PLSC 504 – Fall 2023 Panel/TSCS Data: Unit Effects (+ Dynamics)

October 4, 2023

- "Longitudinal" $\neq$ "Time Series"
- Terminology:
  - "Unit" / "Units" / "Units of observation" / "Panels" $=$ Things we observe repeatedly
  - "Observations" $=$ Each (one) measurement of a unit
  - "Time points" $=$ When each observation on a unit is made
  - $i \in \{1....N\}$ indexes units
  - $t \in \{1...T\}$ or $\{1...T_i\}$ indexes observations / time points
  - If $T_i = T \; \forall i$ then we have "balanced" panels / units
  - $NT =$ Total number of observations (if balanced)
- Averages:
  - Y_{it} indicates a variable that varies over both units and time,
  - $\bar{Y}_i = \frac{1}{T}\sum_{t=1}^{T} Y_{it} =$ the over-time mean of Y for i,
  - $\bar{Y}_t = \frac{1}{N}\sum_{i=1}^{N} Y_{it} =$ the across-unit mean of Y at t, and
  - $\bar{Y} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Y_{it} =$ the grand mean of Y.

More Terminology

- $N >> T \rightarrow$ "panel" data
  - NES panel studies ($N = 2000$, $T = 3$)
  - Panel Study of Income Dynamics ($N =$ large, $T \approx 12$)
- $T >> N$ or $T \approx N \rightarrow$ "time-series cross-sectional" ("TSCS") data
- $N = 1 \rightarrow$ "time series" data

Within- and Between-Unit Variation

Define:

$$\bar{Y}_i = \frac{1}{T_i} \sum_{t=1}^{T_i} Y_{it}$$

Then:

$$Y_{it} = \bar{Y}_i + (Y_{it} - \bar{Y}_i).$$

- The *total* variation in Y_{it} can be decomposed into
- The *between-unit* variation in the $\bar{Y}_i$s, and
- The *within-unit* variation around $\bar{Y}_i$ (that is, $Y_{it} - \bar{Y}_i$).

Model

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

assumes:

- All the usual OLS assumptions, plus
- $\beta_{0i} = \beta_0 \; \forall \; i$s
- $\beta_{1i} = \beta_1 \; \forall \; i$s

For:

$$Y_{it} = \beta_0 + \beta_1 X_{it} + u_{it}$$

... it's the same.

Variable Intercepts

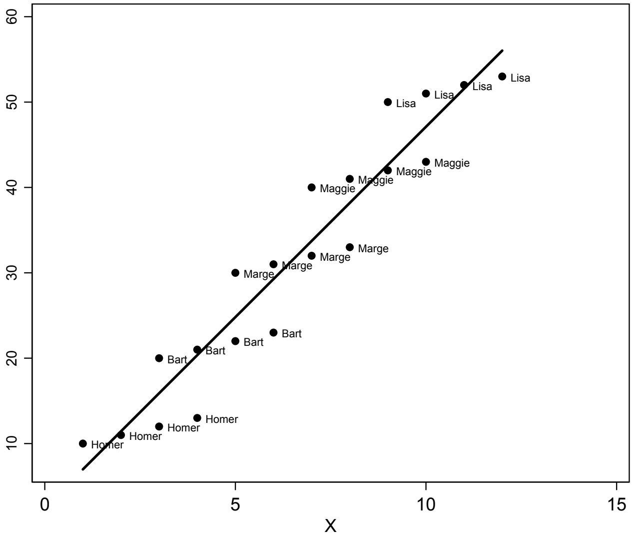
Relax that assumption:

$$Y_{it} = \beta_{0i} + \beta_1 X_{it} + u_{it} \qquad \text{(unit-level)}$$

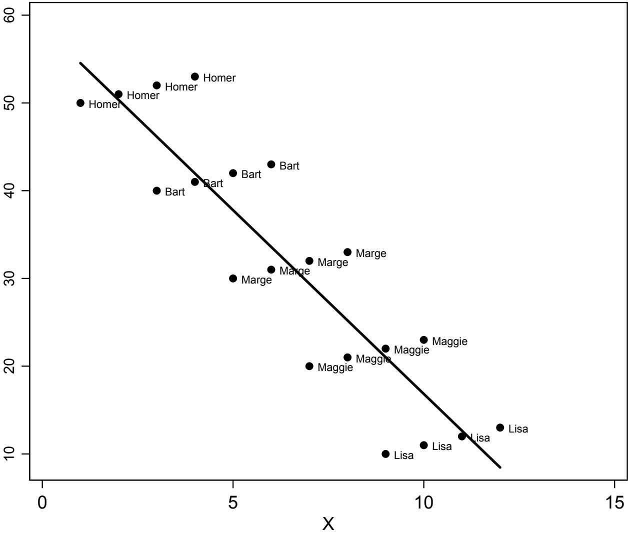
$$Y_{it} = \beta_{0t} + \beta_1 X_{it} + u_{it} \qquad \text{(time-level)}$$

$$Y_{it} = \beta_{0it} + \beta_1 X_{it} + u_{it} \qquad \text{(unit- and time-level)}$$

Varying Intercepts



Varying Intercepts



Varying Slopes (+ Intercepts)

Further relax:

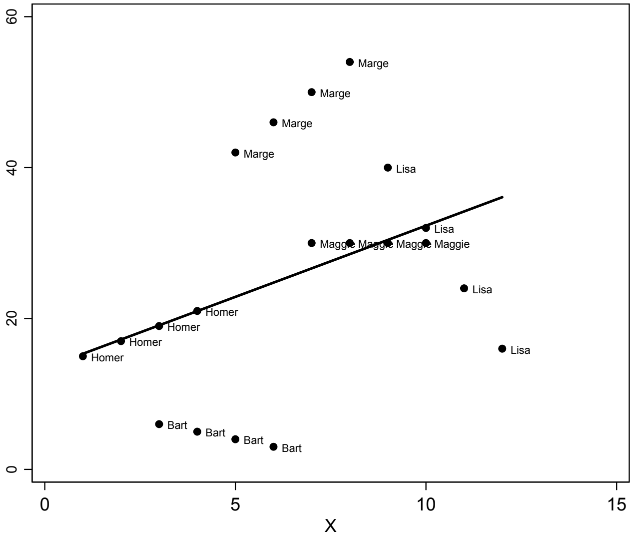
$$Y_{it} = \beta_0 + \beta_{1i} X_{it} + u_{it} \qquad \text{(unit-level slopes)}$$

$$Y_{it} = \beta_{0i} + \beta_{1i} X_{it} + u_{it} \qquad \text{(unit-level slopes and intercepts)}$$

$$Y_{it} = \beta_{0t} + \beta_{1t} X_{it} + u_{it} \qquad \text{(time-level slopes and intercepts)}$$

$$Y_{it} = \beta_{0it} + \beta_{1it} X_{it} + u_{it} \qquad \text{(unit- and time-level slopes and intercepts)}$$

Varying Slopes + Intercepts



"The usual" assumptions require:

$$u_{it} \sim \text{i.i.d.} N(0, \sigma^2) \; \forall \; i, t$$

This means that:

$$\begin{aligned}
\text{Var}(u_{it}) &= \text{Var}(u_{jt}) \; \forall \; i \neq j \text{ (i.e., no cross-unit heteroscedasticity)} \\
\text{Var}(u_{it}) &= \text{Var}(u_{is}) \; \forall \; t \neq s \text{ (i.e., no temporal heteroscedasticity)} \\
\text{Cov}(u_{it}, u_{js}) &= 0 \; \forall \; i \neq j, \; \forall \; t \neq s \text{ (i.e., no auto- or spatial correlation)}
\end{aligned}$$

Two-Way Variation

Two-way variation:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \gamma V_i + \delta W_t + u_{it}$$

where V_i don't vary over time (within a unit), and W_i don't vary across units (for a given time point).

Note that we can write:

$$\alpha_i = \sum(\gamma V_i)$$

and

$$\eta_t = \sum(\delta W_t).$$

So:

$$\begin{aligned} Y_{it} &= \mathbf{X}_{it}\boldsymbol{\beta} + \gamma V_i + \delta W_t + u_{it} \\ &= \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + \eta_t + u_{it} \end{aligned}$$

One- and Two-Way "Unit Effects"

"Two-way" unit effects:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + \eta_t + u_{it}$$

"One-way" effects:

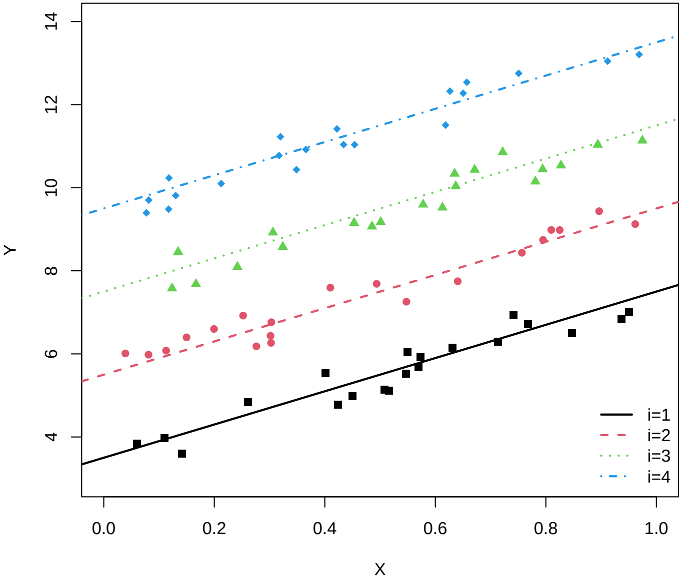
- Assuming $\alpha_i = 0$ (w.l.o.g):

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \eta_t + u_{it} \quad \text{(time)}$$

- Assuming $\eta_t = 0$ (w.l.o.g):

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + u_{it} \quad \text{(units)}$$

Intuition: One-Way Unit Effects



(One-Way) "Fixed" Effects

"Brute force" model fits:

$$\begin{aligned} Y_{it} &= \mathbf{X}_{it}\boldsymbol{\beta}_{FE} + \alpha_i + u_{it} \\ &= \mathbf{X}_{it}\boldsymbol{\beta}_{FE} + \alpha_1 I(i=1)_i + \alpha_2 I(i=2)_i + ... + u_{it} \end{aligned}$$

Alternatively, decompose:

$$\bar{\mathbf{X}}_i = \frac{\sum_{N_i} \mathbf{X}_{it}}{N_i}$$

and

$$\tilde{\mathbf{X}}_{it} = \mathbf{X}_{it} - \bar{\mathbf{X}}_i.$$

Yields:

$$Y_{it} = \bar{\mathbf{X}}_i\boldsymbol{\beta}_B + \tilde{\mathbf{X}}_{it}\boldsymbol{\beta}_W + \alpha_i + u_{it}$$

<u>But!</u>

$$\text{corr}(\bar{\mathbf{X}}_i\boldsymbol{\beta}_B, \alpha_i) = 1.0$$

Means that:

$$\begin{aligned} Y_{it}^{*} &= Y_{it} - \bar{Y}_{i} \\ \mathbf{X}_{it}^{*} &= \mathbf{X}_{it} - \bar{\mathbf{X}}_{i} \end{aligned}$$

gives:

$$Y_{it}^{*} = \mathbf{X}_{it}^{*}\boldsymbol{\beta}_{FE} + u_{it}.$$

$\rightarrow$ **A "Fixed Effects" Model is actually a "Within-Effects" Model.**

Standard F-test for

$$H_0 : \alpha_i = \alpha_j \forall i \neq j$$

versus

$$H_A : \alpha_i \neq \alpha_j \text{ for some } i \neq j$$

is $\sim F_{N-1, NT-(N-1)}$.

