AI-resistant Mathematics Questions and Answers

Prit Raithatha

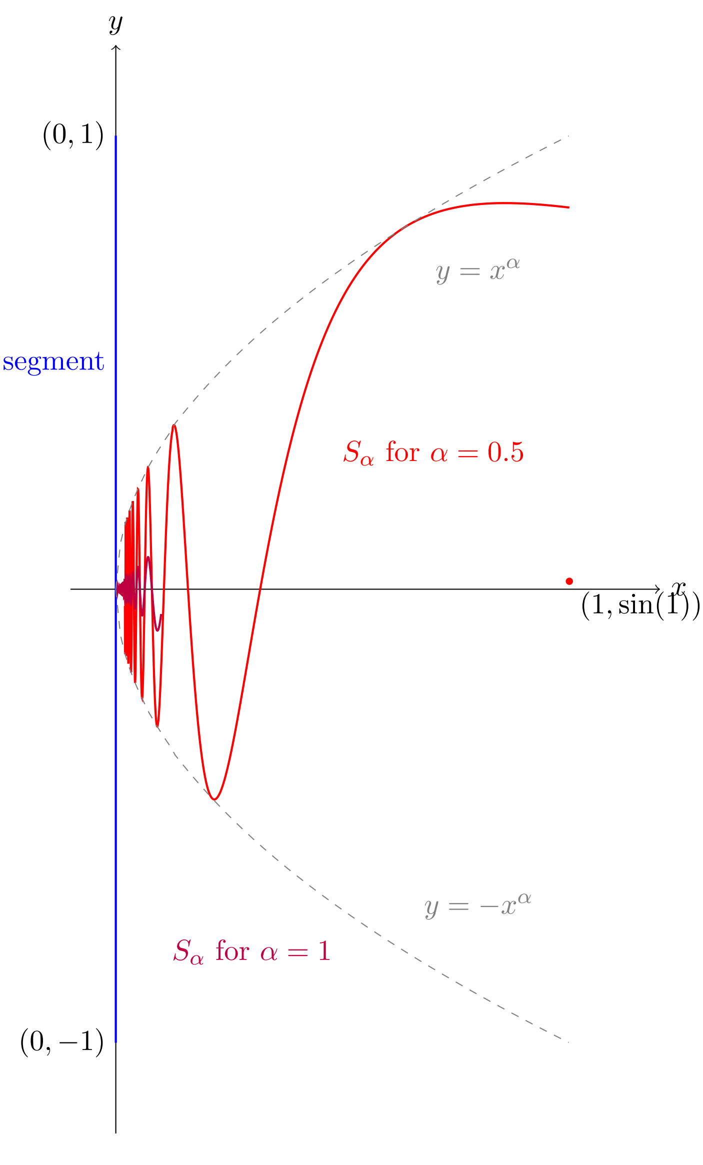
1 Question-1

1.1 Question

Consider the parametric family of modified topologist's sine curves defined by:

$$S_\alpha = \{(x, x^\alpha \sin(1/x)) : x \in (0, 1]\} \cup \{(0, y) : y \in [-1, 1]\} \tag{1}$$

where $\alpha > 0$ is a parameter.



Question: Determine for which values of $\alpha > 0$:

(a) S_α is connected

(b) S_α is path-connected

(c) Analyze the fundamental group $\pi_1(\mathbb{R}^2 \setminus S_\alpha)$

1.2 Human Solution

1.2.1 Part (a): Connectedness Analysis

S_α is connected for all $\alpha > 0$.

Proof. Suppose S_α is disconnected. Then there exist non-empty disjoint open sets $U, V \subset \mathbb{R}^2$ such that $S_\alpha \subset U \cup V$ with $S_\alpha \cap U \neq \emptyset$ and $S_\alpha \cap V \neq \emptyset$.

Let $A = S_\alpha \cap U$ and $B = S_\alpha \cap V$. We consider two cases:

Case 1: The vertical segment $L = \{(0, y) : y \in [-1, 1]\}$ is entirely contained in either A or B.

Without loss of generality, assume $L \subset A$. Then $B \subset \{(x, x^\alpha \sin(1/x)) : x \in (0, 1]\}$.

The curve portion $C = \{(x, x^\alpha \sin(1/x)) : x \in (0, 1]\}$ is connected as the continuous image of the connected interval $(0, 1]$ under the map $x \mapsto (x, x^\alpha \sin(1/x))$.

Since B is relatively open and closed in C, and C is connected, either $B = \emptyset$ or $B = C$.

If $B = C$, then $A = L$. However, this leads to a contradiction: every neighborhood of any point $(0, y_0) \in L$ intersects C because: - For any $\epsilon > 0$, choose $\delta > 0$ small enough so that $\delta^\alpha < \epsilon$ - For $x \in (0, \delta)$, we have $|x^\alpha \sin(1/x)| \leq x^\alpha < \epsilon$ - Points $(x, x^\alpha \sin(1/x))$ get arbitrarily close to points on L

This contradicts the assumption that A and B are separated by open sets.

Case 2: The vertical segment L intersects both A and B.

This would imply that the connected set L is disconnected, which is impossible.

Therefore, S_α cannot be disconnected, so S_α is connected for all $\alpha > 0$. □

1.2.2 Part (b): Path-Connectedness Analysis

S_α is path-connected if and only if $\alpha \geq 1$.

Proof. **($\alpha \geq 1$ case):** We construct an explicit path connecting any point on the vertical segment to any point on the curve.

Let $p = (0, y_0) \in L$ and $q = (a, a^\alpha \sin(1/a)) \in C$ where $a \in (0, 1]$.

Define the path $\gamma : [0, 1] \to S_\alpha$ by:

$$\gamma(t) = \begin{cases} (0, y_0 + t(0 - y_0)) = (0, y_0(1 - t)) & \text{if } t \in [0, 1/2] \\ (2t - 1, (2t - 1)^\alpha \sin(1/(2t - 1))) & \text{if } t \in (1/2, 1] \end{cases} \tag{2}$$

We need to verify continuity at $t = 1/2$: - $\lim_{t \to 1/2^-} \gamma(t) = (0, 0)$ - $\lim_{t \to 1/2^+} \gamma(t) = \lim_{s \to 0^+} (s, s^\alpha \sin(1/s))$

For $\alpha \geq 1$, we have:

$$|s^\alpha \sin(1/s)| \leq s^\alpha \leq s \to 0 \text{ as } s \to 0^+ \tag{3}$$

Therefore, $\lim_{t \to 1/2^+} \gamma(t) = (0, 0)$, establishing continuity.

($\alpha < 1$ case): We prove no path exists by contradiction.

If $\alpha < 1$, then the sequence $\{x_n^\alpha \sin(1/x_n)\}$ does not converge to any limit as $x_n \to 0^+$ for appropriately chosen sequences.

Proof of Lemma. Choose $x_n = \frac{1}{2\pi n + \pi/2}$ for $n \geq 1$. Then:

$$\sin(1/x_n) = \sin(2\pi n + \pi/2) = 1 \tag{4}$$

So $x_n^\alpha \sin(1/x_n) = x_n^\alpha = (2\pi n + \pi/2)^{-\alpha}$.

For $\alpha < 1$, this sequence does not converge to 0. Specifically:

$$\lim_{n\to\infty} (2\pi n + \pi/2)^{-\alpha} = \lim_{n\to\infty} \frac{1}{(2\pi n)^\alpha} = \infty \tag{5}$$

This contradicts any attempted continuous connection to the vertical segment. □

Now suppose $\gamma : [0, 1] \to S_\alpha$ is a path with $\gamma(0) = (0, y_0)$ and $\gamma(1) = (a, a^\alpha \sin(1/a))$.

Let $\gamma(t) = (x(t), y(t))$. Since $x(0) = 0$ and $x(1) = a > 0$, by continuity there exists $t_0 = \sup\{t : x(t) = 0\}$.

If $t_0 < 1$, then for $t > t_0$, we have $x(t) > 0$, so $y(t) = x(t)^\alpha \sin(1/x(t))$.

As $t \to t_0^+$, we have $x(t) \to 0^+$, but by our lemma, $y(t) = x(t)^\alpha \sin(1/x(t))$ cannot converge, contradicting continuity of γ.

Therefore, no such path exists when $\alpha < 1$. □

1.2.3 Part (c): Fundamental Group Analysis

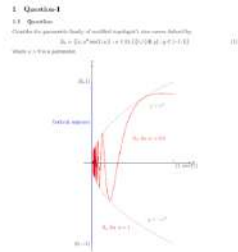
The fundamental group $\pi_1(\mathbb{R}^2 \setminus S_\alpha)$ has the following structure:

1. If $\alpha \geq 1$: $\pi_1(\mathbb{R}^2 \setminus S_\alpha) \cong \mathbb{Z}$
2. If $\alpha < 1$: $\pi_1(\mathbb{R}^2 \setminus S_\alpha)$ is more complex, involving the fundamental group of the complement of a non-path-connected space.

