Index

Unit – 2 ⇝ Cut-Sets & Connectivity

Introduction

This unit focused on

→ Key Question: Which part of a connected graph, when removed, causes the graph to break apart?

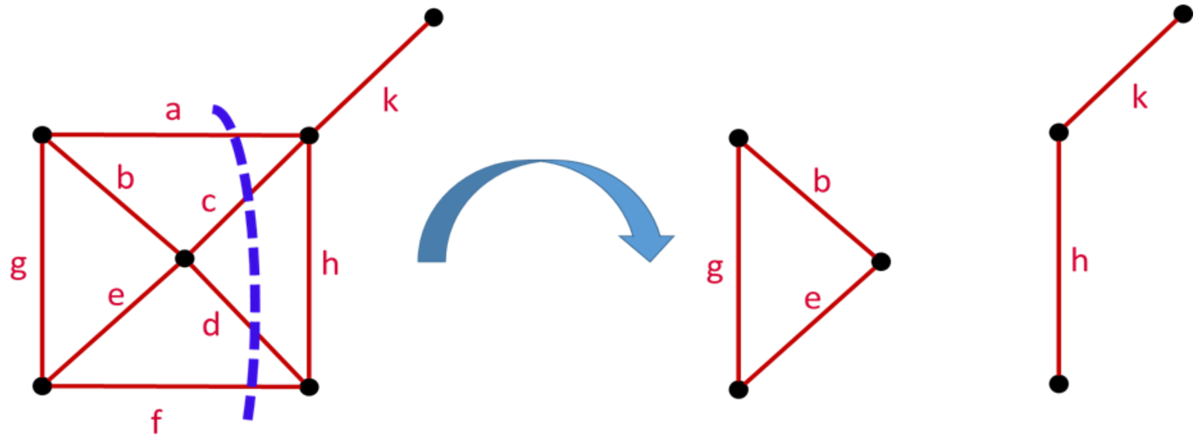
→ To explore this, we studied important concepts such as cut-sets, cut-vertices, connectivity, and related ideas.

→ While a spanning tree connects all vertices without forming cycles, a cut-set identifies edges whose removal separates the graph.

→ We also examined the relationship between spanning trees and cut-sets, especially through fundamental cut-sets derived from spanning trees.

Method – 1 ⇝ Introduction and Properties

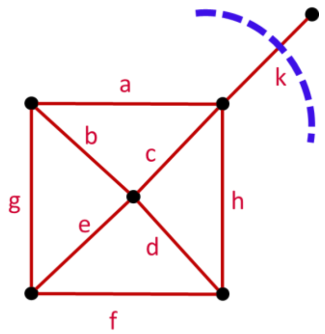
Definition: In a connected graph G, a **cut-set** is a set of edges whose removal from G makes G disconnected, provided removal of no proper subset (fewer edges) of these edges disconnects G.



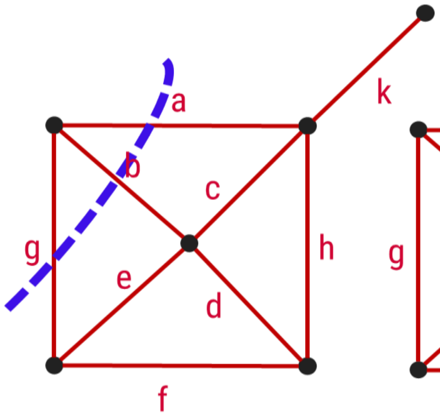
→ The set of edges {a, c, d, f} is a cut-set.

→ Some authors refer to a cut-set as a minimal cut-set, a proper cut-set, or a simple cutset. We shall just use the term **cut-set**.

