Index

Unit – 1 ⇝ Tree & Spanning Tree

Introduction

→ This unit focused on, a special class of connected graphs known as **trees**.

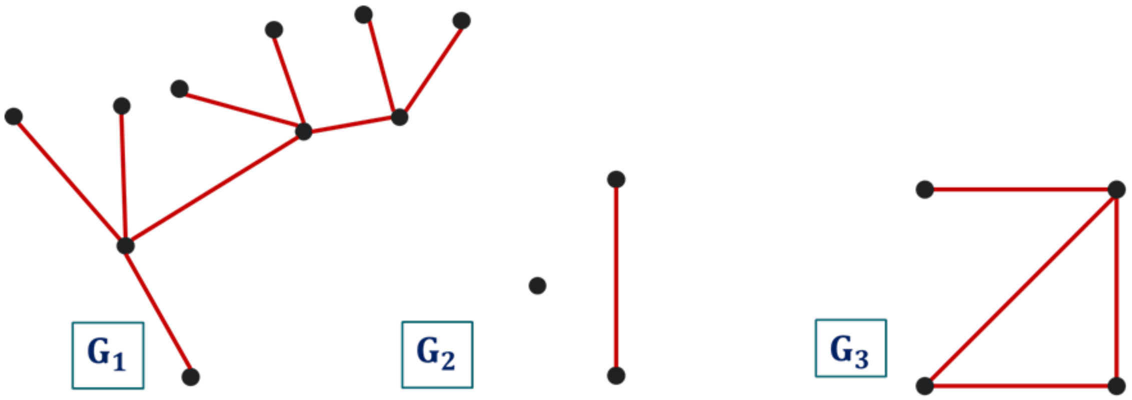
→ **Rooted** and **unrooted** trees, along with their key properties and real-world uses.

→ Trees that serve as subgraphs of a connected graph **G**, while still including **all vertices** of **G**, these are known as **spanning trees** of **G**.

→ Determining **all spanning trees** of a graph (a very useful task in real life).

Method – 1 ⇝ Introduction and Properties

Definition: A tree is a connected graph without any circuits.



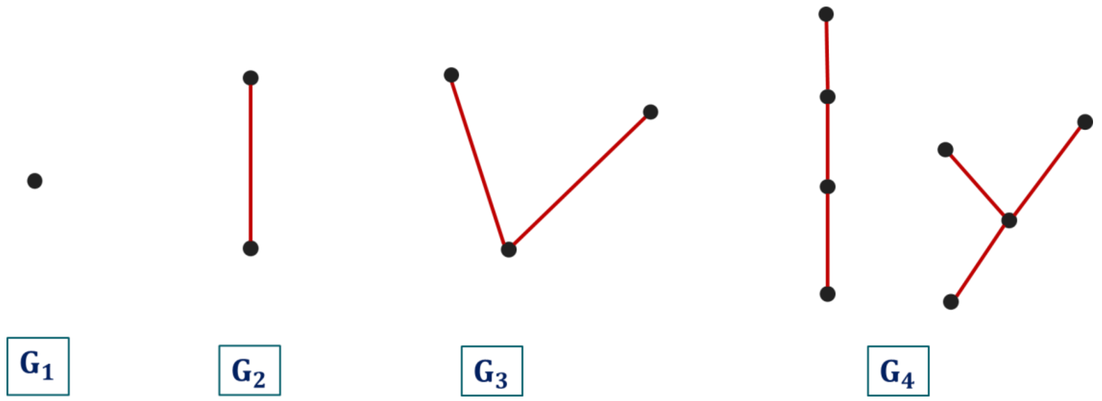
G_1 is a Tree G_2 is not a Tree G_3 is not a Tree

→ Some authors allow the null tree, a tree without any vertices.

→ We are considering only finite trees with at least one vertex.

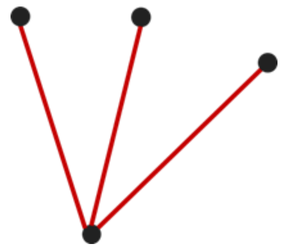
→ From the definition that a tree has to be a simple graph, that is, having neither a self-loop nor parallel edges.