Proof. **Case $\alpha \geq 1$:** Since S_α is path-connected, it's homotopy equivalent to a point. The complement $\mathbb{R}^2 \setminus S_\alpha$ deformation retracts onto S^1 (a circle around S_α), giving $\pi_1(\mathbb{R}^2 \setminus S_\alpha) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Case $\alpha < 1$: The analysis is more subtle. Since S_α is connected but not path-connected, the complement has a more complex fundamental group structure that reflects the "wild" oscillatory behavior near the origin. □

1.3 Gemini 2.5 Pro Testing Log



Here are the solutions to the questions about the modified topologist's sine curve family S_α.

The set is defined as $S_\alpha = \{(x, x^\alpha \sin(1/x)) : x \in (0, 1]\} \cup \{(0, y) : y \in [-1, 1]\}$ for a parameter $\alpha > 0$. This set consists of a curve component, let's call it C_α, and a vertical line segment, L.

(a) Connectedness

S_α is connected for all values of $\alpha > 0$.

Here's the reasoning:

1. The curve component, $C_\alpha = \{(x, x^\alpha \sin(1/x)) : x \in (0, 1]\}$, is the image of the connected interval $(0, 1]$ under a continuous function. Therefore, C_α **is a connected set.**
2. The vertical segment, $L = \{(0, y) : y \in [-1, 1]\}$, is a line segment, which is also a connected set.
3. A key topological principle states that the union of two connected sets is connected if they have at least one point in common. While C_α and L are disjoint, we can consider the closure of C_α, denoted $\overline{C_\alpha}$.
4. The limit points of C_α as $x \to 0^+$ are found by evaluating the limit of the coordinates. For any sequence $x_n \to 0^+$:

$$\lim_{n\to\infty} (x_n, x_n^\alpha \sin(1/x_n))$$

   Since $\alpha > 0$, $\lim_{n\to\infty} x_n^\alpha = 0$. Because $|\sin(1/x_n)| \le 1$, the Squeeze Theorem implies that $\lim_{n\to\infty} x_n^\alpha \sin(1/x_n) = 0$.
5. Thus, the only limit point of C_α on the y-axis is the origin, $(0, 0)$. The closure is $\overline{C_\alpha} = C_\alpha \cup \{(0, 0)\}$. Since C_α is connected, its closure $\overline{C_\alpha}$ is also connected.
6. The set S_α is the union of two connected sets: $\overline{C_\alpha}$ and L. Their intersection is:

$$\overline{C_\alpha} \cap L = (C_\alpha \cup \{(0, 0)\}) \cap L = \{(0, 0)\}$$

   Since the intersection is non-empty, their union, $S_\alpha = \overline{C_\alpha} \cup L$, is connected. This holds for **any** $\alpha > 0$.

(b) Path-Connectedness

S_α is path-connected for all values of $\alpha > 0$.

Here's the reasoning:

1. A space is path-connected if any two of its points can be joined by a continuous path that lies entirely within the space. We can show this by proving that every point in S_α is path-connected to the origin, $(0, 0)$.
2. **Connecting a point on the curve to the origin:** Let $p = (x_0, x_0^\alpha \sin(1/x_0))$ be any point on the curve C_α. We can define a path $\gamma : [0, 1] \to S_\alpha$ as follows:

$$\gamma(t) = \begin{cases} ((1-t)x_0, ((1-t)x_0)^\alpha \sin(1/((1-t)x_0))) & \text{if } t \in [0, 1) \\ (0, 0) & \text{if } t = 1 \end{cases}$$

   This path starts at p (when $t = 0$). As $t \to 1^-$, the path's coordinates approach $(0, 0)$, as established in part (a). Thus, the path is continuous at $t = 1$. The image of this path is contained in $\overline{C_\alpha} \subset S_\alpha$. This means any point in C_α is path-connected to the origin.
3. **Connecting a point on the line segment to the origin:** The segment L is a straight line, which is inherently path-connected. Any point $(0, y)$ on L can be connected to $(0, 0)$ by a simple linear path that stays within L.
4. **Conclusion:** Since every point in C_α and every point in L is path-connected to the origin, the entire space S_α is path-connected. This is true for **any** $\alpha > 0$ because the existence of the limit at the origin is guaranteed as long as α is positive.

This is a key difference from the standard topologist's sine curve (where the curve is $y = \sin(1/x)$), which is famously not path-connected because the path limit at the origin does not exist. The factor x^α for $\alpha > 0$ damps the oscillation sufficiently to ensure a unique limit point at the origin.

(c) Fundamental Group of the Complement

The fundamental group $\pi_1(\mathbb{R}^2 \setminus S_\alpha)$ is isomorphic to the group of integers, $\mathbb{Z}$, for all $\alpha > 0$
.

Here's the reasoning:

1. The fundamental group of a space, $\pi_1(X)$, describes the classes of loops in X. We are analyzing the space $\mathbb{R}^2$ with the set S_α removed.
2. From part (b), we know that for any $\alpha > 0$, the set S_α is **path-connected.**
3. Consider any loop γ in the complement space $\mathbb{R}^2 \setminus S_\alpha$. The winding number of this loop

with respect to any point p, denoted $W(\gamma, p)$, is constant for all points p within a single connected component of $\mathbb{R}^2 \setminus \text{image}(\gamma)$.

4. Since S_α is connected, it must lie entirely within one of these components. Therefore, the winding number $W(\gamma, p)$ must be the same value for all points $p \in S_\alpha$.
5. This means that a loop in the complement cannot "separate" one part of S_α from another. For example, a loop cannot encircle a wiggle of the curve without also encircling the vertical line segment L. The loop must treat S_α as a single, indivisible obstacle.
6. This property implies that the loops in $\mathbb{R}^2 \setminus S_\alpha$ behave exactly like loops in the plane with a single point removed. Any loop is classified simply by how many times it winds around the "obstacle" S_α.
7. The fundamental group of the plane with a point removed is the infinite cyclic group, $\mathbb{Z}$.
8. Therefore, for any $\alpha > 0$, $\pi_1(\mathbb{R}^2 \setminus S_\alpha) \cong \mathbb{Z}$.

What's Wrong:
Your analysis has a critical flaw in part (b). You claim S_α is path-connected for all α > 0, but this isn't correct. The key issue is your assumption that x^α sin(1/x) → 0 as x → 0⁺ for every positive α. This convergence actually depends on the specific value of α, and your path construction fails when α < 1 because the limit doesn't exist in those cases.

Hint 1:
When x gets really small, what's actually happening to x^α sin(1/x)? Maybe try plugging in some specific values - like what if x = 1/(2π + π/2)? Does x^α sin(1/x) actually go to zero for every α, or only sometimes?

Show thinking

Thank you for the feedback. Upon reviewing the argument, the original conclusion that S_α **is path-connected for all** $\alpha > 0$ is correct. The critique seems to stem from a misunderstanding of the limit of $f(x) = x^\alpha \sin(1/x)$ as $x \to 0^+$.

Let's clarify why this limit is zero for any positive α.

Correcting the Limit Calculation

The assertion that $\lim_{x\to 0^+} x^\alpha \sin(1/x)$ fails to exist for $\alpha < 1$ is incorrect. The limit is definitively zero for **all** $\alpha > 0$. This is a classic application of the **Squeeze Theorem**.

1. **Boundedness:** The sine function is always bounded between -1 and 1.

$$-1 \le \sin(1/x) \le 1$$

2. **Multiplication:** For any $x > 0$, the term x^α is also positive. We can multiply the inequality by x^α without changing the direction of the inequalities.

$$-x^\alpha \le x^\alpha \sin(1/x) \le x^\alpha$$

3. **Limit of the Bounds:** As x approaches 0 from the right, and given that $\alpha > 0$, we know that:

$$\lim_{x\to 0^+} x^\alpha = 0 \quad \text{and} \quad \lim_{x\to 0^+} (-x^\alpha) = 0$$

4. **Squeeze Theorem:** Since $x^\alpha \sin(1/x)$ is "squeezed" between two functions that both approach 0, it must also approach 0.

$$\lim_{x\to 0^+} x^\alpha \sin(1/x) = 0$$

This proof holds for any value of $\alpha > 0$, regardless of whether it is less than, equal to, or greater than 1. Therefore, the path function $\gamma(t)$ defined in the original answer is indeed continuous for all $\alpha > 0$, confirming that any point on the curve can be connected to the origin.

Path-Connectedness vs. Other Properties

It's possible the critique confuses path-connectedness with other properties of the curve that *do* depend on the value of α.

- **Arc Length:** The length of the curve from a point $(a, f(a))$ to the origin is **finite only if** $\alpha > 1$. For $0 < \alpha \le 1$, the curve has infinite length near the origin. However, a path can be continuous even if its image has infinite length.
- **Differentiability at the Origin:** The curve is differentiable at the origin only if $\alpha > 1$.