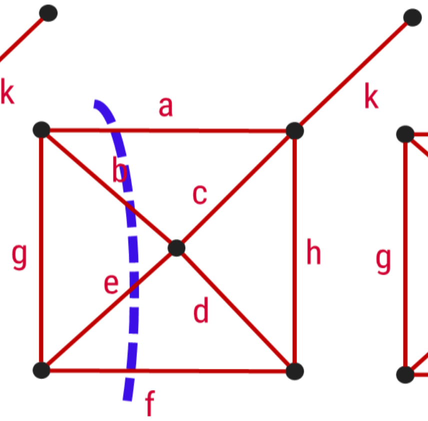
→ Edge {k} alone is also a cut-set.



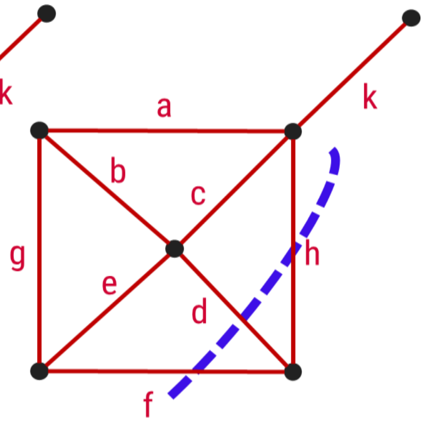
→ There are many other cut-sets, such as



{a, b, g}



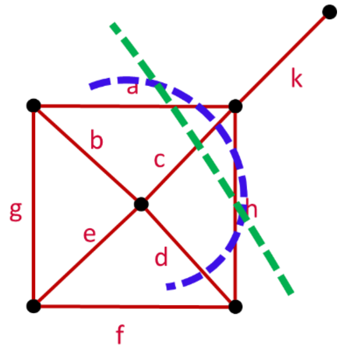
{a, b, e, f}



{d, h, f}

→ The set of edges $\{a, c, h, d\}$, is not a cut-set.

→ As its proper subset of edges $\{a, c, h\}$ is a cut-set.



Properties:

→ No proper subset of a cut-set can be a cut-set.

→ Every edge of tree is a cut-set.

→ A cut-set always "cuts" a graph into two, i.e. whose removal reduces the rank of the graph by one.

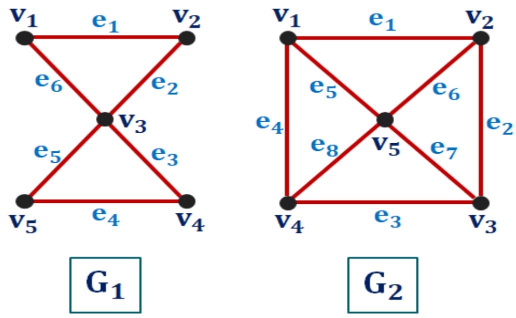
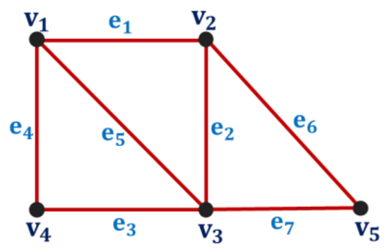
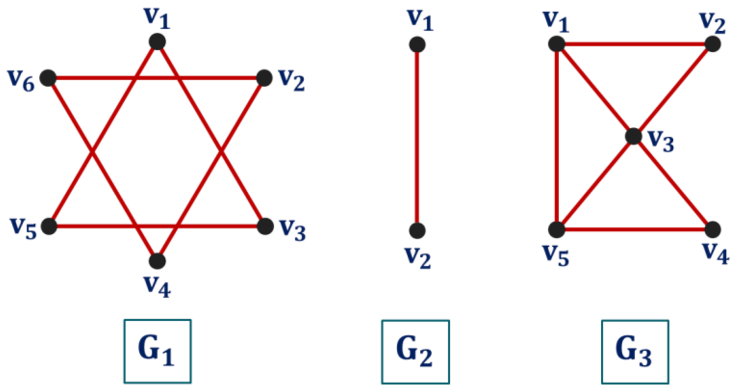
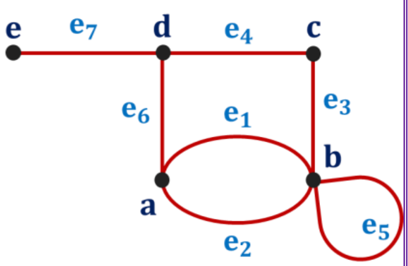
→ Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.