Running Example Data: WDI, 1960-2021

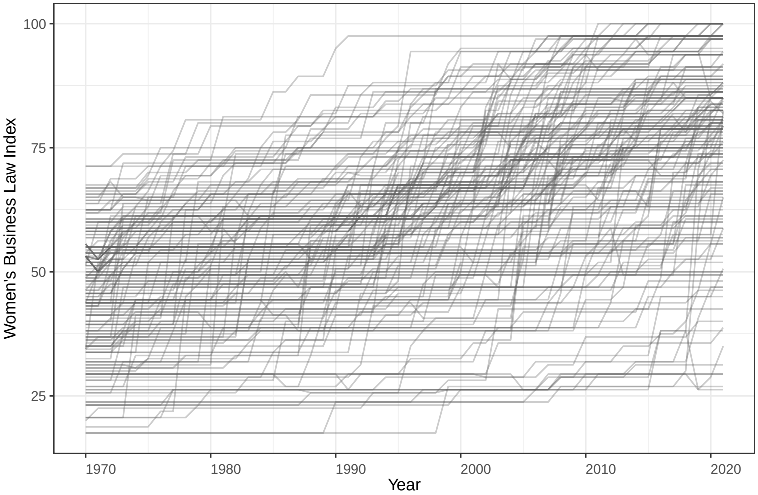
The World Development Indicators

- Cross-national country-level time series data
- $N = 215$ countries, $T = 72$ years (1960-2021) + missingness
- Variables:
  - Geography: land area, arable land
  - Population indicators
  - Demographics: Birth rates, life expectancy, etc.
  - Economics: GDP, inflation, trade, FDI, etc.
  - Governments: expenditures, policies, etc.
- Full descriptions are listed in the Github repo here

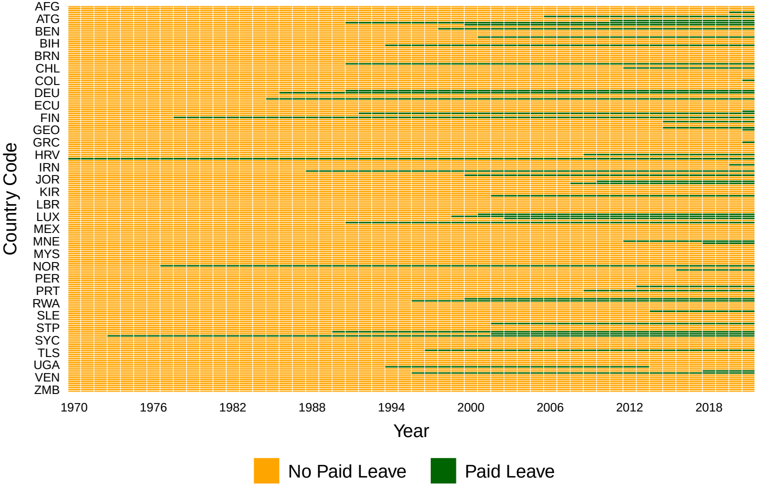
Data Summary

```
> describe(wdi,fast=TRUE,ranges=FALSE,check=TRUE)
                        vars     n            mean               sd             se
ISO3                       1 13330             NaN               NA             NA
Year                       2 13330         1990.50            17.90           0.16
Region                     3 13330             NaN               NA             NA
country                    4 13330             NaN               NA             NA
LandArea                   5 12906       613525.38       1766486.19       15549.43
ArablePercent              6 10935           13.51            13.49           0.13
Population                 7 13073     24638671.91     103129756.53      901978.87
PopGrowth                  8 12856            1.79             1.68           0.01
RuralPopulation            9 13045           48.61            25.74           0.23
UrbanPopulation           10 13045           51.39            25.74           0.23
BirthRatePer1K            11 12112           28.32            13.10           0.12
FertilityRate             12 11847            3.97             2.01           0.02
PrimarySchoolAge          13 10696            6.14             0.62           0.01
LifeExpectancy            14 11829           64.37            11.46           0.11
AgeDepRatioOld            15 11731           10.34             6.36           0.06
CO2Emissions              16  5535            4.28             5.41           0.07
GDP                       17  9585 242308268086.15 1101606170189.79 11252014966.83
GDPPerCapita              18  9582        11685.74         18675.05         190.78
GDPPerCapGrowth           19  9598            1.89             6.21           0.06
Inflation                 20  8275           23.88           332.39           3.65
TotalTrade                21  8363           78.18            54.14           0.59
Exports                   22  8363           36.41            28.85           0.32
Imports                   23  8372           41.78            27.87           0.30
FDIIn                     24  8195            5.57            45.42           0.50
AgriEmployment            25  5394           29.67            24.35           0.33
NetAidReceived            26  8633    453209476.19     867754118.55     9339331.98
MobileCellSubscriptions   27  9849           33.70            50.29           0.51
NaturalResourceRents      28  8745            6.61            10.87           0.12
MilitaryExpenditures      29  7333            2.74             3.23           0.04
GovtExpenditures          30  8012           16.20             8.19           0.09
HIVDeaths                 31  4041         7221.50         19857.71         312.38
WomenBusLawIndex          32  9776           59.23            18.46           0.19
PaidParentalLeave         33  9776            0.10             0.30           0.00
ColdWar                   34 13330            0.48             0.50           0.00
```

Visualization (using `panelView`)



Categorical Variable Visualization



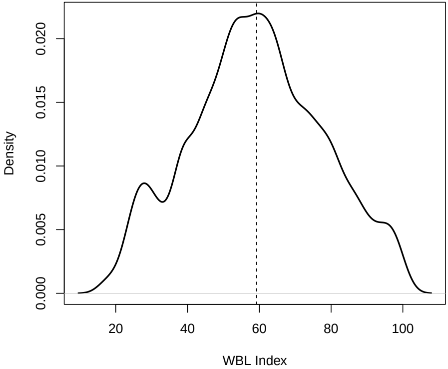
WDI's Women, Business and the Law Index (WBLI)

The basis for a 2021 World Bank report...

- Examines "the laws and regulations that affect women's economic opportunity in 190 economies."
- An index comprising eight indicators "structured around women's interactions with the law as they move through their careers: *Mobility*, *Workplace*, *Pay*, *Marriage*, *Parenthood*, *Entrepreneurship*, *Assets*, and *Pension*."
- The WBL Index:
  - Theoretically ranges from 0 - 100
  - In practice: Lowest values $\approx$ 20
  - Higher values correspond to higher levels of women's empowerment and greater opportunities and support for women, particularly in business
- "Better performance in the areas measured by the Women, Business and the Law index is associated with a more narrow gender gap in development outcomes, higher female labor force participation, lower vulnerable employment, and greater representation of women in national parliaments."

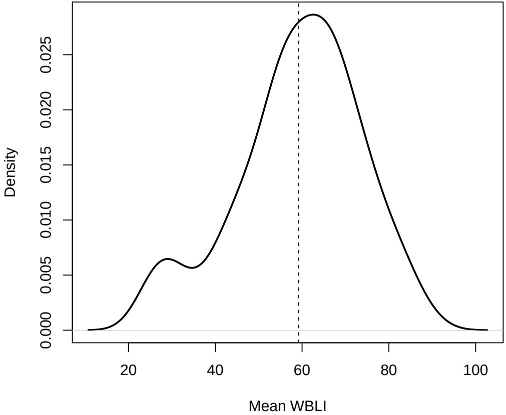
WBLI: Total Variation

```
> WDI<-pdata.frame(wdi)
> WBLI<-WDI$WomenBusLawIndex
> class(WBLI)
[1] "pseries" "numeric"

> describe(WBLI,na.rm=TRUE) # all variation

   vars    n mean   sd median trimmed  mad  min max range skew kurtosis   se
X1    1 9776 59.2 18.5   58.8    59.2 18.5 17.5 100  82.5 0.04    -0.55 0.19
```



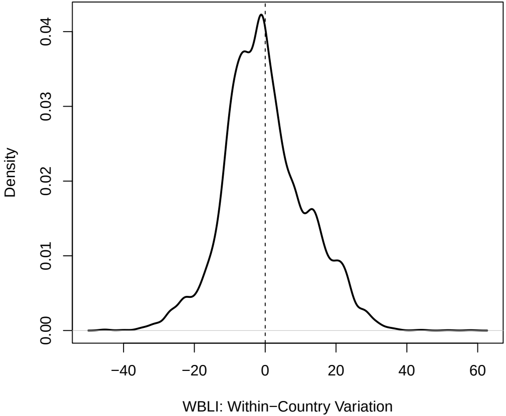
WBLI: “Between” Variation

```
> describe(plm::between(WBLI,effect="individual",na.rm=TRUE)) # "between" variation

   vars   n mean   sd median trimmed  mad  min max range  skew kurtosis   se
X1    1 188 59.2 14.4   60.5    60.1 13.3 23.4  90  66.6 -0.47    -0.13 1.05
```