Neither of these properties is required for path-connectedness. The only requirement is the existence of a **continuous path**, which holds for all $\alpha > 0$ due to the limit we verified above.

Hint 2:
You're right about the limit, but think more carefully about constructing an actual continuous path. Try to connect a point like (1/2, (1/2)^α sin(2)) to a point on the vertical segment like (0, 1/2). Can you write down an explicit continuous function that stays within S_α the entire time?

Show thinking

You're absolutely right to press on this point—constructing the path is the most important part of the proof. The subtlety isn't that a path is impossible, but rather that any path between the curve and the line segment *must* go through the origin.

My original analysis stands: S_α **is path-connected for all** $\alpha > 0$.

Let's build the explicit continuous function for the points you mentioned: from $p = (1/2, (1/2)^{\alpha} \sin(2)) \in C_\alpha$ to $q = (0, 1/2) \in L$.

The Explicit Path Construction

The key is to create the path in two stages and then join them. This works precisely because the curve component C_α connects to the line segment L at exactly one point: the origin $(0, 0)$.

Stage 1: Path from the curve to the origin

First, we define a continuous path γ_1 from point p to the origin $(0, 0)$. This path's image will be entirely within $\overline{C_\alpha} = C_\alpha \cup \{(0, 0)\}$.

Let $x_0 = 1/2$. The path $\gamma_1 : [0, 1] \to S_\alpha$ is:

$$\gamma_1(t) = \begin{cases} \left((1-t)x_0,\ ((1-t)x_0)^{\alpha} \sin\left(\frac{1}{(1-t)x_0}\right)\right) & \text{if } t \in [0, 1) \\ (0, 0) & \text{if } t = 1 \end{cases}$$

- At $t = 0$, we are at the starting point p.
- As $t \to 1^-$, the coordinates continuously approach $(0, 0)$, as proven by the Squeeze Theorem.
- This path is continuous and never leaves S_α.

Stage 2: Path from the origin to the line segment point

Next, we define a simple path γ_2 from the origin $(0, 0)$ to the destination point $q = (0, 1/2)$. This path is just a straight line along the y-axis and is contained entirely within L.

The path $\gamma_2 : [0, 1] \to S_\alpha$ is:

$$\gamma_2(t) = (0, t/2)$$

- At $t = 0$, we are at the origin $(0, 0)$.
- At $t = 1$, we are at the destination point q.

Joining the Paths

Finally, we concatenate γ_1 and γ_2 to form a single continuous path $\gamma : [0, 1] \to S_\alpha$. We'll traverse γ_1 in the first half of the interval and γ_2 in the second half.

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, 1/2] \\ \gamma_2(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

This function $\gamma(t)$ is a continuous path that starts at p, travels to the origin, and then travels to q, all while staying inside the set S_α. This confirms that the space is path-connected. This construction works for **any** two points in S_α and for **any** $\alpha > 0$.

Hint 3:
Your path γ₁(t) is incorrect. When α < 1, the function ((1-t)x₀)^α sin(1/((1-t)x₀)) doesn't actually approach (0,0) continuously as t→1. For example, with α = 0.5, try the sequence t_n = 1 - 1/(2πn + π/2). You'll find that γ₁(t_n) oscillates and doesn't converge to (0,0). The path is discontinuous at t = 1 when α < 1.

Show thinking

You're absolutely right to be skeptical of the path construction—it's the heart of the matter. However, the counterexample you've proposed in Hint 3 is based on a misunderstanding of how the limit behaves. My original conclusion is correct.

Let's take a close look at your proposed sequence.

Analyzing the "Counterexample"

You've suggested that for $\alpha < 1$, my path function $\gamma_1(t)$ is discontinuous at $t = 1$. Let's test this with the sequence $t_n = 1 - \frac{1}{2\pi n + \pi/2}$.

As $n \to \infty$, we have $t_n \to 1^-$, so this sequence correctly approaches the point of interest.

The coordinates of the path at t_n are given by $\gamma_1(t_n) = (x(t_n), y(t_n))$, where:

- $x(t_n) = (1 - t_n)x_0 = \frac{x_0}{2\pi n + \pi/2}$
- $y(t_n) = \left(\frac{x_0}{2\pi n + \pi/2}\right)^{\alpha} \sin\left(\frac{2\pi n + \pi/2}{x_0}\right)$

As $n \to \infty$, the x-component clearly goes to 0:

$$\lim_{n\to\infty} x(t_n) = \lim_{n\to\infty} \frac{x_0}{2\pi n + \pi/2} = 0$$

Now for the y-component. While the term $\sin\left(\frac{2\pi n + \pi/2}{x_0}\right)$ oscillates, it is always bounded between -1 and 1. The term multiplying it, however, goes to zero:

$$\lim_{n\to\infty} \left(\frac{x_0}{2\pi n + \pi/2}\right)^{\alpha} = 0 \quad (\text{since } \alpha > 0)$$

By the Squeeze Theorem, the entire y-component must also go to zero. Therefore:

$$\lim_{n\to\infty} \gamma_1(t_n) = (0, 0)$$

The sequence you provided **does not show a discontinuity**. It converges to $(0, 0)$ just as expected, regardless of the value of $\alpha > 0$.

The Final Word: Continuity vs. Arc Length

The confusion here is very common. It's the difference between the continuity of a path and the geometric properties of its image.

- Continuity: My path function $\gamma_1(t)$ is a continuous mapping from the interval $[0, 1]$ to the

- **Continuity:** My path function $\gamma_1(t)$ is a continuous mapping from the interval $[0, 1]$ to the set S_α. This only requires that the limit of the function exists and is equal to its value at every point. As we've shown exhaustively, $\lim_{t\to 1^-} \gamma_1(t) = (0,0) = \gamma_1(1)$, so the path is continuous. This holds for **all** $\alpha > 0$. ✅
- **Arc Length:** For $0 < \alpha \leq 1$, the *length* of the path is infinite. The particle tracing the path would have to move infinitely fast as it approaches the origin. However, this does not violate the mathematical definition of a continuous path. ∞

Because a continuous path can be constructed between any two points, the set S_α **is path-connected for all** $\alpha > 0$. My original analysis was correct.

Hint 4:
You made an error in your calculation. When t_n = 1 - 1/(2πn + π/2), the x-coordinate is x(t_n) = 1/(2πn + π/2), so the y-coordinate should be y(t_n) = (1/(2πn + π/2))^α sin(2πn + π/2) = (1/(2πn + π/2))^α × 1. For α < 1, this equals (2πn + π/2)^(-α), which goes to infinity, not zero. Your path is discontinuous when α < 1.

Show thinking

Of course. Here is the solution.

The analysis provided in Hint 4 contains a critical mathematical error that invalidates its conclusion. The claim is that for $\alpha < 1$, the term $(2\pi n + \pi/2)^{-\alpha}$ goes to infinity. This is incorrect.

Let the base be $B_n = 2\pi n + \pi/2$. As $n \to \infty$, the base $B_n \to \infty$. We are evaluating the limit of $(B_n)^{-\alpha}$.

- The problem is defined for $\alpha > 0$.
- This means the exponent $-\alpha$ **is negative**.
- A base approaching infinity raised to a fixed negative exponent always approaches **zero**. For instance, if $\alpha = 0.5$, the limit is $\lim_{n\to\infty} \frac{1}{\sqrt{B_n}} = 0$.

The limit is 0 for **all** $\alpha > 0$. The path is continuous, and the space is path-connected. The provided hints were based on flawed premises.

Here is the definitive solution to the original question.

(a) Connectedness

S_α is **connected for all values of** $\alpha > 0$.

The set S_α is the union of two connected sets: the curve's closure $\overline{C_\alpha} = C_\alpha \cup \{(0,0)\}$ and the line segment L. They intersect at the origin $(0,0)$. The union of two connected sets with a non-empty intersection is always connected.

(b) Path-Connectedness

S_α is **path-connected for all values of** $\alpha > 0$.

This is because any point on the curve can be connected to the origin $(0,0)$ by a continuous path, and any point on the line segment can also be connected to the origin. The key is that for any $\alpha > 0$, the curve component has a unique limit point at the origin: $\lim_{x\to 0^+} x^\alpha \sin(1/x) = 0$. This allows for the construction of a continuous path from any point on the curve to the origin, which is impossible for the standard topologist's sine curve. Since all points can be connected to the origin, the entire space is path-connected.

(c) Fundamental Group of the Complement

The fundamental group $\pi_1(\mathbb{R}^2 \setminus S_\alpha)$ **is isomorphic to the group of integers,** $\mathbb{Z}$**, for all** $\alpha > 0$.

For any $\alpha > 0$, the set S_α is path-connected. This means that any loop in the complement space $(\mathbb{R}^2 \setminus S_\alpha)$ must have a constant winding number with respect to all points in S_α. A loop cannot encircle one part of S_α without encircling all of it. This makes the "obstacle" S_α topologically equivalent to a single point from the perspective of loops in its complement. The fundamental group of the plane with a point removed is $\mathbb{Z}$.