→ Trees appear in numerous instances (In fact the term tree comes from family tree), most effective in data searching and sorting, very useful for organizing data that has a natural hierarchy, such as file systems or company hierarchies.



Definition: A path tree, the intermediate vertices have a degree of 2, while the end vertices (leaves) have a degree of 1. (For illustration see graph)

Definition: A star tree, the central vertex has a degree of $n-1$, and each of the remaining vertices (leaves) has a degree of 1. (For illustration see graph)



Observation:

→ A single-vertex tree contains no edges.

→ For any $n \geq 2$, we can construct a path tree with n vertices and $n - 1$ edges.

→ Similarly, we can construct a star tree with n vertices and $n - 1$ edges.

→ For $n = 4$, there are exactly two non-isomorphic trees: the path tree and the star tree.

Properties

→ In a tree T, there is one and only one path between every pair of vertices.

→ If in a graph G there is one and only one path between every pair of vertices, then G is a tree.

→ A tree with n vertices has $n - 1$ edges. (without proof)

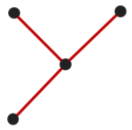
→ Any connected graph with n vertices and $n - 1$ edges is a tree. (without proof)

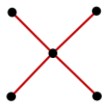
→ A graph is a tree if and only if it is minimally connected. (without proof)

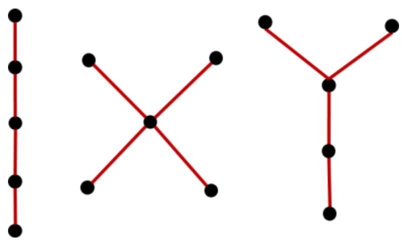
Example of Method-1: Introduction and Properties

C	1	Let G be a graph on n vertices. Then, the following statements are equivalent: a) G is a tree. b) There exists exactly one path between every pair of vertices in G. c) G is a minimally connected graph.
C	2	How many edges does a tree with 7 vertices have? Draw such tree. **Answer: 6**

<table>
<tr><td>C</td><td>3</td><td>Find all non-isomorphic trees with 4 vertices.
Answer:

Path tree
Degree Sequence
$(1,1,2,2)$

Star tree
Degree Sequence
$(1,1,1,3)$</td></tr>
<tr><td>C</td><td>4</td><td>Find all non-isomorphic trees with 5 vertices.
Answer:

Path tree
Degree Sequence
$(1,1,2,2,2)$

Star tree
Degree Sequence
$(1,1,1,1,4)$

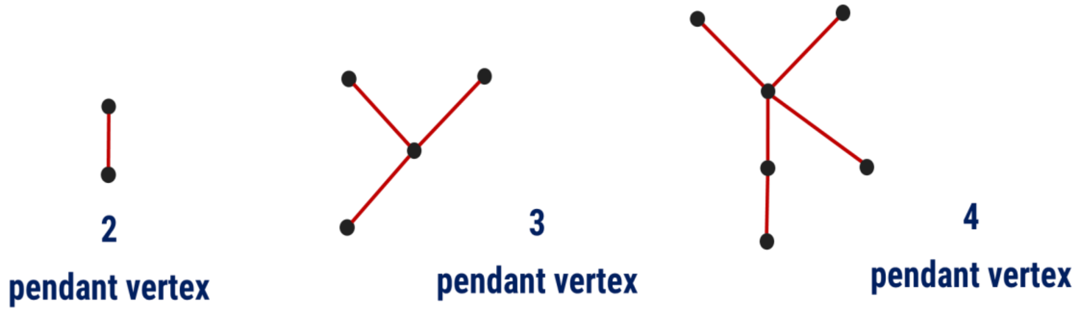
Degree Sequence
$(1,1,1,2,3)$</td></tr>
<tr><td>H</td><td>5</td><td>Define following terms:
Graph, Connected Graph, Degree of Vertex, Circuit, Tree, Path tree, Star tree.</td></tr>
<tr><td>H</td><td>6</td><td>Cite three different situations (games, activities, or problems) that can be represented by trees. Explain.</td></tr>
<tr><td>H</td><td>7</td><td>Take any tree of 5 vertices (different from below tree of 5 vertices) and show that it is isomorphic to below tree.
</td></tr>
<tr><td>H*</td><td>8</td><td>You need to decide how to travel to your college. You have three main options: Car, Public Transport, and Bike. Your goal is to select the best option based on factors like cost, time, and environmental impact.
1. What is your primary factor in choosing transportation?
    ○ Cost
    ○ Time
    ○ Environmental Impact
2. If your priority is cost, which transportation method would you choose?
3. If you care most about saving time, which transportation method would be your choice?
4. If being environmentally friendly is most important to you, which method would you choose?
5. Draw a decision tree based on these factors and your priority.</td></tr>
</table>

