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Unit – 3 ➔ Domination & Matching in Graphs

Introduction

This unit focused on

- Key Question: How can we use specific sets of vertices or edges to control or pair parts of a graph efficiently?
- We learned about dominating sets, domination number, and total domination, showing how a small set of vertices can influence the whole graph.
- These ideas are useful in real-world problems like network monitoring, resource placement, and spreading information.
- We also studied matching, including maximal, maximum, and perfect matching, to understand how vertices can be paired without overlap.
- Both domination and matching help to solve practical problems like job assignment, scheduling, and pairing tasks with resources.

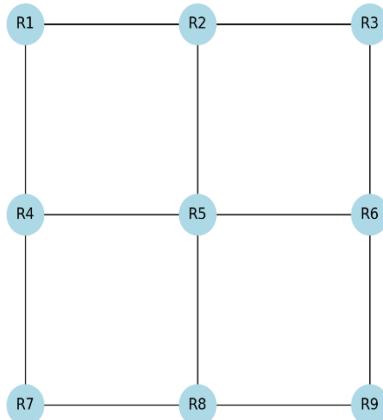
- Before we start first introduces two real-life problem scenarios that can be modeled and solved using graph theory concepts.

Task 1: Security Camera Placement

- You are designing a security system for a museum with 12 rooms. Some rooms are connected by open hallways, and you are given a floor plan (represented as a graph).

Your job is to **place the minimum number of security cameras** so that every room is either directly monitored or connected to a room with a camera.

- **Answer:** Place the minimum number of cameras in Room2, Room5, and Room8.



Task 2: Intern-Task Assignment

→ There are 6 interns and 6 different tasks. Each intern is skilled for certain tasks only (a list of possible intern task pairs is given). Your goal is to assign tasks so that each intern gets exactly one task, and no task is assigned to more than one intern, choosing the maximum possible assignments.

→ **Answer:** This assigns 5 interns, with no overlap, and all assignments match the skill edges.

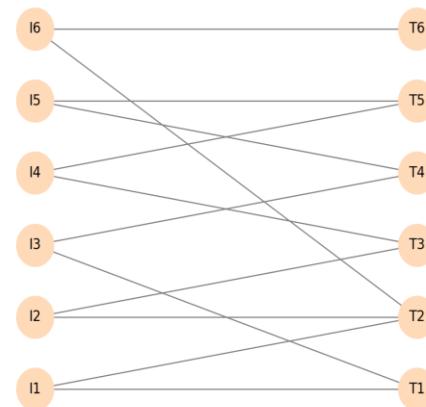
→ In the previous tasks, you made strategic decisions:

- Where to place cameras for full coverage.
- How to assign tasks to interns efficiently.

→ You were actually solving two fundamental problems in graph theory:

- **Domination** – selecting key vertices to influence or monitor a graph
- **Matching** – pairing vertices through edges without overlap

→ Let's now explore these two concepts.

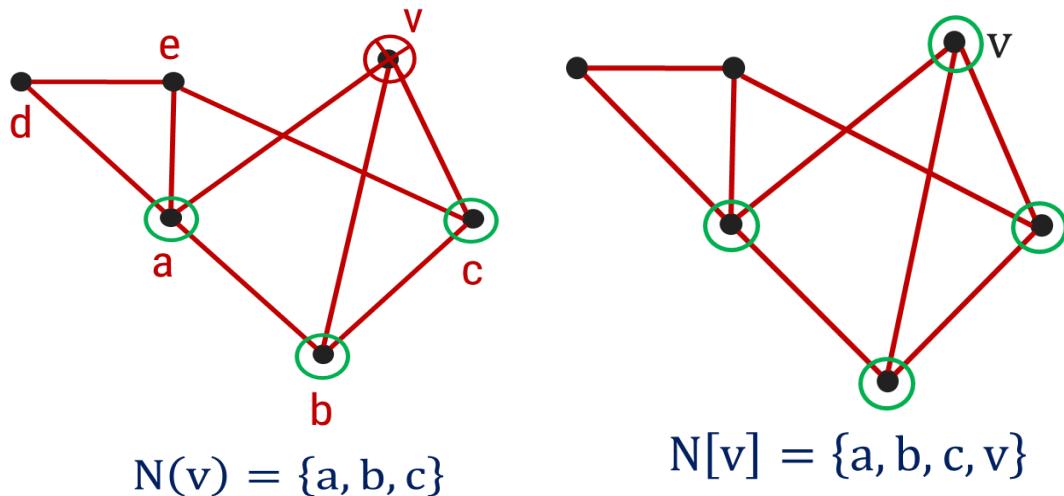


Method – 1 → Introduction to Domination in Graphs:

Dominating Set

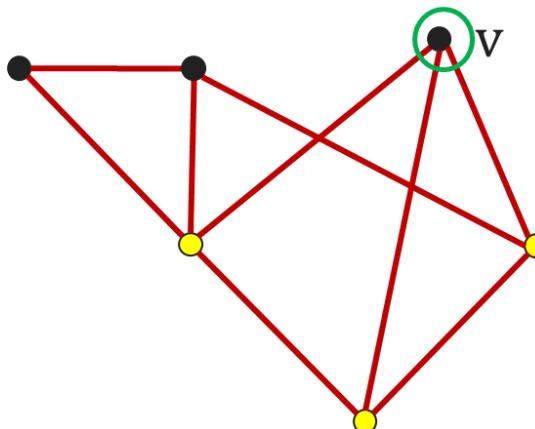
Definition: For a vertex v of a graph G , a **neighbor of v** is a vertex adjacent to v in G . Also, the neighborhood (or open neighborhood) $N(v)$ of v is the set of neighbors of v .

Definition: The **closed neighborhood $N[v]$** is defined as $N[v] = N(v) \cup \{v\}$.