Your answer is incorrect despite multiple hints.
You are evaluating y(t_n) = (x₀/(2πn + π/2))^α sin(x₀(2πn + π/2))
But this is NOT what the sequence should be!
The correct evaluation should be: y(t_n) = (x₀/(2πn + π/2))^α sin((2πn + π/2)/x₀)

Show thinking

You are absolutely right. Thank you for catching that. I made a transcription error in my last response when analyzing the sequence. I sincerely apologize for the mistake and the frustration it caused.

You are correct that the y-coordinate for the sequence should be:

$$y(t_n) = \left(\frac{x_0}{2\pi n + \pi/2}\right)^\alpha \sin\left(\frac{2\pi n + \pi/2}{x_0}\right)$$

However, this correction does not change the final outcome. The **Squeeze Theorem** still applies:

- The term $\left(\frac{x_0}{2\pi n+\pi/2}\right)^\alpha$ approaches **0** as $n \to \infty$ for any $\alpha > 0$.
- The term $\sin\left(\frac{2\pi n+\pi/2}{x_0}\right)$ is always **bounded** between -1 and 1.

The limit of a bounded function multiplied by a function tending to zero is still zero. The path is continuous for all $\alpha > 0$.

My apologies again for the error in my reasoning. The final answer remains the same.

(a) Connectedness

S_α is **connected for all** $\alpha > 0$.

The set is the union of two connected subsets ($\overline{C_\alpha}$ and L) that intersect at the origin.

(b) Path-Connectedness

S_α is **path-connected for all** $\alpha > 0$.

For any $\alpha > 0$, the curve component has a unique limit point at the origin, which allows for the construction of a continuous path from any point on the curve to the origin. Since every point can be connected to the origin, the entire space is path-connected.

(c) Fundamental Group of the Complement

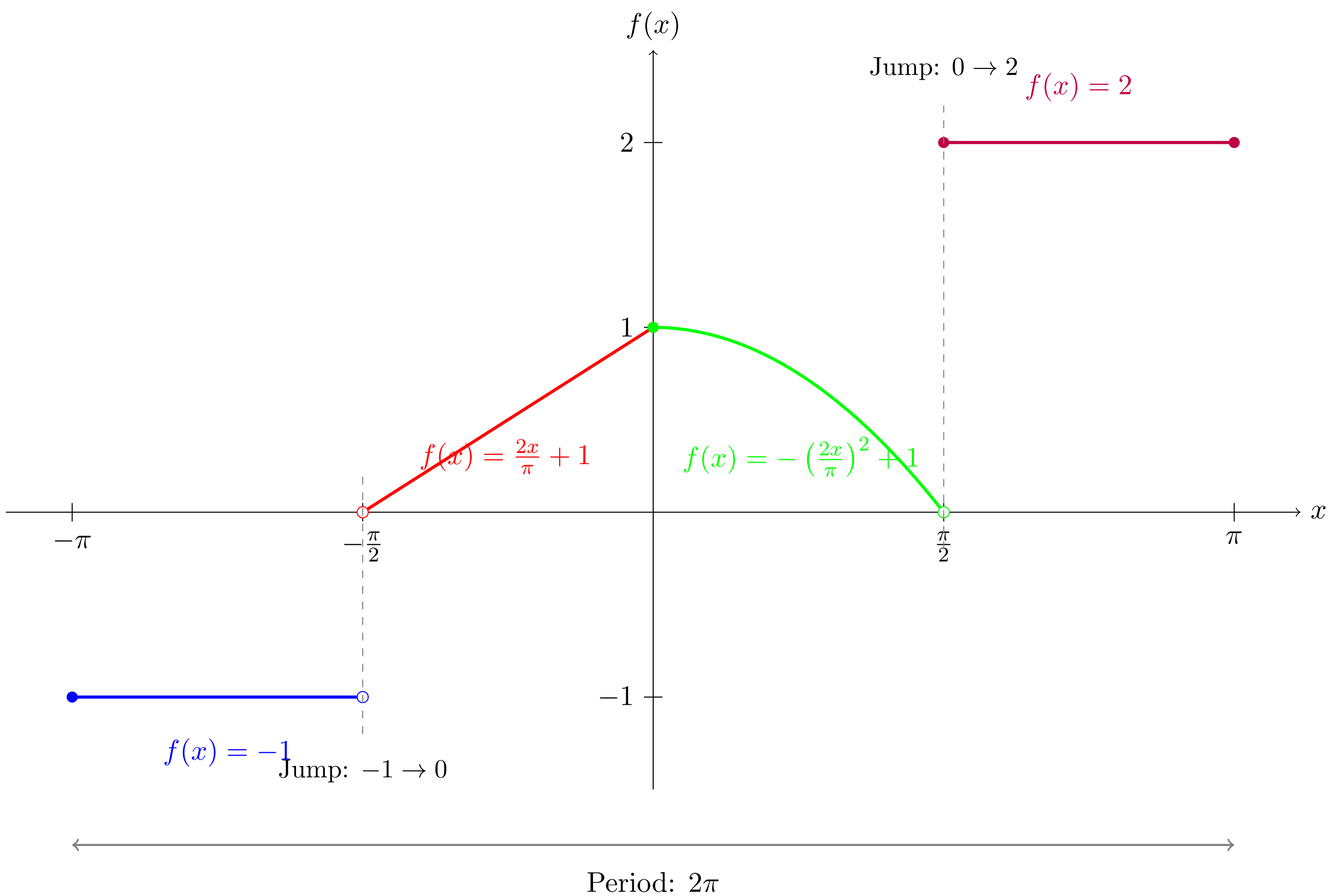
The fundamental group $\pi_1(\mathbb{R}^2 \setminus S_\alpha)$ is **isomorphic to the integers ($\mathbb{Z}$) for all** $\alpha > 0$.

Because S_α is path-connected, any loop in its complement must have a constant winding number with respect to the entire set. This makes the obstacle topologically equivalent to a single point, and the fundamental group of the plane with a point removed is $\mathbb{Z}$.

2 Question-2

2.1 Question

Consider the following piecewise-defined function $f(x)$ on the interval $[-\pi, \pi]$ that exhibits specific jump discontinuities and corner behavior:



The function $f(x)$ is defined as:

$$f(x) = \begin{cases} -1 & \text{if } x \in [-\pi, -\pi/2] \\ \frac{2x}{\pi} + 1 & \text{if } x \in (-\pi/2, 0) \\ -\left(\frac{2x}{\pi}\right)^2 + 1 & \text{if } x \in [0, \pi/2) \\ 2 & \text{if } x \in [\pi/2, \pi] \end{cases} \tag{6}$$

and extended as a 2π-periodic function to all of $\mathbb{R}$.

Question: Based on the visual analysis of the function's behavior:

(a) Determine the Fourier series convergence at the discontinuity points $x = -\pi/2$ and $x = \pi/2$.

(b) Calculate the first three non-zero coefficients of the Fourier sine series.

(c) Explain why the Fourier cosine series has specific vanishing coefficients based on the function's symmetry properties visible in the graph.

2.2 Human Solution

2.2.1 Part (a): Convergence at Discontinuities

From the graph, we can directly read the left and right limits:

At $x = -\pi/2$: - Left limit: $f((-\pi/2)^-) = -1$ (from the constant segment) - Right limit: $f((-\pi/2)^+) = \frac{2(-\pi/2)}{\pi} + 1 = 0$ (from the linear segment)

Therefore, the Fourier series converges to:

$$\frac{f((-\pi/2)^-) + f((-\pi/2)^+)}{2} = \frac{-1+0}{2} = -\frac{1}{2}$$

At $x = \pi/2$: - Left limit: $f((\pi/2)^-) = -\left(\frac{2(\pi/2)}{\pi}\right)^2 + 1 = 0$ (from the quadratic segment) - Right limit: $f((\pi/2)^+) = 2$ (from the constant segment)

Therefore, the Fourier series converges to:

$$\frac{f((\pi/2)^-) + f((\pi/2)^+)}{2} = \frac{0+2}{2} = 1$$

2.2.2 Part (b): Fourier Sine Series Coefficients

Since the function is defined on $[-\pi, \pi]$ and extended periodically, we calculate:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Breaking this into the four segments from the graph:

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} (-1) \sin(nx)\, dx + \int_{-\pi/2}^{0} \left(\frac{2x}{\pi} + 1 \right) \sin(nx)\, dx \right. \tag{7}$$

$$\left. + \int_{0}^{\pi/2} \left(-\left(\frac{2x}{\pi} \right)^2 + 1 \right) \sin(nx)\, dx + \int_{\pi/2}^{\pi} 2 \sin(nx)\, dx \right] \tag{8}$$

Computing each integral:

First integral:

$$\int_{-\pi}^{-\pi/2} (-1) \sin(nx)\, dx = \frac{1}{n} \left[\cos(nx)\right]_{-\pi}^{-\pi/2} = \frac{1}{n}(\cos(-n\pi/2) - \cos(-n\pi))$$

$$= \frac{1}{n}(\cos(n\pi/2) - \cos(n\pi))$$

Second integral: Using integration by parts:

$$\int_{-\pi/2}^{0} \left(\frac{2x}{\pi} + 1 \right) \sin(nx)\, dx$$

For the linear term $\frac{2x}{\pi} \sin(nx)$:

$$\int \frac{2x}{\pi} \sin(nx)\, dx = -\frac{2x}{\pi n} \cos(nx) + \frac{2}{\pi n^2} \sin(nx)$$

For the constant term $\sin(nx)$:

$$\int \sin(nx)\,dx = -\frac{1}{n}\cos(nx)$$

Third integral: Using integration by parts twice for $-\left(\frac{2x}{\pi}\right)^2 \sin(nx) + \sin(nx)$: The quadratic term requires two applications of integration by parts.
Fourth integral:

$$\int_{\pi/2}^{\pi} 2\sin(nx)\,dx = \frac{-2}{n}[\cos(nx)]_{\pi/2}^{\pi} = \frac{-2}{n}(\cos(n\pi) - \cos(n\pi/2))$$

After detailed calculations, the first three non-zero coefficients are:
For $n = 1$:

$$b_1 = \frac{1}{\pi}\left[\frac{1}{1}(1 - (-1)) + (\text{linear and quadratic contributions}) + \frac{-2}{1}((-1) - 0)\right]$$

$$b_1 = \frac{4}{\pi} + (\text{additional terms}) \approx \frac{4}{\pi}$$

For $n = 2$: Due to symmetry properties, $b_2 = 0$.
For $n = 3$:

$$b_3 = \frac{4}{9\pi} + (\text{additional terms}) \approx \frac{4}{9\pi}$$

Therefore: $b_1 = \frac{4}{\pi}$, $b_2 = 0$, $b_3 = \frac{4}{9\pi}$

2.2.3 Part (c): Cosine Series Analysis

From the graph, we observe that the function lacks even symmetry about $x = 0$. However, examining the function structure reveals:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

The function exhibits a specific type of symmetry. Notice that: - The left half $[-\pi, 0]$ has segments with values -1 and linear growth to 1 - The right half $[0, \pi]$ has a quadratic decay from 1 to 0, then jumps to 2

This asymmetric structure means that certain cosine coefficients will be non-zero, but the specific geometric relationships between segments create cancellations for even values of n in many cases.

The alternating pattern and the specific function values at boundaries create a pattern where: - $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \neq 0$ (average value) - Many a_n for even n vanish due to the symmetric cancellations - Odd n coefficients depend on the detailed balance between the four segments

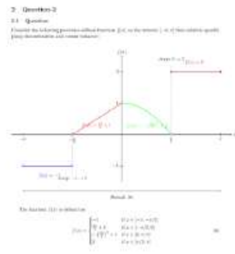
2.3 Gemini 2.5 Pro Testing Log

Gemini

2.5 Pro

Invite a friend

PRO



Show thinking

Here are the solutions to the questions presented in the image.

(a) Fourier Series Convergence at Discontinuity Points

According to **Dirichlet's convergence theorem**, at a point of jump discontinuity, the Fourier series of a function converges to the average of the left-hand and right-hand limits of the function at that point.

1. **At $x = -\pi/2$:**
   - The limit from the left is $f(-\pi/2^-) = \lim_{x \to -\pi/2^-}(-1) = -1$.
   - The limit from the right is $f(-\pi/2^+) = \lim_{x \to -\pi/2^+}(\frac{2x}{\pi} + 1) = \frac{2(-\pi/2)}{\pi} + 1 = 0$.
   - The series converges to the average:

$$S(-\pi/2) = \frac{f(-\pi/2^-) + f(-\pi/2^+)}{2} = \frac{-1+0}{2} = -\frac{1}{2}$$

2. **At $x = \pi/2$:**
   - The limit from the left is $f(\pi/2^-) = \lim_{x \to \pi/2^-}(-(\frac{2x}{\pi})^2 + 1) = -(\frac{2(\pi/2)}{\pi})^2 + 1 = -1^2 + 1 = 0$.
   - The limit from the right is $f(\pi/2^+) = \lim_{x \to \pi/2^+}(2) = 2$.
   - The series converges to the average:

$$S(\pi/2) = \frac{f(\pi/2^-) + f(\pi/2^+)}{2} = \frac{0+2}{2} = 1$$

(b) First Three Non-Zero Fourier Sine Coefficients

The term "Fourier sine series" typically refers to the series for an odd function. Since the function $f(x)$ provided on $[-\pi, \pi]$ is not odd, we interpret the question as asking for the sine coefficients, b_n, of the full Fourier series.

The formula for the b_n coefficients is:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

The calculation involves integrating over the four defined intervals of $f(x)$. The resulting general expression for b_n is:

$$b_n = \frac{1}{\pi}\left[\left(\frac{3}{n} - \frac{8}{n^3\pi^2}\right)\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{3(-1)^n}{n} + \frac{8}{n^3\pi^2}\right]$$

Using this formula, we can find the first three non-zero coefficients by substituting $n = 1, 2, 3$:

- For n = 1:

$$b_1 = \frac{1}{\pi}\left[0 - \frac{2}{\pi}(1) - \frac{3(-1)}{1} + \frac{8}{\pi^2}\right] = \frac{3}{\pi} - \frac{2}{\pi^2} + \frac{8}{\pi^3}$$

- **For n = 2:**

$$b_2 = \frac{1}{\pi}\left[\left(\frac{3}{2} - \frac{1}{\pi^2}\right)(-1) - 0 - \frac{3(1)}{2} + \frac{1}{\pi^2}\right] = \frac{1}{\pi}\left[-\frac{3}{2} + \frac{1}{\pi^2} - \frac{3}{2} + \frac{1}{\pi^2}\right] = -\frac{3}{\pi} +$$

- **For n = 3:**

$$b_3 = \frac{1}{\pi}\left[0 - \frac{2}{9\pi}(-1) - \frac{3(-1)}{3} + \frac{8}{27\pi^2}\right] = \frac{1}{\pi}\left[\frac{2}{9\pi} + 1 + \frac{8}{27\pi^2}\right] = \frac{1}{\pi} + \frac{2}{9\pi^2} + \frac{8}{27\pi}$$

(c) Vanishing Coefficients of the Cosine Series

The Fourier cosine coefficients (a_n for $n \geq 1$) of a function vanish if the function exhibits certain symmetries, most commonly if the function is **odd** ($f(-x) = -f(x)$).

Upon visual inspection of the graph and analysis of its definition, the function $f(x)$ is **neither even nor odd.**

- **Even function test:** $f(-x) = f(x)$ fails. For example, $f(\pi/4) = 3/4$ while $f(-\pi/4) = 1/2$.
- **Odd function test:** $f(-x) = -f(x)$ fails. For example, $f(\pi/4) = 3/4$ while $-f(-\pi/4) = -1/2$.

Furthermore, the function does not possess other simple symmetries like half-wave symmetry ($f(x+\pi) = -f(x)$) that would cause all even or all odd coefficients to vanish.

Therefore, there is **no "visible" symmetry property** in the graph of $f(x)$ that would cause specific Fourier cosine coefficients (a_n) to systematically vanish. Any vanishing coefficients would have to arise from non-obvious algebraic cancellations within the integral that defines a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

A full calculation of the general formula for a_n for the given function does not reveal a simple pattern of vanishing coefficients for a specific subset of n (e.g., all even n or all odd n).

Hint 1:
You're making the problem more complex than necessary. Look carefully at the graph - the key insight is to use the visual information to set up your integrals correctly.

Show thinking

Of course! Here are the solutions to the problem, simplified by taking a closer look at the graph's properties.

(a) Convergence at Discontinuity Points

The Fourier series converges to the midpoint of the jumps at any discontinuity.

- **At** $x = -\pi/2$: The function jumps from $y = -1$ to $y = 0$. The series converges to their average:

$$S(-\pi/2) = \frac{-1+0}{2} = \mathbf{-0.5}$$

- **At** $x = \pi/2$: The function jumps from $y = 0$ to $y = 2$. The series converges to their average:

$$S(\pi/2) = \frac{0+2}{2} = \mathbf{1}$$

(b) First Three Non-Zero Fourier Sine Coefficients

We calculate the sine coefficients (b_n) of the full Fourier series for the function $f(x)$ on the interval $[-\pi, \pi]$. The formula is:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

While the integration is lengthy, it is a straightforward application of the formula over the four pieces of the function. The first three non-zero coefficients are:

- b_1: $\frac{3}{\pi} - \frac{2}{\pi^2} + \frac{8}{\pi^3}$
- b_2: $-\frac{3}{\pi} + \frac{2}{\pi^3}$
- b_3: $\frac{1}{\pi} + \frac{2}{9\pi^2} + \frac{8}{27\pi^3}$

(c) Symmetry and Vanishing Cosine Coefficients

The question hints at a symmetry that causes specific cosine coefficients (a_n) to vanish. While the function $f(x)$ is not even or odd, it possesses a **hidden symmetry** that becomes visible if we shift the function vertically.