Method – 2 ⇝ Pendent Vertices, Distance and Center in Trees

Pendant Vertices in a Tree

Observation:

→ It is observed that each of the trees shown in the figures contains multiple pendant vertices, where a pendant vertex is defined as a vertex with degree one.



Question: Is it true that any tree with $n \geq 2$ vertices always has at least two pendant (leaf) vertices?

Answer: Yes

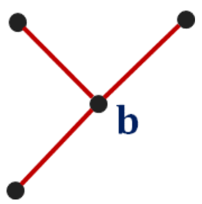
→ The reason is that,

- Since, in a tree of $n \geq 2$ vertices we have $n - 1$ edges.
- Hence, sum of degrees of each vertex (Total Degree) is $2(n - 1)$.
- Which is divided among n vertices. (Since no vertex can be of zero degree)
- We must have at least two vertices of degree one in a tree.

Result: Any tree (with two or more vertices), there are at least two pendant vertices.

Distance and Centers in a tree

→ In the tree shown below, there are four vertices.



→ If you look at it, vertex **b** appears to be more "**central**" than the other three vertices.

→ To understand and formalize this idea of a **center** in a tree, we must first define the concept of distance between vertices.

→ **Definition**: In a connected graph G, the **distance $d(v_i, v_j)$** between two of its vertices $\mathbf{v_i}$ and $\mathbf{v_j}$ is the length of the shortest path (i.e., the number of edges in the shortest path) between them.

→ The definition of distance between any two vertices is valid for any connected graph (not necessarily a tree).

→ In a **graph** that is **not a tree**, there are generally several paths between a pair of vertices.

→ We have to enumerate all these paths and find the length of the shortest one. (There may be several shortest paths.)

→ **In a tree**, since there is exactly one path between any two vertices, the determination of **distance is much easier**

→ **Definition:** The distance from v to the vertex farthest from v is called E(v) **eccentricity** of v. i.e

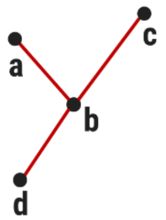
$$E(v) = \max_{v_i \in G} d(v, v_i)$$

→ **Definition:** A vertex with minimum eccentricity in graph G is called a **center** of G.

→ **Definition:** The eccentricity of a center in a tree is called the **radius** of the tree.

→ **Definition:** The length of the longest path in T is called the **diameter** of a tree.

→ In the tree shown below, Distance between each vertex is,



→ $d(a, b) = 1$, $d(a, c) = 2$, $d(a, d) = 2$, $d(b, c) = 1$, $d(b, d) = 1$ and $d(c, d) = 2$.

→ So, **Eccentricity** of $E(a) = 2$, $E(b) = 1$, $E(c) = 2$, and $E(d) = 2$.

→ **Center** of tree is b.

→ **Radius** of tree is 1.

→ **Diameter** is 2 (maximum of eccentricity).

Iterative leaf pruning (eliminating):

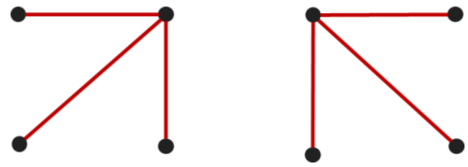
→ Steps to find the Center(s)

- Remove all leaves (vertices of degree 1) from the tree iteratively.
- Stop when only one or two vertices remain — these are the center(s).

