

PLSC 504

Scaling and Item Response Theory

November 1, 2023

- Scaling Overview
- Unidimensional Scaling
- Scale Reliability
- Multidimensional Scaling
- Item Response Theory

Define:

- Always $i \in \{1, 2, \dots, N\}$ observations on $k \in \{1, 2, \dots, K\}$ indicators.
- Z or Z_1, Z_2, \dots will indicate the underlying / **latent** trait(s) / phenomena
- D_1, D_2, \dots, D_K are *dichotomous* indicators
- Y_1, Y_2, \dots, Y_K are *continuous* indicators

(Unidimensional) Scaling: History

Thurstone (1927, 1929)

- Comparative, subject-centered
- “Law of Comparative Judgment”: The degree to which any two stimuli can be discriminated is a direct function of the difference in their status as regards the attribute in question.
- Methods: *paired comparisons*, *successive intervals*, and *equal-appearing intervals*.

Likert (1932)

- Non-comparative, subject-centered
- *Summative + unidimensional* → item construction & selection are key

Guttman (1944, 1950) (“scalogram analysis”)

- Comparative; both subject- and stimulus-centered
- The response to each item is a simple function of the sum score

See Mclver and Carmines (1981) for more...

Scaling: Dissimilarities and Distances

Dissimilarities matrix:

$$\Delta_{K \times K} = \begin{bmatrix} 0 & \delta_{21} & \delta_{31} & \dots & \delta_{K1} \\ \delta_{12} & 0 & \delta_{32} & \dots & \delta_{K2} \\ \delta_{13} & \delta_{23} & 0 & \dots & \delta_{K3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{1K} & \delta_{2K} & \delta_{3K} & \dots & 0 \end{bmatrix}$$

where

$$\delta_{ij} = f(\mathbf{Y}_i - \mathbf{Y}_j)$$

What's a "Distance"?

The function δ_{AB} of A and B is a distance function if it meets three criteria:

- $\delta_{AB} \geq 0$ and $\delta_{AB} = 0$ iff $A = B$
- $\delta_{AB} = \delta_{BA}$ (*symmetry*)
- $\delta_{AC} \leq \delta_{AB} + \delta_{BC}$ (*triangle inequality*)

Euclidean (“L2”) Distance:

$$\delta_{ij} = \sqrt{(Y_{i1} - Y_{j1})^2 + (Y_{i2} - Y_{j2})^2 + \dots (Y_{iN} - Y_{jN})^2}$$

Manhattan (“L1”) Distance:

$$\delta_{ij} = |Y_{i1} - Y_{j1}| + |Y_{i2} - Y_{j2}| + \dots |Y_{iN} - Y_{jN}|$$

Minkowski Order- p (“ L^p ”) Distance:

$$\delta_{ij} = \left(\sum_{\ell=1}^N |Y_{i\ell} - Y_{j\ell}|^p \right)^{1/p}$$

Goal: Locate N points in a low (p)-dimensional space $p \ll N$ such that the (say) Euclidean distances between them approximate Δ .

That is, find a set of $N \times P$ points \mathbf{D} such that $d_{ij}(\mathbf{D}) \approx \delta_{ij} \forall i, j$, where

$$d_{ij}(\mathbf{D}) = \sqrt{\sum_{p=1}^P (Y_{ip} - Y_{jp})^2}$$

and $p = \{1, 2, \dots, P\}$ denotes the dimensionality of the space.

For a $K \times K$ dissimilarity matrix Δ , the *stress* associated with a given set of coordinates \mathbf{D} is:

$$\sigma(\mathbf{D}) = \sum_{i < j} w_{ij} [\delta_{ij} - d_{ij}(\mathbf{D})]^2$$

with weights

$$\mathbf{W} = \begin{bmatrix} 0 & w_{21} & w_{31} & \dots & w_{K1} \\ w_{12} & 0 & w_{32} & \dots & w_{K2} \\ w_{13} & w_{23} & 0 & \dots & w_{K3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{1K} & w_{2K} & w_{3K} & \dots & 0 \end{bmatrix}$$

with the constraint that

$$\sum_{i < j} w_{ij} \delta_{ij}^2 = \frac{N(N-1)}{2}.$$

Key: Transformations of the dissimilarities.

Ratio:

- $\hat{d}_{ij} = \delta_{ij}$ (no transformation)
- A special case of *metric* scaling

Interval:

- $\hat{d}_{ij} = a + b(\delta_{ij})$ (linear transformation)
- Also *metric*; “the ratio of differences of distances should be equal to the corresponding ratio of differences in the data”

Nonlinear:

- $\hat{d}_{ij} = g(\delta_{ij})$ (e.g., splines)
- Also *metric*

Ordinal:

- $\hat{d}_{ij} = f(\delta_{ij})$ such that $\delta_{ij} < \delta_{i'j'} \Rightarrow f(\delta_{ij}) < f(\delta_{i'j'})$
- *Monotone* / rank-preserving transformations
- *Nonmetric*

Unidimensional Scaling: The Sum Score

Simplest approach: the *sum score*:

$$\hat{Z}_i = \sum_{k=1}^K Y_{ik}$$

or

$$\hat{Z}_i = \frac{\sum_{k=1}^{K_i} Y_{ik}}{K_i}.$$

Requires:

- $Var(Y_j) = Var(Y_k) \forall j \neq k$
- $Cov(Y_j, Z) = Cov(Y_k, Z) \forall j \neq k$

For $p = 1$, the stress

$$\sigma(\mathbf{D}) = \sum_{i < j} w_{ij} [\delta_{ij} - d_{ij}(\mathbf{D})]^2$$

can be shown to be a function of the orders of the rank orders of the items (see Mair and deLeeuw 2014 for details). This means:

- No need to chose among distances / similarities.
- Solution is via combinatorics.

