PLSC 504 – Fall 2024

Regression Models for Nominal and Binary Responses

September 4, 2024

Binary Outcomes: Quick Review

Latent:

$$Y_i^* = \mathbf{X}_i\boldsymbol{\beta} + u_i$$

Observed:

$$\begin{aligned} Y_i &= 0 \text{ if } Y_i^* < 0 \\ Y_i &= 1 \text{ if } Y_i^* \geq 0 \end{aligned}$$

So:

$$\begin{aligned} \Pr(Y_i = 1) &= \Pr(Y_i^* \geq 0) \\ &= \Pr(\mathbf{X}_i\boldsymbol{\beta} + u_i \geq 0) \\ &= \Pr(u_i \geq -\mathbf{X}_i\beta) \\ &= \Pr(u_i \leq \mathbf{X}_i\beta) \\ &= \int_{-\infty}^{\mathbf{X}_i\boldsymbol{\beta}} f(u)du \end{aligned}$$

"Standard logistic" PDF:

$$\Pr(u) \equiv \lambda(u) = \frac{\exp(u)}{[1+\exp(u)]^2}$$

CDF:

$$\begin{aligned}
\Lambda(u) &= \int \lambda(u) du \\
&= \frac{\exp(u)}{1+\exp(u)} \\
&= \frac{1}{1+\exp(-u)}
\end{aligned}$$

Logistic → "Logit"

$$
\begin{aligned}
\Pr(Y_i = 1) &= \Pr(Y_i^* > 0) \\
&= \Pr(u_i \leq \mathbf{X}_i\boldsymbol{\beta}) \\
&= \Lambda(\mathbf{X}_i\boldsymbol{\beta}) \\
&= \frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})} \\
\text{(equivalently)} &= \frac{1}{1+\exp(-\mathbf{X}_i\boldsymbol{\beta})}
\end{aligned}
$$

$$
L = \prod_{i=1}^{N} \left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)^{Y_i} \left[1 - \left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right]^{1-Y_i}
$$

$$
\ln L = \sum_{i=1}^{N} Y_i \ln\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right) + (1-Y_i)\ln\left[1 - \left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right]
$$

Normal → "Probit"

$$\begin{aligned} \Pr(Y_i = 1) &= \Phi(\mathbf{X}_i\beta) \\ &= \int_{-\infty}^{\mathbf{X}_i\beta} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\mathbf{X}_i\beta)^2}{2}\right) d\mathbf{X}_i\beta \end{aligned}$$

$$L = \prod_{i=1}^{N} [\Phi(\mathbf{X}_i\beta)]^{Y_i} [1 - \Phi(\mathbf{X}_i\beta)]^{(1-Y_i)}$$

$$\ln L = \sum_{i=1}^{N} Y_i \ln\Phi(\mathbf{X}_i\beta) + (1 - Y_i) \ln[1 - \Phi(\mathbf{X}_i\beta)]$$

Logit and Probit, Explained

Things we talked about at length in PLSC 503 (here and here; code here and here):

- Odds ratios and the random utility model
- Model estimation and interpretation
- Marginal effects, predictions, etc.
- Assessing model fit
- A couple variants (e.g., c-log-log)

Extensions: Two Topics, One Theme

Things:

- Models for dealing with "separation"
- Models for *rare events*

Common Focus: Shortage of information on Y

"Separation" = "perfect prediction" = "monotone likelihood"

Intuition: House votes on the PPACA (3/21/2010)

```
      Dems
Yeas    0   1
   0 178  34
   1   0 219
```

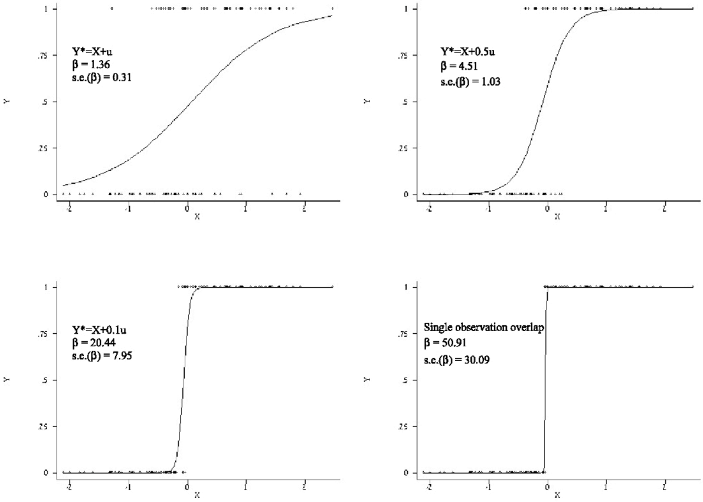
$\Pr(Y = 1 | X = 0) = ?$

"Separation" means that:

- $\hat{\beta}_X = \pm\infty$
- $\widehat{\text{s.e.}}_{\beta} = \infty$
- $\left.\frac{\partial^2 \ln L}{\partial X^2}\right|_{\hat{\beta}} = 0$ (monotone likelihood)

Separation Illustrated

Figure 1: Actual and Predicted Values, Simulated Logistic Regressions



Separation: What Happens

```
> set.seed(7222009)
> Z<-rnorm(500)
> W<-rnorm(500)
> Y<-rbinom(500,size=1,prob=plogis((0.2+0.5*W-0.5*Z)))
> X<-rbinom(500,1,(pnorm(Z)))
> X<-ifelse(Y==0,0,X)  # Induce separation of Y on X

> summary(glm(Y~W+Z+X,family="binomial"))

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)   -0.638      0.133   -4.81  1.5e-06 ***
W              0.653      0.140    4.67  3.0e-06 ***
Z             -1.134      0.146   -7.76  8.3e-15 ***
X             20.915    861.458    0.02     0.98
---
Number of Fisher Scoring iterations: 18


# Change the maximum # of iterations / convergence tolerance:

> summary(glm(Y~W+Z+X,family="binomial",maxit=100,epsilon=1e-16))

Coefficients:
               Estimate  Std. Error z value Pr(>|z|)
(Intercept)      -0.638       0.133   -4.81  1.5e-06 ***
W                 0.653       0.140    4.67  3.0e-06 ***
Z                -1.134       0.146   -7.76  8.3e-15 ***
X                34.915 5978532.779    0.00        1
---
Number of Fisher Scoring iterations: 32

Warning message:
glm.fit: fitted probabilities numerically 0 or 1 occurred
```

One Solution: Exact Logistic Regression

Exact logistic regression (ELR):

- Cox (1970, Ch. 4); Hirji et al. (1987 *JASA*); Mehta & Patel (1995 *Stat. Med.*); Forster et al. (2003 *Stat. & Comp.*); Zamar and Graham (2007 *J. Stat. Soft.*).
- Conditions on permutations of covariate patterns
- $\longrightarrow$ Always has finite solutions for $\hat{\beta}$
- Implementation:
  - `elrm` in R ; `exlogistic` in Stata
  - Fitted via MCMC; see Forster et al. for details
  - In practice, there are often computational issues...

