Digital Image Processing

CLASS NOTES

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Preface

These notes are based on the lectures delivered by **Prof. Rajiv Soundararajan (ECE) and Prof. Soma Biswas (EE)** in the course **E9 241 - Digital Image Processing** at Indian Institute of Science (IISc) Bengaluru - Aug Semester 2025. The notes are intended to be a concise summary of the lectures and are not meant to be a replacement for the lectures.

Disclaimer

These notes are not official and may contain errors. Please refer to the official course material for accurate information.

"I cannot guarantee the correctness of these notes. Please use them at your own risk".

- Mahanth Yalla

Contribution

If you find any errors or have any suggestions, please feel free to open an issue or a pull request on this GitHub repository, I will be happy to incorporate them.

Feedback

If you have any feedback or suggestions on the notes, please feel free to reach out to me via social media or through mail mahanthyalla [at] {iisc [dot] ac [dot] in , gmail [dot] com}.

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Chapter 1

Binary Image Processing

1.1 Introduction

Definition 1.1.1: Image Definition

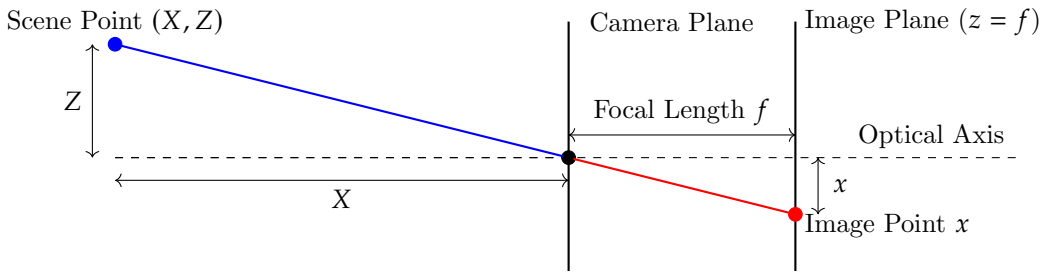
An **image** is a projection of a 3D object onto a 2D surface. This dimensional reduction causes loss of certain 3D information, which is generally very hard to recover.

1.2 Image Formation

The process of **image formation** models how a 3D scene is mapped to a 2D image plane through a camera model. A pinhole camera serves as the simplest abstraction to study this process.

1.2.1 Pinhole Camera Model (2D)

In the 2D case, the relation between a point (X, Z) in the scene and its projection (x) on the image plane is derived by similar triangles:



from the diagram, we have:

$$\frac{x}{f} = \frac{X}{Z} \quad \Rightarrow \quad x = f \cdot \frac{X}{Z}$$

Here, f is the focal length of the pinhole camera.

Note:-

we use Capital letters for 3D coordinates and lowercase for 2D image coordinates and observe that the Z component is lost.

1.2.2 Pinhole Camera Model (3D)

Extending to 3D coordinates (X, Y, Z), the projection onto the image plane gives:

$$x = f \cdot \frac{X}{Z}, \quad y = f \cdot \frac{Y}{Z}$$

This shows how 3D geometry is mapped to 2D via perspective projection.

1.2.3 Homogeneous Coordinates

Homogeneous coordinates are used to express projection as a matrix multiplication:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \frac{1}{Z} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Note:-

Homogeneous coordinates are crucial because they unify perspective projection, translation, and rotation into matrix multiplications.

1.2.4 Intrinsic Matrix

The **intrinsic parameters** capture camera-specific properties such as focal length and principal point offset:

$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

where (f_x, f_y) are focal lengths in pixel units and (c_x, c_y) is the principal point.

1.2.5 Extrinsic Matrix

The **extrinsic parameters** describe the camera's position and orientation in the world:

$$M = [R|t]$$

where R is a 3×3 rotation matrix and t is a 3×1 translation vector.

Note:-

The rotation matrix R can be defined using an angle α in 2D as:

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

This matrix rotates a point by α radians in the plane. In 3D, rotation can be performed about each axis using three matrices:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R_y(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}, \quad R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A general 3D rotation can be represented by multiplying these matrices:

$$R = R_z(\psi) R_y(\phi) R_x(\theta)$$

where θ, ϕ, and ψ are rotation angles about the x, y, and z axes, respectively.

1.2.6 Projection Matrix

The complete mapping from 3D world coordinates to 2D image plane is:

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = KM \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

with scaling factor s.

Claim 1.2.1 Projection Matrix

The projection matrix $P = K@M$ completely defines the mapping from world coordinates to image coordinates, which depends on the intrinsic and extrinsic parameters of the camera. (i.e., focal length, principal point, rotation, and translation vectors.)