→ Consider a spanning tree T in a connected graph G and an arbitrary cutset S in G.

→ It is not possible for S not to have any edge in common with T.

→ Otherwise, removal of the cut-set S from G would not disconnect the graph.

Example of Method-1: Introduction and Properties

C	1	Find cut-set of the following graphs  **Answer: For G_1 $\{e_1, e_6\}$ and for G_2 $\{e_2, e_7, e_3\}$.** **There are many other cutsets, list all.**
C	2	For the given graph, determine whether the following set of edges forms a cut-set. a) $\{e_4, e_1, e_2\}$ b) $\{e_6, e_2, e_5, e_4\}$ c) $\{e_1, e_2, e_7\}$  **Answer: a) not cut set, b) cut set, c) cut set**
C	3	Verify: Removal of cut-sets reduces the rank of the graph by one.
C	4	How many cut-sets does a tree with 10 vertices have? **Answer: 9**
H	5	Find cut-set of the following graph 
H	6	For the given graph, determine whether the following set of edges forms a cut-set. a) $\{e_4, e_1, e_2\}$ b) $\{e_2, e_6, e_4\}$ c) $\{e_1, e_2\}$  **Answer: a) cut set, b) not cut set c) not cut set**

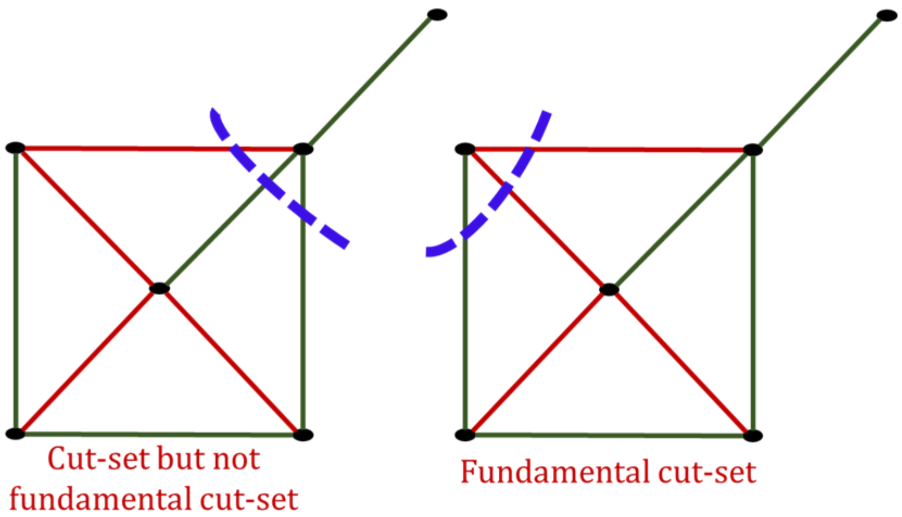
H	7	How many cut-sets does a tree with **n** vertices have? **Answer**: $\mathbf{n - 1}$

Method – 2 ⇝ Fundamental Cut-Sets with respect to a spanning tree

Definition: A cut-set S containing exactly one branch of a spanning tree T is called a **fundamental cut-set** with respect to spanning tree T.

→ Consider graph with spanning tree (Green lines).

→ A cut set represent with blue dotted arc.



→ In first figure cut-set contains 2 branches of spanning tree.

- i.e. it is not fundamental cut-set

→ In second figure cut-set contains exactly one branch of spanning tree

- i.e. it is fundamental cut-set

→ Sometimes a fundamental cut-set is also called a basic cut-set.