Firth's (1993) Correction

Firth proposed:

$$L(\boldsymbol{\beta}|Y)^* = L(\boldsymbol{\beta}|Y)\,|\mathbf{I}(\boldsymbol{\beta})|^{\frac{1}{2}}$$

$$\ln L(\boldsymbol{\beta}|Y)^* = \ln L(\boldsymbol{\beta}|Y) + 0.5\ln|\mathbf{I}(\boldsymbol{\beta})|$$

"Penalized likelihood":

- Is consistent
- Eliminates small-sample bias
- Exist given separation
- To Bayesians, it's "Jeffreys' prior":

$$P(\theta) = \sqrt{\det\left[I(\theta)\right]}$$

- "Profile" (= "concentrated") likelihood
- $\hat{\beta}$ can be asymmetrical...
- $\rightarrow$ can affect "normal" inference...
- Plotting the profile likelihood and calculating alternative C.I.s is recommended

Two directions:

- R
  - `elrm` (exact logistic regression via MCMC)
  - `brlr` ("bias-reduced logistic regression")
  - `logistf` ("Firth's logistic regression")

- Stata
  - `exlogistic` (exact logistic regression)
  - `firthlogit` (Firth corrected logit)

Example: Pets as Family

Some data, and a silly question:

- CBS/NYT Poll, April 1997
- Standard political/demographics, plus
- "Do you consider your pet to be a member of your family, or not?"
- Yes = 84.4%, No = 15.6%

Data:

```
> summary(Pets)

   petfamily          female                  married             partyid              education
 Min.   :0.000   Min.   :0.000   Married       :442   Democrat   :225   < HS          : 71
 1st Qu.:1.000   1st Qu.:0.000   Widowed       : 46   Independent:214   HS diploma   :244
 Median :1.000   Median :1.000   Divorced/Sep:118   GOP          :229   Some college:184
 Mean   :0.844   Mean   :0.556   NBM           :118   NA's         : 58   College Grad:131
 3rd Qu.:1.000   3rd Qu.:1.000   NA's          :  2                      Post-Grad   : 96
 Max.   :1.000   Max.   :1.000
```

Pets as Family: Basic Model

```
> Pets.1<-glm(petfamily~female+as.factor(married)+as.factor(partyid)
+             +as.factor(education),data=Pets,family=binomial)
> summary(Pets.1)

Coefficients:
                                 Estimate Std. Error z value Pr(>|z|)
(Intercept)                        2.0133     0.5388    3.74  0.00019 ***
femaleMale                        -0.6959     0.2142   -3.25  0.00116 **
as.factor(married)Married         -0.0657     0.2911   -0.23  0.82147
as.factor(married)NBM              0.4599     0.3957    1.16  0.24504
as.factor(married)Widowed         -0.1568     0.4921   -0.32  0.75007
as.factor(partyid)Democrat        -0.1241     0.4286   -0.29  0.77213
as.factor(partyid)GOP             -0.0350     0.4321   -0.08  0.93537
as.factor(partyid)Independent     -0.1521     0.4299   -0.35  0.72338
as.factor(education)College Grad   0.2511     0.4121    0.61  0.54228
as.factor(education)HS diploma     0.0595     0.3685    0.16  0.87182
as.factor(education)Post-Grad      0.1946     0.4331    0.45  0.65321
as.factor(education)Some college   0.0587     0.3867    0.15  0.87928
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

    Null deviance: 627.14  on 723  degrees of freedom
Residual deviance: 612.76  on 712  degrees of freedom
AIC: 636.8

Number of Fisher Scoring iterations: 4
```

Pets as Family: More Complicated Model

```
> Pets.2<-glm(petfamily~female+as.factor(married)*female+as.factor(partyid)+
+               as.factor(education),data=Pets,family=binomial)

> summary(Pets.2)

Coefficients:
                                     Estimate Std. Error z value Pr(>|z|)
(Intercept)                            2.2971     0.6166    3.73   0.0002 ***
femaleMale                            -1.1833     0.5305   -2.23   0.0257 *
as.factor(married)Married             -0.3218     0.4470   -0.72   0.4716
as.factor(married)NBM                  0.1854     0.6140    0.30   0.7628
as.factor(married)Widowed             -0.7415     0.5780   -1.28   0.1995
as.factor(partyid)Democrat            -0.1575     0.4297   -0.37   0.7140
as.factor(partyid)GOP                 -0.0445     0.4334   -0.10   0.9182
as.factor(partyid)Independent         -0.1757     0.4312   -0.41   0.6837
as.factor(education)College Grad       0.2332     0.4137    0.56   0.5730
as.factor(education)HS diploma         0.0558     0.3703    0.15   0.8801
as.factor(education)Post-Grad          0.2171     0.4342    0.50   0.6171
as.factor(education)Some college       0.0358     0.3890    0.09   0.9266
femaleMale:as.factor(married)Married   0.4853     0.5908    0.82   0.4114
femaleMale:as.factor(married)NBM       0.5260     0.8051    0.65   0.5136
femaleMale:as.factor(married)Widowed  15.2516   549.3719    0.03   0.9779
---

    Null deviance: 627.14  on 723  degrees of freedom
Residual deviance: 607.42  on 709  degrees of freedom
AIC: 637.4

Number of Fisher Scoring iterations: 14
```

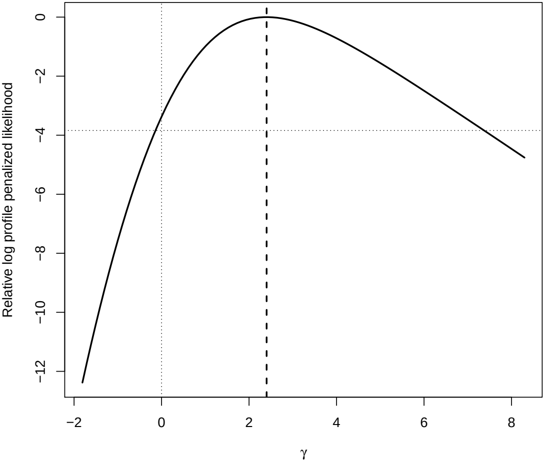
What's Going On?

```
> xtabs(~petfamily+as.factor(married)+female)
, , female = 0

         as.factor(married)
petfamily Married Widowed Divorced/Sep NBM
        0      47       0           11   8
        1     168       7           33  47


, , female = 1

         as.factor(married)
petfamily Married Widowed Divorced/Sep NBM
        0      28       7            7   5
        1     199      32           67  58
```

