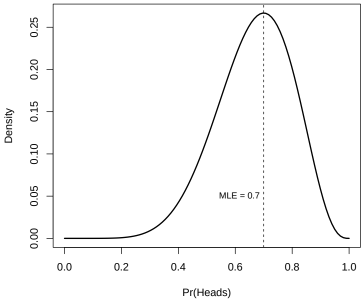
PLSC 504 – Fall 2023
Bayesian Statistics

(A micro-intro)

November 15, 2023

Some characteristics:

- Probability $=$ *Long-run relative frequency*
- $\mathrm{Pr}(X)$ is a *fixed* but *unknown* quantity
- Example: *Likelihood*
  - Suppose we flipped a coin 10 times, and got heads 7 of them.
  - Q: What is $\theta \equiv \widehat{\mathrm{Pr}(\text{Heads})}$?
  - A: The MLE is $\hat{\theta} = \frac{N_{\text{Heads}}}{N} = 7/10 = 0.7$



Components:

- Quantity of interest $= \theta$
- Data $= Y$
- Sampling density $= \Pr(Y|\theta)$
- We want to know $\Pr(\theta|Y)$
- Likelihood $L(\theta|Y) \ \propto \ \Pr(Y|\theta)$

Begin by noting that:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{1}$$

and

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}. \tag{2}$$

That means that:

$$\Pr(A \cap B) = \Pr(B|A)\Pr(A). \tag{3}$$

Substituting (3) into (1), we get

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}. \tag{4}$$

For our data example:

$$
\begin{aligned}
\Pr(\theta|Y) &= \frac{\Pr(\theta \cap Y)}{\Pr(Y)} \\
&= \frac{\Pr(Y|\theta)\Pr(\theta)}{\Pr(Y)}.
\end{aligned}
$$

- $\Pr(\theta|Y)$ is the *posterior density* of θ
- $\Pr(Y|\theta)$ is the *sampling density*
- $\Pr(\theta)$ is the *prior density* of θ
- $\Pr(Y)$ is the marginal probability of Y

Since Y is fixed in a single sample, we can write:

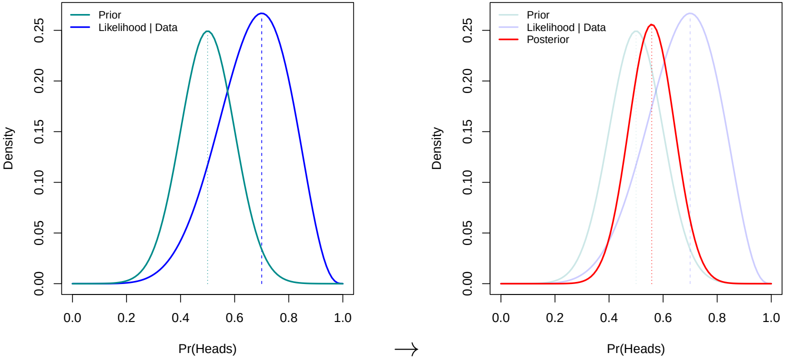
$$\Pr(\theta|Y) \propto \Pr(Y|\theta)\,\Pr(\theta).$$

Bayesian probability:

- ...is **subjective**: Probability is a *belief about the world*...
  - $\Pr(\theta)$ is our prior / "pre-data" estimate of the distribution of θ
  - $\Pr(\theta|Y)$ is our posterior / "post-data" estimate
- ...**updates** our prior beliefs about θ based on the data, in a manner consistent with Bayes' Theorem
- ...presents a probability **density** for $\hat{\theta}$, not just a point estimate

Suppose:

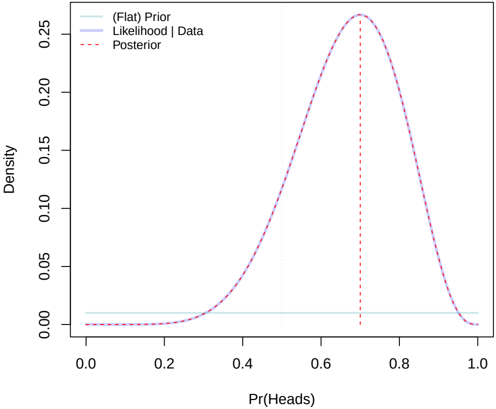
- I believe (with some uncertainty) that a particular coin is "fair" (i.e., that $\theta = 0.5$) *and*...
- ...we then flipped the coin 10 times, and got heads 7 of them.



Bayesian 🤝 Frequentist

Often Bayesian and frequentist results converge when priors are *uninformative* (these are sometimes called “flat priors”).