WBLI: "Within" Variation

```
> describe(Within(WBLI,na.rm=TRUE)) # "within" variation

   vars    n mean   sd median trimmed  mad   min max range skew kurtosis   se
X1    1 9776    0 11.5  -0.93   -0.33 10.4 -45.3  58   103 0.26     0.45 0.12
```



A Regression Model

Regression model:

$$\begin{aligned}\text{WBLI}_{it} &= \beta_0 + \beta_1\text{Population Growth}_{it} + \beta_2\text{Urban Population}_{it} + \beta_3\text{Fertility Rate}_{it} + \\ &\quad \beta_4\ln(\text{GDP Per Capita})_{it} + \beta_5\text{Natural Resource Rents}_{it} + \beta_6\text{Cold War}_t + u_{it}\end{aligned}$$

Descriptive Statistics:

	vars	n	mean	sd	min	max	range	se
WomenBusLawIndex	1	7566	59.99	18.79	17.50	100.00	82.50	0.22
PopGrowth	2	7566	1.70	1.46	-6.77	17.51	24.28	0.02
UrbanPopulation	3	7566	51.19	23.90	2.85	100.00	97.16	0.27
FertilityRate	4	7566	3.67	1.91	0.90	8.61	7.70	0.02
NaturalResourceRents	5	7566	6.82	10.45	0.00	87.51	87.51	0.12
ColdWar	6	7566	0.32	0.47	0.00	1.00	1.00	0.01
lnGDPPerCap	7	7566	8.28	1.44	5.32	11.63	6.31	0.02

Regression: Pooled OLS

```
> OLS<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+                  log(GDPPerCapita)+NaturalResourceRents+ColdWar,
+         data=WDI,model="pooling")

> summary(OLS)
Pooling Model

Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    ColdWar, data = WDI, model = "pooling")

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Residuals:
   Min. 1st Qu.  Median 3rd Qu.    Max.
 -46.94   -8.02    1.05    9.30   49.73

Coefficients:
                     Estimate Std. Error t-value    Pr(>|t|)
(Intercept)           54.8395     1.7261   31.77     < 2e-16 ***
PopGrowth             -3.1926     0.1437  -22.21     < 2e-16 ***
UrbanPopulation       -0.0584     0.0109   -5.37 0.000000083 ***
FertilityRate         -1.7928     0.1652  -10.85     < 2e-16 ***
log(GDPPerCapita)      3.1544     0.1993   15.83     < 2e-16 ***
NaturalResourceRents  -0.3486     0.0162  -21.54     < 2e-16 ***
ColdWar              -11.3437     0.3716  -30.53     < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Total Sum of Squares:    2670000
Residual Sum of Squares: 1280000
R-Squared:      0.519
Adj. R-Squared: 0.519
F-statistic: 1361.01 on 6 and 7559 DF, p-value: <2e-16
```

“Fixed” (Within) Effects

```
> FE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+          log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+          effect="individual",model="within")

> summary(FE)
Oneway (individual) effect Within Model

Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    ColdWar, data = WDI, effect = "individual", model = "within")

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Residuals:
   Min. 1st Qu.  Median 3rd Qu.    Max.
-33.144  -4.832  -0.406   4.802  41.184

Coefficients:
                     Estimate Std. Error t-value Pr(>|t|)
PopGrowth              0.0735     0.1194    0.62    0.538
UrbanPopulation        0.2482     0.0205   12.09   <2e-16 ***
FertilityRate         -2.0657     0.1657  -12.47   <2e-16 ***
log(GDPPerCapita)      9.1607     0.3104   29.51   <2e-16 ***
NaturalResourceRents   0.0353     0.0182    1.94    0.052 .
ColdWar               -7.1920     0.2951  -24.37   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Total Sum of Squares:    934000
Residual Sum of Squares: 434000
R-Squared:      0.535
Adj. R-Squared: 0.523
F-statistic: 1414.37 on 6 and 7374 DF, p-value: <2e-16
```

Table: Models of WBLI

	OLS	FE
Population Growth	-3.190^{***}	0.073
	(0.144)	(0.119)
Urban Population	-0.058^{***}	0.248^{***}
	(0.011)	(0.021)
Fertility Rate	-1.790^{***}	-2.070^{***}
	(0.165)	(0.166)
ln(GDP Per Capita)	3.150^{***}	9.160^{***}
	(0.199)	(0.310)
Natural Resource Rents	-0.349^{***}	0.035^{*}
	(0.016)	(0.018)
Cold War	-11.300^{***}	-7.190^{***}
	(0.372)	(0.295)
Constant	54.800^{***}	
	(1.730)	
Observations	7,566	7,566
R^2	0.519	0.535
Adjusted R^2	0.519	0.523
F Statistic	$1{,}361.000^{***}$ (df = 6; 7559)	$1{,}414.000^{***}$ (df = 6; 7374)

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Time-Period Fixed Effects

The model is:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \eta_t + u_{it}$$

which is estimated via:

$$\begin{aligned} Y_{it}^{**} &= Y_{it} - \bar{Y}_t \\ \mathbf{X}_{it}^{**} &= \mathbf{X}_{it} - \bar{\mathbf{X}}_t \end{aligned}$$

$$Y_{it}^{**} = \boldsymbol{\beta}_{FE}\mathbf{X}_{it}^{**} + u_{it}.$$

Comparison: Unit vs. Time Fixed Effects

Table: FE Models of WBLI (Units vs. Time)

	FE.Units	FE.Time
Population Growth	0.073	-3.630^{***}
	(0.119)	(0.135)
Urban Population	0.248^{***}	-0.054^{***}
	(0.021)	(0.010)
Fertility Rate	-2.070^{***}	-0.657^{***}
	(0.166)	(0.159)
ln(GDP Per Capita)	9.160^{***}	3.550^{***}
	(0.310)	(0.187)
Natural Resource Rents	0.035^{*}	-0.400^{***}
	(0.018)	(0.015)
Cold War	-7.190^{***}	
	(0.295)	
Observations	7,566	7,566
R^2	0.535	0.439
Adjusted R^2	0.523	0.435
F Statistic	$1{,}414.000^{***}$ (df = 6; 7374)	$1{,}175.000^{***}$ (df = 5; 7511)

*p<0.1; **p<0.05; ***p<0.01

The specification:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + u_{it}$$

...suggests that we can use an F-test to examine the hypothesis:

$$H_0 : \alpha_i = 0 \; \forall \; i$$

(and a similar test for $\eta_t = 0$ in the time-centered case).

Arguably better are Lagrange multiplier-based tests:

- Breusch-Pagan (1980)
- King and Wu (1997)
- See (e.g.) Croissant and Millo (2018, §4.1) for details

FE (Country) Model Tests

```
> pFtest(FE,OLS)

F test for individual effects

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
F = 78, df1 = 185, df2 = 7374, p-value <2e-16
alternative hypothesis: significant effects


> plmtest(FE,effect=c("individual"),type=c("bp"))

Lagrange Multiplier Test - (Breusch-Pagan) for unbalanced panels

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
chisq = 44635, df = 1, p-value <2e-16
alternative hypothesis: significant effects


> plmtest(FE,effect=c("individual"),type=c("kw"))

Lagrange Multiplier Test - (King and Wu) for unbalanced panels

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
normal = 211, p-value <2e-16
alternative hypothesis: significant effects
```

Same For Time Effects

```
> pFtest(FE.Time,OLS)

F test for time effects

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
F = 25, df1 = 48, df2 = 7511, p-value <2e-16
alternative hypothesis: significant effects


> plmtest(FE.Time,effect=c("time"),type=c("bp"))

Lagrange Multiplier Test - time effects (Breusch-Pagan) for unbalanced panels

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
chisq = 9920, df = 1, p-value <2e-16
alternative hypothesis: significant effects


> plmtest(FE.Time,effect=c("time"),type=c("kw"))

Lagrange Multiplier Test - time effects (King and Wu) for unbalanced panels

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
normal = 100, p-value <2e-16
alternative hypothesis: significant effects
```

Fixed Effects: Interpretation

FE models are *subject-specific* models, that rely entirely on *within-unit* variability to estimate variable effects.

- This means that:

$$\hat{\beta}_k = \frac{\partial E(Y|\hat{\alpha})}{\partial X_k}$$

- That is, $\hat{\beta}_k$ is *the expected change in E(Y) associated with a one-unit increase in* <u>*observation i's*</u> *value of* X_k
- Key: *within-unit* changes in **X** are associated with *within-unit* expected changes in Y.
- In a linear model, the value of $\hat{\alpha}$ doesn't affect the value of that partial derivative...

Fixed Effects: Interpretation

Mummolo and Peterson (2018) note that:

> *"...because the within-unit variation is always smaller (or at least, no larger) than the overall variation in the independent variable, researchers should use within-unit variation to motivate counterfactuals when discussing the substantive impact of a treatment" (2018, 829).*

Significance:

- Predictors **X** in FE models typically have both cross-sectional and temporal variation
- FE models only consider *within-unit* variation in **X** and Y
- As a result, the degree of actual/observed (within-unit) variation in predictors is almost always less – and sometimes significantly less – than if cross-sectional variation were also considered

Interpretation Example: Urban Population

`UrbanPopulation` – All Variation:

```
> with(WDI, sd(UrbanPopulation,na.rm=TRUE)) # all variation
[1] 25.7
```

`UrbanPopulation` – "Within" Variation:

```
> WDI<-ddply(WDI, .(ISO3), mutate,
+                 UPMean = mean(UrbanPopulation,na.rm=TRUE))
> WDI$UPWithin<-with(WDI, UrbanPopulation-UPMean)
>
> with(WDI, sd(UPWithin,na.rm=TRUE)) # "within" variation
[1] 8.86
```

> *"While the overall variation in the independent variable may be large, the within-unit variation used to estimate β may be much smaller" (M & P 2018, 830).*

Pros and Cons of "Fixed" Effects

Pros:

- Mitigates (Some) Specification Bias
- Simple + Intuitive
- Widely Used/Understood

Cons (see e.g. Collischon and Eberl 2020):

