Freelance Portfolio - 1

Advanced Multimodal Mathematics content to train the advanced reasoning of LLMs

Prit Raithatha

1 Question

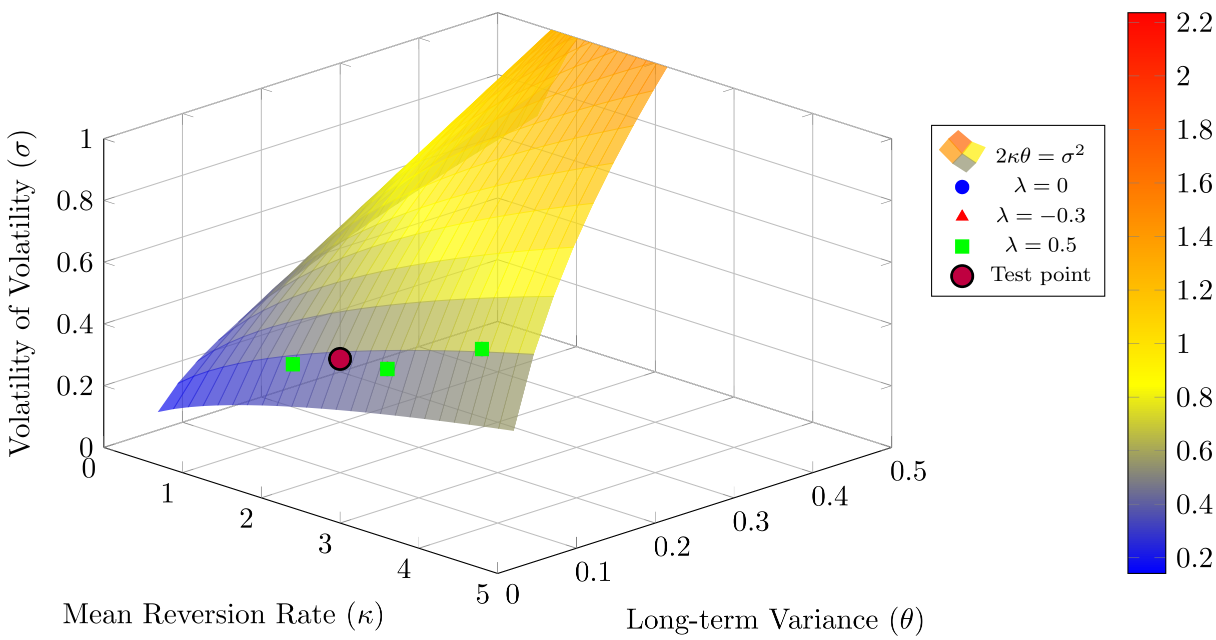

Figure 1: Parameter constraint manifold showing Feller boundary and risk premium effects. The purple point represents $(\kappa, \theta, \sigma) = (2.5, 0.05, 0.45)$.

Consider the Heston stochastic volatility model under the physical probability measure $\mathbb{P}$:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^{(2)}$$

where $(W_t^{(1)}, W_t^{(2)})$ are correlated Brownian motions with $d\langle W^{(1)}, W^{(2)}\rangle_t = \rho dt$.

For risk-neutral pricing, the market price of volatility risk is parameterized as $\lambda_V(t, S, V) = \lambda\sqrt{V}$ where λ is constant. Under the risk-neutral measure $\mathbb{Q}$, the volatility process becomes:

$$dV_t = \kappa^*(\theta^* - V_t)dt + \sigma\sqrt{V_t} d\tilde{W}_t^{(2)}$$

with transformed parameters $\kappa^* = \kappa + \lambda\sigma$ and $\theta^* = \frac{\kappa\theta}{\kappa + \lambda\sigma}$.

(a) Given $\mu = 0.08$ and $r = 0.03$, derive the complete risk-neutral dynamics for both S_t and V_t. Include the Girsanov transformation, Novikov condition verification, and the complete SDE system under $\mathbb{Q}$.

(b) Using Figure 1, analyze the parameter combination $\kappa = 2.5$, $\theta = 0.05$, $\sigma = 0.45$ with $\lambda = -0.3$. Determine:

- Whether the Feller condition $2\kappa\theta \geq \sigma^2$ is satisfied under both $\mathbb{P}$ and $\mathbb{Q}$
- The numerical values of κ^* and θ^*
- Based on the visual manifold, whether this parameter combination ensures PDE well-posedness
- The boundary condition classification at $V = 0$

(c) Derive the complete risk-neutral PDE for European call option pricing $u(t, S, V)$ with strike K and maturity T. Include:

- Multidimensional Itô formula with explicit quadratic variation terms
- The complete second-order PDE with all derivative terms
- Terminal condition and boundary conditions based on part (b)
- Well-posedness verification