Example:



The process:

- Specify a probability model for the data + parameters.
- Posit one's prior beliefs.
- Calculate the posterior distribution using Bayes' Theorem.
- Summarize the posterior density.
- Conduct post-estimation model checking.

"MCMC" = Markov Chain Monte Carlo...

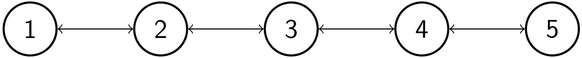
Goal: To characterize a (potentially complex, unknown) probability density in the parameter space.

Note that:

- The posterior density is a combination of the prior(s) and the data / likelihood...
- ...but it is also unknown until we incorporate information from the data.
- High-dimensional multivariate distributions are often mathematically + computationally intractable...

MCMC: Intuition

Intuition:



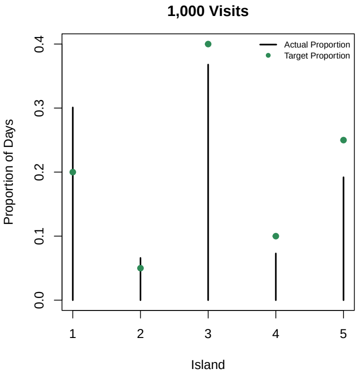
- A legislator campaigns along a chain of five east-west islands...
- Goal: Spend time in each in proportion to their (unknown) population, *but*
- The legislator learns the population of each island when she visits it.
- Algorithm: Each day...
  - ... flip a fair coin: *heads* = east, *tails* = west
  - If the island to the $\{$*east*, *west*$\}$ has a population <u>higher</u> than the current island, go there
  - If the island to the $\{$*east*, *west*$\}$ has a population <u>lower</u> than the current island:
    - Calculate $a = \frac{\text{Population of prospective island}}{\text{Population of current island}}$
    - Move with probability a; stay with probability $1 - a$

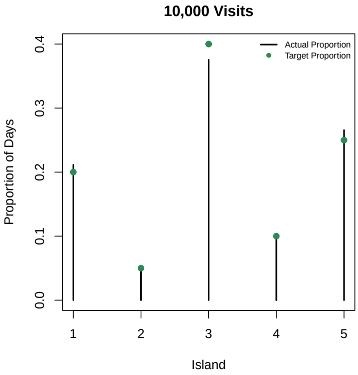
MCMC: Intuition (continued)

Suppose the populations of the islands are:

$$20 \leftrightarrow 5 \leftrightarrow 40 \leftrightarrow 10 \leftrightarrow 25$$

After 1,000 or 10,000 days campaigning:





MCMC...

MCMC samples from the (multivariate) posterior distribution of $\hat{\theta}$...

- It's a *Markov chain*, because it's "memoryless," but
- ...the stable / equilibrium distribution of the chain is the target distribution
- This means that the chain "focuses" (samples more frequently from) places in the parameter space where the target distribution has more density
- $\rightarrow$ the values of the chain at each iteration are (by construction) autocorrelated...
  - "Burn-in": Initializing the chain in the parameter space
  - "Thinning": Taking every *p*th observation from the chain for estimation/inference
- For details, check out this, or this, or this, or this interactive visualization, or this long thing...

Inference: Credible Intervals

Consider that:

Concept	Frequentist...	Bayesian...
Parameter (θ)	*Fixed*	Varies with prior
C.I. $[\theta_L, \theta_H]$	Varies with sample	*Fixed*

This means that...

- ...a Bayesian *credible interval* is an interval within which an unobserved parameter value falls with a particular probability.
- That is, a $k \times 100\%$ credible interval is the interval in which the (unobserved) parameter value falls with probability k.
- For more, read this or this, or this.

Example: Black State Representatives

Data from the Correlates of State Policy Project at MSU...

- Annual state-level data, 2013-2017 ($N = 50$, $T = 5$)
- Outcome: Count of Black representatives in the state legislature's lower house
- Predictor variables:
  - Percentage of the state population that is Black
  - Total state expenditures (logged)
  - Median state household income (logged)
  - State population (logged)
- Models: Linear regression + Poisson / negative binomial