Definition 1.2.1: Camera Calibration

The process of estimating the intrinsic and extrinsic parameters of a camera from observed images of a known calibration pattern.

Calibration ensures accurate geometric measurements from images.

1.3 Image Representation

1.3.1 Pixel Values

A digital image is represented as a 2D array of **pixels (picture elements)**, each encoding intensity (grayscale) or color.

Definition 1.3.1: Image & Pixel

An **image** is a 2D array

$$I : \{0, \ldots, M-1\} \times \{0, \ldots, N-1\} \to \{0, \ldots, K-1\}$$

Each element $I[i, j]$ is a **pixel** with **intensity** (gray level) in $\{0, \ldots, K-1\}$, where, 0 represents black and $K-1$ represents white. For a 8 bit storage, $B = 8$, then maximum intensity can be calculated as $K = 2^B = 2^8 = 256$, 0 represents black and 255 represents white.

For normalized intensity, use

$$I_n[i, j] = \frac{I[i, j]}{(K-1)} \in [0, 1]$$

.

1.3.2 Color Spaces

Different **color spaces** represent pixel values differently:

- RGB: additive primary colors.
- HSV: hue, saturation, value (closer to human perception).
- YCbCr: luminance and chrominance separation. (where Y is intensity)

1.3.3 Feature Extraction and Descriptors

Features capture essential information in images such as edges, corners, or textures. **Descriptors** encode features into numerical representations (e.g., SIFT, HOG) for matching and recognition.

1.3.4 Sampling and Quantization

The ideal irradiance signal is sampled on a discrete grid and quantized to a finite set of gray levels.

- **Sampling:** Selecting discrete spatial points to represent an image.
  choose integer pixel sites (i, j); the image becomes $I[i, j]$ on an $M \times N$ grid.
- **Quantization:** Mapping continuous intensity values into discrete levels
  map real intensities to $\{0, 1, \ldots, K-1\}$, e.g., $K = 256$ for 8-bit grayscale.

Binary images: special case $K = 2$ with intensities $\{0, 1\}$ (or $\{0, 255\}$ in 8-bit storage).

Note:-

Undersampling causes aliasing; inadequate quantization leads to loss of detail.

1.3.5 Image Formats

Typical resolutions include 256×256, 512×512, **1920x1080**, etc.As we can calculate the number of Bytes required to store these images, we find:

- For 256×256 images: $256 \times 256 \times 1 = 65,536$ Bytes (assuming 8-bit grayscale).
- For 512×512 images: $512 \times 512 \times 1 = 262,144$ Bytes (assuming 8-bit grayscale).
- For 1920×1080 images: $1920 \times 1080 \times 3 = 6,220,800$ Bytes (assuming 24-bit RGB).

which is nearly 6.3 MB for a full HD image (1920x1080 3-channel RGB image). Hence, image storage can be quite substantial, necessitating efficient compression techniques.

Few common **Formats** include:

- JPEG: lossy compressed format.
  The most common for photographs to save space, as the human eye is less sensitive to high-frequency details. The compression is achieved by discarding some image data, but the benefit is a significantly reduced file size. (e.g. the full HD image (1920x1080) can be compressed from **6.3 MB** to around less than a **1 MB**)
- PNG: lossless compression, supports transparency.
- BMP: uncompressed raster format.
- TIFF: flexible format supporting various compressions.
- GIF: supports animation and transparency (limited color palette).
- WEBP: modern format providing lossy and lossless compression.
- HEIF: high efficiency image format, supports advanced features.
- AVIF: image format based on AV1 compression, offering high quality at smaller file sizes.
- EXR: high dynamic range (HDR) image format, supports wide color gamuts and high bit depths.
- DNG: raw image format for digital photography, preserves original sensor data.
- PPM: portable pixmap format, simple uncompressed color image format.

1.3.6 Binary Images and Thresholding

Definition 1.3.2: Binary Image

A **binary image** is an image that consists of only two colors, typically black and white. Each pixel in a binary image is represented by a single bit, where 0 represents black and 1 represents white.

The simplest method to obtain binary images is **thresholding**:

$$B(x,y) = \begin{cases} 1 & I(x,y) \geqslant T \\ 0 & I(x,y) < T \end{cases}$$

where T is the threshold.

1.3.7 Gray Level Histograms

The histogram of grayscale values is a fundamental tool to analyze and design thresholding algorithms.

Claim 1.3.1 Histogram-based Thresholding

If the histogram shows two well-separated peaks, the optimal threshold lies near the valley between them.