Pets as Family: Firth Model

```
> Pets.Firth<-logistf(petfamily~female+
+                       as.factor(married)*female+as.factor(partyid)+
+                       as.factor(education),data=Pets)

> Pets.Firth

logistf(formula = petfamily ~ female + as.factor(married) * female +
    as.factor(partyid) + as.factor(education), data = Pets)
Model fitted by Penalized ML
Confidence intervals and p-values by Profile Likelihood

                                        coef se(coef) lower 0.95 upper 0.95    Chisq         p
(Intercept)                          2.15893    0.597      1.054      3.404 16.17636 0.0000577
femaleMale                          -1.13866    0.517     -2.187     -0.145  5.04186 0.0247420
as.factor(married)Married           -0.27387    0.433     -1.192      0.531  0.41518 0.5193531
as.factor(married)NBM                0.15888    0.588     -0.991      1.367  0.07322 0.7867048
as.factor(married)Widowed           -0.72627    0.561     -1.839      0.384  1.67233 0.1959467
as.factor(partyid)Democrat          -0.11818    0.418     -0.992      0.661  0.08159 0.7751592
as.factor(partyid)GOP               -0.00776    0.422     -0.888      0.780  0.00034 0.9852893
as.factor(partyid)Independent       -0.13643    0.419     -1.013      0.646  0.10813 0.7422784
as.factor(education)College Grad     0.23904    0.405     -0.574      1.024  0.34480 0.5570689
as.factor(education)HS diploma       0.07531    0.362     -0.667      0.763  0.04289 0.8359331
as.factor(education)Post-Grad        0.21837    0.425     -0.627      1.050  0.26307 0.6080189
as.factor(education)Some college     0.05240    0.380     -0.721      0.781  0.01888 0.8906980
femaleMale:as.factor(married)Married 0.45582    0.577     -0.661      1.613  0.63550 0.4253467
femaleMale:as.factor(married)NBM     0.52329    0.779     -1.023      2.050  0.45133 0.5017022
femaleMale:as.factor(married)Widowed 2.40167    1.684     -0.139      7.374  3.37453 0.0662116

Likelihood ratio test=17.3 on 14 df, p=0.242, n=724
```

Profile Likelihood Plot



Note: Plot shows estimated profile likelihood for different values of the parameter estimate for the interaction term `femaleMale:as.factor(married)Widowed`. Horizontal dotted line is the likelihood associated with $P \leq 0.05$. Vertical dashed line is $\hat{\gamma}$; vertical dotted line indicates $\hat{\gamma} = 0$.

- Separation is an *estimation* problem...
- Separation $\nrightarrow$ dropping covariates!
- Firth's approach $>$ ELR
- Can also be applied to other sparse-data situations:
  - "Fixed effects" logit models (Cook et al. 2020)
  - Multinomial logit (Cook et al. 2018)
  - Survival models (Anderson et al. 2020)

Finally: Read this twitter thread before it's gone.

If events ("1s") are rare, we can...

- Collect lots of "0s" for a few "1s"
- $\rightarrow$ Classification bias...

Example: Suppose that:

$$\Pr(Y_i) = \Lambda(0 + 1X_i)$$

then:

$$E(\hat{\beta}_0 - \beta_0) \approx \frac{\bar{\pi} - 0.5}{N\bar{\pi}(1 - \bar{\pi})}$$

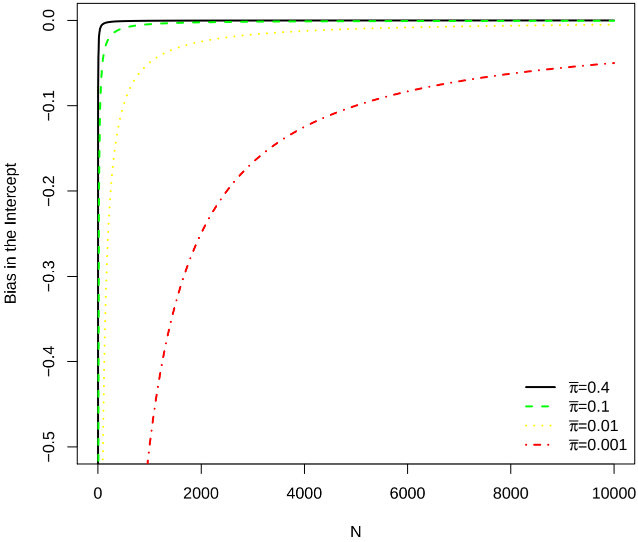
where $\bar{\pi} = \overline{\Pr(Y = 1)}$ is < 0.5.

Bias is:

- always negative,
- worse as $\bar{\pi} \to 0$ (for fixed N),
- disappearing as $N \to \infty$.

Implication: *Logit/probit "work best" around* $\bar{\pi} = 0.5$.

Rare Event Bias, Illustrated



- Calculate $\tau = \frac{N_1 s}{N}$
- Collect data on all "1s"
- Sample from the "0s"
- Estimate a logit*
- *Correct* the estimates ex post...

Sampling...

- $\tau =$ fraction of "1s" in the population
- $\bar{Y} =$ fraction of '1s" in the sample
- K&Z suggest $\bar{Y} \in [0.2, 0.5]$

Weighting...

$$w_1 = \frac{\tau}{\bar{Y}} \text{ (weights for "1s")}$$

$$w_0 = \frac{1-\tau}{1-\bar{Y}} \text{ (weights for "0s")}$$

$$\ln L(\beta|Y) = \sum_{i=1}^{N} w_1 Y_i \ln \Lambda(\mathbf{X}_i\boldsymbol{\beta}) + w_0(1-Y_i)\ln[1-\Lambda(\mathbf{X}_i\boldsymbol{\beta})]$$

Weighting:

- Good under (possible) misspecification, but
- Not as efficient as "prior correction," and
- Gets s.e.s wrong...

Case-Control Data: Prior Correction

$$\hat{\beta}_{0\mathsf{pc}} = \hat{\beta}_0 - \ln\left[\left(\frac{1-\tau}{\tau}\right)\left(\frac{\bar{Y}}{1-\bar{Y}}\right)\right]$$

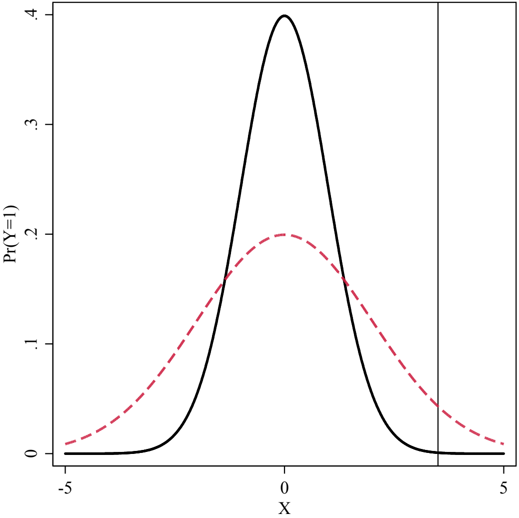
$$\mathsf{bias}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\xi$$

where $\xi = f[w_i, \hat{\pi}_i, \mathbf{X}]$.

Correction is

$$\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} - \mathsf{bias}(\hat{\boldsymbol{\beta}})$$

- Bias correction introduces additional variability...
- Ignoring it yields underpredictions (again).


Pr(Y=1)
.4
.3
.2
.1
0
-5
0
5
X

Post-Correction Adjustments

Use:

$$\Pr(Y_i = 1) \approx \tilde{\pi}_i + C_i$$

where

$$C_i = (0.5 - \tilde{\pi}_i)\tilde{\pi}_i(1 - \tilde{\pi}_i)\mathbf{X}_i \mathbf{V}(\tilde{\beta})\mathbf{X}_i'$$

Puhr et al. (2017) note that Firth's method indices bias (toward 0.5) in predicted probabilities, and that the bias is worse when the baseline $\Pr(Y_i = 1)$ is low.