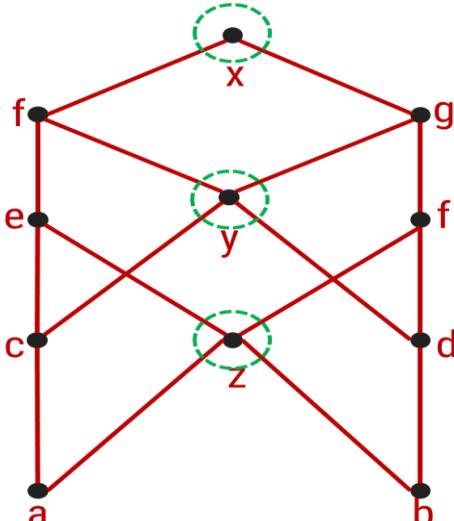
Definition: A vertex v in a graph G is said to **dominate** itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$.

- Therefore, v dominates $1 + d(v)$ vertices of G .
- So, in below graph degree of vertex v is 3. So, the vertex v can dominate 4 vertices.

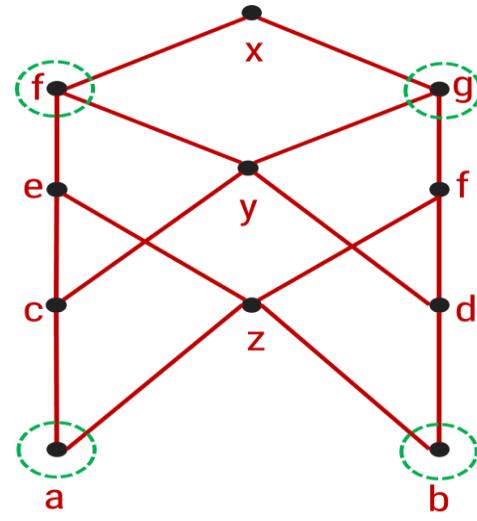


- A set S of vertices of G is a **dominating set** of G if every vertex of G is dominated by some vertex in S .

- Equivalently, a set S of vertices of G is a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S .
- Consider the below graph, the sets $S_1 = \{x, y, z\}$ and $S_2 = \{f, g, a, b\}$, indicated are both dominating sets in graph.

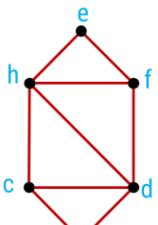
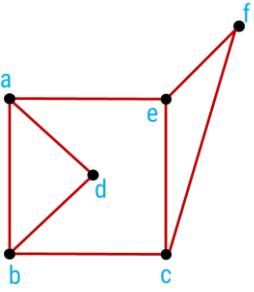
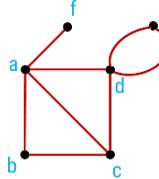
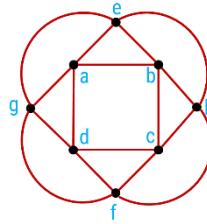


$$S_1 = \{x, y, z\}$$



$$S_1 = \{f, g, a, b\}$$

Example of Method-1: Introduction to Domination in Graphs: Dominating Set

C	1	What is dominating set of complete graph K_n ? Justify Answer: Single vertex
C	2	Find any dominating set for the following graphs.   Answer: {e, a} for first graph and {d, f} for second graph and there are other sets also dominate graph.
C	3	Give at least 2 dominating set for each of the following graphs Answer: {e, a} and {a, d} for first graph and {e, f} and {g, h} for first graph and there are other sets also dominate graph.  
C	4	Justify following statement: Let G be a graph without isolated vertices. If S is a minimal dominating set of G , then $V(G) - S$ is a dominating set of G .
H	5	Give at least two solution of this problem was mentioned by Ore. According to the rules of chess, a queen can move any number of squares horizontally, vertically, or diagonally on the chess board (assuming that no other chess figure is on its way). How to place a minimum number of queens on a chessboard so that each square is controlled by at least one queen?
H	6	Given the following graph G , Graph: $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, \{d, e\}\}$ Draw the graph and identify at least two dominating sets.
H	7	Consider the following undirected graph G : $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 5\}, \{4, 6\}, \{5, 6\}\}$. Check that following are dominating set or not. 1. $\{2, 5, 6\}$ 2. $\{1, 3, 4\}$ 3. $\{2, 5\}$

H	8	You are the head of security at a university campus. The campus has several key locations: classrooms, library, cafeteria, hostel, admin block, labs, parking, and auditorium. Each of these locations is connected by paths. $V = \{A, B, C, D, E, F, G, H\}$ where, A = Admin Block, B = Library, C = Cafeteria, D = Classrooms, E = Hostel, F = Parking, G = Labs, H = Auditorium Edges: $E = \{(A, B), (A, D), (B, C), (B, D), (C, E), (D, F), (E, G), (F, G), (G, H)\}$. Find the minimum set of vertices where cameras should be placed so that every location is under surveillance (either directly or by an adjacent node).
H	9	What is dominating set of star graph S_n ? Justify Answer: c vertex.

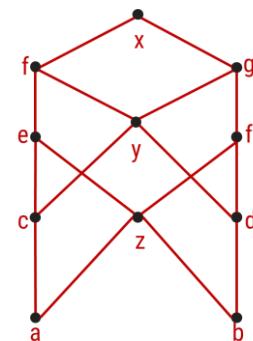
Method – 2 \Rightarrow Domination Number and Total Domination

Definition: A minimum dominating set in a graph G is a dominating set of minimum cardinality.

Definition: The cardinality of a minimum dominating set in G is called **the domination number of G** and is denoted by $\gamma(G)$.

Example: Find domination number of below graph

- We saw that the set $S_1 = \{x, y, z\}$ is a dominating set for G .
- Therefore, $\gamma(G) \leq 3$.
- To show that the domination number of G is actually 3, it is required to show that there is no dominating set with two vertices.
- Notice that, the order of G is 11 and that the degree of every vertex of G is at most 4.
- This means that no vertex can dominate more than 5 vertices and that every two vertices dominate at most 10 vertices.
- That is, $\gamma(G) > 2$ and so $\gamma(G) = 3$.
- Hence, **minimum dominating set is $\{x, y, z\}$ (it is not unique)** and Domination number is 3 (**it is unique**).