Example: Cities

```
> Cities[,c(1,4)]
```

	city	longitude
New York	New York	-74.00594
Los Angeles	Los Angeles	-118.24368
Chicago	Chicago	-87.62980
Houston	Houston	-95.36980
Philadelphia	Philadelphia	-75.16522
Phoenix	Phoenix	-112.07404
San Antonio	San Antonio	-98.49363
San Diego	San Diego	-117.16108
Dallas	Dallas	-96.79699
San Jose	San Jose	-121.88633

Cities: Euclidean Distance Matrix (Δ)

```
> CityLong <- data.frame(t(Cities$longitude)) # longitudes in a row
> colnames(CityLong) <- t(Cities$city) # names
> D1long <- dist(t(CityLong)) # distance object
> D1long
```

	New York	Los Angeles	Chicago	Houston	Philadelphia	Phoenix	San Antonio	San Diego	Dallas
Los Angeles	44.2								
Chicago	13.6	30.6							
Houston	21.4	22.9	7.7						
Philadelphia	1.2	43.1	12.5	20.2					
Phoenix	38.1	6.2	24.4	16.7	36.9				
San Antonio	24.5	19.8	10.9	3.1	23.3	13.6			
San Diego	43.2	1.1	29.5	21.8	42.0	5.1	18.7		
Dallas	22.8	21.4	9.2	1.4	21.6	15.3	1.7	20.4	
San Jose	47.9	3.6	34.3	26.5	46.7	9.8	23.4	4.7	25.1

Unidimensional Scaling

```
> library(smacof)
> UDS <- uniscale(D1long)
> UDS
```

```
Call: uniscale(delta = D1long)
```

```
Final stress value: 4.1e-16
Number of accepted permutations: 180836
Number of possible permutations: 3628800
Number of objects: 10
```

```
> UDS$conf
```

New York	Los Angeles	Chicago	Houston	Philadelphia
-1.04236	0.75350	-0.48929	-0.17508	-0.99530
Phoenix	San Antonio	San Diego	Dallas	San Jose
0.50304	-0.04827	0.70955	-0.11715	0.90137

East-West Locations for the Ten Largest U.S. Cities via UDS



Another Example: SCOTUS Votes (1994-2005)

```
> head(SCOTUS)
```

id	Rehnquist	Stevens	OConnor	Scalia	Kennedy	Souter	Thomas	Ginsburg	Breyer
1	0	1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
4	1	1	1	1	1	1	1	1	1
5	0	1	0	0	1	1	0	1	1
7	1	1	1	0	1	1	0	1	1

SCOTUS: Sum Scores and Distances

```
> SumScores <- colSums(SCOTUS[,2:10],na.rm=TRUE) / nrow(SCOTUS)
```

```
> SumScores
```

	Rehnquist	Stevens	O'Connor	Scalia	Kennedy	Souter	Thomas	Ginsburg	Breyer
	0.2838	0.6906	0.4023	0.2648	0.3672	0.6094	0.2451	0.6130	0.5772

```
> D1SCOTUS <- dist(t(SCOTUS[,2:10]))
```

```
> D1SCOTUS
```

	Rehnquist	Stevens	O'Connor	Scalia	Kennedy	Souter	Thomas	Ginsburg	Breyer
Stevens	25.61								
O'Connor	15.94	22.32							
Scalia	13.49	26.80	18.38						
Kennedy	13.27	23.62	16.31	15.62					
Souter	22.52	15.39	18.73	23.56	20.57				
Thomas	13.45	26.81	18.36	10.15	16.03	23.96			
Ginsburg	22.72	15.10	19.65	24.00	20.88	11.27	24.64		
Breyer	22.29	16.09	18.25	24.02	20.66	13.42	24.78	12.69	

```
> SCOTUS.UDS <- uniscale(D1SCOTUS)
```

```
> SCOTUS.UDS
```

```
Call: uniscale(delta = D1SCOTUS)
```

```
Final stress value: 0.317
```

```
Number of accepted permutations: 347136
```

```
Number of possible permutations: 362880
```

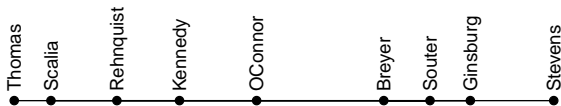
```
Number of objects: 9
```

Sum Scores and UDS

Sum Scores



UDS Results



Reliability: Cronbach's α

For a scale S that is a sum of K separate items Y_1, Y_2, \dots, Y_K ,

$$\alpha = \frac{K}{K-1} \left(1 - \frac{\sum_{k=1}^K \sigma_{Y_k}^2}{\sigma_S^2} \right)$$

where

- $\sigma_{Y_k}^2$ is the variance of item k and
- σ_S^2 is the variance of the scale S .

α : **“The expected correlation of two tests that measure the same construct.”**

Cronbach's α (continued)

α :

- A *lower bound* to reliability.
- If $S = D_1 + D_2 + \dots + D_K$, then $\sigma_{Y_k}^2 = P_k(1 - P_k)$.
- $\alpha \in [0, 1]$ (theoretically)
- Rule of thumb: $\alpha \geq 0.7$ is “adequate”

Limitations (from Sijtsma 2009):

- Requires equal item variances, equal item covariances, and unidimensionality
- Among the “lower bounds,” it's among the smallest
- A better one is the “greatest lower bound” (*glb*), but even it has problems...

Reliability: SCOTUS Data

```
> SCOTUSAlpha <- alpha(SCOTUS[,2:10],check.keys=TRUE)
> SCOTUSAlpha
```

Reliability analysis

Call: alpha(x = SCOTUS[, 2:10], check.keys = TRUE)

```
raw_alpha std.alpha G6(smc) average_r S/N ase mean sd
0.9        0.9        0.93        0.49 8.6 0.0044 0.45 0.35

lower alpha upper      95% confidence boundaries
0.89 0.9 0.9
```

Reliability if an item is dropped:

	raw_alpha	std.alpha	G6(smc)	average_r	S/N	alpha	se
Rehnquist	0.88	0.88	0.91	0.48	7.4	0.0049	
Stevens	0.89	0.89	0.92	0.51	8.4	0.0045	
O'Connor	0.88	0.88	0.92	0.48	7.3	0.0051	
Scalia	0.89	0.89	0.91	0.50	7.9	0.0046	
Kennedy	0.88	0.88	0.92	0.48	7.3	0.0051	
Souter	0.88	0.88	0.91	0.47	7.2	0.0051	
Thomas	0.89	0.89	0.91	0.50	8.0	0.0046	
Ginsburg	0.88	0.88	0.91	0.48	7.4	0.0051	
Breyer	0.88	0.88	0.91	0.49	7.5	0.0050	

Multidimensional Scaling

Types...

- **Metric** - Inputs are interval/ratio-level measures; transformations from δ_{ij} to d_{ij} are cardinal-valued.
- **Non-Metric** - Inputs are binary or ordinal-level; transformations are rank-preserving *only*.