They introduce two modifications to deal with this:

- "Firth's logit with intercept correction" (FLIC)
- "Firth's logit with added covariate" (FLAC)

Through simulations, they show that both remove the bias; they have a slight preference for FLAC, but note that both work well relative to unmodified Firth regression.

An Example

- Washington University's American Panel Study (TAPS)
- $N \approx 1000$ U.S. respondents, 2012-2017
- Outcome: "During the past year, have you ever run out of gas while driving a car or other vehicle?" (`RunOutOfGas`; 0=no, 1=yes)
- Predictors:
    - `Education` – twelve-category ordinal variable with values ranging from 3 to 15;
    - `Income` – a 15-category ordinal variable (each unit roughly corresponds to an increase of $10,000 in annual income);
    - `Age` in years, as of 2016 (divided by 10);
    - `Female` – a binary indicator of sex, naturally-coded;
    - Racial classifications – binary variables for `White`, `Black`, and `Asian` identification;
    - Binary political party variables for `Democrat` and `GOP`; and
    - `Ideology` – a seven-point Likert variable, higher values indicate greater political conservatism

Basic Logit...

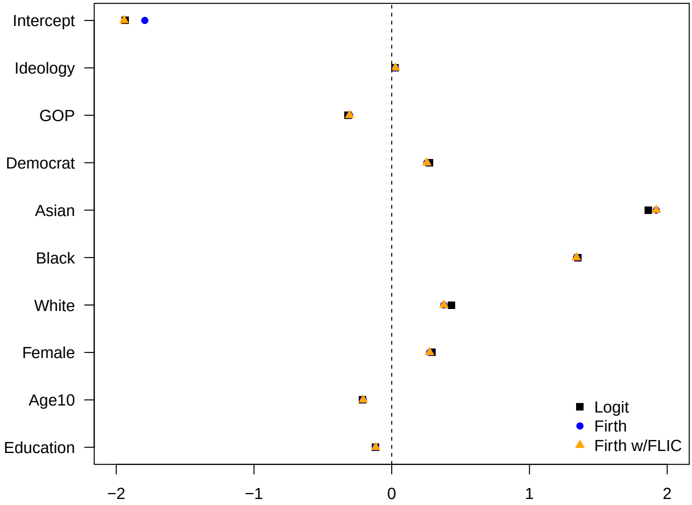
```
> table(TAPS$RunOutOfGas)

  0   1
943  28

> prop.table(table(TAPS$RunOutOfGas))

     0      1
0.9712 0.0288

> ROGlogit<-glm(RunOutOfGas~Education+Age10+Female+White+Black+Asian+
+                      Democrat+GOP+Ideology,data=TAPS,family=binomial)

> summary(ROGlogit)

Deviance Residuals:
   Min      1Q  Median      3Q     Max
-0.661  -0.248  -0.206  -0.170   2.962

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -1.9347     1.8114   -1.07    0.285
Education    -0.1185     0.1118   -1.06    0.289
Age10        -0.2107     0.1341   -1.57    0.116
Female        0.2911     0.3966    0.73    0.463
White         0.4348     0.7260    0.60    0.549
Black         1.3503     0.7602    1.78    0.076 .
Asian         1.8616     0.8717    2.14    0.033 *
Democrat      0.2743     0.4999    0.55    0.583
GOP          -0.3170     0.5926   -0.53    0.593
Ideology      0.0217     0.1097    0.20    0.843
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 253.77  on 970  degrees of freedom
Residual deviance: 238.13  on 961  degrees of freedom
AIC: 258.1
```

Firth Logit (for comparison)

```
> relogit.firth<-logistf(RunOutOfGas~Education+Age10+Female+White+Black+Asian+
+                       Democrat+GOP+Ideology,data=TAPS)

> summary(relogit.firth)

logistf(formula = RunOutOfGas ~ Education + Age10 + Female +
    White + Black + Asian + Democrat + GOP + Ideology, data = TAPS)

Model fitted by Penalized ML
Coefficients:
               coef se(coef) lower 0.95 upper 0.95  Chisq      p method
(Intercept) -1.7929    1.657     -5.362     1.6045 1.0457 0.3065      2
Education   -0.1167    0.103     -0.331     0.1009 1.1154 0.2909      2
Age10       -0.2071    0.124     -0.469     0.0498 2.4952 0.1142      2
Female       0.2749    0.367     -0.478     1.0490 0.5124 0.4741      2
White        0.3782    0.646     -1.007     1.7513 0.2769 0.5987      2
Black        1.3409    0.677     -0.182     2.7141 2.9875 0.0839      2
Asian        1.9202    0.766      0.149     3.4429 4.4610 0.0347      2
Democrat     0.2550    0.464     -0.688     1.2418 0.2767 0.5989      2
GOP         -0.3061    0.546     -1.479     0.7889 0.2969 0.5858      2
Ideology     0.0267    0.101     -0.191     0.2333 0.0613 0.8044      2

Method: 1-Wald, 2-Profile penalized log-likelihood, 3-None

Likelihood ratio test=17.5 on 9 df, p=0.0415, n=971
Wald test = 318 on 9 df, p = 0
```

```
> relogit.flic<-logistf(RunOutOfGas~Education+Age10+Female+White+Black+Asian+
+                          Democrat+GOP+Ideology,data=TAPS,flic=TRUE)

> summary(relogit.flic)

logistf(formula = RunOutOfGas ~ Education + Age10 + Female +
    White + Black + Asian + Democrat + GOP + Ideology, data = TAPS,
    flic = TRUE)

Model fitted by Penalized ML
Coefficients:
               coef se(coef) lower 0.95 upper 0.95  Chisq      p method
(Intercept) -1.9430    1.807     -5.486     1.5995 1.0457 0.3065      1
Education   -0.1167    0.112     -0.331     0.1009 1.1154 0.2909      2
Age10       -0.2071    0.134     -0.469     0.0498 2.4952 0.1142      2
Female       0.2749    0.397     -0.478     1.0490 0.5124 0.4741      2
White        0.3782    0.720     -1.007     1.7513 0.2769 0.5987      2
Black        1.3409    0.756     -0.182     2.7141 2.9875 0.0839      2
Asian        1.9202    0.857      0.149     3.4429 4.4610 0.0347      2
Democrat     0.2550    0.501     -0.688     1.2418 0.2767 0.5989      2
GOP         -0.3061    0.590     -1.479     0.7889 0.2969 0.5858      2
Ideology     0.0267    0.110     -0.191     0.2333 0.0613 0.8044      2

Method: 1-Wald, 2-Profile penalized log-likelihood, 3-None

Likelihood ratio test=17.5 on 9 df, p=0.0415, n=971
Wald test = 299 on 9 df, p = 0
```

Summarizing: $\hat{\beta}$s



Some Final Thoughts

- The key to doing King-Zeng is to be able to conduct C-C sampling *in advance*

- BUT: The R implementation of K&Z (in `Zelig`) is currently a bit buggy (its dependencies are all messed up...)