- Can't Estimate $\boldsymbol{\beta}_B$
- Slowly-Changing **X**s
- (In)Efficiency / Inconsistency (Incidental Parameters)
- Cannot Control for (e.g.) Time-Varying Heterogeneity
- Sensitivity to Measurement Error

“Between” Effects

From the equation above:

$$Y_{it} = \bar{\mathbf{X}}_i \boldsymbol{\beta}_B + \tilde{\mathbf{X}}_{it} \boldsymbol{\beta}_W + \alpha_i + u_{it}$$

...we can derive a “Between Effects” model:

$$\bar{Y}_i = \bar{\mathbf{X}}_i \boldsymbol{\beta}_B + u_{it}$$

This model:

- is essentially cross-sectional,
- is based on N observations,
- considers *only* between-unit (average) differences
- Interpretation:

  $\hat{\beta}_B$ *is the expected difference in* Y *between two units whose values on* $\bar{X}$ *differ by a value of 1.0.*

“Between” Effects

```
> BE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+                  log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+          effect="individual",model="between")

> summary(BE)
Oneway (individual) effect Between Model

Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    ColdWar, data = WDI, effect = "individual", model = "between")

Unbalanced Panel: n = 186, T = 1-50, N = 7566
Observations used in estimation: 186

Residuals:
   Min. 1st Qu.  Median 3rd Qu.    Max.
-30.423  -6.319   0.332   8.067  22.183

Coefficients:
                     Estimate Std. Error t-value     Pr(>|t|)
(Intercept)           50.3106    10.1621    4.95 0.0000016996 ***
PopGrowth             -5.8648     0.9173   -6.39 0.0000000014 ***
UrbanPopulation       -0.0497     0.0549   -0.91        0.366
FertilityRate          0.0633     1.0766    0.06        0.953
log(GDPPerCapita)      3.4052     1.1308    3.01        0.003 **
NaturalResourceRents  -0.3579     0.0888   -4.03 0.0000823194 ***
ColdWar              -12.5095     4.8879   -2.56        0.011 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Total Sum of Squares:    44700
Residual Sum of Squares: 17600
R-Squared:      0.605
Adj. R-Squared: 0.592
F-statistic: 45.7424 on 6 and 179 DF, p-value: <2e-16
```

A Nicer Table (Again)

Table: Models of WBLI

	OLS	FE	BE
Population Growth	-3.190^{***}	0.073	-5.870^{***}
	(0.144)	(0.119)	(0.917)
Urban Population	-0.058^{***}	0.248^{***}	-0.050
	(0.011)	(0.021)	(0.055)
Fertility Rate	-1.790^{***}	-2.070^{***}	0.063
	(0.165)	(0.166)	(1.080)
ln(GDP Per Capita)	3.150^{***}	9.160^{***}	3.400^{***}
	(0.199)	(0.310)	(1.130)
Natural Resource Rents	-0.349^{***}	0.035^{*}	-0.358^{***}
	(0.016)	(0.018)	(0.089)
Cold War	-11.300^{***}	-7.190^{***}	-12.500^{**}
	(0.372)	(0.295)	(4.890)
Constant	54.800^{***}		50.300^{***}
	(1.730)		(10.200)
Observations	7,566	7,566	186
R^2	0.519	0.535	0.605
Adjusted R^2	0.519	0.523	0.592
F Statistic	$1,361.000^{***}$ (df = 6; 7559)	$1,414.000^{***}$ (df = 6; 7374)	45.700^{***} (df = 6; 179)

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Model:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + u_{it}$$

with:

$$u_{it} = \alpha_i + \lambda_t + \eta_{it}$$

and

$$\begin{aligned}
E(\alpha_i) = E(\lambda_t) = E(\eta_{it}) &= 0, \\
E(\alpha_i\lambda_t) = E(\alpha_i\eta_{it}) = E(\lambda_t\eta_{it}) &= 0, \\
E(\alpha_i\alpha_j) &= \sigma^2_\alpha \text{ if } i = j, \ 0 \text{ otherwise}, \\
E(\lambda_t\lambda_s) &= \sigma^2_\lambda \text{ if } t = s, \ 0 \text{ otherwise}, \\
E(\eta_{it}\eta_{js}) &= \sigma^2_\eta \text{ if } i = j, \ t = s, \ 0 \text{ otherwise}, \\
E(\alpha_i\mathbf{X}_{it}) = E(\lambda_t\mathbf{X}_{it}) = E(\eta_{it}\mathbf{X}_{it}) &= 0.
\end{aligned}$$

If those assumptions are met, we can consider the "two-way variance components" model where:

$$\begin{aligned} Var(u_{it}) &= Var(Y_{it}|\mathbf{X}_{it}) \\ &= \sigma^2_\alpha + \sigma^2_\lambda + \sigma^2_\eta \end{aligned}$$

If we assume $\lambda_t = 0$, then we get a model like:

$$Y_{it} = \mathbf{X}_{it}\beta + \alpha_i + \eta_{it}$$

with total error variance:

$$\sigma^2_u = \sigma^2_\alpha + \sigma^2_\eta.$$

"Random" Effects: Estimation

The model above will violate the standard OLS assumptions of uncorrelated errors, because the (compound) "errors" u_{it} within each unit share a common component α_i.

Consider the within-i variance-covariance matrix of the errors $\mathbf{u}$:

$$\begin{aligned}
E(\mathbf{u}_i\mathbf{u}_i') \equiv \mathbf{\Sigma}_i &= \sigma_\eta^2 \mathbf{I}_T + \sigma_\alpha^2 \mathbf{i}\mathbf{i}' \\
&= \begin{pmatrix} \sigma_\eta^2 + \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_\eta^2 + \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\eta^2 + \sigma_\alpha^2 \end{pmatrix}
\end{aligned}$$

Assuming conditional independence across units, we then have:

$$\text{Var}(\mathbf{u}) \equiv \mathbf{\Omega} = \begin{pmatrix} \mathbf{\Sigma}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{\Sigma}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{\Sigma}_N \end{pmatrix}$$

"Random" Effects: Estimation

We can then show that:

$$\boldsymbol{\Sigma}^{-1/2} = \frac{1}{\sigma_\eta}\left[\mathbf{I}_T - \left(\frac{\theta}{T}\mathbf{ii}'\right)\right]$$

where

$$\theta = 1 - \sqrt{\frac{\sigma_\eta^2}{T\sigma_\alpha^2 + \sigma_\eta^2}}$$

is an unknown quantity to be estimated.

Starting with an estimate of $\hat{\theta}$, calculate:

$$\begin{aligned} Y_{it}^* &= Y_{it} - \hat{\theta}\bar{Y}_i \\ X_{it}^* &= X_{it} - \hat{\theta}\bar{X}_i, \end{aligned}$$

then estimate:

$$Y_{it}^* = (1-\hat{\theta})\alpha + X_{it}^*\beta_{RE} + [(1-\hat{\theta})\alpha_i + (\eta_{it} - \hat{\theta}\bar{\eta}_i)]$$

and iterate between the two processes until convergence.

“Random” Effects: An Alternative View

Random Effects

```
> RE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+                  log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+          effect="individual",model="random")

> summary(RE)
Oneway (individual) effect Random Effect Model
   (Swamy-Arora's transformation)

Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    ColdWar, data = WDI, effect = "individual", model = "random")

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Effects:
                var std.dev share
idiosyncratic 58.91    7.68   0.4
individual    88.75    9.42   0.6
theta:
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.368   0.872   0.886   0.876   0.886   0.886

Residuals:
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  -33.3    -5.2    -0.4     0.0     5.5    40.9

Coefficients:
                     Estimate Std. Error z-value       Pr(>|z|)
(Intercept)           2.56506    2.58528    0.99           0.32
PopGrowth            -0.07305    0.12134   -0.60           0.55
UrbanPopulation       0.12838    0.01920    6.69 0.000000000023 ***
FertilityRate        -2.37359    0.16536  -14.35        < 2e-16 ***
log(GDPPerCapita)     7.51273    0.29275   25.66        < 2e-16 ***
NaturalResourceRents  0.00581    0.01820    0.32           0.75
ColdWar              -8.48014    0.28934  -29.31        < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Total Sum of Squares:    982000
Residual Sum of Squares: 471000
R-Squared:      0.52
Adj. R-Squared: 0.52
```

A Nicer Table (Yet Again)

Table: Models of WBLI

	OLS	FE	BE	RE
Population Growth	-3.190^{***}	0.073	-5.870^{***}	-0.073
	(0.144)	(0.119)	(0.917)	(0.121)
Urban Population	-0.058^{***}	0.248^{***}	-0.050	0.128^{***}
	(0.011)	(0.021)	(0.055)	(0.019)
Fertility Rate	-1.790^{***}	-2.070^{***}	0.063	-2.370^{***}
	(0.165)	(0.166)	(1.080)	(0.165)
ln(GDP Per Capita)	3.150^{***}	9.160^{***}	3.400^{***}	7.510^{***}
	(0.199)	(0.310)	(1.130)	(0.293)
Natural Resource Rents	-0.349^{***}	0.035^{*}	-0.358^{***}	0.006
	(0.016)	(0.018)	(0.089)	(0.018)
Cold War	-11.300^{***}	-7.190^{***}	-12.500^{**}	-8.480^{***}
	(0.372)	(0.295)	(4.890)	(0.289)
Constant	54.800^{***}		50.300^{***}	2.560
	(1.730)		(10.200)	(2.580)
Observations	7,566	7,566	186	7,566
R^2	0.519	0.535	0.605	0.520
Adjusted R^2	0.519	0.523	0.592	0.520
F Statistic	$1{,}361.000^{***}$ (df = 6; 7559)	$1{,}414.000^{***}$ (df = 6; 7374)	45.700^{***} (df = 6; 179)	$7{,}910.000^{***}$

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

"Random" Effects: Testing

Intuition:

- RE models require that $\text{Cov}(X_{it}, \alpha_i) = 0$.
- FE models do not.

