbf{X}_\ell\beta)} \\
&= \frac{h_0(t_j)\exp(\mathbf{X}_i\beta)}{h_0(t_j)\sum_{\ell\in R_j}\exp(\mathbf{X}_\ell\beta)} \\
&= \frac{\exp(\mathbf{X}_i\beta)}{\sum_{\ell\in R_j}\exp(\mathbf{X}_\ell\beta)}
\end{aligned}
$$

$$
L = \prod_{i=1}^{N}\left[\frac{\exp(\mathbf{X}_i\beta)}{\sum_{\ell\in R_j}\exp(\mathbf{X}_\ell\beta)}\right]^{C_i}
$$

$$
\ln L = \sum_{i=1}^{N} C_i\left\{\mathbf{X}_i\beta - \ln\left[\sum_{\ell\in R_j}\exp(\mathbf{X}_\ell\beta)\right]\right\}
$$

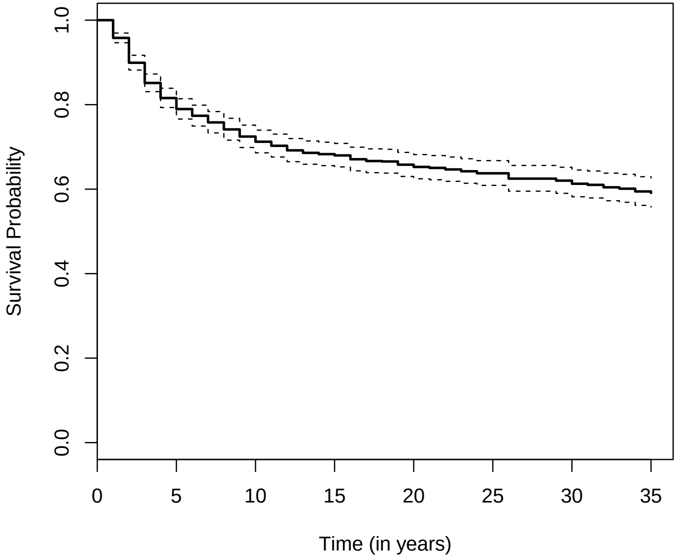
- PL is
  - Consistent
  - Asymptotically normal
  - Slightly inefficient (but asymptotically efficient)
- Considers <u>order</u> of events, but not actual <u>duration</u>
- Censored events: Modify R_j
- No <u>ties</u>

Example: Interstate War, 1950-1985

- Dyad-years for "politically-relevant" dyads
- $N = 827$, $NT = 20448$.
- Covariates:
  - Whether (=1) or not the two countries are *allies*,
  - Whether (=1) or not the two countries are *contiguous*,
  - The *capability ratio* of the two countries,
  - The lower of the two countries' (GDP) *growth* (rescaled),
  - The lower of the two countries' *democracy* (POLITY IV) scores (rescaled to [-1,1]), and
  - The amount of *trade* between the two countries, as a fraction of joint GDP.

```
> summary(OR)

    dyadid            year          start             stop            futime     
 Min.   :  2020   Min.   :1951   Min.   : 0.00   Min.   : 1.00   Min.   : 5.00  
 1st Qu.:100365   1st Qu.:1965   1st Qu.: 5.00   1st Qu.: 6.00   1st Qu.:23.00  
 Median :220235   Median :1972   Median :11.00   Median :12.00   Median :31.00  
 Mean   :253305   Mean   :1971   Mean   :12.32   Mean   :13.32   Mean   :28.97  
 3rd Qu.:365600   3rd Qu.:1979   3rd Qu.:19.00   3rd Qu.:20.00   3rd Qu.:35.00  
 Max.   :900920   Max.   :1985   Max.   :34.00   Max.   :35.00   Max.   :35.00  
   dispute            allies           contig           trade        
 Min.   :0.00000   Min.   :0.0000   Min.   :0.0000   Min.   :0.00000  
 1st Qu.:0.00000   1st Qu.:0.0000   1st Qu.:0.0000   1st Qu.:0.00000  
 Median :0.00000   Median :0.0000   Median :0.0000   Median :0.00020  
 Mean   :0.01981   Mean   :0.3563   Mean   :0.3099   Mean   :0.00231  
 3rd Qu.:0.00000   3rd Qu.:1.0000   3rd Qu.:1.0000   3rd Qu.:0.00120  
 Max.   :1.00000   Max.   :1.0000   Max.   :1.0000   Max.   :0.17680  
    growth             democracy          capratio      
 Min.   :-0.264900   Min.   :-1.0000   Min.   : 0.0100  
 1st Qu.:-0.004800   1st Qu.:-0.8000   1st Qu.: 0.0462  
 Median : 0.014700   Median :-0.7000   Median : 0.2220  
 Mean   : 0.007823   Mean   :-0.3438   Mean   : 1.6677  
 3rd Qu.: 0.027800   3rd Qu.: 0.2000   3rd Qu.: 1.1560  
 Max.   : 0.164700   Max.   : 1.0000   Max.   :78.9296  
```