- In practice: the Firth + FLIC approach is generally superior to King/Zeng (and arguably should *always* be used for binary-response regressions, especially with small-to-medium Ns)

- Also: Remember that as your N gets big, the problem goes away; Paul Allision has a (old, but useful) blog post on that topic.

Other Binary-Response Extensions

Things we'll talk about later:

- Binary responses in panel / longitudinal data
- Multilevel / hierarchical models for binary responses
- Models with (binary) sample selection
- Measurement models for binary outcomes (e.g., item response models)

Things we won't talk about:

- Semi- and non-parametric models (see, e.g., Horowitz and Savin 2001)
- "Heteroscedastic" models (where $\sigma_i^2 \neq \sigma^2 \, \forall \, i$) (see, e.g., Alvarez and Brehm 1995, 1997; Tutz 2018)
- "Bivariate" probit models, where:

  $$\{Y_{1i}, Y_{2i}\} \sim BVN(0, 0, 1, 1, \rho)$$

  (e.g., Zorn 2002)

Nominal Outcomes

Motivation: Discrete *Outcomes*

$$\Pr(Y_i = j) = P_{ij}$$

$$\sum_{j=1}^{J} P_{ij} = 1$$

$$P_{ij} = \exp(\mathbf{X}_i \boldsymbol{\beta}_j)$$

Rescale:

$$\Pr(Y_i = j) \equiv P_{ij} = \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)}$$

Ensures

- $\Pr(Y_i = j) \in (0, 1)$
- $\sum_{j=1}^{J} \Pr(Y_i = j) = 1.0$

Constrain $\boldsymbol{\beta}_1 = \mathbf{0}$; then:

$$\Pr(Y_i = 1) = \frac{1}{1 + \sum_{j=2}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j')}$$

$$\Pr(Y_i = j) = \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j')}{1 + \sum_{j=2}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j')}$$

where $\boldsymbol{\beta}_j' = \boldsymbol{\beta}_j - \boldsymbol{\beta}_1$.

Alternative Motivation: Discrete *Choice*

$$U_{ij} = \mu_i + \epsilon_{ij}$$

$$\mu_i = \mathbf{X}_i \boldsymbol{\beta}_j$$

$$\begin{aligned}
\Pr(Y_i = j) &= \Pr(U_{ij} > U_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\mu_i + \epsilon_{ij} > \mu_i + \epsilon_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\mathbf{X}_i \boldsymbol{\beta}_j + \epsilon_{ij} > \mathbf{X}_i \boldsymbol{\beta}_\ell + \epsilon_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\epsilon_{ij} - \epsilon_{i\ell} > \mathbf{X}_i \boldsymbol{\beta}_\ell - \mathbf{X}_i \boldsymbol{\beta}_j \,\forall\, \ell \neq j \in J)
\end{aligned}$$

$\epsilon \sim$???

- *Type I Extreme Value*
- Density: $f(\epsilon) = \exp[-\epsilon - \exp(-\epsilon)]$
- CDF: $\int f(\epsilon) \equiv F(\epsilon) = \exp[-\exp(-\epsilon)]$
- $\rightarrow$ Multinomial Logit

Estimation

Define:

$$
\begin{aligned}
\delta_{ij} &= 1 \text{ if } Y_i = j, \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

Then:

$$
\begin{aligned}
L_i &= \prod_{j=1}^{J} [\Pr(Y_i = j)]^{\delta_{ij}} \\
&= \prod_{j=1}^{J} \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]^{\delta_{ij}}
\end{aligned}
$$

So:

$$L = \prod_{i=1}^{N} \prod_{j=1}^{J} \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]^{\delta_{ij}}$$

and (of course):

$$\ln L = \sum_{i=1}^{N} \sum_{j=1}^{J} \delta_{ij} \ln \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]$$

It is exactly the same as the multinomial logit model. Period.

CL with choice-varying predictors $\mathbf{Z}_{ij}\gamma$ is:

$$\Pr(Y_{ij} = j) = \frac{\exp(\mathbf{Z}_{ij}\gamma)}{\sum_{j=1}^{J} \exp(\mathbf{Z}_{ij}\gamma)}$$

Combinations: $\mathbf{X}_i\boldsymbol{\beta}$ and $\mathbf{Z}_{ij}\gamma$:

- "Fixed effects" for each possible outcome / choice
- Observation-specific $\mathbf{X}$s
- Interactions...

MNL and CL: Practical Things

The PLSC 503 slides and code include some additional detail, plus a running example (the three-candidate 1992 U.S. presidential election), with discussions of:

- Model estimation (including choosing the baseline/reference outcome),
- Model interpretation and discussion (odds ratios, predicted probabilities, etc.),
- Model fit, and
- Diagnostics.

I've included most of the code for those examples in today's code as well.

Independence of Irrelevant Alternatives ("IIA")

"An individual's choice does not depend on the availability or characteristics of unavailable alternatives."

IIA, Statistically

$$
\begin{aligned}
\frac{\Pr(Y_i = k)}{\Pr(Y_i = \ell)} &= \frac{\frac{\exp(\mathbf{X}_i \beta_k)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \beta_j)}}{\frac{\exp(\mathbf{X}_i \beta_\ell)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \beta_j)}} \\
&= \frac{\exp(\mathbf{X}_i \beta_k)}{\exp(\mathbf{X}_i \beta_\ell)} \\
&= \exp[\mathbf{X}_i(\beta_k - \beta_\ell)]
\end{aligned}
$$

Alternatively:

$$
\frac{\Pr(Y_i = k | S_J)}{\Pr(Y_i = \ell | S_J)} = \frac{\Pr(Y_i = k | S_M)}{\Pr(Y_i = \ell | S_M)} \; \forall \; k, \ell, J, M
$$

IIA, Intuitively

- Initially: $\Pr(\text{Car}) = \Pr(\text{Red Bus}) = 0.5$, $\frac{\Pr(\text{Car})}{\Pr(\text{Red Bus})} = 1$.
- Enter the Blue Bus...
  - Intuitively: $\Pr(\text{Car}) = 0.5$, $\Pr(\text{Red Bus}) = 0.25$, $\Pr(\text{Blue Bus}) = 0.25$
  - IIA requires that $\frac{\Pr(\text{Car})}{\Pr(\text{Red Bus})} = 1$.
  - So, that could be $\Pr(\text{Car}) = \Pr(\text{Red Bus}) = \Pr(\text{Blue Bus}) = 0.33$, or
  - $\Pr(\text{Car}) = \Pr(\text{Red Bus}) = 0.4$ and $\Pr(\text{Blue Bus}) = 0.2$...