→ This is known as iterative leaf pruning (eliminating).

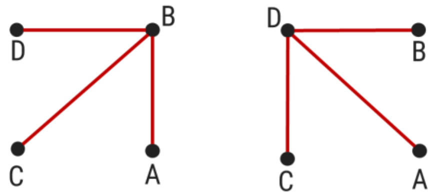
On Counting Trees:

→ We know that the trees shown below are isomorphic, therefore, we count them as a single (unlabeled) tree.



→ However, the situation changes when we assign labels to the vertices of these trees.

→ In that case, even isomorphic trees can become different if their labeling differ.



→ This leads to a natural question:

- What is a labelled tree?
- How many such trees exist for a given number of vertices, say **n**?

→ A graph in which each vertex is assigned a unique name or label (i.e., no two vertices have the same label), is called a labeled graph.

→ The difference between a labeled and an unlabeled graph is very important when we are counting the number of different graphs.

→ The following well-known theorem for counting trees was first stated and proved by Cayley.

Cayley's Theorem: The number of labeled trees with n vertices $(n \geq 2)$ is n^{n-2}.

Question: Draw all labeled tree for vertices $n = 2,\ 3,\ \text{and}\ 4$.

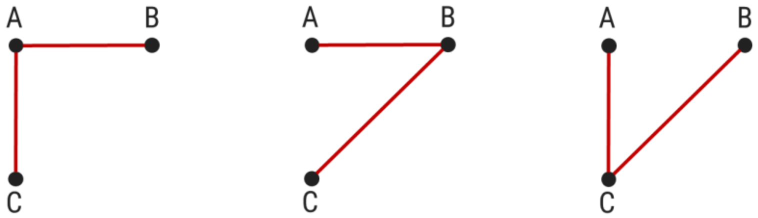
→ Answer:

- By Cayley's theorem,

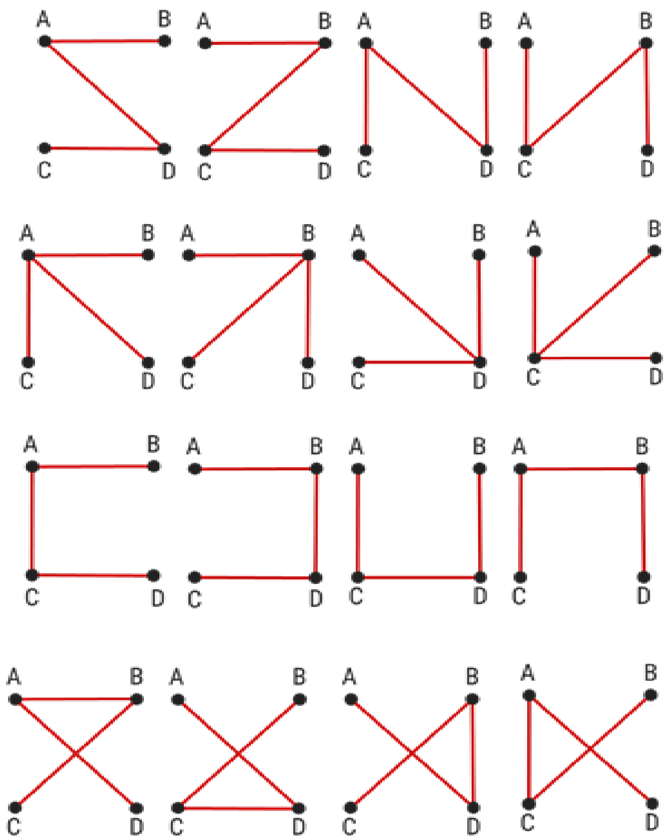
- The number of labeled trees with n vertices is n^{n-2}.
- For $n = 2$ the number of labeled trees is $2^{2-2} = 2^0 = 1$.
- For $n = 3$ the number of labeled trees is $3^{3-2} = 3^1 = 3$.
- For $n = 4$ the number of labeled trees is $4^{4-2} = 4^2 = 16$.