2 Solution

2.1 Part (a): Risk-Neutral Measure Construction

Market Prices of Risk

$$\eta_1(t, S, V) = \frac{\mu - r}{\sqrt{V_t}}, \tag{1}$$

$$\eta_2(t, S, V) = \frac{\lambda}{\sigma}. \tag{2}$$

Girsanov Transformation

$$d\tilde{W}_t^{(1)} = dW_t^{(1)} + \frac{\mu - r}{\sqrt{V_t}}\,dt, \tag{3}$$

$$d\tilde{W}_t^{(2)} = dW_t^{(2)} + \frac{\lambda}{\sigma}\,dt. \tag{4}$$

Novikov Condition. For validity, we require

$$\mathbb{E}\left[\exp\left(\tfrac{1}{2}\int_0^T \left(\frac{(\mu - r)^2}{V_s} + \frac{\lambda^2}{\sigma^2} + \frac{2\rho(\mu - r)\lambda}{\sigma\sqrt{V_s}}\right) ds\right)\right] < \infty.$$

This holds when $2\kappa\theta \geq \sigma^2$ (Feller condition), ensuring $V_t > 0$ almost surely.

Risk-Neutral Dynamics. Under $\mathbb{Q}$:

$$dS_t = rS_t\,dt + \sqrt{V_t}\,S_t\,d\tilde{W}_t^{(1)}, \tag{5}$$

$$dV_t = \kappa^*(\theta^* - V_t)\,dt + \sigma\sqrt{V_t}\,d\tilde{W}_t^{(2)}. \tag{6}$$

2.2 Part (b): Parameter Constraint Analysis

Given Parameters: $\kappa = 2.5$, $\theta = 0.05$, $\sigma = 0.45$, $\lambda = -0.3$

Feller Condition Under $\mathbb{P}$:

$$2\kappa\theta = 2(2.5)(0.05) = 0.25, \qquad \sigma^2 = (0.45)^2 = 0.2025.$$

Since $0.25 > 0.2025$, the condition is satisfied.

Risk-Neutral Parameters:

$$\kappa^* = 2.5 + (-0.3)(0.45) = 2.365, \tag{7}$$

$$\theta^* = \frac{2.5 \times 0.05}{2.365} \approx 0.0528. \tag{8}$$

Feller Condition Under $\mathbb{Q}$:

$$2\kappa^*\theta^* = 2(2.365)(0.0528) \approx 0.25 = 2\kappa\theta.$$

Condition satisfied (invariance property).

Visual Analysis. The test point $(2.5, 0.05, 0.45)$ lies above the Feller boundary surface, confirming analytical results and indicating PDE well-posedness.

Boundary Classification. Since $2\kappa^*\theta^* > \sigma^2$, the boundary $V = 0$ is inaccessible. No boundary condition required.

2.3 Part (c): Complete PDE Derivation

Quadratic Variation Terms. Under $\mathbb{Q}$:

$$(dS_t)^2 = V_t S_t^2\, dt, \tag{9}$$

$$(dV_t)^2 = \sigma^2 V_t\, dt, \tag{10}$$

$$dS_t\, dV_t = \rho\sigma V_t S_t\, dt. \tag{11}$$

Itô Formula Application.

$$du = u_t\, dt + u_S\, dS_t + u_V\, dV_t + \tfrac{1}{2}u_{SS}(dS_t)^2 + \tfrac{1}{2}u_{VV}(dV_t)^2 + u_{SV}\, dS_t dV_t.$$

Risk-Neutral PDE. Since $e^{-rt}u(t, S_t, V_t)$ is a $\mathbb{Q}$-martingale:

$$u_t + rSu_S + \kappa^*(\theta^* - V)u_V + \tfrac{1}{2}VS^2u_{SS} + \tfrac{1}{2}\sigma^2 V u_{VV} + \rho\sigma VSu_{SV} - ru = 0.$$

Boundary Conditions.

- Terminal: $u(T, S, V) = (S - K)^+$
- $S \to 0$: $u(t, 0, V) = 0$
- $S \to \infty$: $u(t, S, V) \sim S - Ke^{-r(T-t)}$
- $V = 0$: No condition (inaccessible)
- $V \to \infty$: Growth condition $|u| \leq C(1 + S + V)^p$

Well-Posedness. The PDE is well-posed because:

1. Elliptic: coefficient matrix has positive determinant $\frac{1}{4}\sigma^2 V^2 S^2 (4 - \rho^2) > 0$
2. $V = 0$ boundary inaccessible under Feller condition
3. Parameters satisfy $\kappa^* > 0$, $\theta^* > 0$, $2\kappa^*\theta^* > \sigma^2$