Random utility model:

$$\begin{aligned} U_{ij} &= \mu_{ij} + \epsilon_{ij} \\ &= \mathbf{X}_i \boldsymbol{\beta}_j + \epsilon_{ij} \end{aligned}$$

... means that:

$$\begin{aligned} \Pr(Y_i = j) &= \Pr(U_{ij} > U_{i\ell}) \forall \ell \neq j \in J \\ &= \Pr(\mathbf{X}_i \boldsymbol{\beta}_j + \epsilon_{ij} > \mathbf{X}_i \boldsymbol{\beta}_\ell + \epsilon_{i\ell}) \forall \ell \neq j \in J \\ &= \Pr(\epsilon_{ij} - \epsilon_{i\ell} > \mathbf{X}_i \boldsymbol{\beta}_\ell - \mathbf{X}_i \boldsymbol{\beta}_j) \forall \ell \neq j \in J \end{aligned}$$

IIA Tests: Hausman/McFadden and Small/Hsiao

$$HM = (\hat{\beta}_r - \hat{\beta}_u)'[\hat{\mathbf{V}}_r - \hat{\mathbf{V}}_u]^{-1}(\hat{\beta}_r - \hat{\beta}_u)$$

$$\widehat{HM} \sim \chi^2_{(J-2)k}$$

$$SH = -2\left[L_r(\hat{\beta}_u^{AB}) - L_r(\hat{\beta}_r^{B})\right]$$

$$\widehat{SH} \sim \chi^2_{k_r}$$

IIA Freedom: Multinomial Probit

$\epsilon_{ij} \sim MVN(0, \Sigma)$, where:

$$\underset{J\times J}{\mathbf{\Sigma}} = \begin{bmatrix} \sigma_1^2 & \dots & \sigma_{1J} \\ \vdots & \ddots & \vdots \\ \sigma_{J1} & \dots & \sigma_J^2 \end{bmatrix}$$

Define $\eta_{ij\ell} = \epsilon_{ij} - \epsilon_{i\ell}$. Then:

$$\begin{aligned} \Pr(Y_i = j) &= \Pr(\eta_{ij\ell} > \mathbf{X}_i\boldsymbol{\beta}_\ell - \mathbf{X}_i\boldsymbol{\beta}_j) \, \forall \, \ell \neq j \in J \\ &= \int_{-\infty}^{\mathbf{X}_i\boldsymbol{\beta}_1 - \mathbf{X}_i\boldsymbol{\beta}_j} \dots \int_{-\infty}^{\mathbf{X}_i\boldsymbol{\beta}_\ell - \mathbf{X}_i\boldsymbol{\beta}_j} \phi_J(\eta_{ij1}, \eta_{ij2}, \dots \eta_{ij\ell}) d\eta_{ij1}, \eta_{ij2}, \dots \eta_{ij\ell} \end{aligned}$$

MNP: Issues and Estimation

- Identification: (Potentially) Fragile
- Estimation:
  - Always hard
  - Via "GHK" algorithm, or
  - Gaussian quadrature, or
  - Simulation (MCMC) (preferred)
- Software:
  - `mlogit` with `probit = TRUE` (Geweke-Hajivassiliou-Keane algorithm)
  - `MNP` package (Bayesian/MCMC)
  - `endogMNP` package (Bayesian with endogenous switching)
  - Others?

IIA Freedom: HEV

$$\begin{aligned} f(\epsilon_{ij}) &= \lambda(\epsilon_{ij}) \\ &= \frac{1}{\theta_j} \exp\left(-\frac{\epsilon_{ij}}{\theta_j}\right) \exp\left[-\exp\left(-\frac{\epsilon_{ij}}{\theta_j}\right)\right] \end{aligned}$$

$$\begin{aligned} F(\epsilon_{ij}) &= \Lambda(\epsilon_{ij}) \\ &= \int_{-\infty}^{z} f(\epsilon_{ij}) d\,\epsilon_{ij} \\ &= \exp\left[-\exp\left(-\frac{\epsilon_{ij}}{\theta_j}\right)\right] \end{aligned}$$

More HEV

Means:

$$\Pr(Y_i = j) = \int_{-\infty}^{\infty} \prod_{\ell \neq j} \Lambda\left(\frac{\mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_\ell + \epsilon_{ij}}{\theta_\ell}\right) \frac{1}{\theta_j} \lambda\left(\frac{\epsilon_{ij}}{\theta_j}\right) d\, \epsilon_{ij}$$

With $w = \frac{\epsilon_{ij}}{\theta_j}$:

$$\Pr(Y_i = j) = \int_{-\infty}^{\infty} \prod_{\ell \neq j} \Lambda\left(\frac{\mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_\ell + \theta_j w}{\theta_\ell}\right) \lambda(w) d\, w$$

MNL $\subset$ HEV: When $\theta_j = 1 \;\; \forall \;\; j \rightarrow$

$$\Pr(Y_i = j) = \int_{-\infty}^{\infty} \prod_{\ell \neq j} \Lambda(\mathbf{X}_i \beta_j - \mathbf{X}_i \beta_\ell + \epsilon_{ij}) \lambda(\epsilon_{ij}) d\, \epsilon_{ij}$$

IIA Freedom: "Mixed Logit"

$$U_{ij} = \mathbf{X}_{ij}\boldsymbol{\beta} + \epsilon_{ij},$$

$$\epsilon_{ij} = \eta_i + \xi_{ij}$$

$$\Pr(Y_i = j|\eta) \equiv \Pr(Y_{ij} = 1|\eta) = \frac{\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \eta_i)}{\sum_{j=1}^{J} \exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \eta_i)}$$

What to do with the ηs?

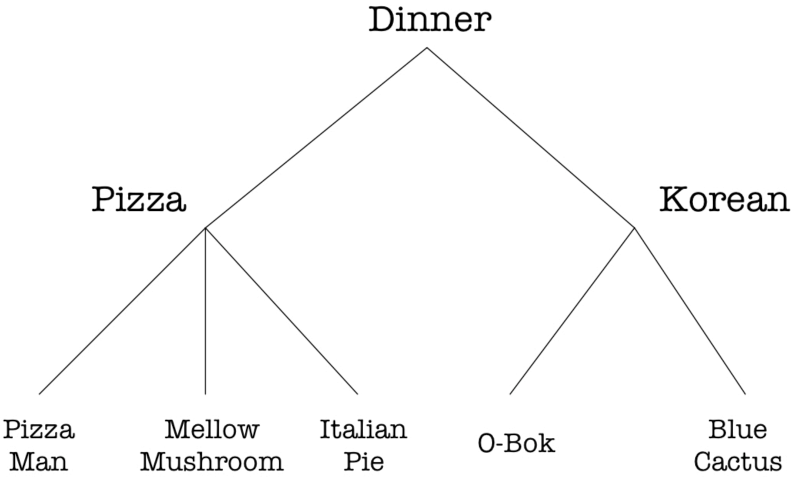
Assume:

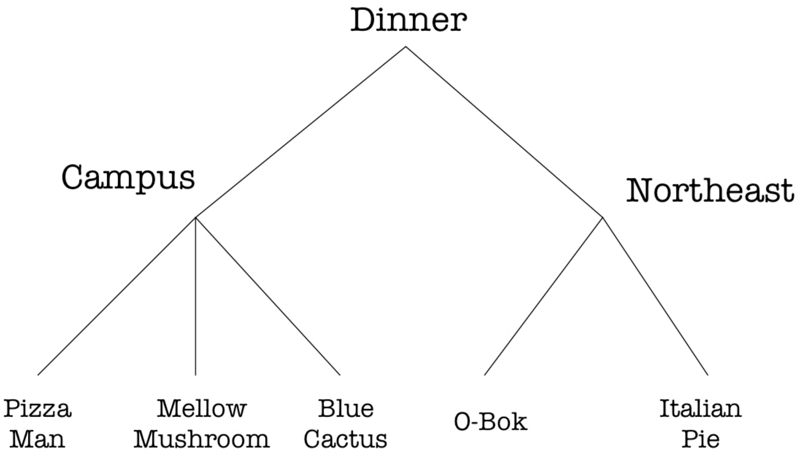
$$\eta_i \sim g(\mathbf{0}, \mathbf{\Omega})$$

Yields:

$$\Pr(Y_i = j) = \int \left[\frac{\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \eta_i)}{\sum_{j=1}^{J} \exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \eta_i)} \right] g(\eta|\mathbf{\Omega})\, d\eta$$

- "Nested" choices
- A priori information about "subsets"
- IIA holds *within* (but not *across*) subsets...