$\boxed{\mathrm{n} = 2 \Rightarrow \text{number of labeled trees} = 1}$

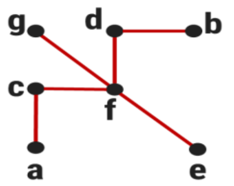
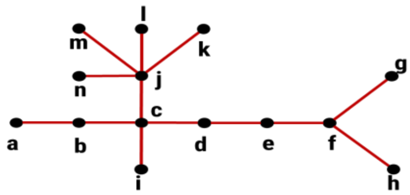
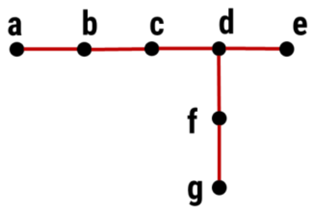
$\boxed{\mathrm{n} = 3 \Rightarrow \text{number of labeled trees} = 3}$



$\boxed{\mathrm{n} = 4 \Rightarrow \text{number of labeled trees} = 16}$



Example of Method-2.2: Pendent Vertices, Distance and Center in Trees

C	1	What is the maximum number of pendant (leaf) vertices that a tree with $n > 2$ vertices can have? Justify your answer. **Answer:** $\mathbf{n-1}$
C	2	Give proper justification. “Every tree has either one or two centers”.
C	3	Find the following for given tree. (i) Distance between two vertices (ii) Eccentricity (iii) Radius (iv) Diameter (v) Center 
C	4	Show that a radius in a tree is not necessarily half of its diameter.
H	5	Define the following terms in the context of trees: Distance between two vertices, Eccentricity, Radius, Diameter, and Center.
H	6	Give an example of a tree in which its diameter is equal to twice the radius.
H	7	Explain how the number of pendant vertices affects the eccentricity of other vertices in a tree.
H	8	Find the following for given tree. (i) Distance between two vertices (ii) Eccentricity (iii) Radius (iv) Diameter (v) Center 
H	9	What is the minimum number of pendant (leaf) vertices that a tree with $n > 2$ vertices can have? Justify your answer
H*	10	A company sets up a wired computer network among 7 computers (nodes), connected with cables in a tree structure (no loops), to minimize cost. The layout of the connections is:  The network administrator wants to: 1. Place antivirus servers on the computers that are at the edges of the network (pendent vertices). 2. Install a main data server at a central computer so that data reaches the farthest computer in the shortest possible time. 3. Determine the radius and diameter of the network for performance metrics. Answer the following: (a) Which computers are pendent vertices? (b) What is the diameter of the network? (c) What is the radius, and which computer(s) can be chosen as the center?

Method – 3 ⇝ Introduction to Spanning Trees and Properties

Introduction

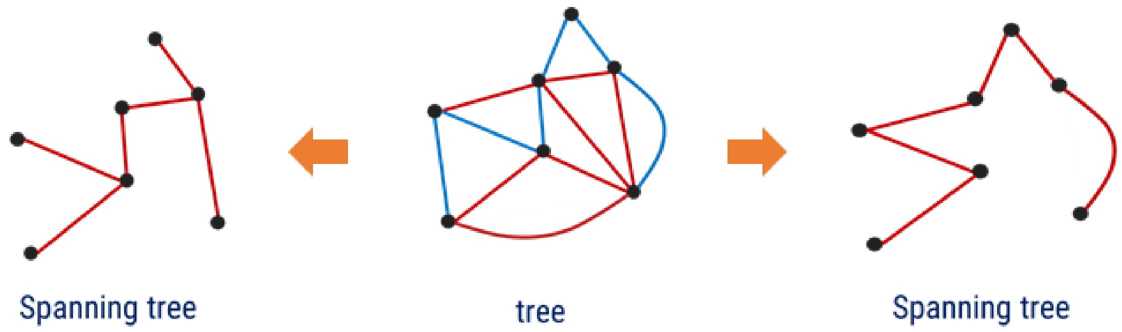
→ We have learned about trees as separate graphs and their properties.

