# LATIN SQUARE COMPLETION

## PROJECT REPORT

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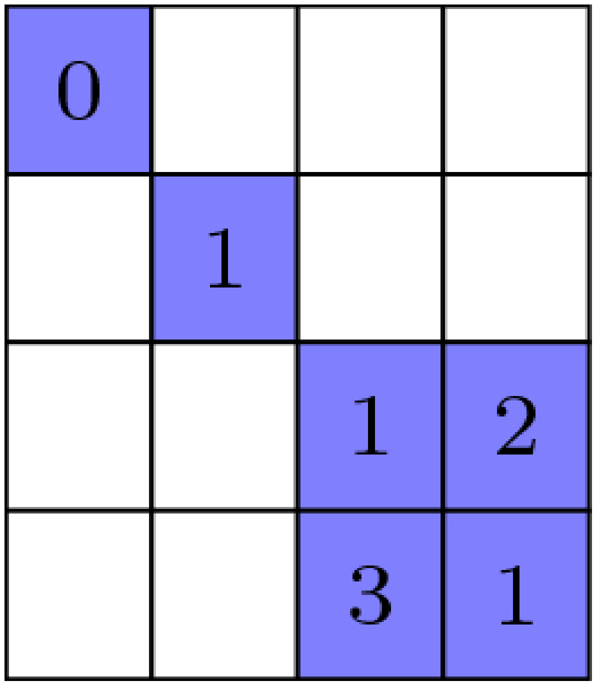
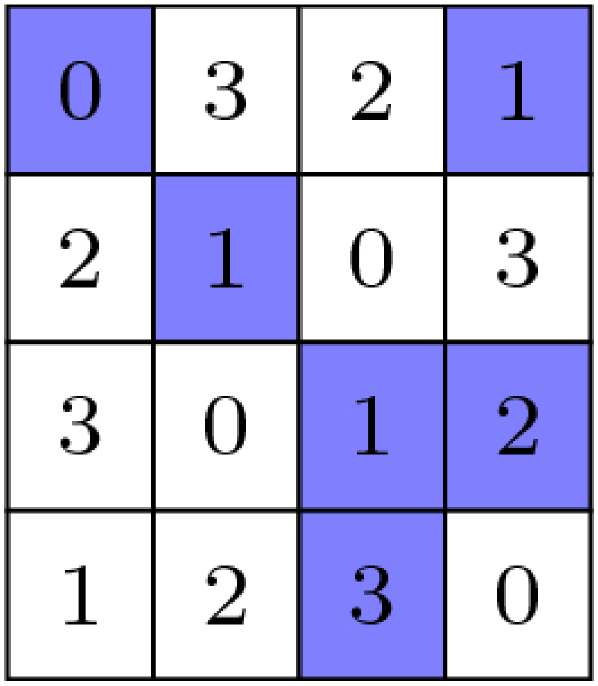
## ABSTRACT

A Latin square of order n is an array of n symbols (i.e., {1, 2, . . . , n}) in which each symbol occurs exactly once in each row and exactly once in each column. If some grids are empty, then the Latin square complete (LSC) problem of order n aims to complete the empty grids with n symbols to obtain an arbitrary legal Latin square.



| | | | |
|---|---|---|---|
| 0 | | | |
| | 1 | | |
| | | 1 | 2 |
| | | 3 | 1 |



| | | | |
|---|---|---|---|
| 0 | 3 | 2 | 1 |
| 2 | 1 | 0 | 3 |
| 3 | 0 | 1 | 2 |
| 1 | 2 | 3 | 0 |

### NP-Complete Problem

Latin Square Completion is a Constraint Satisfaction Problem and hence is NP Complete in nature. This is because it can be solved only in non-deterministic polynomial time, but easy to verify in linear time (O(n)) time complexity.

It can be defined as a Constraint Satisfaction Problem (CSP) as follows:

**Variables**: Each cell in NxN matrix
**Domains**: {1, 2, . . . , n}
**Constraints**: Each symbol (1-n) can occur exactly once in a row and column

In this project, we aim to explore various state space reduction techniques and heuristics which can be used to solve this LSC problem, using graph-coloring representation, to reduce the computational complexity of the problem.