Dinner
Pizza
Korean
Pizza Man
Mellow Mushroom
Italian Pie
O-Bok
Blue Cactus


Dinner
Campus
Northeast
Pizza Man
Mellow Mushroom
Blue Cactus
O-Bok
Italian Pie

Example: 2002 Swedish Election ($N = 6610$)

```
> summary(Sweden)

          partychoice       female            union          leftright    
 Conservatives   :1469   Min.   :0.0000   Min.   :1.000   Min.   :1.000  
 Liberals        :1212   1st Qu.:0.0000   1st Qu.:1.000   1st Qu.:2.000  
 Social Democrats:2975   Median :0.0000   Median :3.000   Median :3.000  
 Left Party      : 954   Mean   :0.4882   Mean   :2.709   Mean   :2.868  
                         3rd Qu.:1.0000   3rd Qu.:4.000   3rd Qu.:4.000  
                         Max.   :1.0000   Max.   :4.000   Max.   :5.000  
      age       
 Min.   :17.00  
 1st Qu.:29.00  
 Median :42.00  
 Mean   :42.93  
 3rd Qu.:55.00  
 Max.   :90.00  
```

Swedish Election: MNL

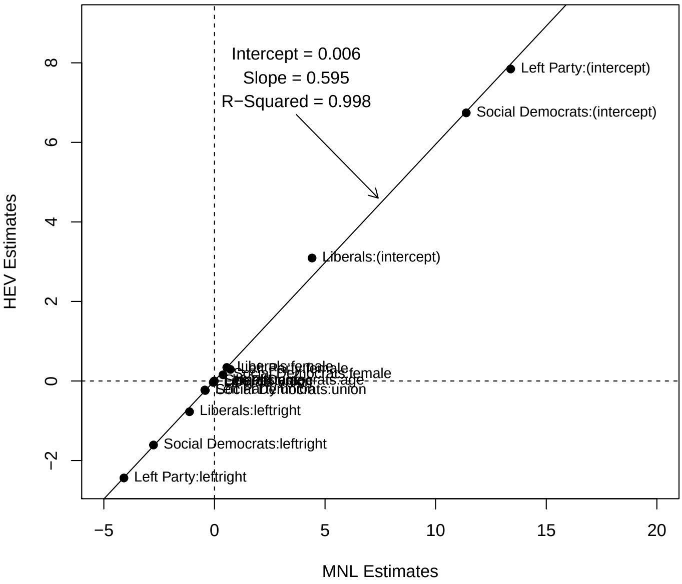
```
> library(mlogit)
> Sweden.Long<-mlogit.data(Sweden,choice="partychoice",shape="wide")
> Sweden.MNL<-mlogit(partychoice~1|female+union+leftright+age,data=Sweden.Long)
> summary(Sweden.MNL)

Frequencies of alternatives:
   Conservatives       Left Party         Liberals Social Democrats
         0.22224          0.14433          0.18336          0.45008

Coefficients :
                              Estimate Std. Error  t-value  Pr(>|t|)
altLeft Party               13.3907039  0.3788540  35.3453 < 2.2e-16 ***
altLiberals                  4.4121638  0.2928137  15.0682 < 2.2e-16 ***
altSocial Democrats         11.3821332  0.3289066  34.6060 < 2.2e-16 ***
altLeft Party:female         0.7211951  0.1218437   5.9190 3.239e-09 ***
altLiberals:female           0.5585172  0.0848597   6.5817 4.652e-11 ***
altSocial Democrats:female   0.3881456  0.0945266   4.1062 4.022e-05 ***
altLeft Party:union         -0.4334637  0.0513499  -8.4414 < 2.2e-16 ***
altLiberals:union           -0.0563136  0.0388720  -1.4487 0.1474228
altSocial Democrats:union   -0.4145682  0.0408153 -10.1572 < 2.2e-16 ***
altLeft Party:leftright     -4.0917135  0.0930610 -43.9681 < 2.2e-16 ***
altLiberals:leftright       -1.1274488  0.0593125 -19.0086 < 2.2e-16 ***
altSocial Democrats:leftright -2.7555009  0.0719411 -38.3022 < 2.2e-16 ***
altLeft Party:age           -0.0277444  0.0038808  -7.1491 8.737e-13 ***
altLiberals:age             -0.0064185  0.0025768  -2.4909 0.0127410 *
altSocial Democrats:age     -0.0105052  0.0029196  -3.5982 0.0003204 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Log-Likelihood: -5627.5
McFadden R^2:  0.33693
Likelihood ratio test : chisq = 5719 (p.value=< 2.22e-16)
```

Hausman-McFadden IIA Test

```
> # Restricted model (omitting Social Democrats)
> Sweden.MNL.Restr<-mlogit(partychoice~1|female+union+leftright+age,
+ Sweden.Long,alt.subset=c("Conservatives","Liberals","Left Party"))
>
> hmftest(Sweden.MNL,Sweden.MNL.Restr)

 Hausman-McFadden test

data:  Sweden.Long
chisq = 19.1137, df = 10, p-value = 0.03884
alternative hypothesis: IIA is rejected
```

```
> Sweden.Het<-mlogit(partychoice~1|female+union+leftright+
+                      age,data=Sweden.Long,heterosc=TRUE)
> summary(Sweden.Het)

Coefficients :
                             Estimate Std. Error z-value Pr(>|z|)
Left Party:(intercept)        7.84569    0.42849   18.31  < 2e-16 ***
Liberals:(intercept)          3.09199    0.30607   10.10  < 2e-16 ***
Social Democrats:(intercept)  6.74242    0.32038   21.04  < 2e-16 ***
Left Party:female             0.29096    0.08057    3.61   0.0003 ***
Liberals:female               0.34113    0.06510    5.24  1.6e-07 ***
Social Democrats:female       0.15572    0.05718    2.72   0.0065 **
Left Party:union             -0.22645    0.03704   -6.11  9.7e-10 ***
Liberals:union               -0.03498    0.02685   -1.30   0.1926
Social Democrats:union       -0.23786    0.03319   -7.17  7.8e-13 ***
Left Party:leftright         -2.43814    0.17450  -13.97  < 2e-16 ***
Liberals:leftright           -0.77255    0.04629  -16.69  < 2e-16 ***
Social Democrats:leftright   -1.60927    0.09462  -17.01  < 2e-16 ***
Left Party:age               -0.01612    0.00338   -4.77  1.9e-06 ***
Liberals:age                 -0.00200    0.00176   -1.14   0.2543
Social Democrats:age         -0.00267    0.00175   -1.53   0.1258
sp.Left Party                 0.90017    0.14304    6.29  3.1e-10 ***
sp.Liberals                   0.59981    0.09925    6.04  1.5e-09 ***
sp.Social Democrats           0.69163    0.10197    6.78  1.2e-11 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Log-Likelihood: -5840
McFadden R^2:  0.312
Likelihood ratio test : chisq = 5300 (p.value = <2e-16)
```