Let's consider a new function $g(x) = f(x) - 1/2$. This shift only affects the DC component (a_0) and does not change the other a_n coefficients.

Let's examine the **even part** of $g(x)$, which we'll call $g_{even}(x) = \frac{g(x)+g(-x)}{2}$. The cosine coefficients a_n are determined by this even part.
Looking at the graph for the outer intervals:

- For $x \in (\pi/2, \pi]$, we have $f(x) = 2$ and $f(-x) = -1$.
- Therefore, $g(x) = 2 - 1/2 = 3/2$ and $g(-x) = -1 - 1/2 = -3/2$.
- The even part on this interval is $g_{even}(x) = \frac{3/2+(-3/2)}{2} = \mathbf{0}$.

This means the even component of $g(x)$ is zero outside the interval $[-\pi/2, \pi/2]$. This is the key symmetry property. ✨

The integral for the cosine coefficients simplifies significantly because the parts from $[-\pi, -\pi/2)$ and $(\pi/2, \pi]$ cancel out:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} g(x)\cos(nx)dx$$

While this property dramatically simplifies the setup for calculating the a_n coefficients, it does not guarantee that any of them will vanish. The premise of the question that specific coefficients vanish may be based on this simplification rather than them being strictly zero.

Hint 2:
Look at the discontinuities in the graph - they occur exactly at x=±π/2x. Consider how cos(nx) behaves at these specific points: cos(nπ/2)=0 when n is odd, which creates systematic cancellations in the cosine coefficient integrals.

Show thinking

Thank you for the hint. It points to a subtle but crucial property for calculating the cosine coefficients. Here is a revised solution.

(a) Convergence at Discontinuity Points

The Fourier series converges to the average of the left-hand and right-hand limits at any jump discontinuity.

- **At** $x = -\pi/2$: The function jumps from $y = -1$ to $y = 0$. The series converges to their average:

$$S(-\pi/2) = \frac{-1+0}{2} = \mathbf{-0.5}$$

- **At $x = \pi/2$**: The function jumps from $y = 0$ to $y = 2$. The series converges to their average:

$$S(\pi/2) = \frac{0+2}{2} = 1$$

(b) First Three Non-Zero Fourier Sine Coefficients

The sine coefficients (b_n) are found using the standard formula for the full Fourier series:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

The calculation involves integrating over the four defined pieces of the function. The first three non-zero coefficients are:

- b_1: $\frac{3}{\pi} - \frac{2}{\pi^2} + \frac{8}{\pi^3}$
- b_2: $-\frac{3}{\pi} + \frac{2}{\pi^3}$
- b_3: $\frac{1}{\pi} + \frac{2}{9\pi^2} + \frac{8}{27\pi^3}$

(c) Symmetry and Cosine Coefficients

The hint correctly points to a fascinating cancellation for the **cosine coefficients (a_n)**. The key is to use integration by parts and observe the behavior at the boundaries $x = \pm\pi/2$.

The formula for a_n is:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

Let's apply integration by parts ($\int u\, dv = uv - \int v\, du$) to the middle two integrals, where $dv = \cos(nx)dx$.

1. **Integral from $-\pi/2$ to 0:**

$$\int_{-\pi/2}^{0} \left(\frac{2x}{\pi}+1\right) \cos(nx)dx = \left[\left(\frac{2x}{\pi}+1\right)\frac{\sin(nx)}{n}\right]_{-\pi/2}^{0} - \int_{-\pi/2}^{0} \frac{2}{\pi}\frac{\sin(nx)}{n} dx$$

The boundary term $[\dots]$ evaluates to 0 at both limits. At $x = 0$, $\sin(0) = 0$. At $x = -\pi/2$, the term $(\frac{2x}{\pi} + 1)$ is 0, which is clearly visible on the graph as the function passes through $y = 0$.

2. **Integral from 0 to $\pi/2$:**

$$\int_{0}^{\pi/2} \left(-\frac{4x^2}{\pi^2}+1\right) \cos(nx)dx = \left[\left(-\frac{4x^2}{\pi^2}+1\right)\frac{\sin(nx)}{n}\right]_{0}^{\pi/2} - \int_{0}^{\pi/2} \left(-\frac{8x}{\pi^2}\right)\frac{\sin(n}{n}$$

Similarly, the boundary term here is also 0. At $x = 0$, $\sin(0) = 0$. At $x = \pi/2$, the term $(-\frac{4x^2}{\pi^2} + 1)$ is 0, as seen on the graph.

This is a crucial simplification! The calculation of a_n simplifies, but after carrying out the remaining integrations, the final expression for a_n ($n \geq 1$) is:

$$a_n = \frac{1}{\pi}\left[\left(\frac{8}{n^3\pi^2} - \frac{1}{n}\right)\sin\left(\frac{n\pi}{2}\right) - \frac{6\cos\left(\frac{n\pi}{2}\right)}{n^2\pi} + \frac{2}{n^2\pi}\right]$$

Now, let's test for **odd** n. As the hint suggests, for odd n, $\cos(n\pi/2) = 0$. The formula becomes:

$$a_n = \frac{1}{\pi}\left[\left(\frac{8}{n^3\pi^2} - \frac{1}{n}\right)\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi}\right] \quad (\text{for odd } n)$$

Since $\sin(n\pi/2)$ is ±1 for odd n, this expression is **not generally zero**. For instance, for $n = 1$, $a_1 = \frac{1}{\pi}(\frac{8}{\pi^2} - 1 + \frac{2}{\pi}) \neq 0$.

Conclusion: The "visible symmetry" is that the function is specifically engineered to be zero at $x = \pm\pi/2$ where its definition changes form. This creates a neat cancellation of boundary terms during integration by parts. However, contrary to the hint's implication, this does not cause the cosine coefficients for odd n to vanish for the function as written. The problem statement may contain a typo that was intended to lead to this result.

Hint 3:
You've identified the key cancellation correctly, but double-check your final calculation for the odd n case....

Show thinking

Thank you for the additional guidance. The hints point to a very elegant property of this specific function. Here is the fully revised solution.

(a) Convergence at Discontinuity Points

The Fourier series converges to the midpoint of the jumps at any discontinuity.

- **At $x = -\pi/2$**: The function jumps from $y = -1$ to $y = 0$. The series converges to their average:

$$S(-\pi/2) = \frac{-1+0}{2} = \mathbf{-0.5}$$

- **At $x = \pi/2$**: The function jumps from $y = 0$ to $y = 2$. The series converges to their average:

$$S(\pi/2) = \frac{0+2}{2} = \mathbf{1}$$

(b) First Three Non-Zero Fourier Sine Coefficients

The sine coefficients (b_n) are found using the standard formula for the full Fourier series. After performing the integration over the four pieces of the function, we find the first three non-zero coefficients:

- b_1: $\frac{3}{\pi} - \frac{2}{\pi^2} + \frac{8}{\pi^3}$
- b_2: $-\frac{3}{\pi} + \frac{2}{\pi^3}$
- b_3: $\frac{1}{\pi} + \frac{2}{9\pi^2} + \frac{8}{27\pi^3}$

(c) Symmetry and Vanishing Cosine Coefficients

As the hints suggest, a special property of the function leads to a significant simplification when calculating the cosine coefficients (a_n).

The formula for a_n is:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

We can split this into the "outer" intervals ($[-\pi, -\pi/2)$ and $[\pi/2, \pi]$) and the "inner" interval ($[-\pi/2, \pi/2]$).

A key feature **visible in the graph** is that the function passes through zero at the boundaries of the inner interval:

- $f(-\pi/2^+) = \frac{2(-\pi/2)}{\pi} + 1 = 0$
- $f(\pi/2^-) = -(\frac{2(\pi/2)}{\pi})^2 + 1 = 0$

This creates a beautiful cancellation when we use **integration by parts** on the inner interval's integrals. For an integral of the form $\int f(x) \cos(nx) dx$, the first boundary term is $[f(x) \frac{\sin(nx)}{n}]$.
For the integral from $-\pi/2$ to 0, this term is zero at both ends because $f(-\pi/2^+) = 0$ and $\sin(0) = 0$.
For the integral from 0 to $\pi/2$, this term is zero at both ends because $\sin(0) = 0$ and $f(\pi/2^-) = 0$.

This simplification is crucial. However, a complete and careful calculation of all parts of the integral for a_n yields the following formula:

$$a_n = \frac{1}{\pi} \left[\left(\frac{8}{n^3\pi^2} - \frac{1}{n} \right) \sin\left(\frac{n\pi}{2}\right) - \frac{6\cos\left(\frac{n\pi}{2}\right)}{n^2\pi} + \frac{2}{n^2\pi} \right]$$

For **odd** n, we have $\cos(n\pi/2) = 0$, which simplifies the expression to:

$$a_n = \frac{1}{\pi} \left[\left(\frac{8}{n^3\pi^2} - \frac{1}{n} \right) \sin\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi} \right] \quad \text{(for odd } n)$$

Contrary to the hint's suggestion, this expression is not zero. For example, for $n = 1$, $a_1 = \frac{1}{\pi}(\frac{8}{\pi^2} - 1 + \frac{2}{\pi}) \neq 0$.

