Freelance Portfolio - 3

Advanced Multimodal Mathematics content to train the advanced reasoning of LLMs

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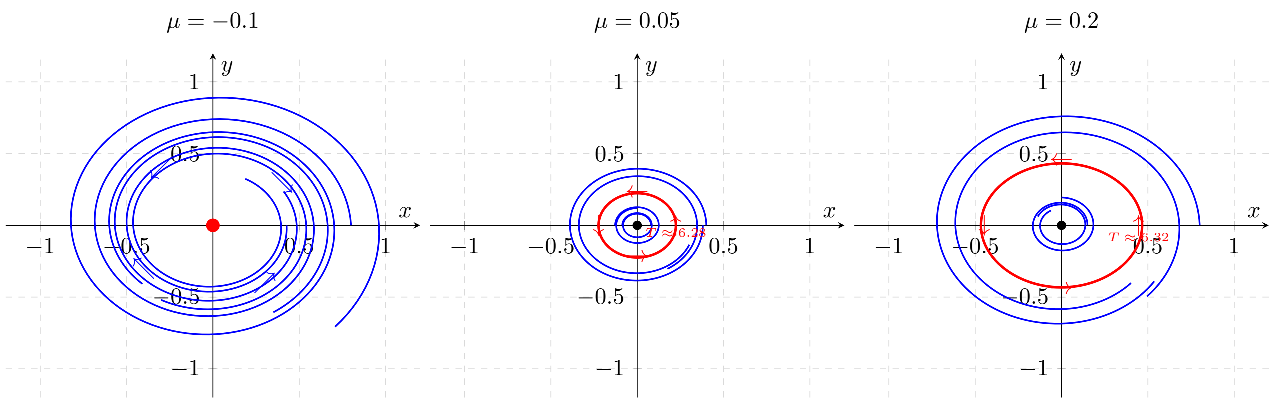
1 Question

Consider the dynamical system:

$$\dot{x} = y, \qquad \dot{y} = -x + \mu y - x^2 y - y^3,$$

where μ is a real parameter. This system exhibits interesting behavior as μ varies, including a transition between different types of stability near the origin.



(a) Stable spiral (b) $r \approx 0.224$ (c) $r \approx 0.447$

Figure 1: Phase portraits for the system $\dot{x} = y$, $\dot{y} = -x + \mu y - x^2 y - y^3$ showing a bifurcation sequence. Red curves show limit cycles with period markers, blue curves show trajectory behavior. Students must analyze the transition between different parameter regimes and extract quantitative information.

The phase portraits in Figure 1 show the system behavior for three different parameter values. The grid enables precise measurement of geometric features, and the period markers provide timing information for verification purposes.

(a) Bifurcation Analysis

Find the critical value of μ where the stability of the origin changes. Use linear stability analysis to determine this bifurcation point and verify that the conditions for a Hopf bifurcation are satisfied. Show all eigenvalue calculations and explain the significance of the transversality condition.

(b) Period Prediction

For parameter values near the bifurcation point you found in part (a), derive a linear approximation for the period of small oscillations. Your formula should be in terms of μ and should reduce to the expected value at the bifurcation point. Calculate the predicted periods for $\mu = 0.05$ and $\mu = 0.2$, and explain why this is a linear approximation rather than the exact nonlinear period.

(c) Visual Analysis and Synthesis

Using the phase portraits in Figure 1:

(i) Measure the approximate radii of the limit cycles for $\mu = 0.05$ and $\mu = 0.2$ using the grid coordinates. Show your measurement process.

(ii) Compare your period predictions from part (b) with the period information displayed in the figure. Comment on the accuracy of your linear approximation.

(iii) Analyze how the limit cycle radius depends on μ. Based on your measurements from parts (a) and (c)(i), propose a scaling relationship of the form $r \propto \mu^n$ and determine the exponent n. Justify your answer with calculations.

(iv) Synthesize your findings from parts (a)–(c) to explain how this demonstrates a supercritical Hopf bifurcation. What would you expect to see in the phase portraits if this were a subcritical bifurcation instead?