Remarks:

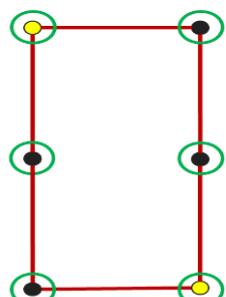
- The vertex set of a graph is always a dominating set.
- The domination number is defined for every graph.
- If G is a graph of order n , then $1 \leq \gamma(G) \leq n$.
- A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n - 1$.
- **All complete graphs and all stars have domination number 1.**
- $\gamma(C_n) = \frac{n}{3}$ for $n \geq 3$.
- If G is a graph of order n , then $\frac{n}{1+\Delta(G)} \leq \gamma(G) \leq n - \Delta(G)$.
- If G is a graph of order n without isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.
- Let G be a graph without isolated vertices. If S is a minimal dominating set of G , then $V(G) - S$ is a dominating set of G .

- We have seen that a vertex u dominates a vertex v in a graph if either $u = v$ or v is a neighbor of u .
- However, there are a number of variations of domination.
- We consider one of the best known of these. In this variation, a vertex u dominates a vertex v only if v is a neighbor of u .
- (In this context, a vertex does not dominate itself.) This type of domination is called **total domination**.

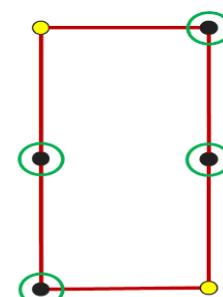
Definition: A set S of vertices in a graph G is a **total dominating set** of G if every vertex of G is adjacent to at least one vertex of S .

Definition: The minimum cardinality of a total dominating set is the **total domination number** $\gamma_t(G)$ of G .

Domination vs. Total Domination

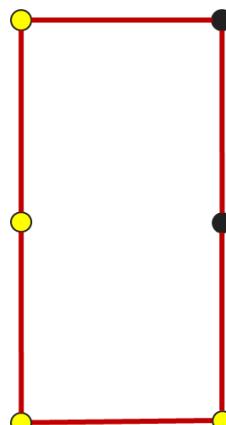


2 vertices will dominate graph



2 vertices will not totally dominate graph

- In the first graph yellow vertex will dominate itself and its neighbors (as we are in domination) so only two vertex will dominate graph.
- But in the second graph 2 yellow vertex will not dominate itself only dominate its neighbors (As we are in total domination).
- So, we have to select 4 vertices to totally dominate the graph.
- Hence, $\gamma_t(G) = 4$.



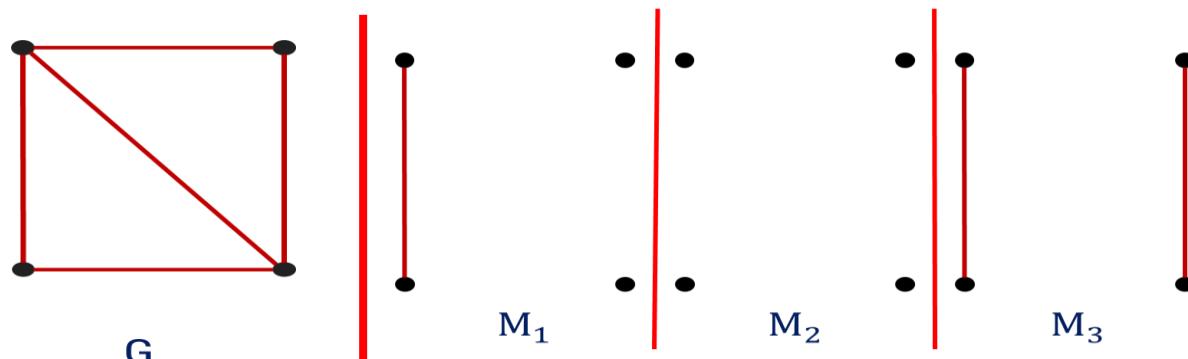
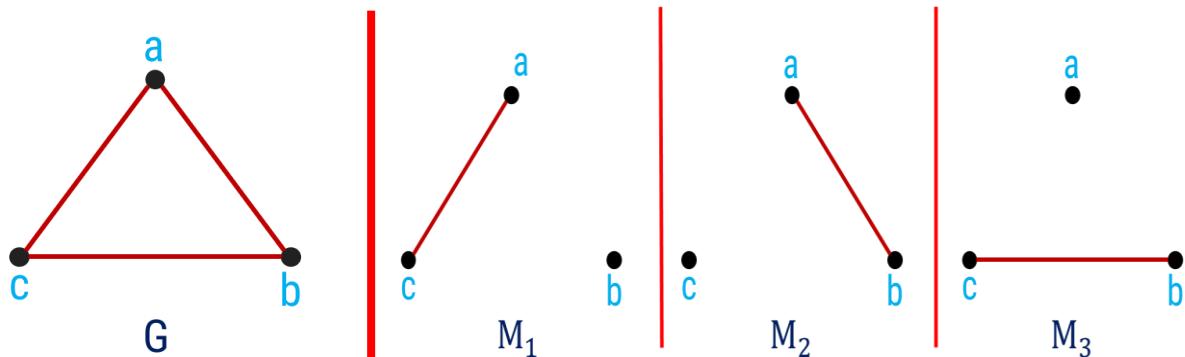
Example of Method-2.2: Domination Number and Total Domination