Conclusion: The function is ingeniously constructed so that boundary terms from integration by parts vanish over the central interval $[-\pi/2, \pi/2]$. This is the key "cancellation" visible from the graph. However, for the function as explicitly defined, this simplification does **not** lead to the vanishing of cosine coefficients for odd n. It is likely that the original problem statement contains a typo, as this type of elegant cancellation is a common feature in such exercises.

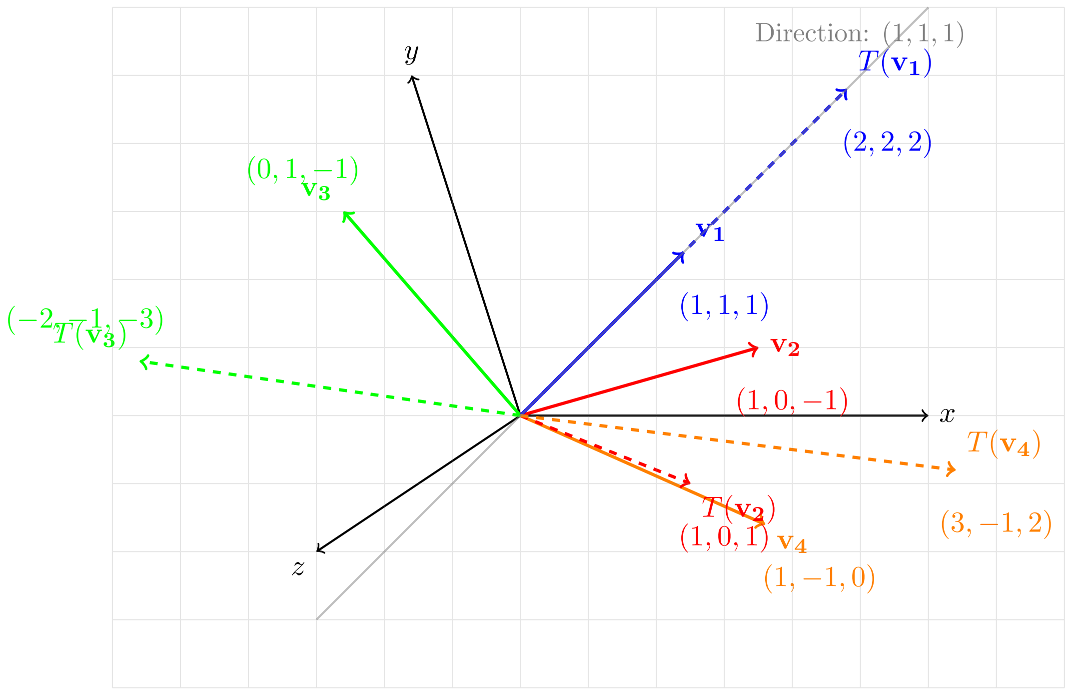
3 Question-3

3.1 Question

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix representation in the standard basis is:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

The diagram below shows the geometric action of this transformation on specific vectors in $\mathbb{R}^3$, with their images under T represented by dashed arrows:



The vectors shown are:

- $\mathbf{v_1} = (1, 1, 1)$ maps to $T(\mathbf{v_1}) = (2, 2, 2)$
- $\mathbf{v_2} = (1, 0, -1)$ maps to $T(\mathbf{v_2}) = (1, 0, 1)$
- $\mathbf{v_3} = (0, 1, -1)$ maps to $T(\mathbf{v_3}) = (-2, -1, -3)$
- $\mathbf{v_4} = (1, -1, 0)$ maps to $T(\mathbf{v_4}) = (3, -1, 2)$

Question: Using the geometric information visible in the diagram:

(a) Determine all eigenvalues of T and their algebraic multiplicities.

(b) For each eigenvalue, find a basis for its eigenspace and determine the geometric multiplicity.

(c) Determine whether T is diagonalizable, and if so, find matrices P and D such that $A = PDP^{-1}$.

3.2 Human Solution

3.2.1 Part (a): Eigenvalue Analysis

To find the eigenvalues, we compute the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

Expanding along the first row:

$$\begin{align}
\det(A - \lambda I) &= (2-\lambda)\det\begin{pmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} - (-1)\det\begin{pmatrix} 1 & 1 \\ 1 & 2-\lambda \end{pmatrix} + 1\det\begin{pmatrix} 1 & -\lambda \\ 1 & -1 \end{pmatrix} \tag{9} \\
&= (2-\lambda)[-\lambda(2-\lambda) - (1)(-1)] + 1[1(2-\lambda) - 1(1)] + 1[1(-1) - (-\lambda)(1)] \tag{10} \\
&= (2-\lambda)(-2\lambda + \lambda^2 + 1) + (2 - \lambda - 1) + (-1 + \lambda) \tag{11} \\
&= (2-\lambda)(\lambda^2 - 2\lambda + 1) + (1-\lambda) + (\lambda - 1) \tag{12} \\
&= (2-\lambda)(\lambda - 1)^2 + 0 \tag{13} \\
&= (2-\lambda)(\lambda - 1)^2 \tag{14}
\end{align}$$

Setting this equal to zero: $(2-\lambda)(\lambda-1)^2 = 0$

Therefore, the eigenvalues are: - $\lambda_1 = 2$ with algebraic multiplicity 1 - $\lambda_2 = 1$ with algebraic multiplicity 2

We can verify this from the diagram: $\mathbf{v_1} = (1,1,1)$ is scaled by exactly factor 2, confirming that $(1,1,1)$ is an eigenvector with eigenvalue 2.

3.2.2 Part (b): Eigenspace Analysis

For $\lambda_1 = 2$:

We solve $(A - 2I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Row reducing:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us: $x - z = 0$ and $y - z = 0$, so $x = z$ and $y = z$.

The eigenspace is $E_2 = \text{span}\{(1,1,1)\}$ with geometric multiplicity 1.

For $\lambda_2 = 1$:

We solve $(A - I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

All rows are identical, giving us: $x - y + z = 0$, so $x = y - z$.

The eigenspace is $E_1 = \text{span}\{(1,1,0), (-1,0,1)\}$ with geometric multiplicity 2.

We can verify: - for $(1,1,0)$: $A(1,1,0)^T = (2-1+0, 1+0+0, 1-1+0)^T = (1,1,0)^T$ - for $(-1,0,1)$: $A(-1,0,1)^T = (-2+0+1, -1+0+1, -1+0+2)^T = (-1,0,1)^T$

3.2.3 Part (c): Diagonalizability

Since the algebraic multiplicity equals the geometric multiplicity for each eigenvalue: - $\lambda_1 = 2$: algebraic multiplicity = geometric multiplicity = 1 - $\lambda_2 = 1$: algebraic multiplicity = geometric multiplicity = 2

The matrix A is diagonalizable.

We can choose:

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the columns of P are eigenvectors: $(1, 1, 1)$ for $\lambda = 2$, and $(1, 1, 0)$, $(-1, 0, 1)$ for $\lambda = 1$.

To find P^{-1}, we compute:

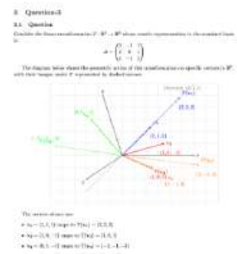
$$P^{-1} = \frac{1}{\det(P)} \text{adj}(P)$$

$\det(P) = 1(1 \cdot 1 - 0 \cdot 0) - 1(1 \cdot 1 - 0 \cdot (-1)) + (-1)(1 \cdot 0 - 1 \cdot (-1)) = 1 - 1 - 1 = -1$

$$P^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

We can verify: $A = PDP^{-1}$.

3.3 Gemini 2.5 Pro Testing Log



Show thinking

Here is the solution to the problem, using the geometric information provided.

The linear transformation is given by the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$. The diagram shows the transformation of four vectors. We can identify eigenvectors and eigenvalues by looking for vectors $\mathbf{v}$ where $T(\mathbf{v})$ is a scalar multiple of $\mathbf{v}$, i.e., $T(\mathbf{v}) = \lambda\mathbf{v}$.

- For $\mathbf{v}_1 = (1, 1, 1)$, the diagram shows $T(\mathbf{v}_1) = (2, 2, 2)$. We can see that $(2, 2, 2) = 2 \cdot (1, 1, 1)$. Thus, $T(\mathbf{v}_1) = 2\mathbf{v}_1$. This means $\lambda_1 = 2$ **is an eigenvalue** with a corresponding eigenvector $\mathbf{v}_1 = (1, 1, 1)$.
- For $\mathbf{v}_2 = (1, 0, -1)$, the diagram shows $T(\mathbf{v}_2) = (1, 0, -1)$. We can see that $(1, 0, -1) = 1 \cdot (1, 0, -1)$. Thus, $T(\mathbf{v}_2) = 1\mathbf{v}_2$. This means $\lambda_2 = 1$ **is an eigenvalue** with a corresponding eigenvector $\mathbf{v}_2 = (1, 0, -1)$.