Linear Regression via OLS

```
> OLSfit<-lm(BlackHouseMembers~PercentBlackPop+log(TotalExpenditures)+
               log(MedianHHIncome)+log(Population),data=DF)
> summary(OLSfit)

Call:
lm(formula = BlackHouseMembers ~ PercentBlackPop + log(TotalExpenditures) +
    log(MedianHHIncome) + log(Population), data = DF)

Residuals:
    Min      1Q  Median      3Q     Max
-15.215  -2.576   0.397   2.141  17.234

Coefficients:
                       Estimate Std. Error t value Pr(>|t|)
(Intercept)            -46.7852    18.5221   -2.53  0.01218 *
PercentBlackPop          0.9929     0.0335   29.59  < 2e-16 ***
log(TotalExpenditures)  -3.0865     1.3635   -2.26  0.02448 *
log(MedianHHIncome)      2.4575     1.7896    1.37  0.17096
log(Population)          4.7667     1.2504    3.81  0.00018 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.49 on 240 degrees of freedom
Multiple R-squared:  0.846,Adjusted R-squared:  0.843
F-statistic:  329 on 4 and 240 DF,  p-value: <2e-16
```

Linear Regression via MCMC (using bayesm)

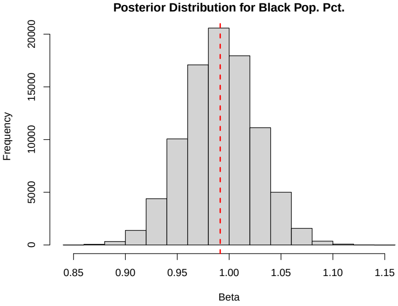
```
> DF$logTE<-log(DF$TotalExpenditures)
> DF$logHHInc<-log(DF$MedianHHIncome)
> DF$logPop<-log(DF$Population)
> DF$One<-1  # constant...

> # Model:
>
> Data<-list(y=DF$BlackHouseMembers,X=as.matrix(DF[,c(11,4,8,9,10)]))
> MCMC<-list(R=1e6,keep=10,nprint=0)

> BayesFit<-runireg(Data=Data,Mcmc=MCMC)

Starting IID Sampler for Univariate Regression Model
  with  245  observations

Prior Parms:
betabar
[1] 0 0 0 0 0
A
     [,1] [,2] [,3] [,4] [,5]
[1,] 0.01 0.00 0.00 0.00 0.00
[2,] 0.00 0.01 0.00 0.00 0.00
[3,] 0.00 0.00 0.01 0.00 0.00
[4,] 0.00 0.00 0.00 0.01 0.00
[5,] 0.00 0.00 0.00 0.00 0.01
nu =  3  ssq=  128.4295

MCMC parms:
R=  1e+06  keep=  10  nprint=  0
```

Bayesian Linear Regression Summary

```
> summary(BayesFit$betadraw)

Summary of Posterior Marginal Distributions
Moments
    mean std dev  num se rel eff sam size
1 -39.89  17.624 0.06097    1.08    45000
2   0.99   0.034 0.00011    0.99    90000
3  -2.99   1.399 0.00486    1.09    45000
4   1.83   1.717 0.00594    1.08    45000
5   4.66   1.281 0.00446    1.09    45000

Quantiles
    2.5%      5%    50%     95% 97.5%
1 -74.39 -68.85 -39.88 -11.01 -5.44
2   0.92   0.93   0.99   1.05  1.06
3  -5.74  -5.29  -2.99  -0.68 -0.24
4  -1.54  -1.00   1.82   4.64  5.19
5   2.14   2.55   4.66   6.77  7.19
   based on 90000 valid draws (burn-in=10000)
```

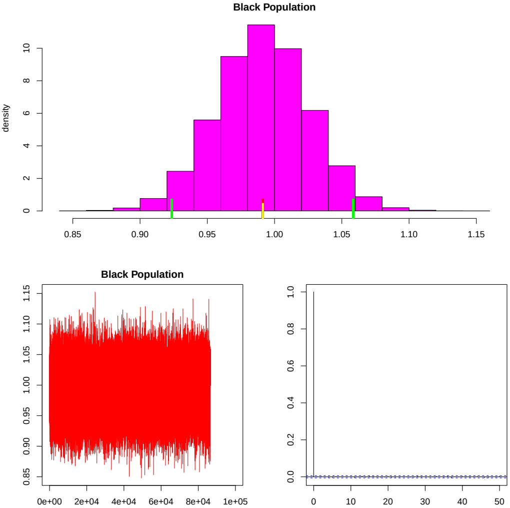
Plotting the Posterior ("By Hand")

```
B<-10000 # Discard 10K burn-in draws...
ND<-1e5  # Total kept draws after thinning...

hist(BayesFit$betadraw[B:ND,1],xlab="Beta",
  main="Posterior Distribution for Black Pop. Pct.")
abline(v=median(BayesFit$betadraw[B:ND,1]),
  lty=2,lwd=2,col="red")
```