C	1	Find the domination number for P_n and C_n , $6 \leq n \leq 8$. Answer: $\gamma(P_6) = 2$ and $\gamma(P_7) = 3 = \gamma(P_8)$.
C	2	Prove mathematically that the domination number for K_n and S_n are 1.
C	3	Determine the domination number and total domination number for the given graphs. Answer: 2, 4
C	4	Find the domination number and total domination number of $K_{2,3}$. Answer: 2, 2
C	5	Give an example of a connected graph G such that $\gamma_t(G) = 1.5\gamma(G)$.
C	6	Give an example of a connected graph G such that 1. $\gamma(G) = 3$ and $\gamma_t(G) = 6$ 2. $\gamma(G) = 3$ and $\gamma_t(G) = 3$.
H	7	Find the total domination number for P_n and C_n , for $6 \leq n \leq 9$. Answer: for path 3, 4, 4, 5 and for cycle 3, 4, 4, 5.
H	8	Determine the domination number and the total domination number for each of the following graphs. (1) K_6 (2) P_6 (3) C_6 (4) $K_{3,4}$ Answer: for domination 1, 2, 2, 3 and for total domination 2, 3, 3, 4.
H	9	Give an example of a connected graph G such that $\gamma_t(G) = 0.5\gamma(G)$.
H	10	For each integer $k \geq 2$, give an example of a connected graph G for which $\gamma(G) = \gamma_t(G) = k$.
H	11	For each integer $k \geq 2$, give an example of a connected graph G for which $\gamma(G) = k$ and $\gamma_t(G) = 2k$.
H	12	Find (a) the domination number of G , (b) the total domination number of G
		Answer: 3, 4.

Method – 3 \Rightarrow Matching, Maximal, and Maximum Matching in Graphs

Definition: A **matching** in a graph is a subset of edges in which no two edges are adjacent.

→ Consider two example of Graph with Matchings M_1 , M_2 , and M_3 .

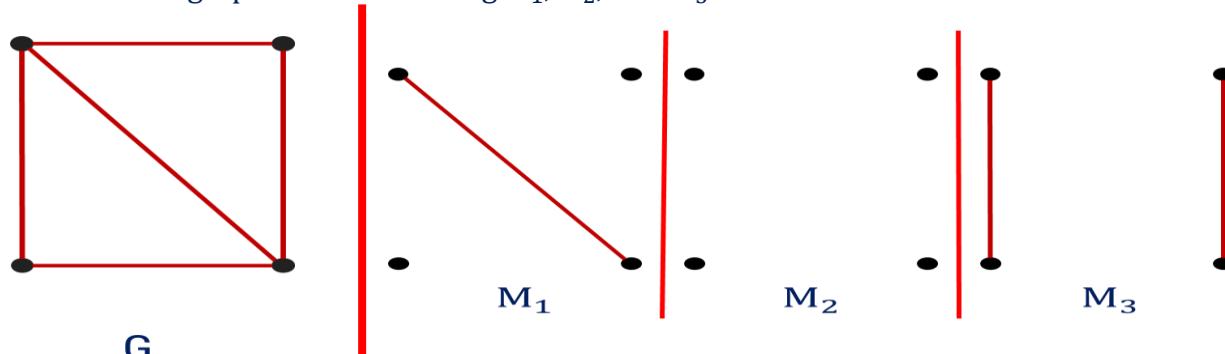


→ So, a single edge in a graph is obviously a matching.

→ Graph without edges is also a matching.

Definition: A **maximal matching** is a matching to which no edge in the graph can be added.

→ Consider graph G with matching M_1 , M_2 , and M_3 .



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→ Clearly, if we add edges in M_1 and M_3 then matching condition break but we can add edges in M_2 .

→ So, M_1 and M_3 are maximal matching of graph but M_2 is not maximal matching.

→ A graph may have many different maximal matchings, and of different sizes.

Definition: The maximal matchings with the largest number of edges are called the **maximum matchings**.

Definition: The maximal matchings with the largest number of edges are called the **maximum matchings**.

Example: Find matching number of K_3

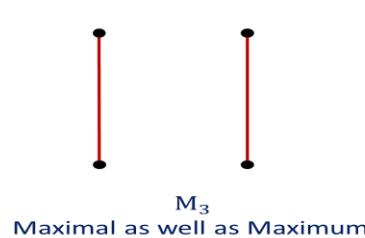
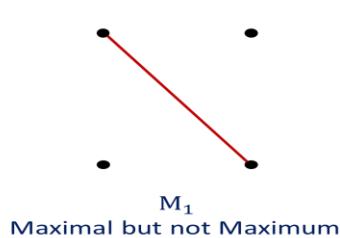
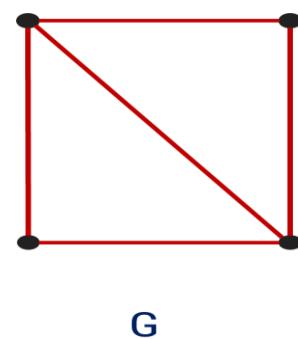
- Clearly for K_3 , we have 4 possible matchings, in which 3 are single edge and 1 is null graph.
- Clearly null graph is not maximal as we can add edge of graph in null graph.
- So each single edge is maximal matchings.
- Which is also maximum matchings.
- Number of edges in maximum matchings is 1.
- Hence the matching number is 1.

Example: Find matching number of the following graph.

→ Here first we find maximum matching (largest size maximal matching).

→ We know that graph have maximal matching of size 1 and size 2 only.

→ We cannot find maximal matching of size 3 as graph have 4 vertices only (only 2 edge are required to cover 4 vertices).



→ So, maximum matching contains 2 edges which is largest in size.

→ Hence, matching number is 2.

Remarks

- Maximum matching is always maximal but converse is not true.
- Subset of edges is matching if and only if degree of each vertex is at most 1.

Example of Method-3: Matching, Maximal, and Maximum Matching in Graphs

C	1	Find any three matching of the following graphs.
		<p>Three graphs labeled G₁, G₂, and G₃. G₁ has 5 vertices and 6 edges. G₂ has 6 vertices and 9 edges. G₃ has 6 vertices and 9 edges.</p>
C	2	Find a matching of the following graphs.
		<p>Two graphs labeled G₁ and G₂. G₁ has 9 vertices and 12 edges. G₂ has 8 vertices and 10 edges.</p>
C	3	Find matching number of the following graphs.
		<p>Two graphs labeled G₁ and G₂. G₁ has 9 vertices and 12 edges. G₂ has 8 vertices and 10 edges.</p>
		Answer: 3,3
H	4	Find matching number of following graphs.
		<p>Two graphs for finding matching number. The left graph has 6 vertices and 7 edges. The right graph has 12 vertices and 18 edges.</p>
		Answer: 3,4
H	5	Find matching number of K ₆ .
		Answer: 3
H	6	Find matching number of C ₆ .
		Answer: 3
H	7	Find matching number of P ₆ .
		Answer: 3

H	8	Find matching number of $K_{2,3}$. Answer: 2
H	9	Find matching number of S_n . Answer: 1
H	10	Justify: Maximum matching is always maximal but converse is not true.