## PROBLEM REPRESENTATION

An arbitrary legal Latin square $L_n$ is an n × n array filled with n different symbols, each occurring exactly once in each row and exactly once in each column. If some grids are empty, then $L_n^p$ is called a partial Latin square. The Latin square completion (LSC) problem aims to fill symbols (i.e., {1, 2, . . . , n}) to empty grids of $L_n^p$ to obtain an arbitrary legal Latin square.

Jin and Hao (2019) defined a Latin square graph G = (V, E) to intuitively show a partial Latin square $L_n^p$, where $V = \{v_{ij} \mid 1 \le i \le n, 1 \le j \le n\}$ represents all grids and vertex $v_{ij}$ denotes a grid on the i-th row and the j-th column.

- If two grids u, w ∈ V are in the same row or column, then we say (u, w) ∈ E. Thus, $|V| = n^2$ and $|E| = n^2(n - 1)$.
- For each vertex v ∈ V , the neighbors of vertex v is defined as

$$N(v) = \{u \in V \mid (v, u) \in E\}$$

  and the degree of vertex is $d(v) = |N(v)| = 2(n - 1)$.

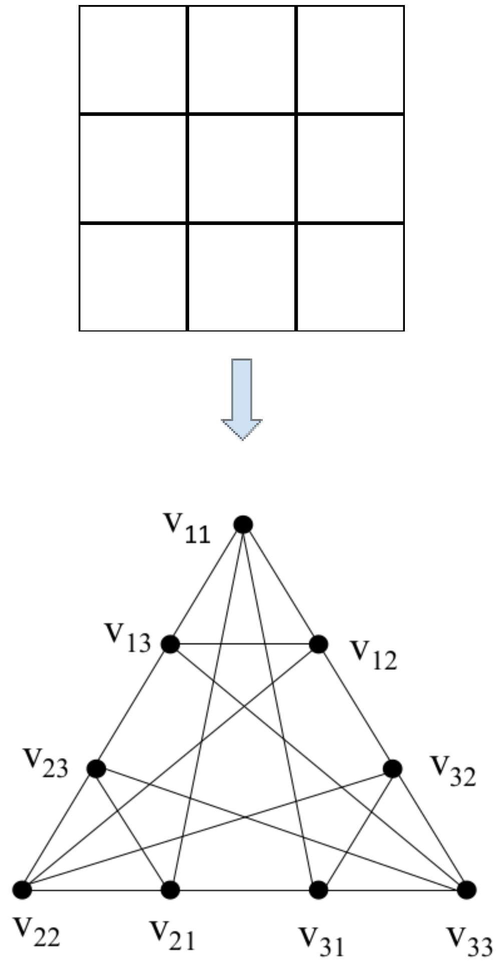
- For ∀vij ∈ V , its row vertex set is denoted as

$$RN(v_{ij}) = \{v_{ik} \mid 1 \le k \le n, k \ne j\},$$

  while its column vertex set is denoted as

$$CN(v_{ij}) = \{v_{kj} \mid 1 \le k \le n, k \ne i\}.$$

  Hence, $N(v) = RN(v) \cup CN(v)$.