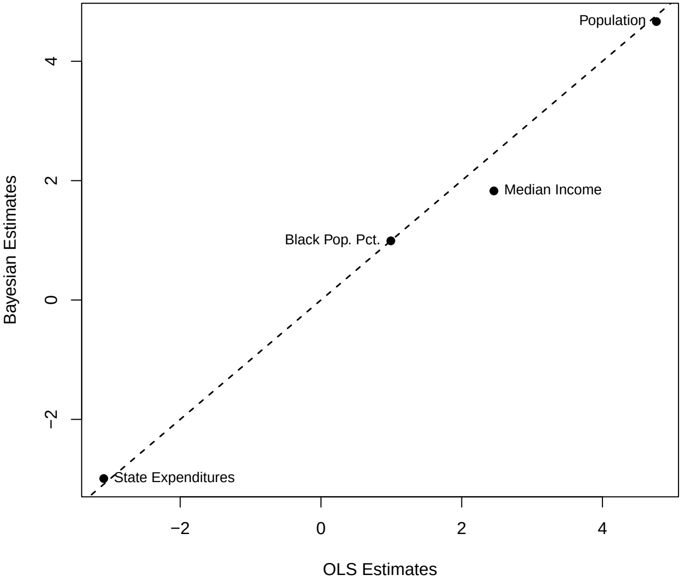


Plotting the Posterior (using `plot.bayesm`)

```
plot.bayesm.mat(BayesFit$betadraw[,1],names="Black Population")
```



Comparing Bayes & OLS $\hat{\beta}$s



Negative Binomial (via MLE)

```
> NBfit<-glm.nb(BlackHouseMembers~PercentBlackPop+log(TotalExpenditures)+
+              log(MedianHHIncome)+log(Population),data=DF,init.theta=4,maxit=1e4)

> summary(NBfit)

Call:
glm.nb(formula = BlackHouseMembers ~ PercentBlackPop + log(TotalExpenditures) +
    log(MedianHHIncome) + log(Population), data = DF, maxit = 10000,
    init.theta = 75.45975019, link = log)

Coefficients:
                        Estimate Std. Error z value Pr(>|z|)
(Intercept)            -10.22986    1.42113   -7.20  6.1e-13 ***
PercentBlackPop          0.07961    0.00223   35.63  < 2e-16 ***
log(TotalExpenditures)  -0.24002    0.12054   -1.99    0.046 *
log(MedianHHIncome)      0.31709    0.13851    2.29    0.022 *
log(Population)          0.77393    0.11124    6.96  3.5e-12 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for Negative Binomial(75.46) family taken to be 1)

    Null deviance: 2476.51  on 244  degrees of freedom
Residual deviance:  334.61  on 240  degrees of freedom
AIC: 1192

Number of Fisher Scoring iterations: 1

              Theta:  75.5
          Std. Err.:  63.7

 2 x log-likelihood:  -1179.7
```

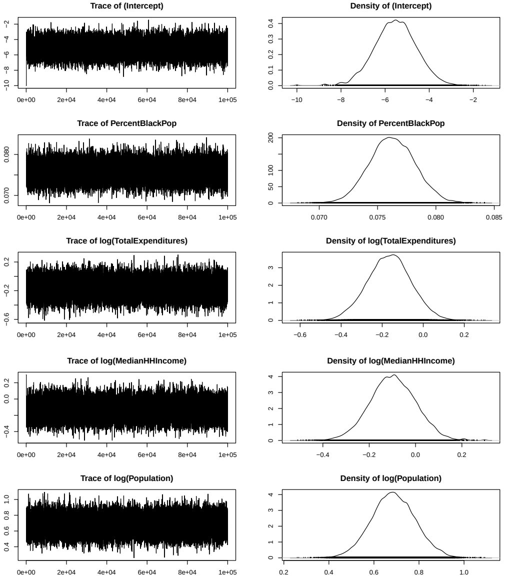
$\rightarrow$ Suggests Poisson...

```
> PoisFit<-glm(BlackHouseMembers~PercentBlackPop+log(TotalExpenditures)+
+                   log(MedianHHIncome)+log(Population),data=DF,family=poisson)

> summary(PoisFit)

Call:
glm(formula = BlackHouseMembers ~ PercentBlackPop + log(TotalExpenditures) +
    log(MedianHHIncome) + log(Population), family = poisson,
    data = DF)

Coefficients:
                        Estimate Std. Error z value Pr(>|z|)
(Intercept)            -10.00217    1.25320   -7.98  1.4e-15 ***
PercentBlackPop          0.07857    0.00199   39.51  < 2e-16 ***
log(TotalExpenditures)  -0.21925    0.10728   -2.04    0.041 *
log(MedianHHIncome)      0.29760    0.12153    2.45    0.014 *
log(Population)          0.75093    0.09811    7.65  1.9e-14 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 2823.42  on 244  degrees of freedom
Residual deviance:  365.67  on 240  degrees of freedom
AIC: 1191

Number of Fisher Scoring iterations: 5
```