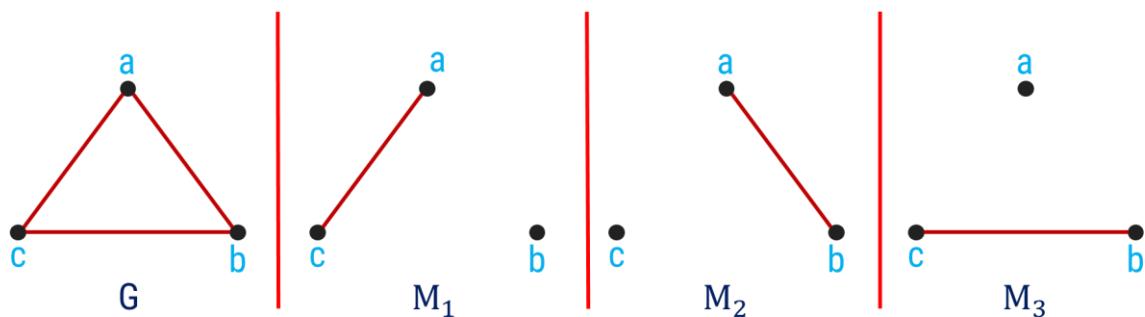
Method – 4 ↗ Perfect Matching and Hall's Theorem (without proof)

Definition: A matching of a graph in which every vertex is matched is called **perfect matching**.

OR

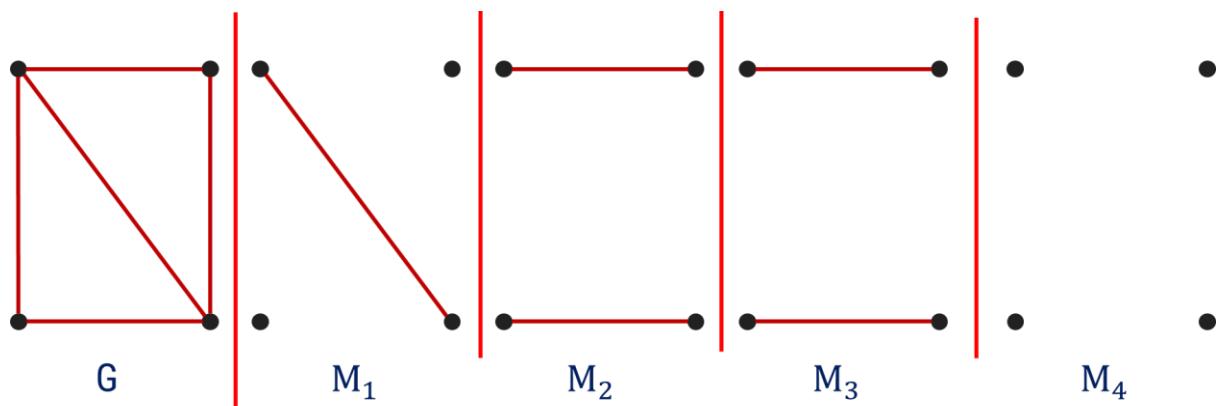
A matching for which degree of each vertex is 1 is called **perfect matching**.

→ Consider following graph K_3 with three of its matching.



→ So, for K_3 , have matching but we cannot find perfect matching for K_3 .

→ Consider another example with two of its perfect matching.



As degree of each vertex in M₂ and M₃ is 1.

So, M₂ and M₃ are perfect matching while M₁ and M₄ are not perfect matching.

Question: Is perfect matching possible for graph with odd order?

Answer:

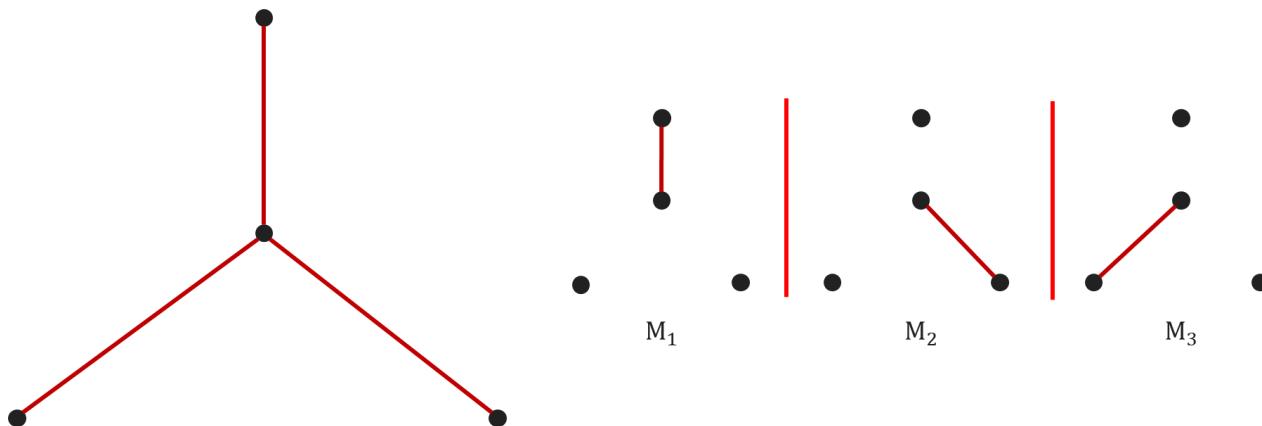
From definition,

- In a perfect matching, the degree of each vertex is 1.
- So, a perfect matching of a graph is a matching where every vertex of the graph is connected to exactly one edge in the matching.
- That means, a perfect matching has $\frac{n}{2}$ edges (where n is the number of vertices).
- Therefore, perfect matchings are only possible in graphs with an even number of vertices.
- Hence, perfect matching is not possible for graph with odd order.