→ Now, we will look at trees that are part of a bigger graph (as subgraphs).

→ A graph can have many subgraphs, and some of them will be trees.

→ Among these trees, we are mainly interested in special ones called **spanning trees**— as defined next.

Definition: A tree T is said to be a **spanning tree** of a connected graph G if T is a subgraphs of G and T contains all vertices of G.



→ Spanning trees are the **largest trees** among all trees in G.

→ Spanning tree is defined only for a **connected graph.**

→ **For finding a spanning tree of a connected graph G.**

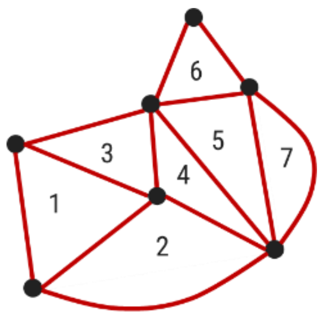
- If the graph G has **no circuit** (loop), then G **itself** is a spanning tree.
- If G has **a circuit**, **remove one edge from that circuit**.
- If there are **more circuits**, keep **removing one edge from each circuit** until no circuits are left.

→ What remains is **a connected graph with no circuits —** this is the spanning tree.

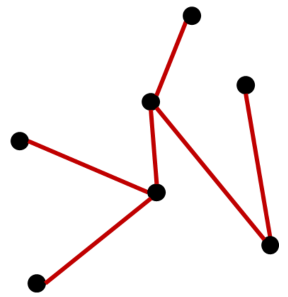
→ **Example**: Finding a spanning tree of a connected graph G.

- **Step 1:** If the graph G has no circuit (loop), then G itself is a spanning tree. **(Which is not applicable in our case)**
- **Step 2:** If G has a circuit, remove one edge from that circuit.

- **Step 3:** If there are more circuits, keep removing one edge from each circuit until no circuits are left.



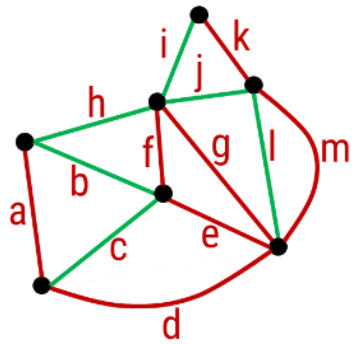
- **Step 4:** What remains is a **connected graph with no circuits —** this is the spanning tree.



Definition: An edge in a spanning tree T is called a branch.

Definition: An edge of G that is not in a spanning tree T is called a chord.

Example: For given graph find branch and chord with respect to given spanning tree {c, b, h, i, j, l}.



→ **Branch:** c, b, h, i, j, l

→ **Chord:** a, d, e, f, g, k, m

→ Note: It must be kept in mind that branches and chords are defined only with respect to a given spanning tree.

Properties

→ A connected graph G can have **at least one** spanning tree.

→ All possible spanning trees of graph G, have **the same number of edges and vertices.**

→ With respect to any of its spanning trees, a connected graph of **n vertices** and **e edges** has **n − 1 tree branches.**

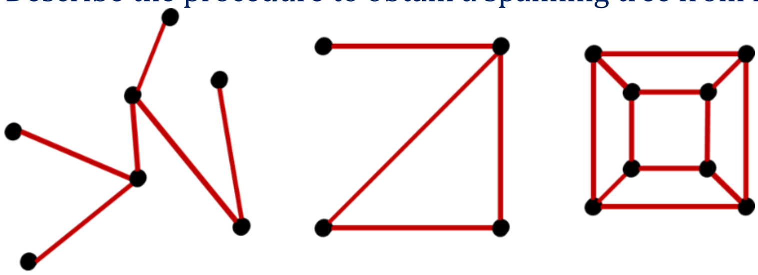
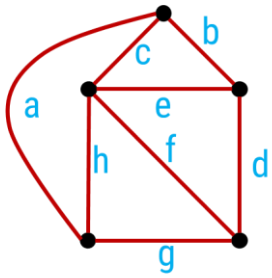
→ With respect to any of its spanning trees, a connected graph of **n vertices** and **e edges** has **e − n + 1 chords.**