General steps:

- Generate a dissimilarity matrix Δ (choosing distance metric)
- Choose the dimensionality p
- Choose the type of scaling
- Fit model + interpret the results
- Assess model fit + conduct diagnostics

MDS: Model Fit / Diagnostics

Recall: Stress is:

$$\sigma(\mathbf{D}) = \sum_{i < j} w_{ij} [\delta_{ij} - d_{ij}(\mathbf{D})]^2$$

Kruskal's rule of thumb for stress:

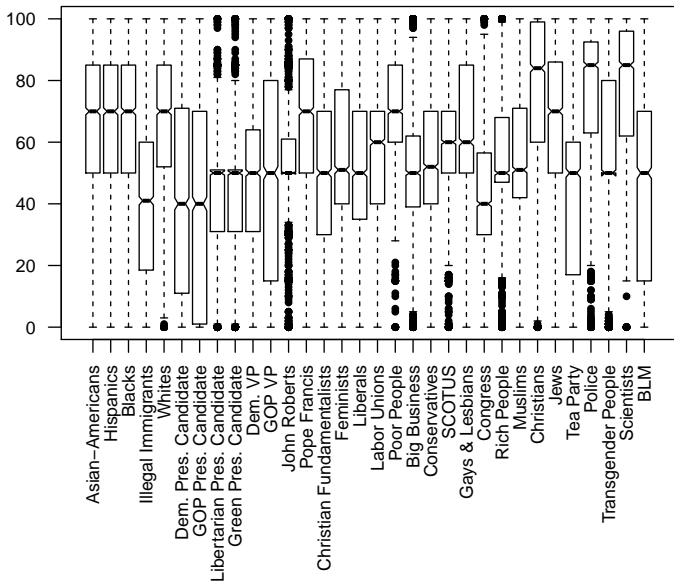
- $\sigma(\mathbf{D}) = 0.20 \rightarrow$ “poor”
- $\sigma(\mathbf{D}) = 0.10 \rightarrow$ “fair”
- $\sigma(\mathbf{D}) = 0.05 \rightarrow$ “good”
- $\sigma(\mathbf{D}) = 0.025 \rightarrow$ “excellent”
- $\sigma(\mathbf{D}) = 0 \rightarrow$ “perfect”

Key diagnostic: **Shepard plot**: a scatterplot of δ_{ij} vs. \hat{d}_{ij} ...

- Illustrates model “fit”
- Also illustrates the transformation of the δ_{ij} s

Also **permutation tests** for model fit...

Example: 2016 ANES Thermometer Scores

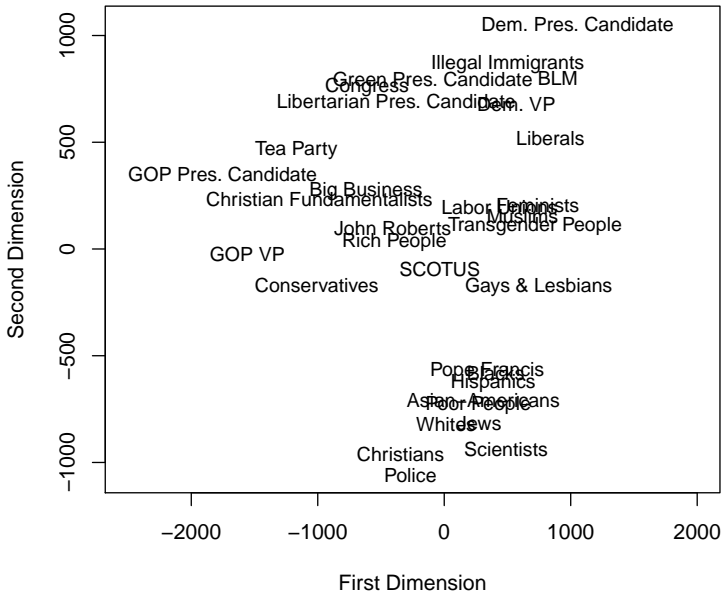


```
> MDS2.alt <- cmdscale(ThermDist,k=2)
```

```
> head(MDS2.alt)
```

	[,1]	[,2]
Asian-Americans	309.90	-709.9
Hispanics	387.51	-627.6
Blacks	407.03	-580.7
Illegal Immigrants	501.74	866.3
Whites	14.72	-821.5
Dem. Pres. Candidate	1054.53	1053.2

MDS Plot, using cmdscale



MDS using mds (ratio scaling)

```
> MDS2 <- mds(ThermDist, ndim=2)  
> MDS2
```

Call:

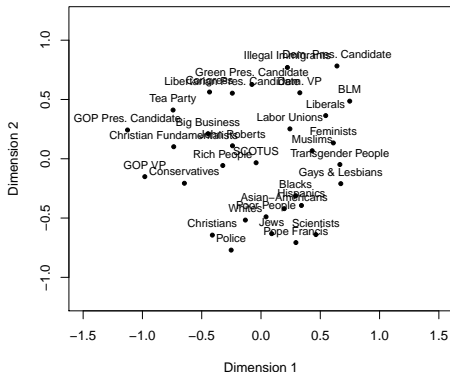
```
mds(delta = ThermDist, ndim = 2)
```

Model: Symmetric SMACOF

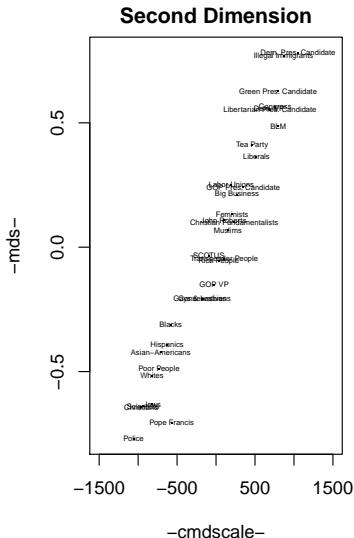
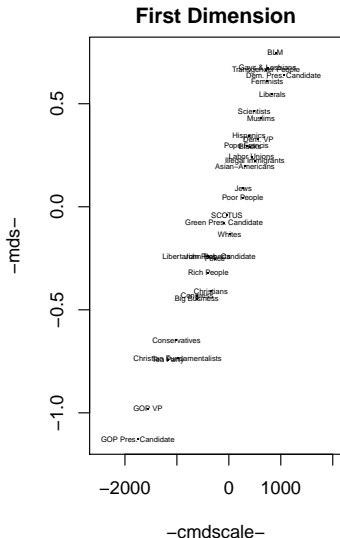
Number of objects: 32