This means that:

	Reality	
Model	$\text{Cov}(X_{it}, \alpha_i) = 0$	$\text{Cov}(X_{it}, \alpha_i) \neq 0$
Fixed Effects	Consistent, Inefficient	Consistent, Efficient
Random Effects	Consistent, Efficient	Inconsistent

The Hausman Test

Hausman test (FE vs. RE):

$$\hat{W} = (\hat{\boldsymbol{\beta}}_{\mathbf{FE}} - \hat{\boldsymbol{\beta}}_{\mathbf{RE}})'(\hat{\mathbf{V}}_{\mathbf{FE}} - \hat{\mathbf{V}}_{\mathbf{RE}})^{-1}(\hat{\boldsymbol{\beta}}_{\mathbf{FE}} - \hat{\boldsymbol{\beta}}_{\mathbf{RE}})$$

$$W \sim \chi^2_k$$

Null: The RE model is consistent ($\text{Cov}(X_{it}, \alpha_i) = 0$).

Issues:

- Asymptotic
- No guarantee $(\hat{\mathbf{V}}_{\mathbf{FE}} - \hat{\mathbf{V}}_{\mathbf{RE}})^{-1}$ is positive definite
- A general specification test...

Hausman Test Results

Hausman test (FE vs. RE):

```
> phtest(FE, RE)

Hausman Test

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
chisq = 946, df = 6, p-value <2e-16

alternative hypothesis: one model is inconsistent
```

Practical “Fixed” vs. “Random” Effects

Factors to consider:

- “Panel” vs. “TSCS” Data
- Nature of the Data-Generating Processes
- Importance of Non-Time-Varying Covariate Effects
- Dimension/Location of the Variance in the Data (N vs. T)

Connections: Hierarchical Linear Models

HLM Starting Points

Begin by considering a two-level "nested" data structure, with:

$$\begin{aligned} i &\in \{1, 2, ...N\} \text{ indexing first-level units, and} \\ j &\in \{1, 2, ...J\} \text{ indexing second-level groups.} \end{aligned}$$

A general two-level HLM is an equation of the form:

$$Y_{ij} = \beta_{0j} + \mathbf{X}_{ij}\boldsymbol{\beta}_j + u_{ij} \quad (1)$$

where β_{0j} is a "constant" term, $\mathbf{X}_{ij}$ is a $NJ \times K$ matrix of K covariates, $\boldsymbol{\beta}_j$ is a $K \times 1$ vector of parameters, and $u_{ij} \sim$ i.i.d. $N(0, \sigma_u^2)$ is the usual random-disturbance assumption.

Each of the $K+1$ "level-one" parameters is then allowed to vary across Q "level-two" variables $\mathbf{Z}_j$, so that:

$$\beta_{0j} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \varepsilon_{0j} \tag{2}$$

for the "intercept" and

$$\beta_{kj} = \gamma_{k0} + \mathbf{Z}_j\gamma_k + \varepsilon_{kj} \tag{3}$$

for the "slopes" of $\mathbf{X}$. The εs are typically assumed to be distributed multivariate Normal, with parameters for the variances and covariances selected by the analyst. Substitution of (2) and (3) into (1) yields:

$$Y_{ij} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \mathbf{X}_{ij}\gamma_{k0} + \mathbf{X}_{ij}\mathbf{Z}_j\gamma_k + \mathbf{X}_{ij}\varepsilon_{kj} + \varepsilon_{0j} + u_{ij} \tag{4}$$

The form is essentially a model with "saturated" interaction effects across the various levels, as well as "errors" which are multivariate Normal.

HLM Details

Model Assumptions

- Linearity / Additivity
- Normality of *u*s
- Homoscedasticity
- Residual Independence:
  - $\text{Cov}(\varepsilon_{\cdot j}, u_{ij}) = 0$
  - $\text{Cov}(u_{ij}, u_{i\ell}) = 0$

Model Fitting

- MLE
- "Restricted" MLE ("RMLE")
- Choosing:
  - MLE is biased in small samples, especially for estimating variances
  - RMLE is not, but prevents use of LR tests when the models do not have identical fixed effects
  - In general: RMLE is better with small sample sizes, but MLE is fine in larger ones

Note that if we specify:

$$\beta_{0j} = \gamma_{00} + \varepsilon_{0j}$$

and

$$\beta_{kj} = \gamma_{k0}$$

we get:

$$Y_{ij} = \gamma_{00} + \mathbf{X}_{ij}\gamma_{k0} + \varepsilon_{0j} + u_{ij}$$

which is the RE model above (formally, a "one-level random-intercept" HLM).

In addition:

- HLMs can be expanded to 3- and 4- and higher-level models
- One can include cross-level interactions...

HLMs are widely used in education, psychology, ecology, etc. (less so in economics, political science). Also, there are many, many excellent books, websites, etc. that address HLMs

Random Effects Remix (using lmer)

```
> library(lme4)
> AltRE<-lmer(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+             log(GDPPerCapita)+NaturalResourceRents+ColdWar+(1|ISO3),
+             data=WDI)

> summary(AltRE)
Linear mixed model fit by REML ['lmerMod']
Formula: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +
    log(GDPPerCapita) + NaturalResourceRents + ColdWar + (1 |      ISO3)
   Data: WDI

REML criterion at convergence: 53322

Scaled residuals:
   Min     1Q Median     3Q    Max
-4.321 -0.632 -0.058  0.636  5.367

Random effects:
 Groups   Name        Variance Std.Dev.
 ISO3     (Intercept) 323      17.97
 Residual              59       7.68
Number of obs: 7566, groups:  ISO3, 186

Fixed effects:
                     Estimate Std. Error t value
(Intercept)          -11.5141     2.9322   -3.93
PopGrowth              0.0323     0.1191    0.27
UrbanPopulation        0.2056     0.0199   10.31
FertilityRate         -2.1709     0.1643  -13.21
log(GDPPerCapita)      8.5722     0.3022   28.37
NaturalResourceRents   0.0269     0.0181    1.49
ColdWar               -7.6610     0.2906  -26.36

Correlation of Fixed Effects:
            (Intr) PpGrwt UrbnPp FrtltR l(GDPP NtrlRR
PopGrowth    0.051
UrbanPopltn -0.183 -0.014
FertilityRt -0.399 -0.288  0.436
lg(GDPPrCp) -0.764 -0.059 -0.305  0.093
NtrlRsrcRnt -0.010 -0.113 -0.012 -0.053 -0.015
ColdWar     -0.109  0.013  0.219 -0.422  0.101  0.055
```

Q: Are They The Same? [A: More Or Less]

Table: RE and HLM Models of WBLI

	RE	AltRE
Population Growth	−0.073	0.032
	(0.121)	(0.119)
Urban Population	0.128***	0.206***
	(0.019)	(0.020)
Fertility Rate	−2.370***	−2.170***
	(0.165)	(0.164)
ln(GDP Per Capita)	7.510***	8.570***
	(0.293)	(0.302)
Natural Resource Rents	0.006	0.027
	(0.018)	(0.018)
Cold War	−8.480***	−7.660***
	(0.289)	(0.291)
Constant	2.560	−11.500***
	(2.580)	(2.930)
Observations	7,566	7,566
R^2	0.520	
Adjusted R^2	0.520	
Log Likelihood		-26,661.000
Akaike Inf. Crit.		53,340.000
Bayesian Inf. Crit.		53,402.000
F Statistic	7,910.000***	

*p<0.1; **p<0.05; ***p<0.01

For more, see here.

HLM with Country-Level Random βs for `ColdWar`

```
> HLM1<-lmer(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+              log(GDPPerCapita)+NaturalResourceRents+ColdWar+(ColdWar|ISO3),
+              data=WDI,control=lmerControl(optimizer="bobyqa"))

> summary(HLM1)
Linear mixed model fit by REML ['lmerMod']
Formula: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +
    log(GDPPerCapita) + NaturalResourceRents + ColdWar + (ColdWar |      ISO3)
   Data: WDI
Control: lmerControl(optimizer = "bobyqa")

REML criterion at convergence: 50165

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 ISO3     (Intercept) 560.7    23.68
          ColdWar     131.6    11.47    -0.20
 Residual              35.4     5.95
Number of obs: 7566, groups:  ISO3, 186

Fixed effects:
                      Estimate Std. Error t value
(Intercept)          -29.248209   3.374962   -8.67
PopGrowth             -0.137795   0.099032   -1.39
UrbanPopulation        0.285673   0.023264   12.28
FertilityRate         -4.130147   0.177531  -23.26
log(GDPPerCapita)     10.946306   0.339078   32.28
NaturalResourceRents  -0.000932   0.015744   -0.06
ColdWar               -2.647474   0.995578   -2.66

Correlation of Fixed Effects:
            (Intr) PpGrwt UrbnPp FrtltR l(GDPP NtrlRR
PopGrowth    0.052
UrbanPopltn -0.132 -0.037
FertilityRt -0.479 -0.197  0.480
lg(GDPPrCp) -0.722 -0.056 -0.384  0.168
NtrlRsrcRnt  0.031 -0.079  0.058 -0.011 -0.095
ColdWar     -0.089 -0.012  0.056 -0.118 -0.003  0.011
```

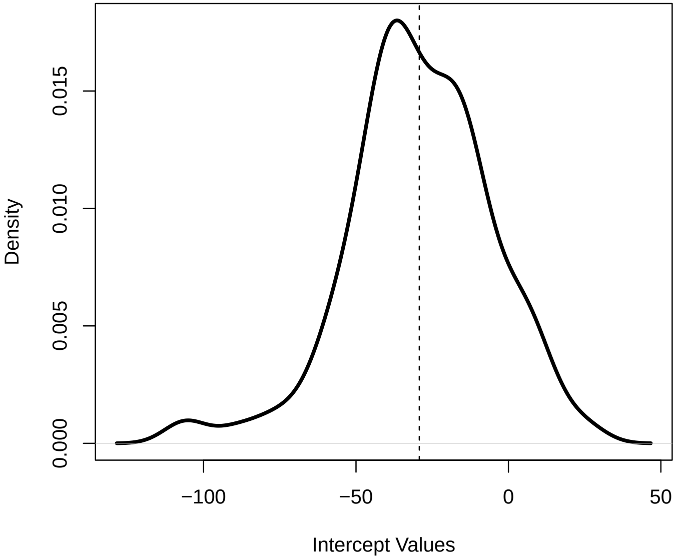
Testing

```
> anova(AltRE,HLM1)

refitting model(s) with ML (instead of REML)
Data: WDI

Models:
AltRE: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +
       log(GDPPerCapita) + NaturalResourceRents + ColdWar + (1 | ISO3)
HLM1: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +
       log(GDPPerCapita) + NaturalResourceRents + ColdWar + (ColdWar | ISO3)

      npar   AIC   BIC logLik deviance Chisq Df Pr(>Chisq)
AltRE    9 53323 53386 -26653    53305
HLM1    11 50174 50250 -25076    50152  3154  2     <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1


> VarCorr(HLM1)

 Groups   Name        Std.Dev. Corr
 ISO3     (Intercept) 23.68
          ColdWar     11.47    -0.20
 Residual              5.95
```