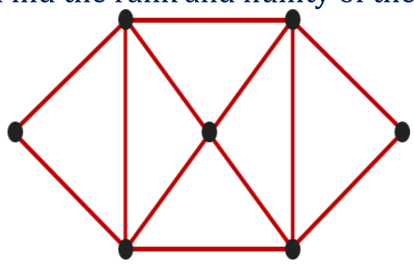
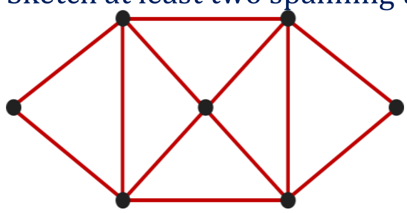
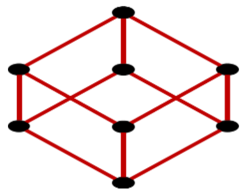
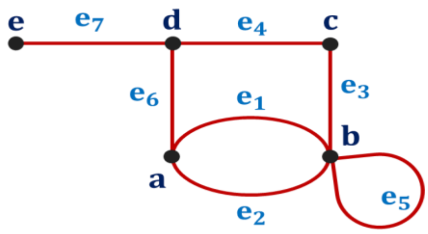
→ **Rank** of connected graph G (Number of branches in any spanning tree of G) $= n - 1$.

→ **Nullity** of connected graph G (Number of chords in G) $= e - n + 1$.

→ Rank + Nullity (Number of edges in G) $= e$.

Example of Method-3: Introduction to Spanning Trees and Properties

C	1	Describe the procedure to obtain a spanning tree from following graph. 
C	2	For each of the following standard graphs with **n** vertices, find the rank and nullity. a. Complete graph $\mathbf{K_n}$ b. Cycle graph $\mathbf{C_n}$ c. Path graph $\mathbf{P_n}$ d. Star tree $\mathbf{S_n}$
C	3	For given graph find branch and chord with respect to spanning tree of your choice. 

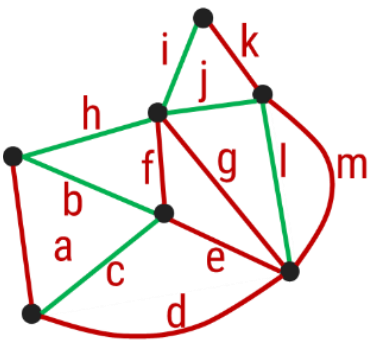
C	4	Find the rank and nullity of the following graph.  **Answer: Rank = 6, Nullity = 6**
H	5	Sketch at least two spanning tree of following graph.  
H	6	Show that a path is its own spanning tree.
H	7	For given graph find branch and chord with respect to spanning tree of your choice. 
H	8*	A company has 5 office buildings that need to be connected with network cables. Each pair of buildings can be connected, but the cost is high, so the company wants to: Ensure all buildings are connected, Use the least amount of cable, Avoid loops in the network. Solve following. a) Represent this problem as a graph. b) Explain how a spanning tree helps solve this problem. c) How many cables (edges) will be needed? d) What will happen if we use more than that number of cables?

Method – 4 ⇝ Fundamental Circuits and finding all Spanning Trees of Graphs

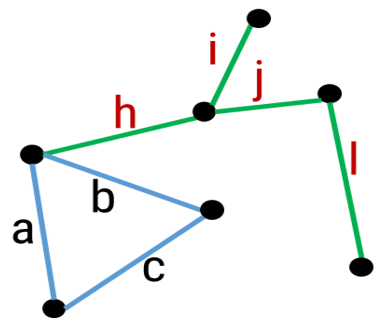
→ **Definition**: A circuit, formed by adding a chord to a spanning tree, is called a **fundamental circuit**.

→ Consider graph with spanning tree {c, b, h, i, j, l}.

→ Here Branches are **c**, **b**, **h**, i, **j**, **l** and chords are **a**, **d**, **e**, **f**, **g**, **m**, **k**.