The Data (Kaplan-Meier plot)



R:

- `coxph` in `survival` (preferred)
- `cph` in `design`
- Plots: `plot(survfit(PHobject))`

Stata:

- Basic command = `stcox`
- `stset` first
- Options: `robust`, various methods for ties, postestimation commands

```
> ORCox.br<-coxph(OR.S~allies+contig+capratio+growth+democracy+trade,
                  data=OR,na.action=na.exclude,method="breslow")

> summary(ORCox.br)

  n= 20448, number of events= 405

              coef exp(coef) se(coef)      z Pr(>|z|)
allies    -0.34849   0.70576  0.11096 -3.141 0.001686 **
contig     0.94861   2.58213  0.12173  7.793 6.55e-15 ***
capratio  -0.22303   0.80009  0.05164 -4.319 1.57e-05 ***
growth    -3.69487   0.02485  1.19950 -3.080 0.002068 **
democracy -0.38194   0.68254  0.09915 -3.852 0.000117 ***
trade     -3.22857   0.03961  9.45588 -0.341 0.732776
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
.
.
.
```

```
.
.
.
          exp(coef) exp(-coef) lower .95 upper .95
allies      0.70576     1.4169 5.678e-01 8.772e-01
contig      2.58213     0.3873 2.034e+00 3.278e+00
capratio    0.80009     1.2499 7.231e-01 8.853e-01
growth      0.02485    40.2402 2.368e-03 2.608e-01
democracy   0.68254     1.4651 5.620e-01 8.289e-01
trade       0.03961    25.2436 3.540e-10 4.433e+06

Concordance= 0.714  (se = 0.015 )
Rsquare= 0.01   (max possible= 0.234 )
Likelihood ratio test= 210.3  on 6 df,   p=0
Wald test            = 159.8  on 6 df,   p=0
Score (logrank) test = 185.8  on 6 df,   p=0
```

Interpretation: Hazard Ratios

$$HR = \exp[(\mathbf{X}_j - \mathbf{X}_k)\hat{\beta}]$$

Means:

- $HR = 1 \; \leftrightarrow \; \hat{\beta} = 0$
- $HR > 1 \; \leftrightarrow \; \hat{\beta} > 0$
- $HR < 1 \; \leftrightarrow \; \hat{\beta} < 0$

  Percentage difference $= 100 \times \{\exp[(\mathbf{X}_j - \mathbf{X}_k)\hat{\beta}] - 1\}$.

Example: Hazard Ratios

From above:

	exp(coef)	exp(-coef)	lower .95	upper .95
allies	0.70576	1.4169	5.678e-01	8.772e-01
contig	2.58213	0.3873	2.034e+00	3.278e+00
capratio	0.80009	1.2499	7.231e-01	8.853e-01
growth	0.02485	40.2402	2.368e-03	2.608e-01
democracy	0.68254	1.4651	5.620e-01	8.289e-01
trade	0.03961	25.2436	3.540e-10	4.433e+06

Interpretation:

- Countries which are *allies* have an expected $(0.706 - 1) \times 100) = 29.4$ percent lower hazard of conflict than those that are not.
- *Contiguous* countries have $(2.582 - 1) \times 100 = 158$ percent higher hazards of conflict than non-contiguous ones.
- A one-unit increase in *democracy* corresponds to a $(0.683 - 1) \times 100 = 31.7$ percent decrease in the expected hazard of conflict.