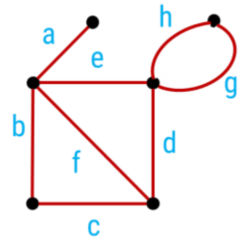
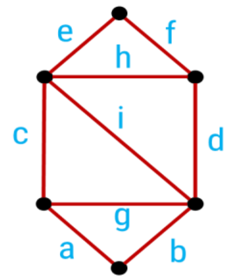
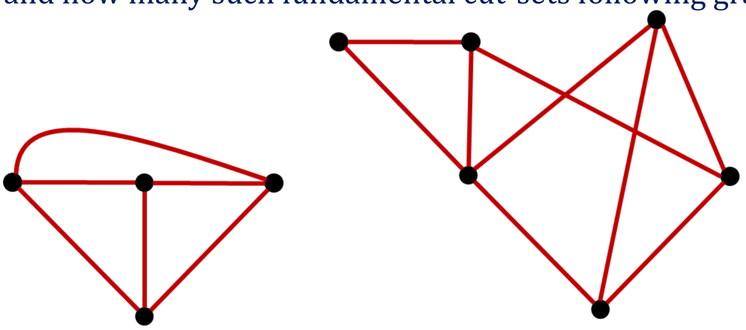
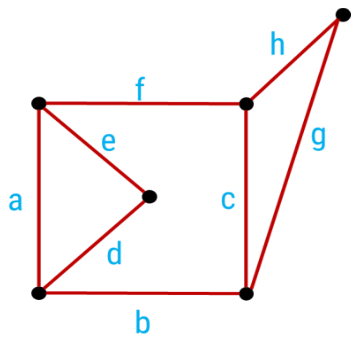
→ Kept in mind that the term fundamental cut-set has meaning only with respect to a given spanning tree.

→ Just as every chord of a spanning tree defines **a unique fundamental circuit** every branch of a spanning tree defines **a unique fundamental cut-set**.

→ **How many fundamental cut-sets does a graph have?**

- Exactly as many as the number of branches $(= \ n - 1)$ of spanning tree of graph.

Example of Method-2.2: Fundamental Cut-Sets with respect to a spanning tree

C	1	Find any fundamental cut-set with respect to spanning tree of your choice and how many such fundamental cut-sets following graph have? 
C	2	Find all fundamental cut-sets with respect to spanning tree $\{f, h, i, g, a\}$ for following graph.  **Answer**: $\{\mathbf{e}, \mathbf{f}\}$ **There are other 5 fundamental cut sets, list all**
H	3	Find any fundamental cut-set with respect to spanning tree of your choice and how many such fundamental cut-sets following graph have? 
H	4	Find all fundamental cut-sets with respect to spanning tree $\{\mathbf{d}, \mathbf{a}, \mathbf{f}, \mathbf{c}, \mathbf{g}\}$ for following graph.  **Answer**: $\{\mathbf{h}, \mathbf{g}\}$ **There are other 4 fundamental cutsets, list all**
H	5	Consider a graph which is generated from minimally connected graph by removing an edge and we know the rank of graph is 9, then how many vertices graph have? Justify.
H	6	Find all fundamental circuits and fundamental cutsets for $\mathbf{K_5}$.

H	7	Explain with example that in a connected graph G the complement of a cut-set in G does not contain a spanning tree and the complement of a spanning tree does not contain a cut-set.

Method – 3 ⇝ Relationship between Fundamental Cut-Set and Fundamental Circuit

→ As we know, **fundamental circuits** and **fundamental cut-sets** are key concepts derived from a **spanning tree** of a connected graph.