→ If we add chord a to given spanning tree we will get fundamental circuit {b, c, a}.



→ In our case, we have eight chords: **a**, d, **e**, **f**, **g**, **m**, and **k**. Therefore, there are only eight fundamental circuits, each formed by one chord together with the branches of the tree.

→ **How many fundamental circuits does a graph have?**

- Exactly as many as the number of chords (= $e - n + k$).

→ **Keep in mind:**

- A circuit is a fundamental circuit only with respect to a given spanning tree.
- A given circuit may be fundamental with respect to one spanning tree, but not with respect to a different spanning tree of the same graph.
- Although the number of fundamental circuits in a graph is fixed, the circuits that become fundamental change with the spanning trees.

Finding all Spanning Trees of a Graph

→ In a given connected graph there are a large number of spanning trees:

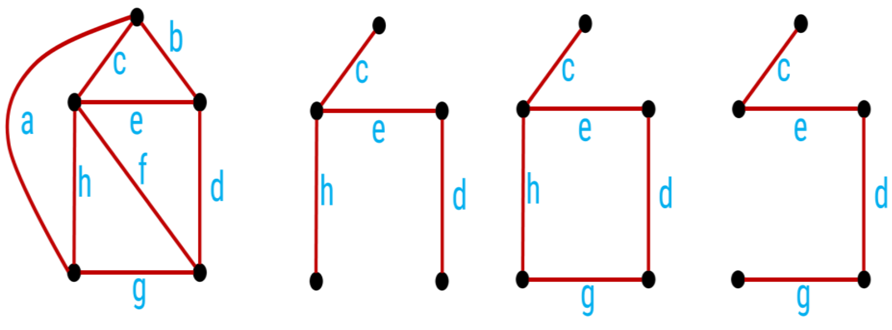
→ To find the exact number of spanning trees, **Kirchhoff's Matrix-Tree Theorem (**The **determinant** of that minor of the Laplacian matrix is the **number of spanning trees).**

→ Here instead of this theorem we will use following procedure for finding all spanning tree of a graph.

- Step 1: Start with any spanning tree of the graph. Let's say the tree is $\mathbf{T_1}$ (e, c, h, d).
- Step 2: **Add a chord** to the tree. For example, add edge g. This will create a fundamental circuit — (e, h, g, d).
- Step 3: **Remove one branch** from this fundamental circuit. For example, remove edge h.

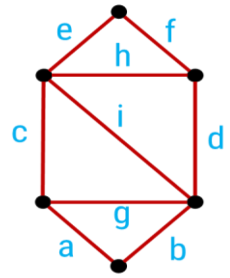
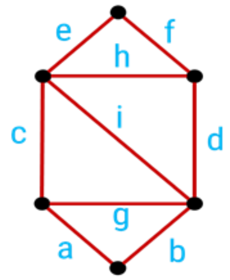
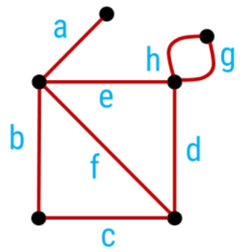
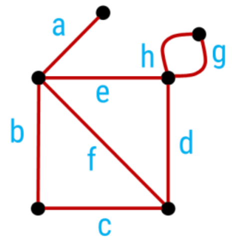
→ The result is a new spanning tree, called T_2.

→ This method of creating a new spanning tree from an existing one by adding a chord and removing one branch from the resulting fundamental circuit is called a cyclic interchange or an elementary tree transformation.



T_1 Fundamental circuit T_2

Example of Method-4: Total Probability and Bayes' Theorem

C	1	Find any fundamental circuit with respect to spanning tree of your choice and how many such fundamental circuits following graph have?	
C	2	Generate any 2 spanning tree using **procedure of finding all spanning tree** for following graph.	
H	3	Find any fundamental circuit with respect to spanning tree of your choice and how many such fundamental circuits following graph have?	
H	4	Generate any 2 spanning tree using **procedure of finding all spanning tree** for following graph.	

*** * * * * End of the Unit * * * * ***