$\hat{\beta}$s: MNL vs. HEV



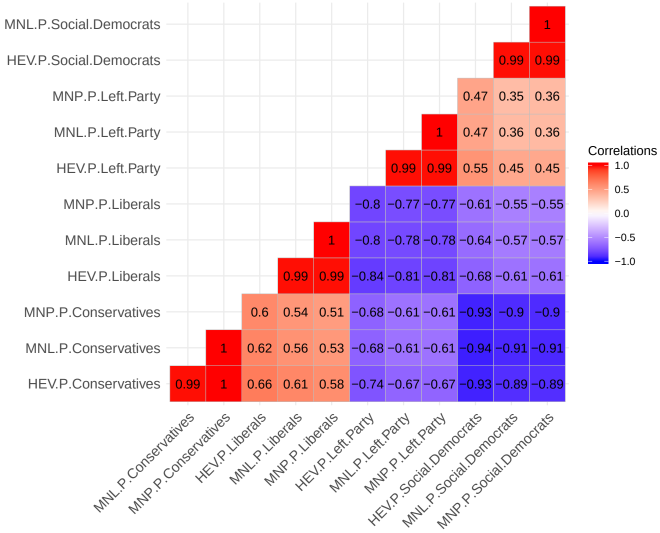
Tests:

```
> MNL.HEV.Wald <- waldtest(Sweden.Het, heterosc = FALSE) # Wald test
> MNL.HEV.Wald

 Wald test

data:  homoscedasticity
chisq = 20, df = 3, p-value = 0.0004

> MNL.HEV.LR <- lrtest(Sweden.Het)         # LR test
> MNL.HEV.LR
Likelihood ratio test

Model 1: partychoice ~ 1 | female + union + leftright + age
Model 2: partychoice ~ 1 | female + union + leftright + age
  #Df LogLik Df Chisq Pr(>Chisq)
1  18  -5836
2  15  -5627 -3   416     <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

> MNL.HEV.Score <- scoretest(Sweden.MNL, heterosc = TRUE)   # score test
> MNL.HEV.Score

 score test

data:  heterosc = TRUE
chisq = 20, df = 3, p-value = 0.00002
alternative hypothesis: heteroscedastic model
```

Swedish Election: MNP

```
> library(MNP)
> Sweden.MNP<-mnp(partychoice~female+union+leftright+age, data=Sweden)
> summary(Sweden.MNP)

Coefficients:
                                  mean  std.dev.      2.5%     97.5%
(Intercept):Liberals          3.964677  0.879442  0.983572     4.669
(Intercept):Social Democrats  7.993453  1.495732  3.986961     9.812
(Intercept):Left Party       10.342468  2.082971  4.845935    12.714
female:Liberals               0.293136  0.046373  0.204654     0.382
female:Social Democrats       0.290311  0.079166  0.124746     0.447
female:Left Party             0.613163  0.163673  0.289974     0.944
union:Liberals               -0.083366  0.036782 -0.140052     0.024
union:Social Democrats       -0.275696  0.059260 -0.369943    -0.145
union:Left Party             -0.346922  0.087131 -0.489992    -0.148
leftright:Liberals           -0.913247  0.168331 -1.045781    -0.350
leftright:Social Democrats   -1.920076  0.362403 -2.371245    -0.977
leftright:Left Party         -3.409277  0.750701 -4.308455    -1.576
age:Liberals                 -0.003350  0.001490 -0.006264 -0.000409
age:Social Democrats         -0.007171  0.002630 -0.012327    -0.002
age:Left Party               -0.025595  0.007323 -0.039641    -0.011

Covariances:
                                    mean std.dev.   2.5%  97.5%
Liberals:Liberals                 1.0000   0.0000 1.0000  1.000
Liberals:Social Democrats         1.4083   0.3925 0.2116  1.830
Liberals:Left Party               2.4450   1.0779 0.6731  3.988
Social Democrats:Social Democrats 2.6696   0.9215 0.5630  3.898
Social Democrats:Left Party       4.4852   2.1846 0.3521  7.524
Left Party:Left Party             9.4811   5.0787 1.1682 17.095

Base category: Conservatives
Number of alternatives: 4
Number of observations: 6610
Number of estimated parameters: 20
Number of stored MCMC draws: 5000
```

How I Stopped Worrying and Learned To Love MNL...



	MNL.P.Conservatives	MNP.P.Conservatives	HEV.P.Liberals	MNL.P.Liberals	MNP.P.Liberals	HEV.P.Left.Party	MNL.P.Left.Party	MNP.P.Left.Party	HEV.P.Social.Democrats	MNL.P.Social.Democrats	MNP.P.Social.Democrats
MNL.P.Social.Democrats											1
HEV.P.Social.Democrats										0.99	0.99
MNP.P.Left.Party									0.47	0.35	0.36
MNL.P.Left.Party								1	0.47	0.36	0.36
HEV.P.Left.Party							0.99	0.99	0.55	0.45	0.45
MNP.P.Liberals						−0.8	−0.77	−0.77	−0.61	−0.55	−0.55
MNL.P.Liberals					1	−0.8	−0.78	−0.78	−0.64	−0.57	−0.57
HEV.P.Liberals				0.99	0.99	−0.84	−0.81	−0.81	−0.68	−0.61	−0.61
MNP.P.Conservatives			0.6	0.54	0.51	−0.68	−0.61	−0.61	−0.93	−0.9	−0.9
MNL.P.Conservatives		1	0.62	0.56	0.53	−0.68	−0.61	−0.61	−0.94	−0.91	−0.91
HEV.P.Conservatives	0.99	1	0.66	0.61	0.58	−0.74	−0.67	−0.67	−0.93	−0.89	−0.89

Software

Model	Stata	SAS	R
Multinomial Logit	`mlogit`	`proc catmod`	`vglm`, `mlogit`, `multinom`*
Conditional Logit	`clogit`	`proc mdc`	`clogit`, `mlogit`
Multinomial Probit	`mprobit` / `asmprobit`	`proc mdc`	`mnp`*, `mlogit`
Heteroscedastic Extreme Value	No(?)	`proc mdc`	`mlogit`
Mixed Logit	`mixlogit`	`proc mdc`	`mlogit`
Nested Logit	`nlogit`	`proc mdc`	`mlogit`

* See also `bayesm`.

Things To Read

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