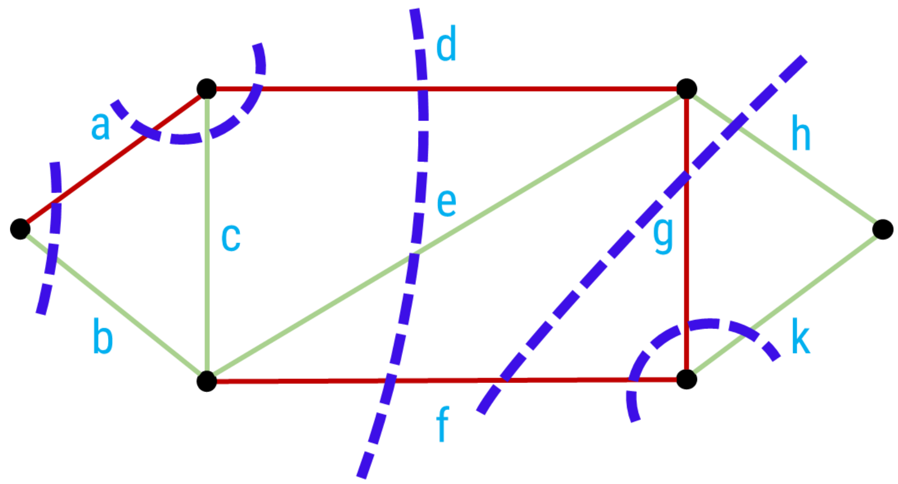
→ The **relationship** between them provides insight into how connectivity and cycles interact in a graph.

Properties:

With respect to a given spanning tree T,

→ **A chord c_i that determines a fundamental circuit C occurs in every fundamental cut-set associated with the branches in C and in no other.**

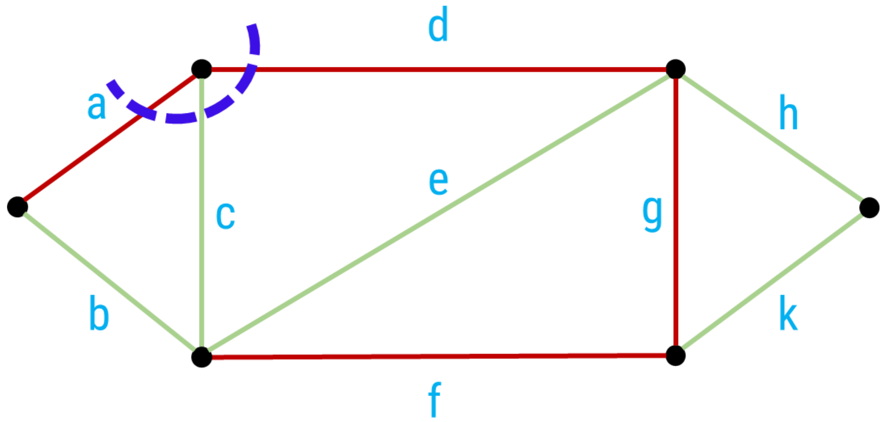
→ Consider the graph below, where the **green lines** represent the edges of a **spanning tree**, and the **blue dotted lines** indicate the corresponding **fundamental cut-sets** with respect to that spanning tree.



- Here the spanning tree is {b, c, e, h, k}.
- The fundamental circuit made by chord f is {f, e, h, k}.
- The three fundamental cutsets determined by the three branches e, h, and k are
  - Determined by branch e: {d, e, f}.
  - Determined by branch h: {f, g, h}.
  - Determined by branch k: {f, g, k}.
- Chord f occurs in each of these three fundamental cutsets, and there is no other fundamental cut-set that contains f.

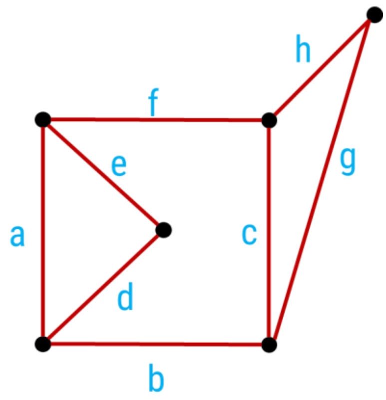
→ **A branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S, and in no others.**

→ Consider the graph below, where the **green lines** represent the edges of a **spanning tree**, and the **blue dotted lines** indicate the corresponding **fundamental cut-sets** with respect to that spanning tree.