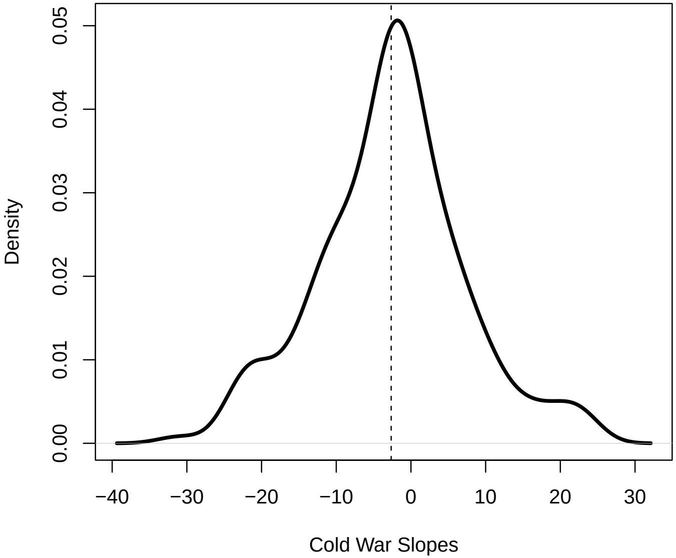
Random Coefficients

```
> Bs<-data.frame(coef(HLM1)[1])

> head(Bs)
    ISO3..Intercept. ISO3.PopGrowth ISO3.UrbanPopulation ISO3.FertilityRate
AFG            -19.2         -0.138                0.286              -4.13
AGO            -13.5         -0.138                0.286              -4.13
ALB            -11.3         -0.138                0.286              -4.13
ARE           -105.3         -0.138                0.286              -4.13
ARG            -49.2         -0.138                0.286              -4.13
ARM            -24.6         -0.138                0.286              -4.13
    ISO3.log.GDPPerCapita. ISO3.NaturalResourceRents ISO3.ColdWar
AFG                   10.9                 -0.000932       -3.632
AGO                   10.9                 -0.000932      -14.877
ALB                   10.9                 -0.000932       -7.589
ARE                   10.9                 -0.000932       -0.729
ARG                   10.9                 -0.000932      -22.651
ARM                   10.9                 -0.000932       -3.106

> mean(Bs$ISO3..Intercept.)
[1] -29.2

> mean(Bs$ISO3.ColdWar)
[1] -2.65
```

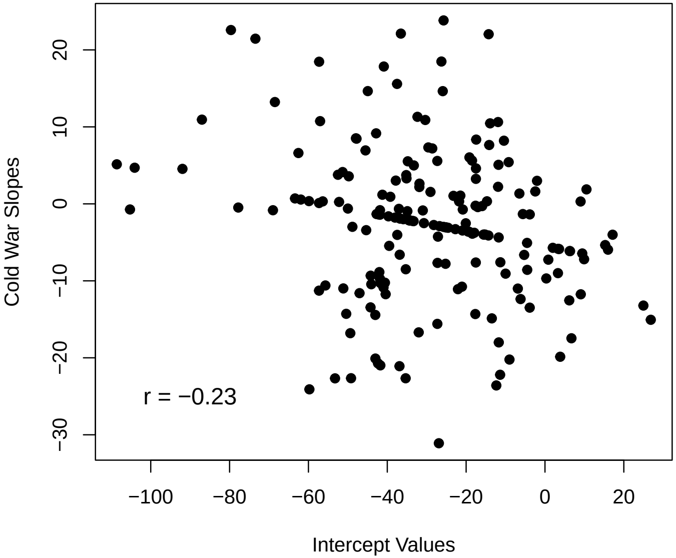
Random Intercepts (Plotted)



Random Slopes for `ColdWar` (Plotted)



Scatterplot: Random Intercepts and Slopes



Separating Within and Between Effects

Recall that we can decompose the variance of our regression model as:

$$Y_{it} = \bar{\mathbf{X}}_i \boldsymbol{\beta}_B + (\mathbf{X}_{it} - \bar{\mathbf{X}}_i)\boldsymbol{\beta}_W + u_{it} \tag{5}$$

This raises the possibility of fitting the model in (5) directly...

- Simple to estimate (via OLS or other)
- Relatively easy interpretation
- Easy to test $\hat{\boldsymbol{\beta}}_B = \hat{\boldsymbol{\beta}}_W$

Example data: Separate effects for within- and between-country *Natural Resource Rents*...

Combining Within- and Between-Effects

Table: BE + WE Model of WBLI

	WEBE.OLS
Population Growth	-3.000^{***}
	(0.141)
Urban Population	-0.046^{***}
	(0.011)
Fertility Rate	-1.470^{***}
	(0.163)
ln(GDP Per Capita)	3.210^{***}
	(0.195)
Within-Country Nat. Resource Rents	0.090^{***}
	(0.030)
Between-Country Nat. Resource Rents	-0.515^{***}
	(0.019)
Cold War	-11.800^{***}
	(0.365)
Constant	53.500^{***}
	(1.700)
Observations	7,566
R^2	0.538
Adjusted R^2	0.537
Residual Std. Error	12.800 (df = 7558)
F Statistic	$1{,}256.000^{***}$ (df = 7; 7558)

*p$<$0.1; **p$<$0.05; ***p$<$0.01

Our original decomposition considered "two-way" effects:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + \eta_t + u_{it}$$

This implies that we can use (e.g.) an F-test to examine the hypothesis:

$$H_0 : \alpha_i = \eta_t = 0 \; \forall \; i, \; t$$

...that is, whether adding the (two-way) effects improves the model's fit.

We can also consider the partial hypotheses:

$$H_0 : \alpha_i = 0 \; \forall \; i$$

and

$$H_0 : \eta_t = 0 \; \forall \; t$$

separately.

Two-Way Effects: Good & Bad

The Good:

- Addresses both time-invariant, unit-level heterogeneity and cross-sectionally-invariant, time-point-specific heterogeneity
- May be "fixed" or "random"...
- Two-way FE is equivalent to differences-in-differences when $X \in \{0, 1\}$ and $T = 2$ (more on that later)

The (Potentially) Bad:

- Ability to control for unobserved heterogeneity depends on (probably invalid) functional form assumptions (Imai and Kim 2020)
- FE *requires* predictors that vary both within and between both units and time points
- RE requires the (additional) assumption that $\text{Cov}(\mathbf{X}_{it}, \eta_t) = Cov(\alpha_i, \eta_t) = 0$
- Two-way effects models ask a *lot* of your data (effectively fits $N + T + k$ parameters using NT observations)

Example: Two-Way Fixed Effects

```
> TwoWayFE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+               log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+               effect="twoway",model="within")

> summary(TwoWayFE)
Twoways effects Within Model

Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    ColdWar, data = WDI, effect = "twoway", model = "within")

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Residuals:
   Min. 1st Qu.  Median 3rd Qu.    Max.
-31.756  -3.866   0.251   4.100  29.715

Coefficients:
                     Estimate Std. Error t-value Pr(>|t|)
PopGrowth            -0.26475    0.10023   -2.64   0.0083 **
UrbanPopulation      -0.00333    0.01798   -0.19   0.8529
FertilityRate         1.03517    0.15072    6.87    7e-12 ***
log(GDPPerCapita)     2.50015    0.28544    8.76   <2e-16 ***
NaturalResourceRents -0.00420    0.01588   -0.26   0.7913
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Total Sum of Squares:    306000
Residual Sum of Squares: 300000
R-Squared:      0.0179
Adj. R-Squared: -0.0141
F-statistic: 26.7123 on 5 and 7326 DF, p-value: <2e-16
```

Two-Way Effects: Testing

```
> # Two-way effects:

> pFtest(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+        log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+        effect="twoway",model="within")

F test for twoways effects

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
F = 103, df1 = 233, df2 = 7326, p-value <2e-16
alternative hypothesis: significant effects

> plmtest(TwoWayFE,c("twoways"),type=("kw"))

Lagrange Multiplier Test - two-ways effects (King and Wu) for unbalanced panels

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
normal = 187, p-value <2e-16
alternative hypothesis: significant effects



> # One-way effects in the two-way model:
>
> plmtest(TwoWayFE,c("individual"),type=("kw"))

Lagrange Multiplier Test - (King and Wu) for unbalanced panels

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
normal = 211, p-value <2e-16
alternative hypothesis: significant effects

> plmtest(TwoWayFE,c("time"),type=("kw"))

Lagrange Multiplier Test - time effects (King and Wu) for unbalanced panels

data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  ...
normal = 100, p-value <2e-16
alternative hypothesis: significant effects
```

Two-Way Fixed Effects via `lm`

```
> TwoWayFE.BF<-lm(WomenBusLawIndex~PopGrowth+UrbanPopulation+
+                 FertilityRate+log(GDPPerCapita)+NaturalResourceRents+
+                 factor(ISO3)+factor(Year),data=WDI)

> summary(TwoWayFE.BF)

Call:
lm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    factor(ISO3) + factor(Year), data = WDI)

Residuals:
   Min     1Q Median     3Q    Max
-31.76  -3.87   0.25   4.10  29.72

Coefficients:
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)          -17.26971    2.50039   -6.91  5.4e-12 ***
PopGrowth             -0.26475    0.10023   -2.64   0.0083 **
UrbanPopulation       -0.00333    0.01798   -0.19   0.8529
FertilityRate          1.03517    0.15072    6.87  7.0e-12 ***
log(GDPPerCapita)      2.50015    0.28544    8.76  < 2e-16 ***
NaturalResourceRents  -0.00420    0.01588   -0.26   0.7913
factor(ISO3)AGO       29.30071    2.01487   14.54  < 2e-16 ***
factor(ISO3)ALB       51.72646    2.02286   25.57  < 2e-16 ***
factor(ISO3)ARE       -5.95970    2.54027   -2.35   0.0190 *
.
.
.
factor(Year)1978       4.82791    0.88259    5.47  4.6e-08 ***
factor(Year)1979       5.23708    0.88928    5.89  4.1e-09 ***
 [ reached getOption("max.print") -- omitted 40 rows ]
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 6.4 on 7326 degrees of freedom
  (5764 observations deleted due to missingness)
Multiple R-squared:  0.888,Adjusted R-squared:  0.884
F-statistic:  242 on 239 and 7326 DF,  p-value: <2e-16
```