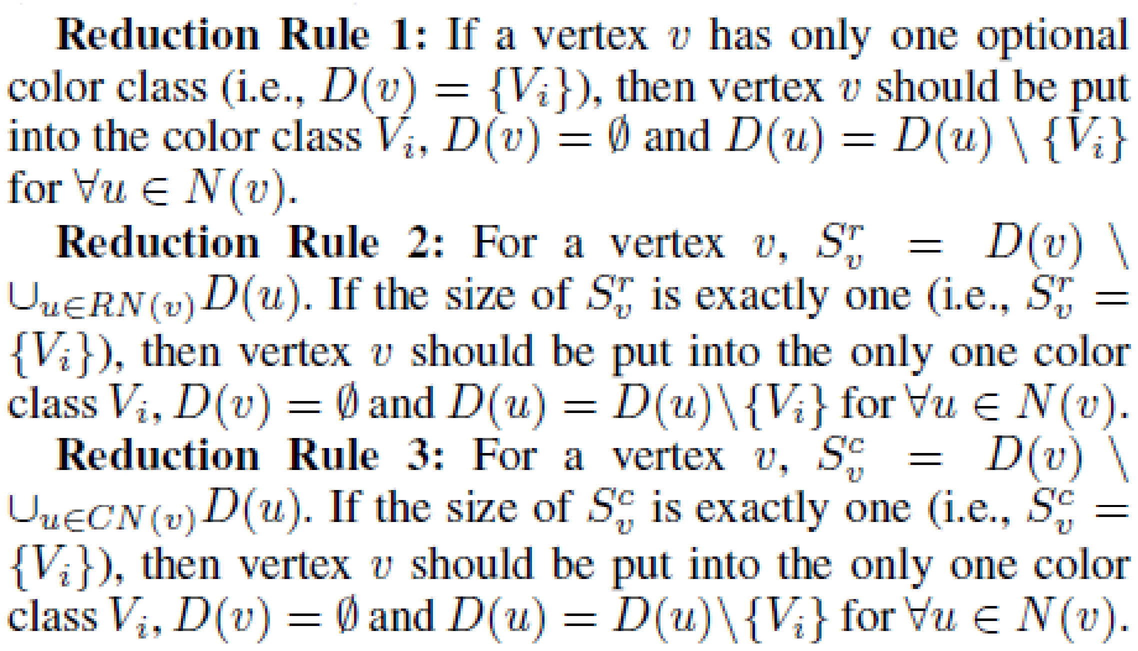


A legal n-coloring is a partition of V into n independent sets (color classes), i.e., $V_l^n = \{V_i \mid 1 \le i \le n\}$. We also denote the domain of each grid as $D(u) = \{V_1,....,V_n\}$.

Thus, the aim of the LSC problem can be seen as finding a legal n-coloring where the number of vertices for each color class is exactly n. During the search process, V is used to denote the current set of color classes.

## Reduction or Smart Solving

The paper proposes a reduction method which allows us to reduce the number of candidate variables. The following reduction rules are proposed in the paper:

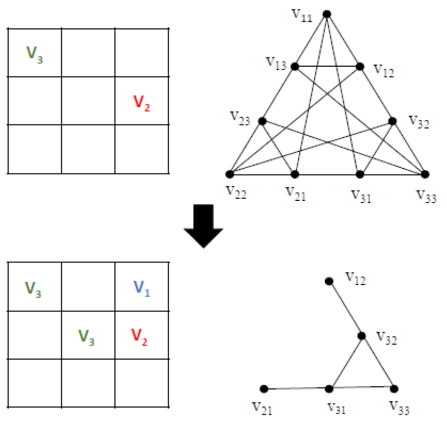


**Reduction Rule 1:** If a vertex $v$ has only one optional color class (i.e., $D(v) = \{V_i\}$), then vertex $v$ should be put into the color class $V_i$, $D(v) = \emptyset$ and $D(u) = D(u) \setminus \{V_i\}$ for $\forall u \in N(v)$.

**Reduction Rule 2:** For a vertex $v$, $S_v^r = D(v) \setminus \cup_{u \in RN(v)} D(u)$. If the size of $S_v^r$ is exactly one (i.e., $S_v^r = \{V_i\}$), then vertex $v$ should be put into the only one color class $V_i$, $D(v) = \emptyset$ and $D(u) = D(u) \setminus \{V_i\}$ for $\forall u \in N(v)$.

**Reduction Rule 3:** For a vertex $v$, $S_v^c = D(v) \setminus \cup_{u \in CN(v)} D(u)$. If the size of $S_v^c$ is exactly one (i.e., $S_v^c = \{V_i\}$), then vertex $v$ should be put into the only one color class $V_i$, $D(v) = \emptyset$ and $D(u) = D(u) \setminus \{V_i\}$ for $\forall u \in N(v)$.

This reduction is analogous to how we solve sudoku. We don't start solving sudoku by trial and error, but by looking at those cells where answers can be obtained directly. For eg. if a row in latin square of order 3 already has 1 and 2, then the next cell has to be 3. This direct analysis allows us to reduce the candidate set and hence significantly reduce the time complexity and steps required.