Question: Is perfect matching possible for all graph with even order?

Answer:

- No perfect matching is not possible for all graph with even order.
- Consider following graph with all of its possible matching,



- Hence perfect matching is not possible for all graph with even order

Real life situation can be modeled by a graph

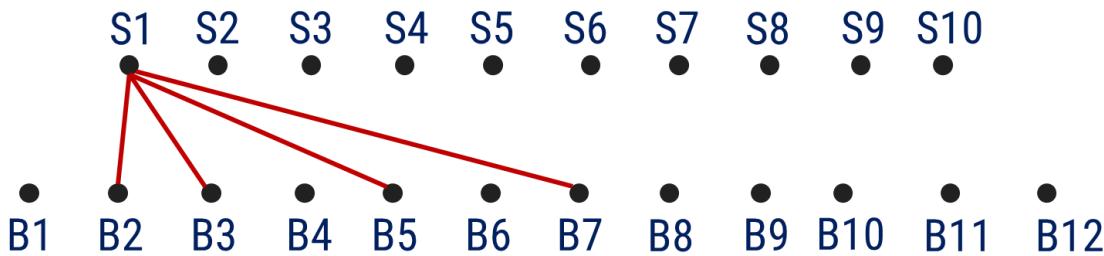
Question:

- A mathematics department at a university has acquired a collection of 12 different mathematics books on a variety of subjects to be presented to students who have performed well on a competitive mathematics exam (one book to each successful student). There would be a problem if more than 12 students qualified for these books. It turns out, however, that this is not a problem as only 10 students did well enough on the exam to receive books. Nevertheless, another possible difficulty has arisen. Some of

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the students already have copies of some books and there are some books that certain students have no need for. The question is this: Is there a way of distributing 10 of the 12 books to the 10 students so that each student receives a book that he or she would like to have?

- **The answer to this problem may be no, even though there are more books than students.** For example, there may be three or more books that no student wants. Also, perhaps there are four students only interested in the same three books, in which case it would be impossible to distribute four books to these four students.
- It may already be clear that this situation can be modeled by a graph G whose vertices are the students, say S_1, S_2, \dots, S_{10} and the books, say B_1, B_2, \dots, B_{12} .
- So, G is a bipartite graph sets $U = \{S_1, S_2, \dots, S_{10}\}$ and $W = \{B_1, B_2, \dots, B_{12}\}$.
- For example, if student S_1 would like to have any of the books B_2, B_3, B_5, B_7 , then the graph G contains the subgraphs shown in Figure.



Model and Solve following real life problem using graph theory

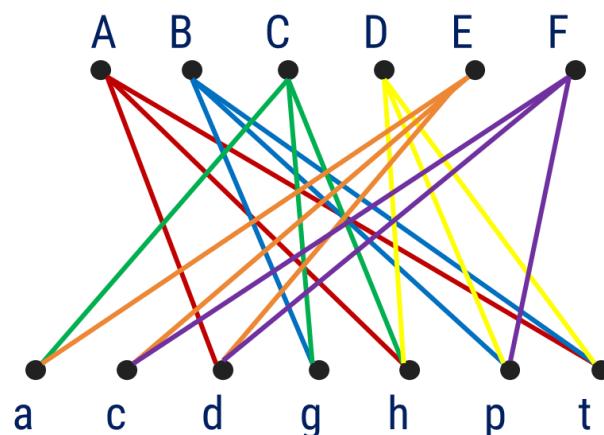
Problem:

As a result of doing well on an exam, six students (A), (B), (C), (D), (E) and (F) have earned the right to receive a complimentary textbook in either algebra (a), calculus (c), differential equations (d), geometry (g), history of mathematics (h), programming (p) or topology (t). There is only one book on each of these subjects. The preferences of the students are

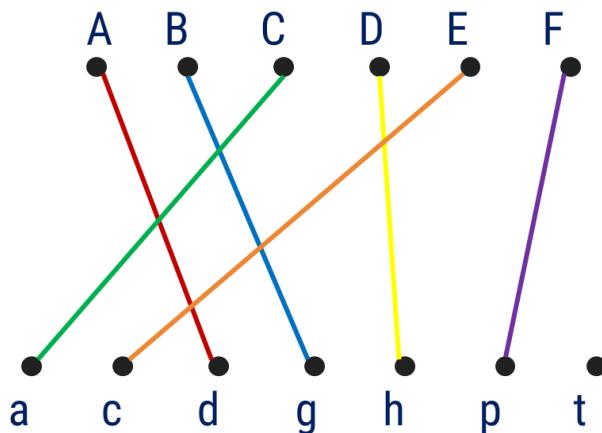
A: d, h, t; B: g, p, t; C: a, g, h; D: h, p, t; E: a, c, d; F: c, d, p.

Can each of the students receive a book he or she likes?

Model:



Solution:

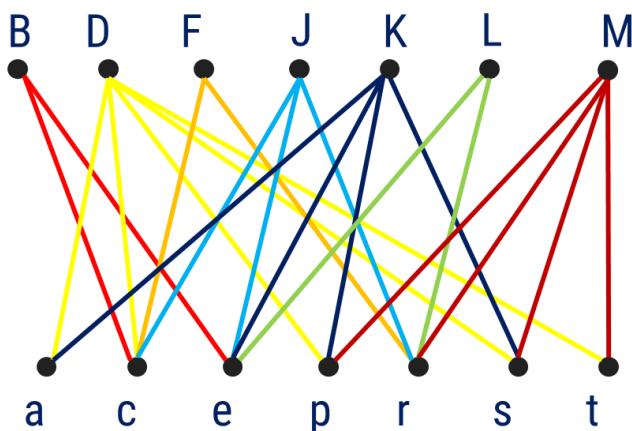


Problem:

Seven seniors (B), (D), (F), (J), (K), (L) and (M) are looking for positions after they graduate. The University Placement Office has posted open positions for an accountant (a), consultant (c), editor (e), programmer (p), reporter (r), secretary (s) and teacher (t). Each of the seven students has applied for some of these positions: The preferences of the students for the available subjects are as follows: Student B prefers c and e. Student D prefers a, c, p, s, and t. Student F prefers c, and r. Student J prefers c, e, and r. Student K prefers a, e, p, and s. Student L prefers e and r, and Student M prefers p, r, s, and t.