Stress-1 value: 0.221

Number of iterations: 87



cmdscale and mds Comparison



SCOTUS Redux ($p = 2$)

```
> SCR <- mds(D1SCOTUS, ndim=2, type="ratio")  
> SCR
```

Call:

```
mds(delta = D1SCOTUS, ndim = 2, type = "ratio")
```

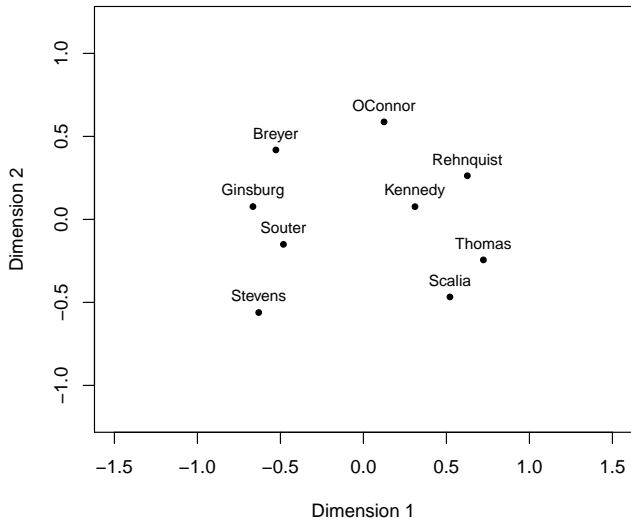
Model: Symmetric SMACOF

Number of objects: 9

Stress-1 value: 0.184

Number of iterations: 73

SCOTUS MDS Plot



Useful References

- Kruskal, J.B., and M. Wish 1978. *Multidimensional Scaling*. Sage.
- McIver, John, and Edward C. and Carmines. 1981. *Unidimensional Scaling*. Sage Publications.
- Davison, Mark L. *Multidimensional Scaling*. 1983. New York: Wiley.
- Cox, Trevor F. and Michael A. A. Cox. 2000. *Multidimensional Scaling*, 2nd Ed. New York: Chapman & Hall.
- Borg, Ingwer, and Groenen, Patrick. 2005. *Modern Multidimensional Scaling: Theory and Applications*, 2nd Ed. Berlin: Springer-Verlag.
- Borg, Ingwer, Patrick Groenen, and Patrick Mair. 2013. *Applied Multidimensional Scaling*. Berlin: Springer-Verlag.

Useful R Packages and Routines

Distances, Proximities, etc.

- `dist` function (base R)
- `distances` package
- `proxy` package

Scaling

- `stats::cmdscale` (classical MDS, in base R)
- `smacof` (state-of-the-art MDS package)
- `vegan` (ecology package; has some good MDS routines)
- Others...

- `smacof` documentation.
- Seven ways to do MDS in R.
- Jan De Leeuw's website.

Item Response Theory (IRT)

Item Response Theory (“IRT”)

- Origins in psychometrics / testing
- *Measurement* model – (typically) *no* **X**
- *Unidimensional*
- *Discrete* responses **Y**
- Equally descriptive and inferential

Y^* = latent trait (“ability”)

Y = observed measures

- $i \in \{1, 2 \dots N\}$ indexes *subjects* / *units*, and
- $j \in \{1, 2, \dots J\}$ indexes *items* / *measures*.

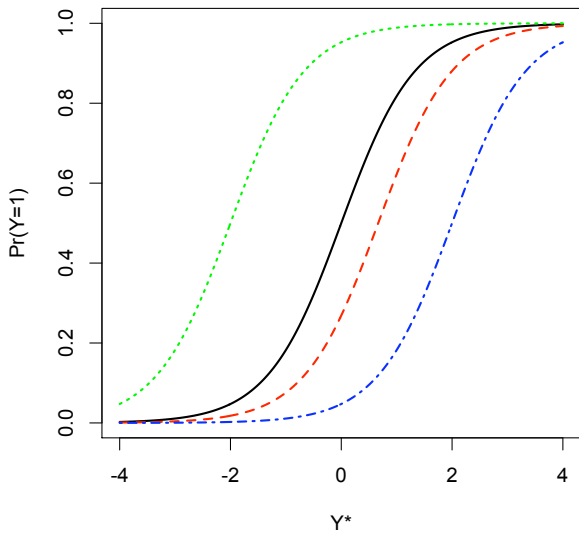
$$Y_{ij} = \begin{cases} 0 & \text{if subject } i \text{ gets item } j \text{ “incorrect,”} \\ 1 & \text{if subject } i \text{ gets item } j \text{ “correct.”} \end{cases}$$

One-Parameter Logistic Model (“1PLM”)

$$\Pr(Y_{ij} = 1) = \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)}$$

Here,

- θ_i = respondent i 's *ability*,
- β_j = item j 's *difficulty*.
- $\beta_j \equiv$ value of Y^* where $\Pr(Y_{ij} = 1) = 0.50$



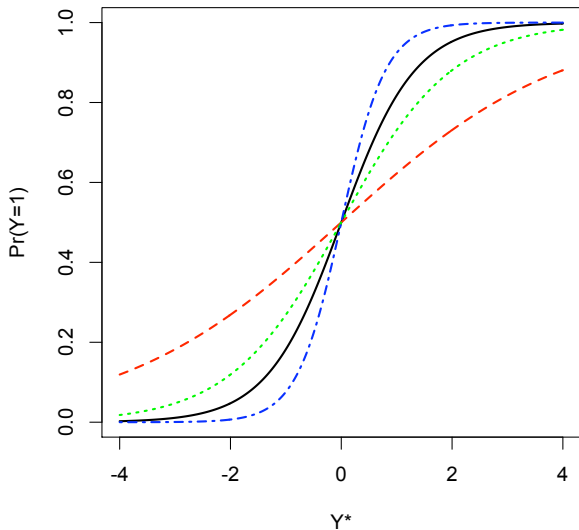
- a.k.a. “Rasch” model (Rasch 1960)
- Implicit “slope” = 1.0
- Implies items are equally “discriminating”
- If not...