- Here the spanning tree is {b, c, e, h, k}.
- The fundamental cut-set made by branch c is {a, c, d}.
- The two fundamental circuits determined by the two chords a, and d are
  - Determined by chord a: {a, b, c}.
  - Determined by chord d: {d, c, e}.
- Branch c occurs in each of these two fundamental circuits, and there is no other fundamental circuits that contains c.

Example of Method-3: Relationship between Fundamental Cut-Set and Fundamental Circuit

C	1	**Explain property with example:** A chord $\mathbf{c_i}$ that determines a fundamental circuit C occurs in every fundamental cut-set associated with the branches in C and in no other.
C	2	**Explain property with example:** A branch $\mathbf{b_i}$ that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S, and in no others.
H	3	Explain Relationship between Fundamental Cut-Set and Fundamental Circuit in below example. 

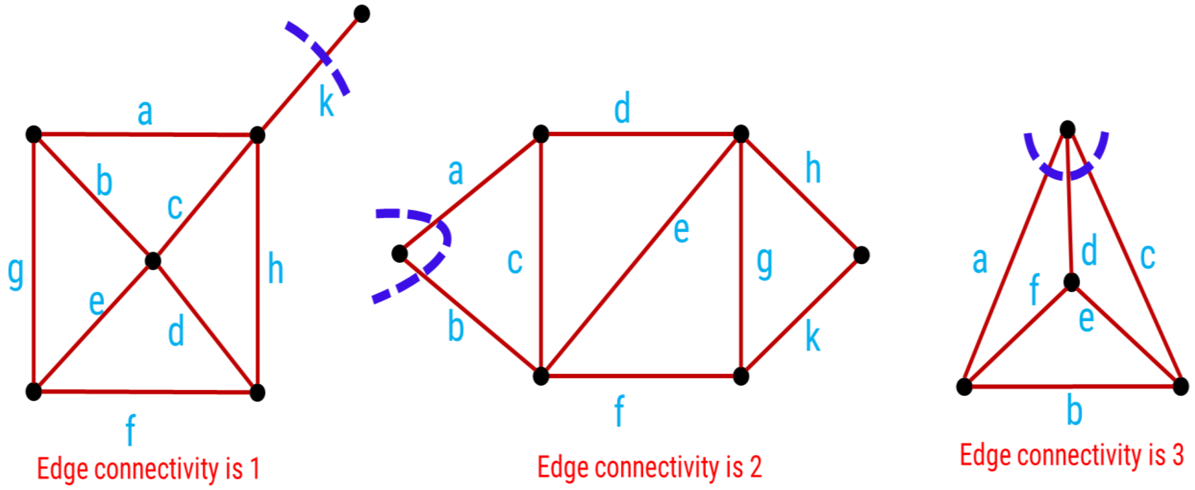
Method – 4 ⇝ Connectivity and Separability

Definition: The edge connectivity of a connected graph can be defined as the number of edges in the smallest cut-set (i.e., cutset with fewest number of edges).

→ Equivalently, is defined as the minimum number of edges whose removal (i.e., deletion) reduces the rank of the graph by one.

→ The edge connectivity of a tree, for instance, is one.

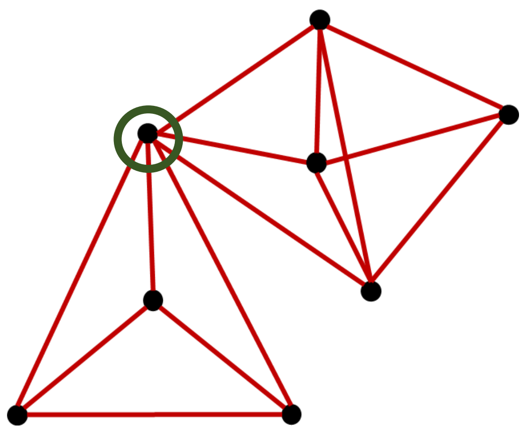
→ The edge connectivity's of the below graphs are one, two, and three, respectively.