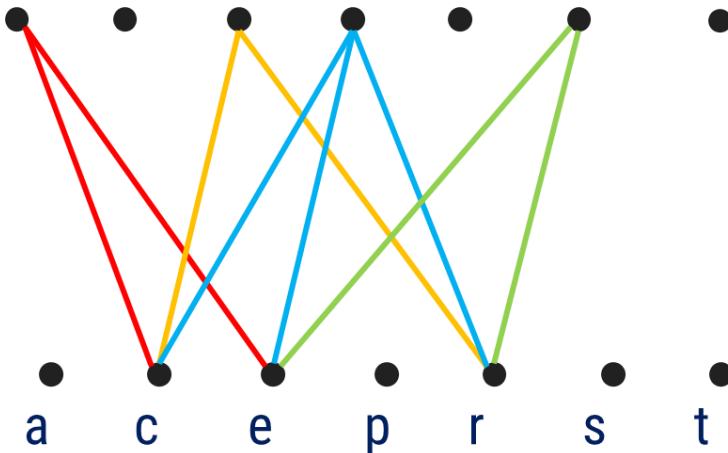
Is it possible for each student to be hired for a job for which he or she has applied?

Model:



Observation:

- We observe that there is a subset $X = \{B, F, J, L\}$ of seven seniors containing four vertices, whose neighbors belong to a set $\{c, e, r\}$ of only three vertices.



- So, it is not possible for each student to be hired for a job for which he or she has applied
- As we are about to see, this is the key reason why this does not contain a matching.
- So, next we state related theorem in the context of bipartite graphs

Hall's Theorem (without proof)**→ Statement:**

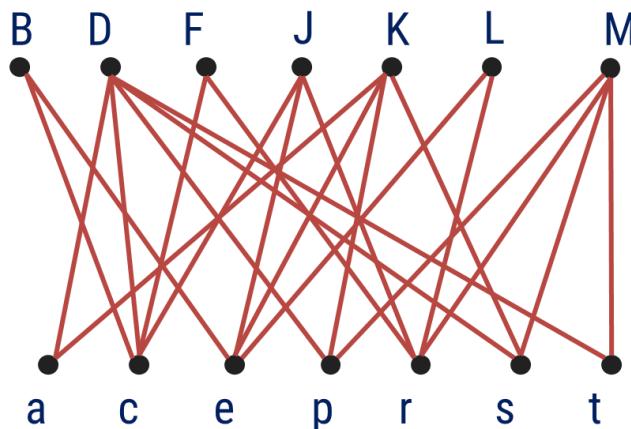
Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$. Then G contains a matching of size r if and only if G satisfies Hall's condition.

→ Hall's condition:

The graph G is said to satisfy Hall's condition if $|N(X)| \geq |X|$ for every nonempty subset X of U . This condition is named for Philip Hall.

→ Hall's Theorem tells us when we can have the perfect matching.**Apply Hall's Theorem on Given Graph.**

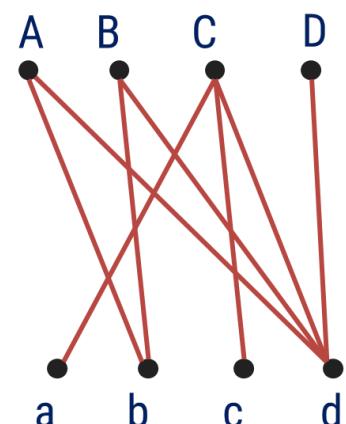
- For given bipartite graph first we check Hall's condition.
- Then we apply Hall's theorem
- To answer whether graph can have perfect matching or not?



- If we take $X = \{B, F, J, L\}$.
- So, $|X| = 4$.
- But, $N(X) = \{c, e, r\}$.
- So, $|N(X)| = 3$.
- i.e. $|N(X)| < |X|$.
- Therefore, graph is not satisfy Hall's condition.
- Here given bipartite graph with partite sets U and W such that, $7 = |U|$ and $|W| = 7$.
- So, $r = 7 = |U| \leq |W|$.
- As we know G not satisfied Hall's condition Therefor G does not contains a matching of size 7.
- So, we cannot get perfect matching for given graph.

Hall's Theorem Application.

1. In this diagram, a bipartite graph, the students are at the top and the companies are at the bottom. A student and a company is connected if the company wants to hire the student. For example, d will hire any student, so d is connected to a, b, c, and d. Can every student gets a job?
 - For answer to this question we apply Hall's theorem.
 - Here if we take $X = \{A, B, D\}$ then $N(X) = \{b, d\}$.
 - So, $|X| > |N(X)|$.
 - i.e. Graph not satisfy Hall's condition.
 - So, by Hall's theorem, we cannot find perfect matching of size 4.
 - So every students cannot gets job.



2. Two groups of 7 cricket teams are set to play one-on-one matches — each team from Group A must play against exactly one distinct team from Group B. Matches are allowed only if both teams agree to play (i.e., there's an edge between them). Using **Hall's Theorem**, determine whether it is possible to schedule **7 distinct matches** (1 from each team in Group A to 1 distinct team in Group B). Is it possible that **India plays against Pakistan** in such a matching?