Two-Parameter Logistic Model (“2PLM”)

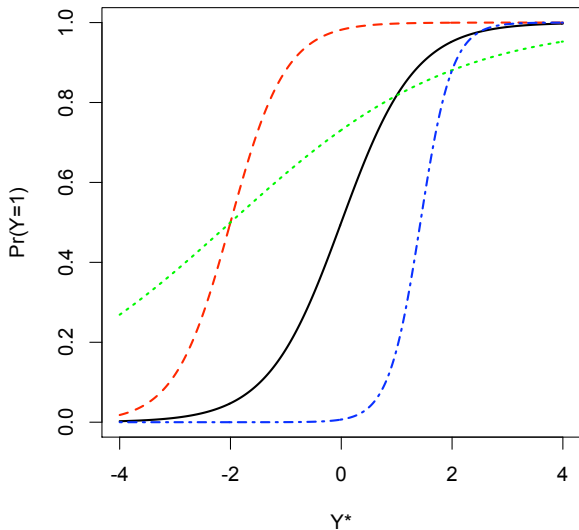
$$\Pr(Y_{ij} = 1) = \frac{\exp[\alpha_j(\theta_i - \beta_j)]}{1 + \exp[\alpha_j(\theta_i - \beta_j)]}$$

- θ_i = respondent i 's *ability*,
- β_j = item j 's *difficulty*,
- α_j = item j 's *discrimination*.

Identical Difficulty, Different Discrimination



Different Difficulty & Discrimination



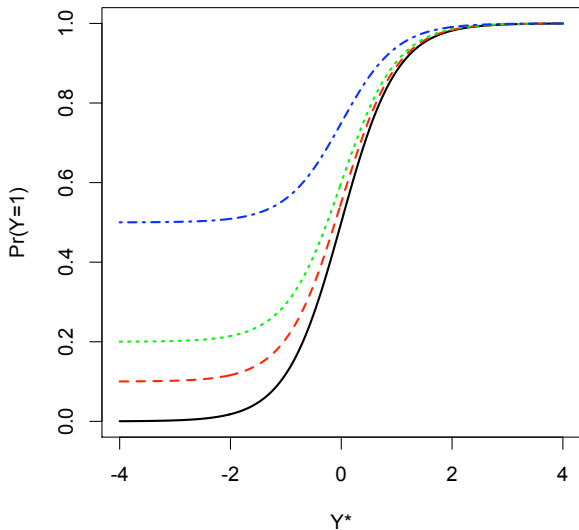
- Due to Birnbaum (1968)
- Similar to “typical” logit...
- Nests the 1PLM as a special case ($\alpha_j = 1 \forall j$)

Three-Parameter Logistic Model (“3PLM”)

$$\Pr(Y_{ij} = 1) = \delta_j + (1 - \delta_j) \left\{ \frac{\exp[\alpha_j(\theta_i - \beta_j)]}{1 + \exp[\alpha_j(\theta_i - \beta_j)]} \right\}$$

- θ_i = respondent i 's *ability*,
- β_j = item j 's *difficulty*,
- α_j = item j 's *discrimination*.
- δ_j = *lower asymptote* of $\Pr(Y_{ij} = 1)$ (incorrectly: “guessing” parameter).

3PLM, Constant α & β , Varying δ



The Two Big Assumptions

- *Unidimensionality*
- *Local Item Independence* (“No LID”):

$$\text{Cov}(Y_{ij}, Y_{ik} | \theta_i) = 0 \quad \forall j \neq k$$

$$P_{ij} = \Pr(Y_{ij} = 1),$$

$$\begin{aligned} Q_{ij} &= \Pr(Y_{ij} = 0) \\ &= 1 - \Pr(Y_{ij} = 1), \end{aligned}$$

$$\psi = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_J \\ \alpha_1 \\ \vdots \\ \alpha_J \\ \delta_1 \\ \vdots \\ \delta_J \end{pmatrix}.$$

Known $\Psi = \alpha, \beta, \delta$:

$$L(\mathbf{Y}|\Psi) = \prod_{j=1}^J P_{ij}^{Y_{ij}} Q_{ij}^{1-Y_{ij}}.$$

Known θ :

$$L(\mathbf{Y}|\theta) = \prod_{i=1}^N P_{ij}^{Y_{ij}} Q_{ij}^{1-Y_{ij}}.$$

$$L(\mathbf{Y}|\Psi, \theta) = \prod_{i=1}^N \prod_{j=1}^J P_{ij}^{Y_{ij}} Q_{ij}^{1-Y_{ij}}$$

$$\ln L(\mathbf{Y}|\Psi, \theta) = \sum_{i=1}^N \sum_{j=1}^J Y_{ij} \ln P_{ij} + (1 - Y_{ij}) \ln Q_{ij}.$$

- $N + J$ parameters in the 1PLM,
- $N + 2J$ parameters in the 2PLM,
- $N + 3J$ parameters in the 3PLM.

But...

- NJ observations,
- Asymptotics as $N \rightarrow \infty, J \rightarrow \infty \dots$

Estimation: Conditional Likelihood

Total score is:

$$T_i = \sum_{j=1}^J Y_{ij} \in \{0, 1, \dots, J\}$$

$$L = \prod_{i=1}^N \frac{\exp[\alpha_j(\theta_t - \beta_j)]}{1 + \exp[\alpha_j(\theta_t - \beta_j)]}$$

θ_t are “score-group” parameters corresponding to the $J + 1$ possible values of T .

Estimation: Conditional Likelihood

- Equivalent to fitting a conditional logit model:

$$\Pr(Y_{ij} = 1) = \frac{\exp(\mathbf{Z}_{ij}\gamma)}{\sum_{j=1}^J \exp(\mathbf{Z}_{ij}\gamma)}$$

with \mathbf{Z}_{ij} = “item dummies.”

- Useful only for 1PLM (since T_i is a sufficient statistic for θ_i).

Estimation: Marginal Likelihood

$$L(\mathbf{Y}|\Psi, \theta) = \prod_{i=1}^N \left[\int_{-\infty}^{\infty} \prod_{j=1}^J P_{ij}^{Y_{ij}} Q_{ij}^{1-Y_{ij}} d\theta \right]$$

- Analogous to “random effects” ...
- Eliminates inconsistency as $N \rightarrow \infty$, *but*
- Requires *strong* exogeneity of θ and Ψ .

Estimation: Bayesian Approaches

- Place priors on θ, Ψ ;
- Estimate via sampling from posteriors, via MCMC.
- Eliminates problems with $\hat{\alpha}, \hat{\beta}, \hat{\theta} = \infty$ (see below).
- Easily extensible to other circumstances (hierarchical/multilevel, etc.)