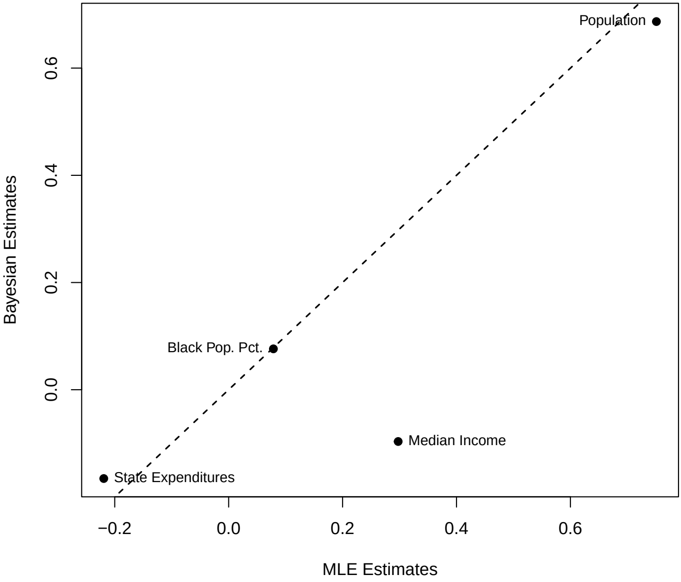
Bayesian Poisson (using bpr)

```
> BayesPois<-sample_bpr(BlackHouseMembers~PercentBlackPop+log(TotalExpenditures)+
+                       log(MedianHHIncome)+log(Population),data=DF,
+                       iter=1e5,burnin=5e2,thin=5,perc_burnin=0)

Running MH sampler with a gaussian prior distribution.
Chains initialized at the maximum likelihood estimates.

Sampling 1e+05 iterations

Sampling completed in 6.65438 secs

> summary(BayesPois)

Call:
 sample_poisreg( formula =  BlackHouseMembers ~ PercentBlackPop + log(TotalExpenditures) +
  log(MedianHHIncome) + log(Population),prior= gaussian,algorithm=Metropolis-
Hastings )

Coefficients:
                            Mean Std. Error    Median Lower CI Upper CI
(Intercept)            -5.528200  0.9431559 -5.523252  -7.4372 -
3.744*
PercentBlackPop         0.076275  0.0019659  0.076260   0.0725   0.0801*
log(TotalExpenditures) -0.166010  0.1060195 -0.164954  -0.3731   0.0440
log(MedianHHIncome)    -0.096640  0.0984735 -0.097365  -0.2842   0.0989
log(Population)         0.687099  0.0969245  0.686325   0.4979   0.8784*
---
 '*' if 95% credible interval does not include zero.

Algorithm:
 Posterior estimates computed on 19900 iterations after discarding the first 500 iterations
    as burn-in, with thinning frequency = 5.
 Mean effective sample size is equal to 9728.
 Acceptance rate is 0.3682.
```

Posteriors for β...



Comparing Bayes & MLE $\hat{\beta}$s



The Good:

- Directly quantifies uncertainty
- Provides direct quantities of interest to researchers.
- Logically consistent and intuitive
- Allow the incorporation of prior information
- Allow the fitting of (very) complex models
- Flexibility

The (Potentially) Bad:

- Inherent subjectivity of choosing priors
- Computational complexity
- Difficulty in knowing when estimates have converged
- Lack of software[1]

[1] This is less and less a concern.

Bayesian Statistics in R

Note: The `Bayesian` CRAN Task View is usually a good place to start...

Packages:

- `arm` – from Gelman et al.'s book *Data Analysis Using Regression and Multilevel/Hierarchical Models*
- `bayesm` – lots of regression-like models (built for marketing, but quite general)
- `MCMCpack` – general + specific Bayesian models, fit via MCMC
- Other packages for causal inference, hierarchical / multilevel models, IRT/measurement, machine learning, time series, network models, spatial models, more...
- Learning tools: `LearnBayes`, `BayesDA`, `BaM`, `rethinking`, others...