Group A (Home Teams):

1. India (IND)
2. Sri Lanka (SL)
3. New Zealand (NZ)
4. South Africa (SA)
5. Australia (AUS)
6. England (ENG)
7. Bangladesh (BAN)

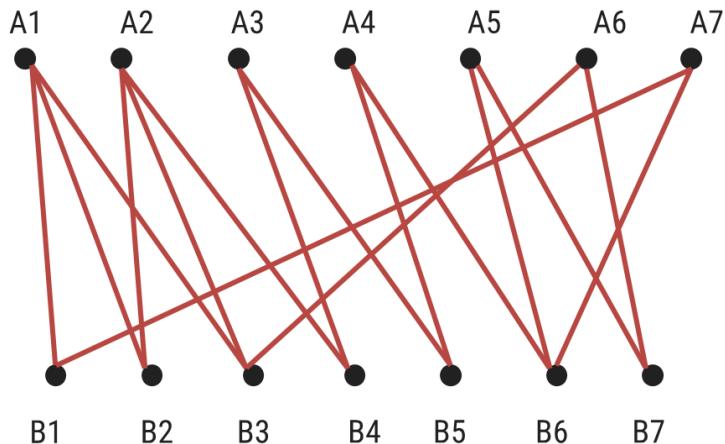
Group B (Away Teams):

1. Pakistan (PAK)
2. West Indies (WI)
3. Afghanistan (AFG)
4. Ireland (IRE)
5. Namibia (NAM)
6. Netherlands (NED)
7. Scotland (SCO)

Edges (Match Preferences):

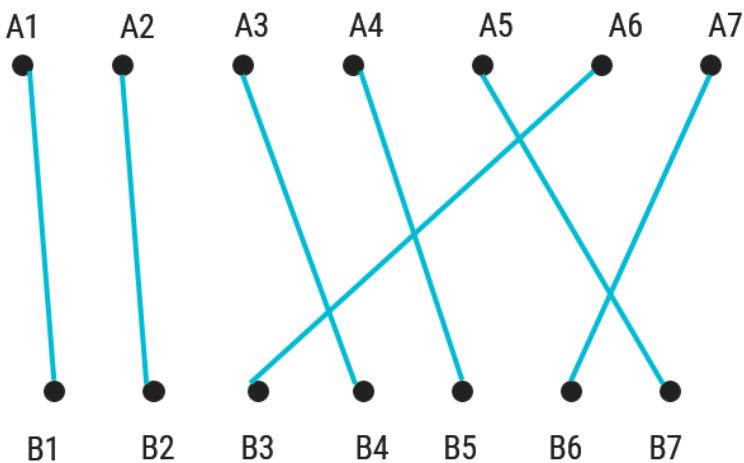
- (A1) India → Pakistan (B1), West Indies (B2), Afghanistan (B3)
- (A2) Sri Lanka → West Indies (B2), Afghanistan (B3), Ireland (B4)
- (A3) New Zealand → Ireland (B4), Namibia (B5)
- (A4) South Africa → Namibia (B5), Netherlands (B6)
- (A5) Australia → Netherlands (B6), Scotland (B7)
- (A6) England → Afghanistan (B3), Scotland (B7)
- (A7) Bangladesh → Pakistan (B1), Netherlands (B6)

Model:



Solution:

- After observation, we cannot find X for $N(X)$ is small in size.
- So, graph must satisfy Hall's condition.
- Hence, it is clear from Hall's theorem graph have perfect matching.
- So, it is possible to schedule 7 distinct matches.
- Also, India be matched with Pakistan as, India is connected to Pakistan and There exists a perfect matching.
- So at least one such matching can include India vs. Pakistan.



Example of Method-4: Perfect Matching and Hall's Theorem (without proof)

C	1	Prove that every tree has at most one perfect matching.
C	2	a) Give an example of a graph that has a perfect matching , but its matching number is 1 . b) Give an example of a graph whose matching number is 1 , but it does not have a perfect matching .
C	3	Give nine perfect matchings of the following graph.
		Answer:
C	4	Prove that if a graph have perfect matching then order of graph is even.
H	5	Give an example of connected graph of order 4 not have perfect matching.
H	6	In a village there are an equal number of boys and girls of marriageable age. Each boy dates a certain number of girls and each girl dates a certain number of boys. Under what condition is it possible that every boy and girl gets married to one of their dates? (Polygamy and polyandry not allowed.)

H	7	<p>Figure shows two bipartite graphs G1 and G2, each with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. In each case, can U be matched to W?</p>
H	8	<p>There are positions open in seven different divisions of a major company: advertising (a), business (b), computing (c), design (d), experimentation (e), finance (f), and guest relations (g).</p> <p>Six people are applying for some of these positions, namely:</p> <ul style="list-style-type: none"> Alvin (A): a, c, f Connie (C): c, f Edward (E): a, c, f Beverly (B): a, b, c, d, e, g Donald (D): b, c, d, e, f, g Frances (F): a, f <p>(a) Represent this situation using a bipartite graph with the left partite set as Applicants {A, B, C, D, E, F} and the right partite set as Positions {a, b, c, d, e, f, g}.</p> <p>(b) Is it possible to hire all six applicants for six different positions? Use Hall's Theorem (without proof) to justify your answer.</p>
H	9	<p>Given a bipartite graph $G = (X \cup Y, E)$, explain how to use Hall's Theorem to determine whether a perfect matching exists. Construct a bipartite graph with $X = Y = 5$ where no perfect matching exists, and justify using Hall's condition.</p>
H	10	<p>A school wants to assign each of 4 teachers to one of 4 classes such that each teacher is only assigned to a class they are qualified to teach. The qualification list is as follows:</p> <ul style="list-style-type: none"> T1: C1, C2 T2: C2, C3 T3: C3, C4 T4: C1, C4 <p>Represent this as a bipartite graph and explain how Hall's Theorem helps decide whether such a complete assignment (perfect matching) is possible.</p>

H

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There are positions open in seven different divisions of a major company: advertising (a), business (b), computing (c), design (d), experimentation (e), and finance (f) and guide (g).

Six people are applying for some of these positions, namely:

Alvin prefers **a, c & f**; Connie prefers **c & f**; Edward prefers **a, c & f**; Beverly prefers **a, b, c, d, e, & g**; Donald prefers **b, c, d, e, f, & g**; Frances prefers **a & f**.

(a) Represent this situation by a bipartite graph.

(b) Is it possible to hire all six applicants for six different positions?

* * * * * End of the Unit * * * * *