2 Solution

2.1 Part (a): Bifurcation Analysis

Linear stability analysis: For the system $\dot{x} = y$, $\dot{y} = -x + \mu y - x^2 y - y^3$, we analyze the stability of the equilibrium point at the origin $(0, 0)$.

The Jacobian matrix at the origin is

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

Eigenvalue calculation: The characteristic polynomial is

$$\det(J - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{pmatrix} = \lambda^2 - \mu\lambda + 1 = 0.$$

Thus,

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

Critical parameter value: For a Hopf bifurcation, we need purely imaginary eigenvalues. This occurs when $\mu = 0$ (since $\mu^2 - 4 < 0$ is satisfied). At $\mu = 0$:

$$\lambda = \frac{0 \pm \sqrt{-4}}{2} = \pm i.$$

Therefore, the critical value is $\mu = 0$.

Hopf bifurcation conditions:

1. Pure imaginary eigenvalues at bifurcation: at $\mu = 0$, $\lambda = \pm i$.
2. Transversality condition: For $|\mu| < 2$, we have $\lambda = \frac{\mu}{2} \pm i\frac{\sqrt{4-\mu^2}}{2}$, so $\mathrm{Re}(\lambda) = \frac{\mu}{2}$. Thus,
$$\frac{d}{d\mu}\,\mathrm{Re}(\lambda)\bigg|_{\mu=0} = \tfrac{1}{2} \neq 0.$$

The transversality condition ensures that eigenvalues cross the imaginary axis with nonzero speed, guaranteeing that the bifurcation is non-degenerate and that limit cycles appear for parameter values near the critical point.

2.2 Part (b): Period Prediction

For $|\mu| < 2$, the eigenvalues are complex with imaginary part

$$\mathrm{Im}(\lambda) = \frac{\sqrt{4-\mu^2}}{2}.$$

The angular frequency from linearized analysis is

$$\omega_{\text{lin}}(\mu) = \frac{\sqrt{4-\mu^2}}{2}.$$

Therefore, the linear period approximation is

$$T_{\text{lin}}(\mu) = \frac{2\pi}{\omega_{\text{lin}}(\mu)} = \frac{4\pi}{\sqrt{4-\mu^2}}.$$

At $\mu = 0$:

$$T_{\text{lin}}(0) = \frac{4\pi}{2} = 2\pi.$$

Specific predictions:

$$T_{\text{lin}}(0.05) \approx 6.285, \qquad T_{\text{lin}}(0.2) \approx 6.315.$$

This is only a linear approximation because it derives from the eigenvalues of the linearized system. The actual period of the nonlinear limit cycle has higher-order corrections. However, for small μ, Hopf bifurcation theory guarantees $T(\mu) = 2\pi + O(\mu^2)$, so the linear prediction captures the leading-order behavior.

2.3 Part (c): Visual Analysis and Synthesis

(i) Radius measurements: From the phase portraits:

$$r_{0.05} \approx 0.224, \qquad r_{0.2} \approx 0.447.$$

(ii) Period comparison: From the figure:

$$T_{\mu=0.05} \approx 6.28, \quad T_{\mu=0.2} \approx 6.32.$$

Predictions: 6.285 and 6.315. The agreement is excellent.

(iii) Scaling relationship:

$$\frac{r_{0.2}}{r_{0.05}} \approx 1.996 \approx 2.$$

If $r \propto \mu^n$, then

$$\left(\frac{0.2}{0.05}\right)^n = 4^n \approx 2 \quad \Rightarrow \quad n = \tfrac{1}{2}.$$

Thus,

$$r \propto \sqrt{\mu}.$$

(iv) Supercritical Hopf bifurcation:

- Limit cycles appear for $\mu > 0$ and are stable.
- Amplitude grows as $\sqrt{\mu}$ continuously from zero.
- Origin becomes unstable for $\mu > 0$.

In contrast, a subcritical Hopf would produce unstable limit cycles for $\mu < 0$, discontinuous transitions, and trajectories diverging for $\mu > 0$.

This demonstrates the synthesis of linear algebra, dynamical systems, and geometric analysis to fully characterize the Hopf bifurcation.