Two Issues:

- *Scale* invariance: $L(\hat{\Psi}) = L(\hat{\Psi} + c)$
- *Rotational* invariance: $L(\hat{\Psi}) = L(-\hat{\Psi})$

Fixes:

- Set one (arbitrary) $\beta_j = 0$, and another (arbitrary) $\beta_k > 0$, or
- Fix two θ_i s at specific values.

Further (Potential) Concerns

- $Y_{ij} = 0/1 \ \forall i \rightarrow \beta_j = \pm\infty$.
- $Y_{ij} = 0/1 \ \forall j \rightarrow \theta_i = \pm\infty$.
- Separation / “empty cells” $\rightarrow \alpha_j = \pm\infty$.
- Problematic for joint and conditional approaches; more easily dealt with in the Bayesian framework.

- Estimates of $\hat{\alpha}$ s, $\hat{\beta}$ s, and/or $\hat{\delta}$ s, plus $\hat{\theta}$ s
- Associated s.e.s / c.i.s
- “Scale-free” quantities of interest...

- Library `ltm` (marginal estimation)
 - `rasch` (1PLM)
 - `ltm` (2PLM)
 - `tpm` (3PLM)
- Library `MCMCpack` (Bayesian estimation)
 - 1 and 2PLM
 - Standard, hierarchical, dynamic, multidimensional
- `ideal` (in library `pscl`) (Bayesian estimation)
 - 1 and 2PLM
 - k -dimensional
 - takes a `rollcall` object
- Other packages: `eRm`, `irtoys`, `irtProb`, `MiscPsycho`, etc.

Example: SCOTUS Voting, 1994-2005

```
> summary(SCOTUS)
```

id	Rehnquist	Stevens	OConnor	Scalia
Min. : 1	Min. :0.00	Min. :0.00	Min. :0.0	Min. :0.00
1st Qu.: 377	1st Qu.:0.00	1st Qu.:0.00	1st Qu.:0.0	1st Qu.:0.00
Median : 753	Median :0.00	Median :1.00	Median :0.0	Median :0.00
Mean : 753	Mean :0.28	Mean :0.69	Mean :0.4	Mean :0.27
3rd Qu.:1129	3rd Qu.:1.00	3rd Qu.:1.00	3rd Qu.:1.0	3rd Qu.:1.00
Max. :1505	Max. :1.00	Max. :1.00	Max. :1.0	Max. :1.00
	NA's :49	NA's :51	NA's :55	NA's :41

Kennedy	Souter	Thomas	Ginsburg	Breyer
Min. :0.00	Min. :0.0	Min. :0.00	Min. :0.00	Min. :0.00
1st Qu.:0.00	1st Qu.:0.0	1st Qu.:0.00	1st Qu.:0.00	1st Qu.:0.00
Median :0.00	Median :1.0	Median :0.00	Median :1.00	Median :1.00
Mean :0.37	Mean :0.6	Mean :0.25	Mean :0.61	Mean :0.57
3rd Qu.:1.00	3rd Qu.:1.0	3rd Qu.:0.00	3rd Qu.:1.00	3rd Qu.:1.00
Max. :1.00	Max. :1.0	Max. :1.00	Max. :1.00	Max. :1.00
NA's :32	NA's :37	NA's :44	NA's :39	NA's :61

1PLM Using rasch

```
> # 1PLM / Rasch Model:  
> require(ltm)  
> OnePLM<-rasch(SCOTUS[c(2:10)])  
> summary(OnePLM)
```

Model Summary:

log.Lik	AIC	BIC
-5529	11079	11132

Coefficients:

	value	std.err	z.vals
Dffclt.Rehnquist	0.46	0.040	11.5
Dffclt.Stevens	-0.59	0.030	-19.8
Dffclt.OConnor	0.14	0.030	4.6
Dffclt.Scalia	0.52	0.041	12.5
Dffclt.Kennedy	0.21	0.032	6.5
Dffclt.Souter	-0.36	0.027	-13.1
Dffclt.Thomas	0.60	0.043	13.8
Dffclt.Ginsburg	-0.37	0.027	-13.4
Dffclt.Breyer	-0.26	0.027	-9.9
Dscrmn	3.74	0.130	28.9

Integration:

method: Gauss-Hermite
quadrature points: 21

Optimization:

Convergence: 0
max(|grad|): 0.0027
quasi-Newton: BFGS

Converted to $\Pr(\widehat{Y_i = 1} | \theta_i = 0)$

```
> # Convert to probabilities given theta=0  
>  
> coef(OnePLM, prob=TRUE, order=TRUE)
```

	Dffc1t	Dscrmn	P(x=1 z=0)
Stevens	-0.59	3.7	0.900
Ginsburg	-0.37	3.7	0.797
Souter	-0.36	3.7	0.791
Breyer	-0.26	3.7	0.729
O'Connor	0.14	3.7	0.373
Kennedy	0.21	3.7	0.311
Rehnquist	0.46	3.7	0.151
Scalia	0.52	3.7	0.126
Thomas	0.60	3.7	0.096

Alternative Model Constraining $\alpha = 1.0$

```
> AltOnePLM<-rasch(IRTData, constraint=cbind(length(IRTData)+1,1))  
> summary(AltOnePLM)
```

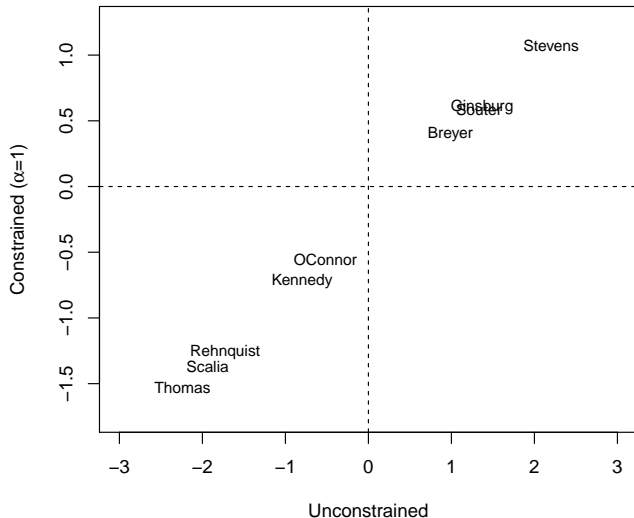
Model Summary:

log.Lik	AIC	BIC
-6452	12923	12971

Coefficients:

	value	std.err	z.vals
Dffclt.Rehnquist	1.26	0.073	17.3
Dffclt.Stevens	-1.07	0.071	-15.1
Dffclt.OConnor	0.56	0.069	8.1
Dffclt.Scalia	1.37	0.074	18.6
Dffclt.Kennedy	0.72	0.069	10.4
Dffclt.Souter	-0.58	0.068	-8.6
Dffclt.Thomas	1.53	0.075	20.3
Dffclt.Ginsburg	-0.61	0.068	-8.9
Dffclt.Breyer	-0.40	0.068	-5.9
Dscrmn	1.00	NA	NA

