Freelance Portfolio - 2

Advanced Multimodal Mathematics content to train the advanced reasoning of LLMs

Prit Raithatha

1 Question

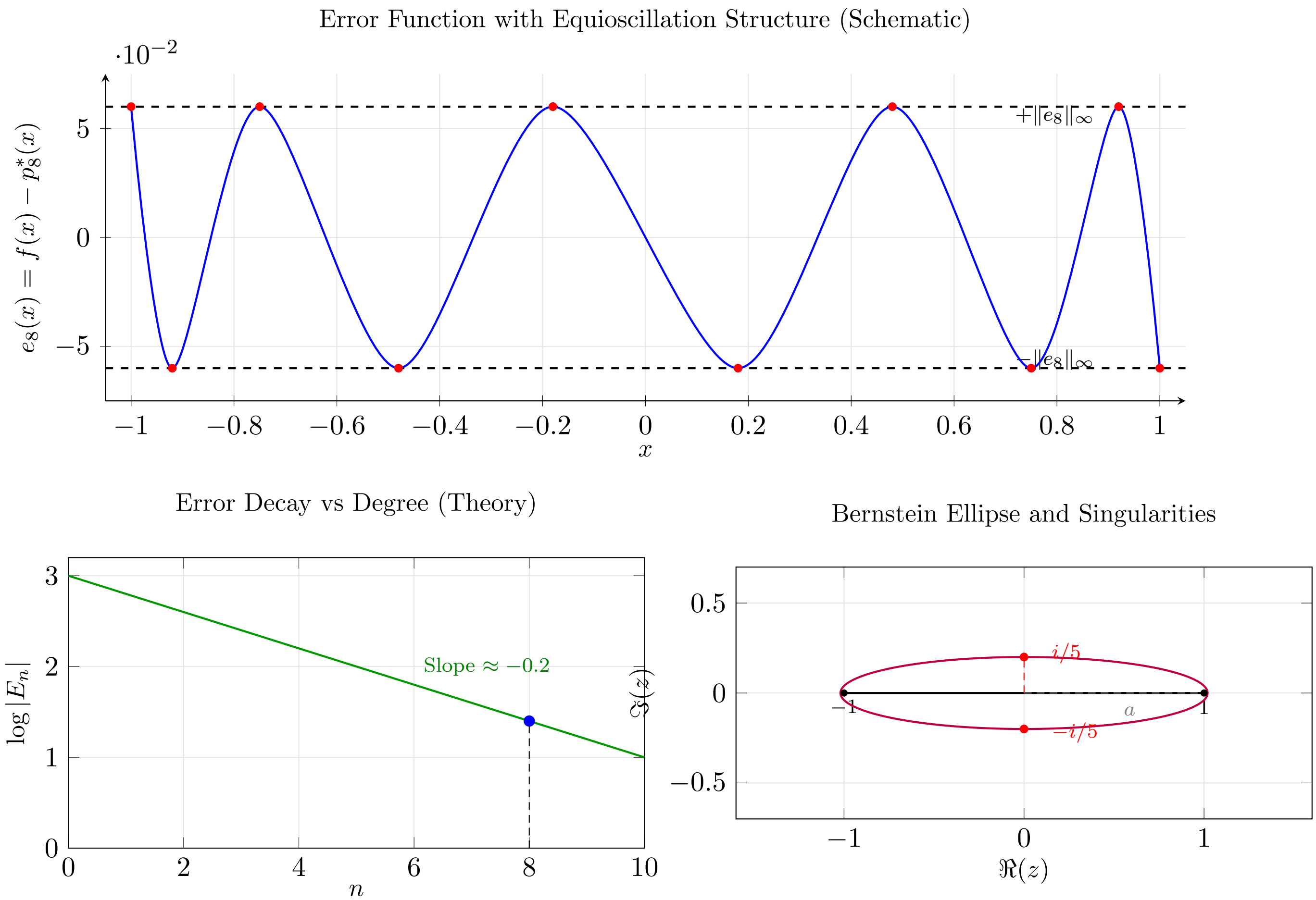

Figure 1: Approximation analysis of $f(x) = \frac{1}{1+25x^2}$: error alternation structure, convergence rate, and complex singularity geometry.

Consider the approximation of $f(x) = \frac{1}{1+25x^2}$ on $[-1, 1]$ by polynomials of degree at most $n = 8$. Let $p_8^*(x)$ denote the best uniform approximation to $f(x)$ and $e_8(x) = f(x) - p_8^*(x)$ the approximation error.