→ An edge e of a graph G is said to be a **cut edge** if whose removal will disconnect the graph.

→ For above graph with edge connectivity is 1, have cut edge {k}.

→ On examining the below graph we find that although removal of no single edge (or even a pair of edges) disconnects the graph.

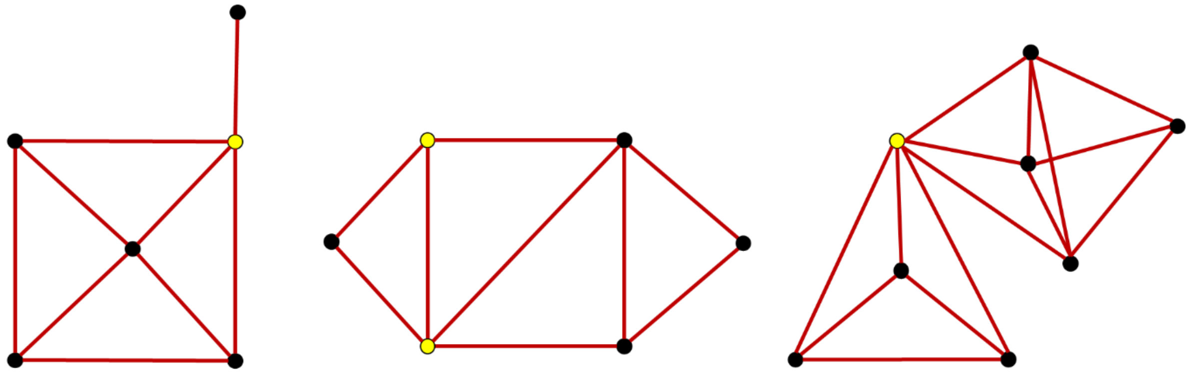


→ The removal of the single vertex does!

→ Therefore, we define another analogous term called vertex connectivity.

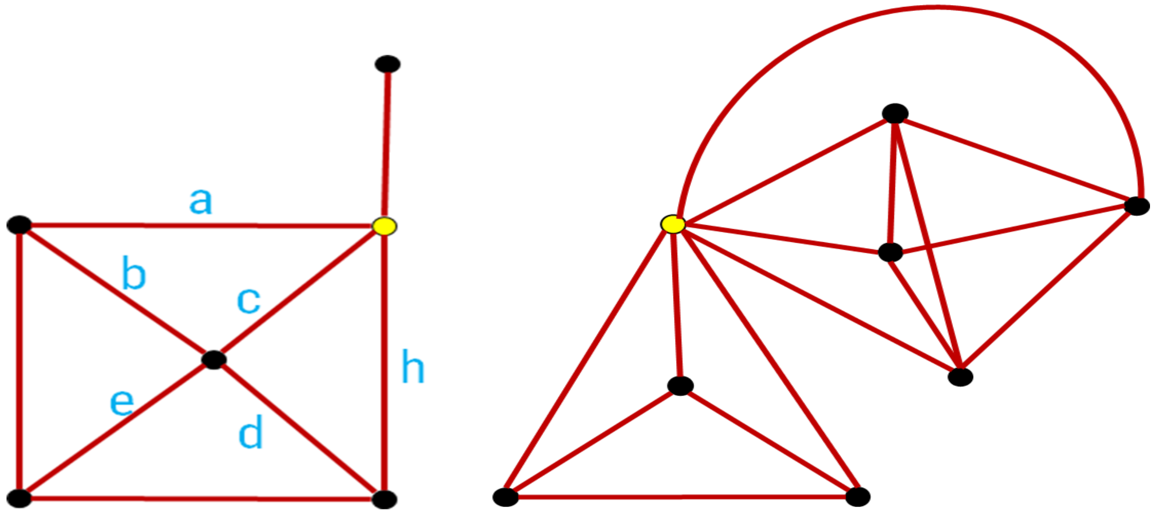
Definition: The vertex connectivity (or simply connectivity) of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.