Constrained and Unconstrained 1PLM $\hat{\beta}$ s



```
> TwoPLM<-ltm(IRTData ~ z1)
> summary(TwoPLM)
```

Coefficients:

	value	std.err	z.vals
Dffc1t.Rehnquist	0.44	0.035	12.3
Dffc1t.Stevens	-0.63	0.038	-16.7
Dffc1t.OConnor	0.14	0.026	5.6
Dffc1t.Scalia	0.59	0.042	14.1
Dffc1t.Kennedy	0.20	0.028	7.2
Dffc1t.Souter	-0.27	0.025	-10.7
Dffc1t.Thomas	0.68	0.044	15.2
Dffc1t.Ginsburg	-0.29	0.025	-11.8
Dffc1t.Breyer	-0.24	0.025	-9.6
Dscrmn.Rehnquist	4.77	0.377	12.7
Dscrmn.Stevens	2.46	0.165	14.9
Dscrmn.OConnor	4.14	0.341	12.1
Dscrmn.Scalia	2.82	0.188	15.0
Dscrmn.Kennedy	4.74	0.448	10.6
Dscrmn.Souter	6.69	0.535	12.5
Dscrmn.Thomas	2.84	0.190	14.9
Dscrmn.Ginsburg	5.83	0.439	13.3
Dscrmn.Breyer	3.76	0.253	14.9

2PLM: Probabilities and Testing

```
> coef(TwoPLM, prob=TRUE, order=TRUE)
```

	Dffc1t	Dscrmn	P(x=1 z=0)
Stevens	-0.63	2.5	0.82
Ginsburg	-0.29	5.8	0.85
Souter	-0.27	6.7	0.86
Breyer	-0.24	3.8	0.71
OConnor	0.14	4.1	0.35
Kennedy	0.20	4.7	0.28
Rehnquist	0.44	4.8	0.11
Scalia	0.59	2.8	0.16
Thomas	0.68	2.8	0.13

```
> anova(OnePLM, TwoPLM)
```

Likelihood Ratio Table						
	AIC	BIC	log.Lik	LRT	df	p.value
OnePLM	11079	11132	-5529			
TwoPLM	10882	10978	-5423	212.7	8	<0.001

```
> ThreePLM<-tpm(IRTData)
> summary(ThreePLM)
```

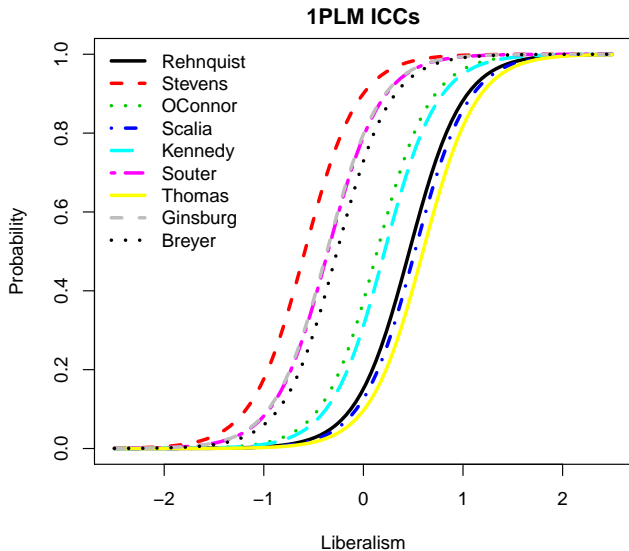
Coefficients:

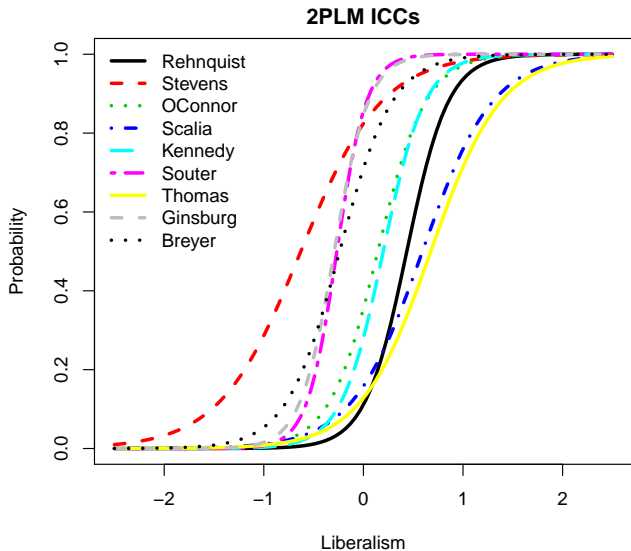
	value	std.err	z.vals
Gussng.Rehnquist	0.049	0.008	6.260
Gussng.Stevens	0.000	0.001	0.018
Gussng.OConnor	0.043	0.013	3.415
Gussng.Scalia	0.097	0.011	9.119
Gussng.Kennedy	0.071	0.014	5.162
Gussng.Souter	0.011	0.029	0.386
Gussng.Thomas	0.087	0.010	8.900
Gussng.Ginsburg	0.000	0.000	0.009
Gussng.Breyer	0.000	0.000	0.004
Dffclt.Rehnquist	0.716	0.030	23.511
Dffclt.Stevens	-0.630	0.038	-16.434
Dffclt.OConnor	0.340	0.040	8.537
Dffclt.Scalia	0.759	1.766	0.430
Dffclt.Kennedy	0.500	0.041	12.170
Dffclt.Souter	-0.294	0.063	-4.642
Dffclt.Thomas	0.808	10.610	0.076
Dffclt.Ginsburg	-0.329	0.030	-10.970
Dffclt.Breyer	-0.232	0.031	-7.439
Dscrmn.Rehnquist	8.735	4.259	2.051
Dscrmn.Stevens	2.577	0.181	14.214
Dscrmn.OConnor	3.979	0.439	9.068
Dscrmn.Scalia	26.537	578.889	0.046
Dscrmn.Kennedy	4.408	0.588	7.498
Dscrmn.Souter	6.698	1.416	4.731
Dscrmn.Thomas	34.074	2779.161	0.012
Dscrmn.Ginsburg	5.800	0.509	11.394
Dscrmn.Breyer	3.538	0.231	15.335