The image depicts how reduction can allow us to reduce our graph coloring problem. Consider cell (0,2). Here, we are sure that V2 and V3 can not be the answers. Hence, we immediately assign it V1.

The entire reduction runs in polynomial time or O(n^3) time complexity.

**CONFLICT VALUE SELECTION HEURISTIC**

Consider the current color class set as $V^n$, and $V_k \in V^n$. If v, u $\in V_k$ and (v, u) $\in$ E, then (v, u) is called a conflict edge. We use $CL(V_n)$ to denote the total number of conflict edges in $V_n$.

Moving a vertex v from color class $V_i$ to another class $V_j$ is denoted as Move(v, $V_i$, $V_j$), which will result into a new color set class $V_c^n$

Hence, the heuristic is defined as follows

$$pscore(v, V_i, V_j) = CL(\mathcal{V}^n) - CL(\mathcal{V}_c^n)$$

Note that this heuristic intuitively reflects the effects of the moving operation for the current color class set

Hence, the selection rule is as follows: pick an operation Move(v, $V_i$, $V_j$) with the biggest conflict heuristic value

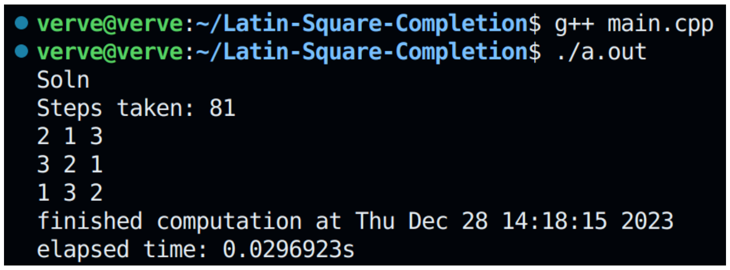
**GRAPH CONSTRUCTION - ALGORITHM**

```
Algorithm 1: $Construct(G)$
Input: A Latin square graph $G = (V, E)$ where $|V| = n^2$
Output: The set of color classes $\mathcal{V}^n$
initialize a color domain $D(v)$ for $\forall v \in V$;
initialize a color class $V_i$ for $\forall V_i \in \mathcal{V}^n$;
$CandSet := V$;
while $CandSet$ is not empty do
    if there exists vertex $v_1 \in CandSet$ satisfying any
      reduction rule or as a filed grid then
        $V := V \setminus \{v_1\}$;
        put vertex $v_1$ into a color class $V_i$ based on
          reduction rule;
        $CandSet := CandSet \setminus \{v_1\}$;
        $D(u) := D(u) \setminus \{V_i\}$ for $\forall u \in N(v_1)$;
    else break ;
while $CandSet$ is not empty do
    select a random vertex $v_2$ from $CandSet$;
    $CandSet := CandSet \setminus \{v_2\}$;
    put vertex $v_2$ into a random color class selected from
      $D(v_2)$;
return $\mathcal{V}^n$;
```