Example: Two-Way Random Effects

```
> TwoWayRE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+               log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+               effect="twoway",model="random")

> summary(TwoWayRE)

Twoways effects Random Effect Model
   (Swamy-Arora's transformation)

Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    ColdWar, data = WDI, effect = "twoway", model = "random")

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Effects:
                 var std.dev share
idiosyncratic 40.993   6.403  0.31
individual    89.185   9.444  0.68
time           0.612   0.782  0.00
theta:
       Min. 1st Qu. Median  Mean 3rd Qu.  Max.
id    0.439   0.893  0.905 0.896   0.905 0.905
time  0.372   0.423  0.469 0.449   0.477 0.479
total 0.330   0.422  0.468 0.448   0.476 0.478

Residuals:
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  -61.3    -6.4     2.4     0.4    10.6    32.2

Coefficients:
                      Estimate Std. Error z-value  Pr(>|z|)
(Intercept)           21.29333    0.35391   60.17   < 2e-16 ***
PopGrowth             -0.23814    0.01549  -15.37   < 2e-16 ***
UrbanPopulation        0.06588    0.00254   25.97   < 2e-16 ***
FertilityRate         -0.58746    0.02216  -26.51   < 2e-16 ***
log(GDPPerCapita)      5.02484    0.03939  127.55   < 2e-16 ***
NaturalResourceRents  -0.01087    0.00239   -4.54 0.0000056 ***
ColdWar              -12.66453    0.05391 -234.91   < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Total Sum of Squares:    2670000
Residual Sum of Squares: 1740000
R-Squared:      0.351
Adj. R-Squared: 0.351
Chisq: 161011 on 6 DF, p-value: <2e-16
```

A Prettier Table

Table: Models of WBLI

	OLS	FE	BE	RE	TwoWayFE	TwoWayRE
Population Growth	-3.190^{***}	0.073	-5.870^{***}	-0.073	-0.265^{***}	-0.238^{***}
	(0.144)	(0.119)	(0.917)	(0.121)	(0.100)	(0.015)
Urban Population	-0.058^{***}	0.248^{***}	-0.050	0.128^{***}	-0.003	0.066^{***}
	(0.011)	(0.021)	(0.055)	(0.019)	(0.018)	(0.003)
Fertility Rate	-1.790^{***}	-2.070^{***}	0.063	-2.370^{***}	1.030^{***}	-0.587^{***}
	(0.165)	(0.166)	(1.080)	(0.165)	(0.151)	(0.022)
ln(GDP Per Capita)	3.150^{***}	9.160^{***}	3.400^{***}	7.510^{***}	2.500^{***}	5.030^{***}
	(0.199)	(0.310)	(1.130)	(0.293)	(0.285)	(0.039)
Natural Resource Rents	-0.349^{***}	0.035^{*}	-0.358^{***}	0.006	-0.004	-0.011^{***}
	(0.016)	(0.018)	(0.089)	(0.018)	(0.016)	(0.002)
Cold War	-11.300^{***}	-7.190^{***}	-12.500^{**}	-8.480^{***}		-12.700^{***}
	(0.372)	(0.295)	(4.890)	(0.289)		(0.054)
Constant	54.800^{***}		50.300^{***}	2.560		21.300^{***}
	(1.730)		(10.200)	(2.580)		(0.354)
Observations	7,566	7,566	186	7,566	7,566	7,566
R^2	0.519	0.535	0.605	0.520	0.018	0.351
Adjusted R^2	0.519	0.523	0.592	0.520	-0.014	0.351

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Other Variations: FEIS

“Fixed Effects Individual Slope” models

- Cite: Bruederl, Josef, and Volker Ludwig. 2015. “Fixed-Effects Panel Regression.” In *The Sage Handbook of Regression Analysis and Causal Inference*, Eds. Henning Best and Christof Wolf. Los Angeles: Sage, pp. 327-357.
- FE + unit-level slopes for (some / all) predictor variables
- Equivalent to including $N - 1$ interactions between a predictor $\mathbf{X}$ and each of the α_is
- Also can test for homogeneity of estimated slopes (Hausman-like test)
- See the `feisr` R package, and its accompanying vignette, or `xtfeis` in Stata

Unit Effects Models: Software

R :

- the `plm` package; `plm` command
  - Fits one- and two-way FE, BE, RE models
  - Also fits first difference (FD) and instrumental variable (IV) models
- the `fixest` package; fast/scalable FE estimation for OLS and GLMs
- the `lme4` package; command is `lmer`
- the `nlme` package; command `lme`
- the `Paneldata` package

Stata : `xtreg`

- option `re` (the default) = random effects
- option `fe` = fixed (within) effects
- option `be` = between-effects
- Stata package `fect` = two-way models

Dynamics

Issues with Unit Roots in Panel Data

In general, in panel / TSCS data:

- Short series + Asymptotic tests $\rightarrow$ "borrow strength"
- Requires uniform unit roots across cross-sectional units
- Many tests require balanced panels...
- Various alternatives:
    - Maddala and Wu (1999)
    - Hadri (2000)
    - Levin, Lin and Chu (2002)
    - Im, Pesaran, and Shin (2003)
- What to do?
    - Difference the data...
    - Error-correction models

Panel Unit Root Tests: R

```
[data wrangling...]

> purtest(WBLI.W,exo="trend",test="levinlin",pmax=2)

Levin-Lin-Chu Unit-Root Test (ex. var.: Individual Intercepts and
Trend)

data:  WBLI.W
z = -2.7, p-value = 0.003
alternative hypothesis: stationarity

> purtest(WBLI.W,exo="trend",test="hadri",pmax=2)

Hadri Test (ex. var.: Individual Intercepts and Trend)
(Heterosked. Consistent)

data:  WBLI.W
z = 189, p-value <2e-16
alternative hypothesis: at least one series has a unit root

> purtest(WBLI.W,exo="trend",test="madwu",pmax=2)

Maddala-Wu Unit-Root Test (ex. var.: Individual Intercepts and
Trend)

data:  WBLI.W
chisq = 332, df = 376, p-value = 0.9
alternative hypothesis: stationarity

> purtest(WBLI.W,exo="trend",test="ips",pmax=2)

Im-Pesaran-Shin Unit-Root Test (ex. var.: Individual Intercepts
and Trend)

data:  WBLI.W
Wtbar = 3.6, p-value = 1
alternative hypothesis: stationarity
```

A Better Table

Table: Panel Unit Root Tests: WBRI

Test	Alternative	Statistic	Estimate	P-Value
Levin-Lin-Chu Test	stationarity	z	-2.286	0.0111
Hadri Test (Heterosked. Consistent)	$\geq$ one series has a unit root	z	192.036	< 0.001
Maddala-Wu Test	stationarity	χ^2	782.604	< 0.001
Im-Pesaran-Shin Test	stationarity	$\bar{W}_t$	3.342	0.9996

Note: All assume individual intercepts and trends.

Consider a model like this:

$$Y_{it} = \phi Y_{it-1} + \mathbf{X}_{it}\boldsymbol{\beta}_{LDV} + \epsilon_{it}$$

If ϵ_{it} is perfect...

- $\hat{\beta}_{LDV}$ is biased (but consistent),
- O(bias) $= \frac{-1+3\boldsymbol{\beta}_{LDV}}{T}$

If ϵ_{it} is autocorrelated...

- $\hat{\beta}_{LDV}$ is biased and <u>inconsistent</u>
- IV is one (bad) option...

Can rewrite:

$$\begin{aligned} Y_{it} &= \mathbf{X}_{it}\boldsymbol{\beta}_{AR} + u_{it} \\ u_{it} &= \phi u_{it-1} + \eta_{it} \end{aligned}$$

as

$$\begin{aligned} Y_{it} &= \mathbf{X}_{it}\boldsymbol{\beta}_{AR} + \phi u_{it-1} + \eta_{it} \\ &= \mathbf{X}_{it}\boldsymbol{\beta}_{AR} + \phi(Y_{it-1} - \mathbf{X}_{it-1}\boldsymbol{\beta}_{AR}) + \eta_{it} \\ &= \phi Y_{it-1} + \mathbf{X}_{it}\boldsymbol{\beta}_{AR} + \mathbf{X}_{it-1}\psi + \eta_{it} \end{aligned}$$

where $\psi = \phi\beta_{AR}$ and $\psi = 0$ (by assumption).

Lagged Ys and World Domination

In:

$$Y_{it} = \phi Y_{it-1} + \mathbf{X}_{it}\boldsymbol{\beta}_{LDV} + \epsilon_{it}$$

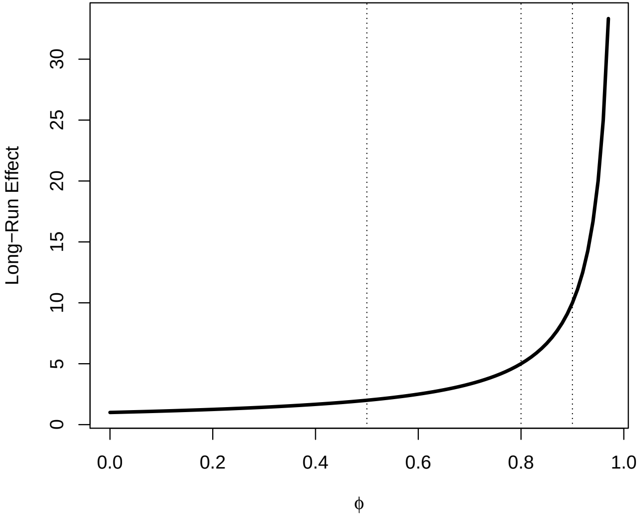
Achen: Bias "deflates" $\hat{\boldsymbol{\beta}}_{LDV}$ relative to $\hat{\phi}$, "suppress" the effects of $\mathbf{X}$...