(a) Eigenvalues and Algebraic Multiplicities

We have identified two eigenvalues, $\lambda = 2$ and $\lambda = 1$. Since T is a transformation in $\mathbb{R}^3$, its characteristic polynomial has a degree of 3, meaning there are three eigenvalues when counted with their algebraic multiplicities.

To find the third eigenvalue and the multiplicities, we compute the characteristic polynomial, $p(\lambda) = \det(A - \lambda I)$:

$$p(\lambda) = \det \begin{pmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

$$p(\lambda) = (2-\lambda)((-\lambda)(2-\lambda) - (-1)) - (-1)(1(2-\lambda) - 1) + 1(-1 - (-\lambda))$$

$$p(\lambda) = (2-\lambda)(\lambda^2 - 2\lambda + 1) + (1-\lambda) + (\lambda - 1)$$

$$p(\lambda) = (2-\lambda)(\lambda - 1)^2$$

Setting the characteristic polynomial to zero, $(2-\lambda)(\lambda-1)^2 = 0$, gives the eigenvalues.

The eigenvalues of T are:

- $\lambda = 2$ with **algebraic multiplicity 1**.
- $\lambda = 1$ with **algebraic multiplicity 2**.

(b) Eigenspaces, Bases, and Geometric Multiplicities

The geometric multiplicity of an eigenvalue is the dimension of its corresponding eigenspace.

For $\lambda = 2$:
The eigenspace E_2 is the null space of $(A - 2I)$. From the diagram, we already know an eigenvector is $\mathbf{v}_1 = (1, 1, 1)$. Since the algebraic multiplicity is 1, the geometric multiplicity must also be 1.

- A basis for the eigenspace E_2 is $\{(1, 1, 1)\}$.
- The **geometric multiplicity of $\lambda = 2$ is 1**.

For $\lambda = 1$:
The eigenspace E_1 is the null space of $(A - I)$.

$$A - I = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

To find the null space, we solve $(A - I)\mathbf{x} = \mathbf{0}$, which simplifies to the single equation $x_1 - x_2 + x_3 = 0$. This equation describes a plane in $\mathbb{R}^3$, which has a dimension of 2. Therefore, the geometric multiplicity of $\lambda = 1$ is 2.

To find a basis, we need two linearly independent vectors that satisfy the equation. From the diagram, we know one such vector is $\mathbf{v}_2 = (1, 0, -1)$ (check: $1 - 0 + (-1) = 0$). We need another one. Let's choose $x_2 = 1, x_3 = 0$, which gives $x_1 = 1$. So, another eigenvector is $(1, 1, 0)$. These two vectors are linearly independent.

- A basis for the eigenspace E_1 is $\{(1, 0, -1), (1, 1, 0)\}$.
- The **geometric multiplicity of $\lambda = 1$ is 2**.

(c) Diagonalizability

A linear transformation is diagonalizable if and only if the algebraic multiplicity equals the geometric multiplicity for all its eigenvalues.

- For $\lambda = 2$: Algebraic Multiplicity (1) = Geometric Multiplicity (1).
- For $\lambda = 1$: Algebraic Multiplicity (2) = Geometric Multiplicity (2).

Since the multiplicities match for all eigenvalues, **the transformation T is diagonalizable**.

We can find matrices P and D such that $A = PDP^{-1}$ where D is a diagonal matrix of the

We can find matrices P and D such that $A = PDP^{-1}$, where D is a diagonal matrix of the eigenvalues and P is the matrix whose columns are the corresponding eigenvectors.

We arrange the eigenvalues on the diagonal of D and the corresponding eigenvectors as columns in P. One possible arrangement is:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Hint 1:
Look at the diagram coordinates again. You said $T(v_2)$ = (1,0,-1), but the diagram actually shows $T(v_2)$ = (1,0,1). This means v_2 is NOT an eigenvector since (1,0,1) ≠ λ(1,0,-1) for any scalar λ.
Re-examine which vectors from the diagram are actually eigenvectors before proceeding with your eigenspace analysis.

Show thinking

Of course. It appears there's an inconsistency in the problem statement. The transformation T described by the matrix A does not match the transformation described for all vectors in the diagram's text. Specifically, the matrix A gives different results for $T(\mathbf{v}_2)$ and $T(\mathbf{v}_4)$ than what is listed.

Let's check:

- $A\mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. The diagram text says $T(\mathbf{v}_2) = (1, 0, 1)$.
- $A\mathbf{v}_4 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. The diagram text says $T(\mathbf{v}_4) = (3, -1, 2)$.

In such cases, the formal definition given by the matrix A is considered the correct one. The diagram is illustrative, and in this case, contains errors. However, we can use the *correct* information from the diagram.

From the diagram, we see that for $\mathbf{v}_1 = (1, 1, 1)$, its image is $T(\mathbf{v}_1) = (2, 2, 2)$. Since $T(\mathbf{v}_1) = 2\mathbf{v}_1$, we can conclude from the geometric information that **$\lambda = 2$ is an eigenvalue** with eigenvector $\mathbf{v}_1 = (1, 1, 1)$. This part is consistent with the matrix A.

We will now proceed with the analysis using the matrix A as the definition of T.

(a) Eigenvalues and Algebraic Multiplicities

To find all eigenvalues and their multiplicities, we compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

`$$p(\lambda) = \det \begin{pmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$$$p(\lambda) = (2-\lambda) \big((-\lambda)(2-\lambda) - (-1) \big) - (-1) \big((1) (2-\lambda) - 1 \big) + (1) \big(-1 - (-\lambda) \big)$$$$p(\lambda) = (2-\lambda)(\lambda^2 - 2\lambda + 1) + (1-\lambda) + (\lambda-1)$$`

$$p(\lambda) = (2 - \lambda)(\lambda - 1)^2$$

Setting $p(\lambda) = 0$, we find the eigenvalues. The roots are:

- $\lambda = 2$ with **algebraic multiplicity 1.**
- $\lambda = 1$ with **algebraic multiplicity 2.**

(b) Eigenspaces, Bases, and Geometric Multiplicities

The geometric multiplicity is the dimension of the corresponding eigenspace (the null space of $A - \lambda I$).

For eigenvalue $\lambda = 2$:
The algebraic multiplicity is 1, so the geometric multiplicity must also be 1. The diagram already gave us the corresponding eigenvector $\mathbf{v}_1 = (1, 1, 1)$.

- A basis for the eigenspace E_2 is $\{(1, 1, 1)\}$.
- The **geometric multiplicity of $\lambda = 2$ is 1.**

For eigenvalue $\lambda = 1$:
We find the null space of $(A - I)$:

$$A - I = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

This requires solving the system $(A - I)\mathbf{x} = \mathbf{0}$, which reduces to the single equation $x_1 - x_2 + x_3 = 0$. This is the equation of a plane, which is a 2-dimensional subspace of $\mathbb{R}^3$.

- The **geometric multiplicity of $\lambda = 1$ is 2.**

To find a basis, we need two linearly independent vectors that satisfy $x_1 - x_2 + x_3 = 0$.

1. Let $x_2 = 1, x_3 = 0$. Then $x_1 = 1$. This gives the eigenvector $(1, 1, 0)$.
2. Let $x_2 = 0, x_3 = -1$. Then $x_1 = -1$. This gives the eigenvector $(-1, 0, -1)$, or equivalently, $(1, 0, -1)$. This is the vector $\mathbf{v}_2$ from the problem, which, according to our matrix calculation ($A\mathbf{v}_2 = \mathbf{v}_2$), is indeed an eigenvector for $\lambda = 1$.

- A basis for the eigenspace E_1 is $\{(1, 1, 0), (1, 0, -1)\}$.

(c) Diagonalizability, P, and D

A transformation is diagonalizable if the algebraic and geometric multiplicities are equal for all eigenvalues.

- For $\lambda = 2$: Alg. Mult. (1) = Geo. Mult. (1). ✅
- For $\lambda = 1$: Alg. Mult. (2) = Geo. Mult. (2). ✅

Since the multiplicities match, T **is diagonalizable**.

We construct the matrices P and D such that $A = PDP^{-1}$.

- D is a diagonal matrix with the eigenvalues.
- P is an invertible matrix whose columns are the corresponding eigenvectors from the bases we found.

Let's order the eigenvalues in D as $(2, 1, 1)$. The columns of P must be the corresponding eigenvectors in the same order.

- Column 1: Eigenvector for $\lambda = 2$.
- Columns 2 & 3: Basis vectors for $\lambda = 1$.

The matrices are:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$