## RESULTS

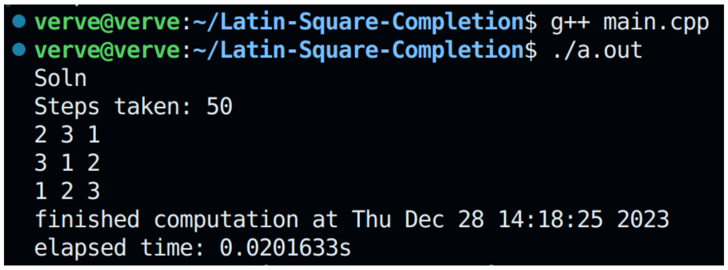
### *With State Space reduction*

BFS



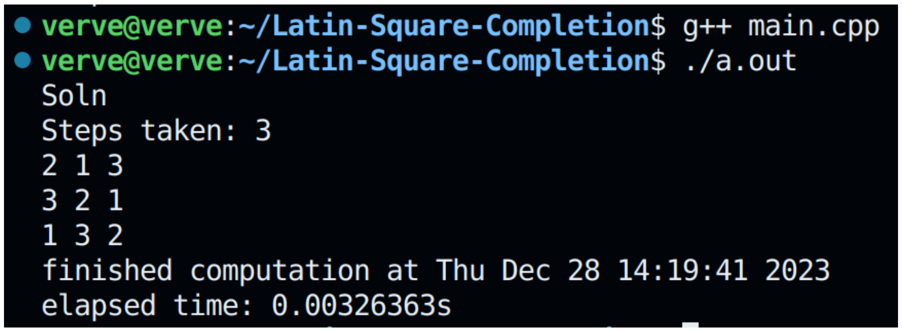
```
● verve@verve:~/Latin-Square-Completion$ g++ main.cpp
● verve@verve:~/Latin-Square-Completion$ ./a.out
  Soln
  Steps taken: 81
  2 1 3
  3 2 1
  1 3 2
  finished computation at Thu Dec 28 14:18:15 2023
  elapsed time: 0.0296923s
```

DFS



```
● verve@verve:~/Latin-Square-Completion$ g++ main.cpp
● verve@verve:~/Latin-Square-Completion$ ./a.out
  Soln
  Steps taken: 50
  2 3 1
  3 1 2
  1 2 3
  finished computation at Thu Dec 28 14:18:25 2023
  elapsed time: 0.0201633s
```

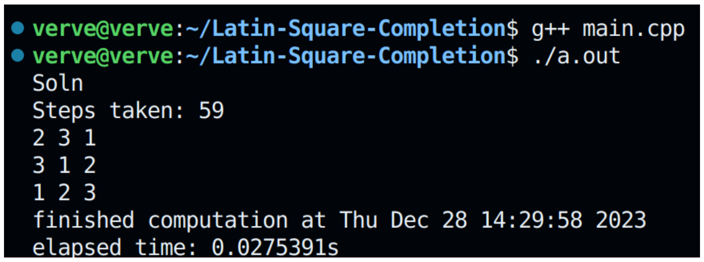
BestFS



```
● verve@verve:~/Latin-Square-Completion$ g++ main.cpp
● verve@verve:~/Latin-Square-Completion$ ./a.out
  Soln
  Steps taken: 3
  2 1 3
  3 2 1
  1 3 2
  finished computation at Thu Dec 28 14:19:41 2023
  elapsed time: 0.00326363s
```

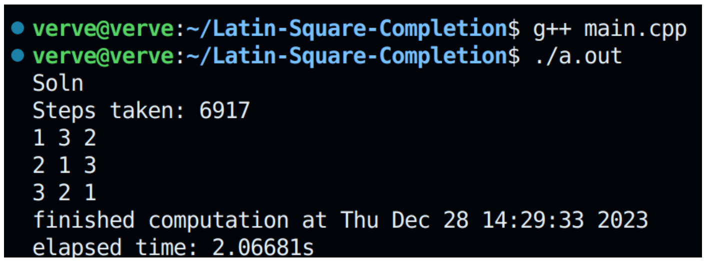
### *Without State Space reduction*

BFS



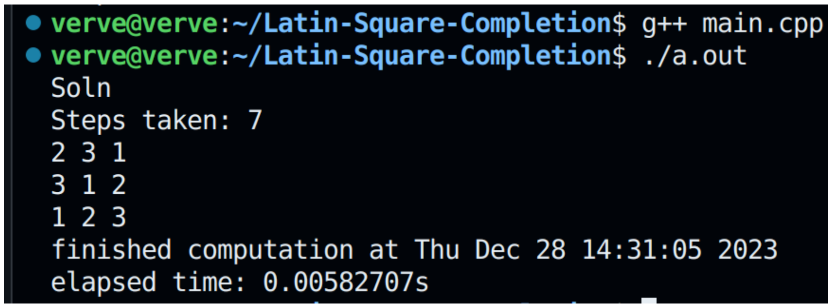
```
● verve@verve:~/Latin-Square-Completion$ g++ main.cpp
● verve@verve:~/Latin-Square-Completion$ ./a.out
  Soln
  Steps taken: 59
  2 3 1
  3 1 2
  1 2 3
  finished computation at Thu Dec 28 14:29:58 2023
  elapsed time: 0.0275391s
```