Keele & Kelly (2006):

- Contingent on ϵs having autocorrelation
- Key: In LDV, *long-run impact of a unit change in X is*:

$$\hat{\beta}_{LR} = \frac{\hat{\beta}_{LDV}}{1 - \hat{\phi}}$$

Long-Run Impact for $\hat{\beta} = 1$



Lagged Ys and Unit Effects

Consider:

$$Y_{it} = \phi Y_{it-1} + \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + u_{it}.$$

If we omit the unit effects, we have:

$$Y_{it} = \phi Y_{it-1} + \mathbf{X}_{it}\boldsymbol{\beta} + u^*_{it}$$

with

$$u^*_{it} = \alpha_i + u_{it}$$

Lagging yields:

$$Y_{it-1} = \phi Y_{it-2} + \mathbf{X}_{it-1}\boldsymbol{\beta} + \alpha_i + u_{it-1}$$

which means

$$\text{Cov}(Y_{it-1}, u^*_{it}) \neq 0. \;\rightarrow\; \text{bias in } \hat{\phi}, \hat{\boldsymbol{\beta}}$$

"Nickell" Bias

Omitting fixed effects in a model with Y_{it-1} yields bias in $\hat{\phi}$ that is:

- toward zero when $\phi > 0$,
- increasing in ϕ.

Including unit effects still yields bias in $\hat{\phi}$ of $O(\frac{1}{T})$, and bias in $\hat{\boldsymbol{\beta}}$.

Solutions:

- Difference/GMM estimation
- Bias correction approaches

First Difference Estimation

$$
\begin{aligned}
Y_{it} - Y_{it-1} &= \phi(Y_{it-1} - Y_{it-2}) + (\mathbf{X}_{it} - \mathbf{X}_{it-1})\boldsymbol{\beta} + (\alpha_i - \alpha_i) + (u_{it} - u_{it-1}) \\
\Delta Y_{it} &= \phi \Delta Y_{it-1} + \Delta \mathbf{X}_{it}\boldsymbol{\beta} + \Delta u_{it}
\end{aligned}
$$

Anderson/Hsiao: If $\nexists$ autocorrelation, then use ΔY_{it-2} or Y_{it-2} as instruments for ΔY_{it-1}...

- Consistent in theory,
- in practice, the former is preferred, and both have issues if ϕ is high;
- both are inefficient.

A&H $\rightarrow$ A&B

Arellano & Bond (also Wawro): Use *all* lags of Y_{it} and $\mathbf{X}_{it}$ from $t-2$ and before.

- "Good" estimates, better as $T \rightarrow \infty$,
- Easy to handle higher-order lags of Y,
- Easy software (`plm` in R , `xtabond` in Stata).
- Model *is* fixed effects...
- $\mathbf{Z}_i$ has $T-p-1$ rows, $\sum_{i=p}^{T-2} i$ columns $\rightarrow$ difficulty of estimation declines in p, <u>grows in T</u>.

Bias-Correction Models

Kiviet (1995, 1999; Bun and Kiviet 2003; Bruno 2005a,b): Derive the bias in $\hat{\phi}$ and $\hat{\beta}$, then correct it...

- More accurate than the instrumental-variables/GMM estimators of A&H/A&B...
- ...especially when T is small; but not as T gets reasonably large ($T \approx 20$)

Some Dynamic Models

	Lagged Y	First Difference	FE	Lagged Y + FE
Intercept	2.260*	0.641*		
	(0.335)	(0.040)		
Lagged WBLI	0.986*			0.948*
	(0.002)			(0.004)
Population Growth	−0.051	0.035	0.073	0.011
	(0.027)	(0.077)	(0.119)	(0.037)
Urban Population	0.002	−0.040	0.248*	0.009
	(0.002)	(0.062)	(0.021)	(0.007)
Fertility Rate	−0.085*	−1.023*	−2.066*	−0.292*
	(0.030)	(0.373)	(0.166)	(0.052)
ln(GDP Per Capita)	−0.036	0.780	9.161*	0.276*
	(0.037)	(0.476)	(0.310)	(0.102)
Natural Resource Rents	−0.010*	0.020*	0.035	−0.003
	(0.003)	(0.008)	(0.018)	(0.006)
Cold War	−0.298*	−0.021	−7.192*	−0.445*
	(0.072)	(0.204)	(0.295)	(0.094)
R^2	0.984	0.003	0.535	0.956
Adj. R^2	0.984	0.002	0.523	0.954
Num. obs.	7463	7380	7566	7463

$^*p < 0.05$

Trends!

What if Y is *trending* over time?

- First Question: Why?
  - Organic growth (e.g., populations)
  - Temporary / short-term factors
  - Covariates...

- Second question: Should we care?
  (A: Yes, usually... $\rightarrow$ "spurious regressions")

- Third question: What to do?
  - Ignore it...
  - Include a counter / trend term...

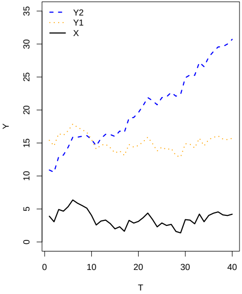
In general, adding a trend term will *decrease* the magnitudes of $\hat{\beta}$...

Trends Matter, Illustrated

Data generating processes:

$$Y_{1t} = 10 + (1 \times X_t) + u_t$$

$$Y_{2t} = 5 + (1 \times X_t) + (0.5 \times T) + u_t$$



	Y_1	Y_2	
		No Trend	Trend
X	0.921***	−0.382	0.874***
	(0.245)	(0.786)	(0.255)
T			0.482***
			(0.026)
Constant	10.300***	20.200***	5.860***
	(0.917)	(2.950)	(1.200)
Observations	40	40	40
R^2	0.272	0.006	0.905
Adjusted R^2	0.253	−0.020	0.900
Residual Std. Error	1.800 (df = 38)	5.790 (df = 38)	1.810 (df = 37)
Note:			*p<0.1; **p<0.05; ***p<0.01

Trends Matter, Part II

Table: FE Models of WBLI

	FE	FE.trend	FE.intx
Population Growth	0.073	-0.287^{***}	-0.242^{**}
	(0.119)	(0.100)	(0.100)
Urban Population	0.248^{***}	-0.024	-0.003
	(0.021)	(0.018)	(0.018)
Fertility Rate	-2.066^{***}	1.080^{***}	1.018^{***}
	(0.166)	(0.150)	(0.149)
ln(GDP Per Capita)	9.161^{***}	2.867^{***}	2.585^{***}
	(0.310)	(0.283)	(0.283)
Natural Resource Rents	0.035^{*}	0.009	0.008
	(0.018)	(0.015)	(0.015)
Cold War	-7.192^{***}	1.660^{***}	9.300^{***}
	(0.295)	(0.293)	(0.944)
Trend (1950=0)		0.749^{***}	0.783^{***}
		(0.013)	(0.014)
Cold War x Trend			-0.220^{***}
			(0.026)
Observations	7,566	7,566	7,566
R^2	0.535	0.674	0.678
Adjusted R^2	0.523	0.666	0.669
F Statistic	$1{,}414.000^{***}$ (df = 6; 7374)	$2{,}182.000^{***}$ (df = 7; 7373)	$1{,}937.000^{***}$ (df = 8; 7372)

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Another Approach: Orthogonalization

Note: We're rarely substantively interested in the fixed effects $\hat{\alpha}$...

- $\rightarrow$ reparameterize the αs so that they are *information-orthogonal* to the other parameters in the model (including the βs and ϕ)
- Key idea: Transform the αs so that (for example):

$$\mathrm{E}\left(\frac{\partial^2 L_i}{\partial\alpha\partial\beta}\right) = 0$$

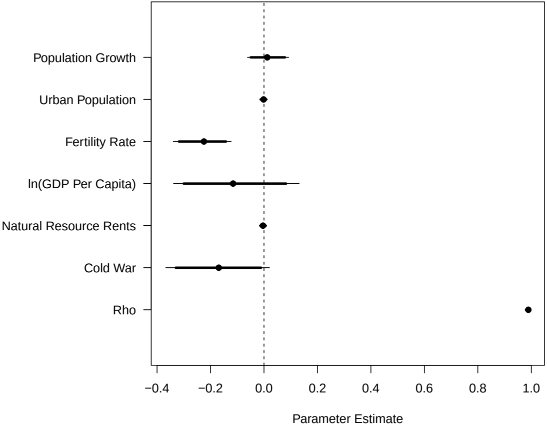
- Can do this via imposition of priors, in a Bayesian framework...
- **In general**, this approach is less assumption-laden and more efficient than the IV/GMM-based approaches discussed above.
- Provides consistent-in-N estimates for T as low as 2...

<u>References</u>:

- Lancaster, T. 2002. "Orthogonal Parameters and Panel Data." *Review of Economic Studies* 69:647-666.
- Pickup et al. (2017) [the "orthogonalized panel model" ("OPM")]

FE + Dynamics Using Orthogonalization

```
> library(OrthoPanels)
> set.seed(7222009)
> OPM.fit <- opm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
            lnGDPPerCap+NaturalResourceRents+ColdWar,data=WDI,
            index=c("ISO3","Year"),n.samp=1000)
```



OPM Results: Short- and Long-Run Effects

For $\hat{\phi} \approx 0.98$:

Parameter	Short-Run	Long-Run
Population Growth	0.0122	0.9148
Urban Population	-0.0016	-0.1420
Fertility Rate	-0.2247	-19.0090
ln(GDP Per Capita)	-0.1155	-9.9996
Natural Resource Rents	-0.0037	-0.3086
Cold War	-0.1691	-14.3630

Dynamic Models: Software

R :

- the plm package (purtest for unit roots; plm for first-difference models)
- the panelAR package (GLS-ARMA models)
- the gls package (GLS)
- the pgmm package (A&B)
- the dynpanel package (A&H, A&B)

Stata :

- xtgls (GLS)
- xtpcse (PCSEs)
- xtabond / xtdpd (A&H A&B dynamic models)

- N vs. T...
- Are dynamics nuisance or substance?
- What problem(s) do you *really* care about?