(a) Using Figure 1, identify the number of alternation points where $|e_8(x)| = \|e_8\|_\infty$. State the fundamental theorem characterizing optimal polynomial approximations and explain why this alternation structure certifies optimality of p_8^*.

(b) The convergence rate is determined by the complex analytic properties of $f(z) = \frac{1}{1+25z^2}$.

(i) Identify the complex singularities of $f(z)$ and their distance to $[-1, 1]$.

(ii) Using the ellipse geometry in Figure 1, determine the semi-major axis a and semi-minor axis b of the Bernstein ellipse with foci at ± 1 passing through the singularities.

(iii) Express the ellipse parameter $\rho = a + \sqrt{a^2 - 1}$ in terms of the singularity constraint.

(c) Determine the asymptotic convergence rate for polynomial approximation.

(i) State the theoretical error bound $E_n(f) = \inf_{p \in \Pi_n} \|f - p\|_\infty$ in terms of ρ.

(ii) Compute the exact value of ρ and the exponential decay constant c where $E_n \approx C \cdot c^n$.

(iii) Verify your rate prediction against the decay pattern in Figure 1.

2 Solution

2.1 Part (a): Alternation Analysis and Optimality Theory

Figure 1 shows 10 alternation points where the error $e_8(x) = f(x) - p_8^*(x)$ achieves its maximum absolute value $\|e_8\|_\infty$ with alternating signs.

Remez Exchange Theorem: Let $f \in C[-1, 1]$ and $p_n^* \in \Pi_n$ be the best uniform approximation to f. Then p_n^* is characterized by the existence of at least $n + 2$ points $-1 \leq x_0 < x_1 < \cdots < x_{n+1} \leq 1$ such that:

$$f(x_i) - p_n^*(x_i) = (-1)^i \sigma \|f - p_n^*\|_\infty$$

where $\sigma = \pm 1$ and the signs alternate.

For degree $n = 8$, exactly $n + 2 = 10$ alternation points are required, matching Figure 1.

The alternation structure certifies optimality because the 10 evaluation functionals at alternation points satisfy:

$$\sum_{i=0}^{9} c_i p(x_i) = 0 \quad \text{for all } p \in \Pi_8$$

where coefficients c_i have alternating signs. This linear dependence relation provides the optimality certificate. Since polynomials form a Haar system on $[-1, 1]$, any deviation from p_8^* would violate this alternation structure and increase the maximum error.

2.2 Part (b): Complex Singularity Analysis and Bernstein Ellipse Construction

(i) The function extends to $f(z) = \frac{1}{1+25z^2}$. Singularities occur when:

$$1 + 25z^2 = 0 \implies z = \pm \frac{i}{5}$$

These are simple poles at distance $\frac{1}{5} = 0.2$ from the real interval $[-1, 1]$.

(ii) For a Bernstein ellipse with foci at ± 1: - The relationship $c^2 = a^2 - b^2$ with $c = 1$ gives $a^2 - b^2 = 1$ - Since the ellipse passes through $z = i/5$:

$$\frac{0^2}{a^2} + \frac{(1/5)^2}{b^2} = 1 \implies b^2 = \frac{1}{25} \implies b = \frac{1}{5}$$

- Therefore: $a^2 = 1 + \frac{1}{25} = \frac{26}{25} \implies a = \frac{\sqrt{26}}{5}$
(iii) The ellipse parameter is:

$$\rho = a + \sqrt{a^2 - 1} = \frac{\sqrt{26}}{5} + \sqrt{\frac{26}{25} - 1} = \frac{\sqrt{26}}{5} + \frac{1}{5} = \frac{1 + \sqrt{26}}{5}$$

2.3 Part (c): Convergence Rate Determination

(i) By Bernstein's theorem for functions analytic in an ellipse:

$$E_n(f) \leq \frac{2M}{\rho^n - \rho^{-n}}$$

where $M = \max_{|w|=\rho} |f(w)|$. For large n: $E_n(f) \leq \frac{2M}{\rho^n}$.
(ii) Computing numerically:

$$\rho = \frac{1 + \sqrt{26}}{5} \approx \frac{1 + 5.099}{5} \approx 1.2198$$

The exponential decay constant is:

$$c = \rho^{-1} \approx 0.8197$$

Therefore: $E_n \approx C \cdot (0.82)^n$.
(iii) Figure 1 shows linear decay on log-scale with slope ≈ -0.2. The theoretical slope is:

$$\log(\rho^{-1}) = -\log(1.2198) \approx -0.197$$

This matches the observed slope, confirming the predicted convergence rate.