DFS



```
● verve@verve:~/Latin-Square-Completion$ g++ main.cpp
● verve@verve:~/Latin-Square-Completion$ ./a.out
  Soln
  Steps taken: 6917
  1 3 2
  2 1 3
  3 2 1
  finished computation at Thu Dec 28 14:29:33 2023
  elapsed time: 2.06681s
```

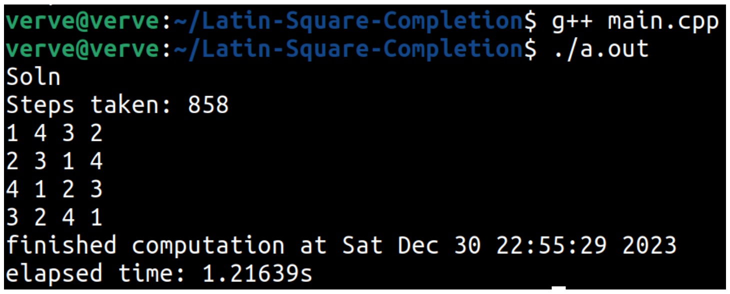
BestFS



```
● verve@verve:~/Latin-Square-Completion$ g++ main.cpp
● verve@verve:~/Latin-Square-Completion$ ./a.out
  Soln
  Steps taken: 7
  2 3 1
  3 1 2
  1 2 3
  finished computation at Thu Dec 28 14:31:05 2023
  elapsed time: 0.00582707s
```

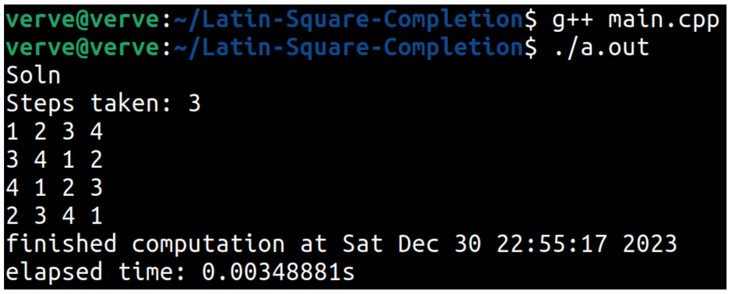
*With 4x4 grids*

BFS



```
verve@verve:~/Latin-Square-Completion$ g++ main.cpp
verve@verve:~/Latin-Square-Completion$ ./a.out
Soln
Steps taken: 858
1 4 3 2
2 3 1 4
4 1 2 3
3 2 4 1
finished computation at Sat Dec 30 22:55:29 2023
elapsed time: 1.21639s
```

BestFS



```
verve@verve:~/Latin-Square-Completion$ g++ main.cpp
verve@verve:~/Latin-Square-Completion$ ./a.out
Soln
Steps taken: 3
1 2 3 4
3 4 1 2
4 1 2 3
2 3 4 1
finished computation at Sat Dec 30 22:55:17 2023
elapsed time: 0.00348881s
```

## APPLICATIONS

- Scheduling Problems: LSC has potential applications in scheduling problems, where tasks or resources need to be assigned to time slots or locations while satisfying constraints. It can help find optimal schedules that minimize conflicts or maximize efficiency.
- DNA Sequencing: LSC has been explored for DNA sequencing, where it can aid in reconstructing DNA sequences from fragments. It can also be used for designing experiments to study gene interactions.
- Coding Theory: LSC finds applications in error correction codes, which are used to detect and correct errors in data transmission or storage. LSC can be used to construct efficient codes with high error-correcting capabilities.

## REFERENCES

- Shiwei Pan, Yiyuan Wang, Minghao Yin. 2022. “A Fast Local Search Algorithm for the Latin Square Completion Problem”. The Thirty-Sixth AAAI Conference on Artificial Intelligence (AAAI-22).
- Jin, Y.; and Hao, J.-K. 2019. Solving the Latin square completion problem by memetic graph coloring. IEEE Transactions on Evolutionary Computation, 23(6): 1015–1028.