→ The vertex connectivity's of the below graphs are one, two, and one, respectively.



Vertex connectivity is 1 Vertex connectivity is 2 Vertex connectivity is 1

Definition: A connected graph is said to be **separable** if its vertex connectivity is one. Otherwise we call it non-separable graph



Separable graph

→ In a separable graph a vertex whose removal disconnects the graph is called **a cut-vertex**, a cut-node, or an articulation point.

→ For example, in the above example of separable graph the yellow vertex is a cut-vertex,

Remarks:

→ The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

→ The vertex connectivity of any graph G can never exceed the edge connectivity of G.

→ **From the way we have defined it vertex connectivity is meaningful only for graphs that have three or more vertices and are not complete.**

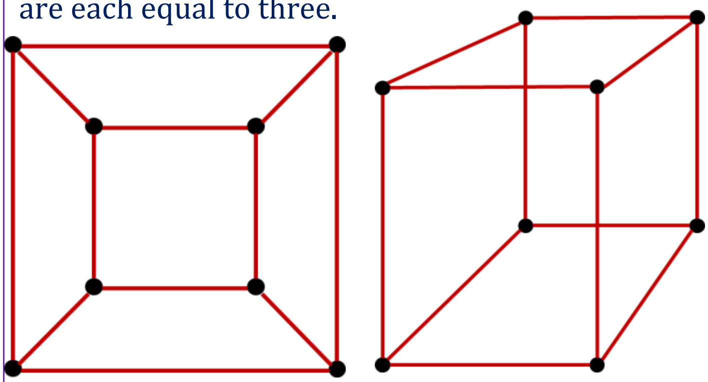
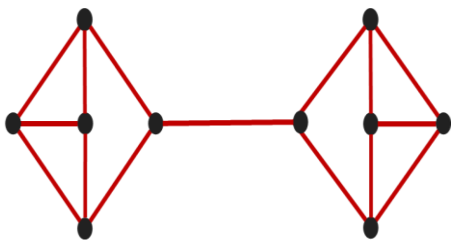
→ Removal of a vertex may increase the number of components in a graph by **at least one**.

→ Removal of an edge may increase the number of components by **at most one**.

→ The end vertices of a cut edge are cut vertices if their degree is more than one.

→ Every non-pendant vertex of a tree is a cut vertex.

Example of Method-4: Total Probability and Bayes' Theorem

C	1	Show that the edge connectivity and vertex connectivity of the below graph are each equal to three. 
C	2	What is the edge connectivity of the complete graph and cycle graph of n vertices? **Answer: For K_n it is $n - 1$ and for C_n it is 2.**
C	3	Prove that a nonseparable graph has a nullity $\mu = 1$ if and only if the graph is a circuit.
C	4	Is every regular graph of degree ($d \geq 3$) nonseparable? If not, give a simple regular graph of degree three that is separable. **Answer: No** 
H	5	Prove that in a tree every vertex of degree greater than one is a cut vertex.
H	6	Construct a graph G with the following properties: Edge connectivity of G = 4, vertex connectivity of G = 3, and degree of every vertex of G ≥ 5.

H	7	Define vertex connectivity and edge connectivity of a graph with a simple example.
H	8	Find the vertex connectivity and edge connectivity of the star graph $K_{1,n}$. **Answer: both are 1**
H	9	Explain with an example: What is a cut-edge (bridge) ? Is every tree guaranteed to have at least one cut-edge?
H	10	Consider regular graph G of degree 2 on $n \geq 3$ vertices. What is the vertex and edge connectivity of G? **Answer: both are 2**

*** * * * * End of the Unit * * * * ***