Hazard Ratios: Scaling Covariates

It is good for one-unit changes to be meaningful / realistic...

```
> OR$growthPct<-OR$growth*100
> summary(coxph(OR.S~allies+contig+capratio+growthPct+democracy+trade,
                data=OR,na.action=na.exclude, method="breslow"))
.
.
.
          exp(coef) exp(-coef) lower .95 upper .95
allies      0.70576     1.4169 5.678e-01 8.772e-01
contig      2.58213     0.3873 2.034e+00 3.278e+00
capratio    0.80009     1.2499 7.231e-01 8.853e-01
growthPct   0.96373     1.0376 9.413e-01 9.867e-01
democracy   0.68254     1.4651 5.620e-01 8.289e-01
trade       0.03961    25.2436 3.540e-10 4.433e+06
```

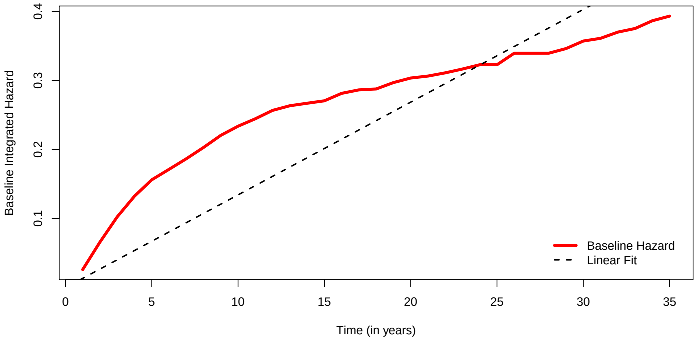
Note:

- Previous HR for `growth` $= 0.02485 \rightarrow$ 97.5 percent decrease in $\hat{h}(t)$
- HR for `growthPct` is now 0.964; 1 unit increase $\rightarrow$ 4% decrease in $\hat{h}(t)$
- Same result, proportionally: $0.96373^{100} = 0.02485$

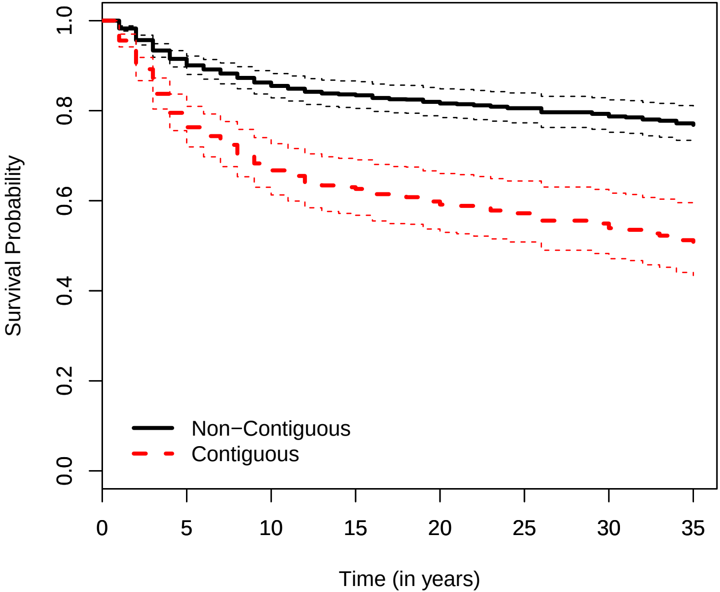
Baseline Hazards

Because the Cox model is semiparametric, it uses a conventional / univariate (Nelson-Aalen) estimate of the "baseline" hazard:

```
OR.BH<-basehaz(ORCox.br,centered=FALSE)
```



Comparing Survival Curves



Conceptual considerations:

- Theory
- Nature of $h(t)$
- Relative importance: Bias vs. efficiency
- Need / willingness for out-of-sample predictions / forecasting

Cox, On His Model

Reid: "What do you think of the cottage industry that's grown up around [the Cox model]?"

Cox: "In the light of further results one knows since, I think I would normally want to tackle the problem parametrically... I'm not keen on non-parametric formulations normally."

Reid: "So if you had a set of censored survival data today, you might rather fit a parametric model, even though there was a feeling among the medical statisticians that that wasn't quite right."

Cox: "That's right, but since then various people have shown that the answers are very insensitive to the parametric formulation of the underlying distribution. And if you want to do things like predict the outcome for a particular patient, it's much more convenient to do that parametrically."

– From Reid (1994).

Survival Model Variants and Extensions...

- Discrete-Time Models
- Stratification
- Cox Models for *repeated events*
- Models with “frailties”
- Competing risks
- Models for “cured” subpopulations
- Joint Models for Survival and Longitudinal Outcomes
- Complex sampling schemes
- Multilevel / spatial / etc. models for survival outcomes