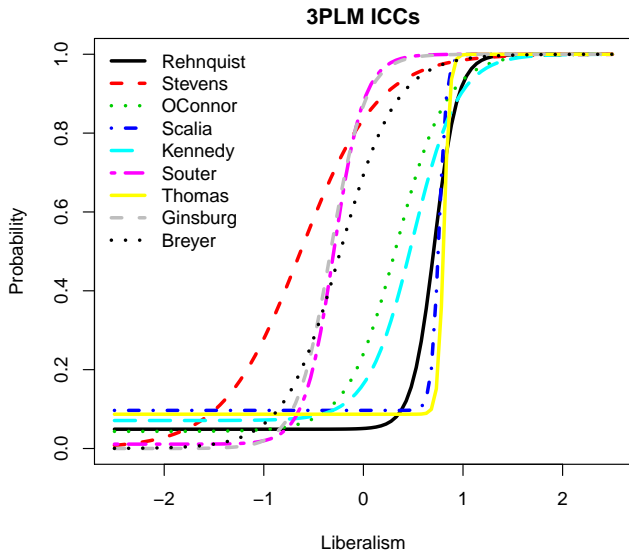
```
> anova(TwoPLM, ThreePLM)
```

Likelihood Ratio Table

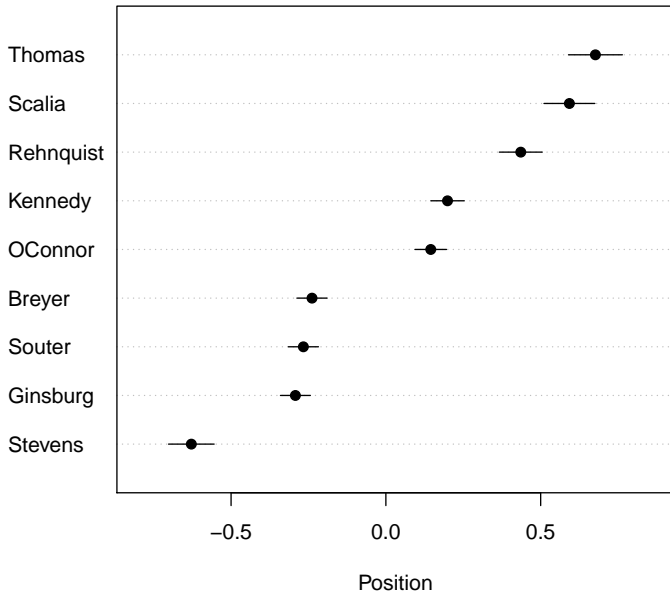
	AIC	BIC	log.Lik	LRT	df	p.value
TwoPLM	10882	10978	-5423			
ThreePLM	10737	10881	-5342	162.94	9	<0.001



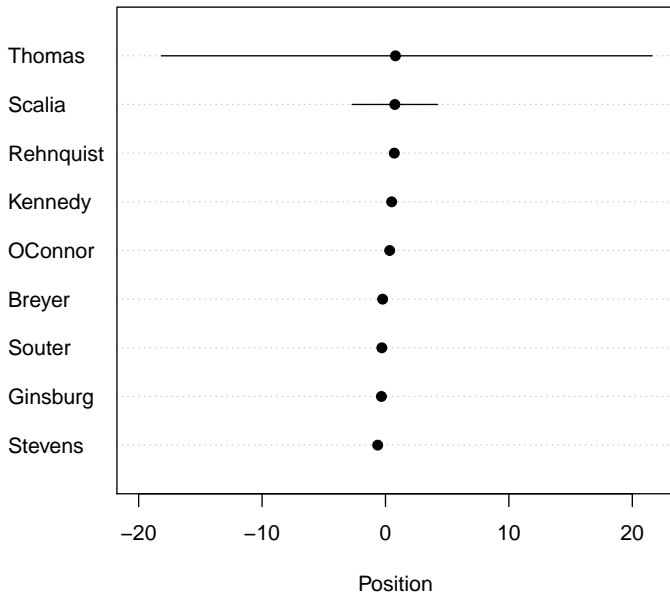




Presenting Measures: Ladderplots (2PLM)



3PLM Ladderplot (#wtf)



Miscellaneous Things, I: Dimensionality

- Usually, *unidimensional*
- Sometimes, *two-dimensional*
- Tests:
 - Tetrachoric correlations among items
 - DIMTEST (Stout & Zhang, etc.)
 - Yen's Q_3
 - 1-D vs. 2-D comparisons (LR tests, etc.)

Miscellaneous Things, II: “DIF”

- *Differential item functioning*
- Formally,

$$\Pr(Y_{ij} = 1) = \Lambda[\alpha_j(\theta_i - \mathbf{X}_i\beta_j)].$$

- \rightarrow violates *local item independence*

- Nominal/Multinomial Y
- Ordinal Y :
 - *Graded response model* (“GRM”) (Samejima 1969)
 - *Partial credit model* (Masters 1982)
 - *Generalized partial credit model* (Muraki 1992)
- Models for mixed response types (Thissen and Wainer 2001, 2003)
- Hierarchical IRT models (e.g. Bolt and Kim 2005)
- Models with covariates (e.g., DeBoeck and Wilson 2004)

Further Reading / Useful References

Hambleton, Ronald K., H. Swaminathan, and H. Jane Rogers. 1991. *Fundamentals of Item Response Theory*. Newbury Park CA: Sage Publications.

Fahrmeier, L., and G. Tutz. 2000. *Multivariate Statistical Modelling Based on Generalized Linear Models*. Berlin: Springer-Verlag.

De Boeck, Paul, and Mark Wilson, Eds. 2004. *Explanatory Item Response Models: A Generalized Linear and Nonlinear Approach*. New York: Springer.

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Baker, Frank B., and Seock-Ho Kim. 2017. *The Basics of Item Response Theory Using R*. New York: Springer.

van der Linden, Wim J., Ed. 2018. *Handbook of Item Response Theory* (3 vol.). New York: Chapman & Hall CRC.