1.4 Histogram of an Image

Definition 1.4.1: Gray-Level Histogram

A **gray-level histogram** is a representation of the distribution of pixel intensities in a grayscale image. It counts the number of pixels for each intensity level, providing insights into the image's contrast and brightness.

Mathematical Formulation Let a grayscale image be

$$I : \{0,\dots,M-1\} \times \{0,\dots,N-1\} \to \{0,\dots,K-1\}$$

. The *histogram* counts occurrences at each gray level, given as a function as

$$H : \{0,\dots,K-1\} \to \{0,\dots,MN\}$$

such that

$$H(k) = \text{no of occurrences of gray level } k$$

where $k \in \{0,\dots,K-1\}$

$$H(k) \;=\; \#\{(i,j) : I[i,j] = k\} \qquad k = 0,\dots,K-1$$

also,

$$\sum_{k=0}^{K-1} H(k) = MN.$$

The *normalized histogram* (PMF) is

$$p(k) = \tfrac{H(k)}{MN}$$

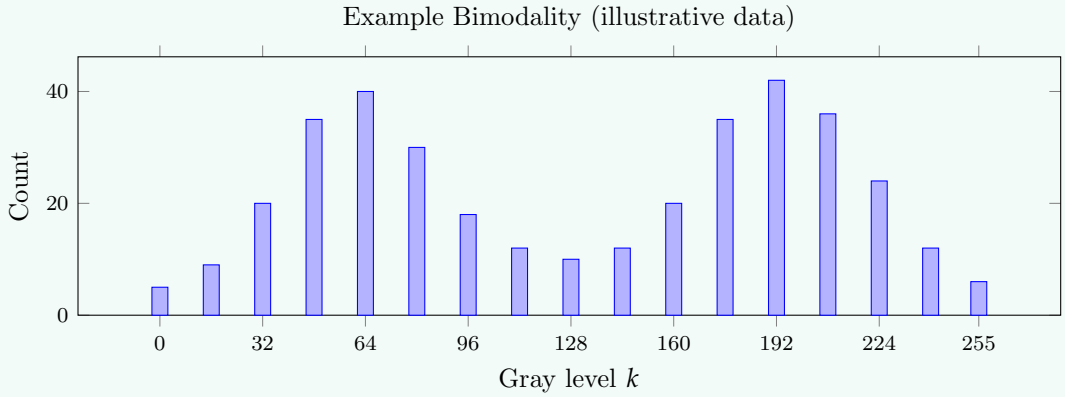
and

$$\sum_{k} p(k) = 1$$

Example 1.4.1 (Interpreting Histogram Shapes)

Given an image whose histogram exhibits a tall peak near intensity 40 (dark shades), and another smaller, broad peak near 200 (bright shades), we deduce the image features a dark background with a bright foreground—ideal for segmentation via thresholding.

- **Dark image:** $p(k)$ concentrated near $k \approx 0$.
- **Bright image:** $p(k)$ concentrated near $k \approx K-1$.
- **Bimodal image:** two peaks (e.g., dark foreground on bright background) $\Rightarrow$ suitable for a *single* global threshold.



Note:-

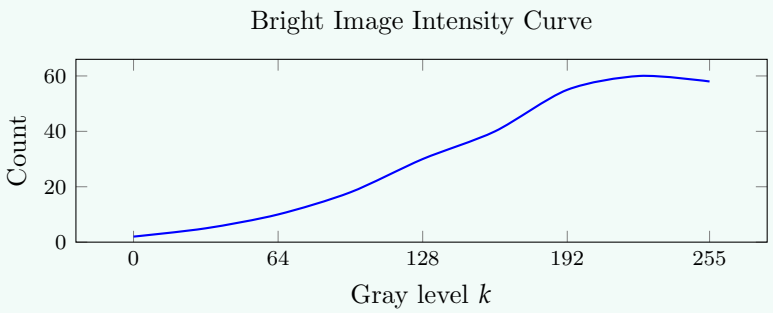
Understanding the histogram profile assists in identifying whether an image is underexposed, overexposed, well-contrasted, or subject to other illumination artifacts.

Types of Histograms and Their Interpretation Different histogram profiles relate directly to the visual impression and underlying properties of an image:

- **Bright Images:** Most pixel values clustered towards the higher end of the gray level range.
- **Dark Images:** Values crowded in the lower end, leading to overall darker visual output.
- **Dual Peak Model:** Exhibits two pronounced peaks, often corresponding to distinct foreground and background regions; common in images fit for binarization.
- **Flat Histogram:** Pixels uniformly distributed across gray levels—rare in natural images, may occur in images heavily corrupted with noise.
- **Equal Histograms:** Result from histogram equalization operations to improve contrast.

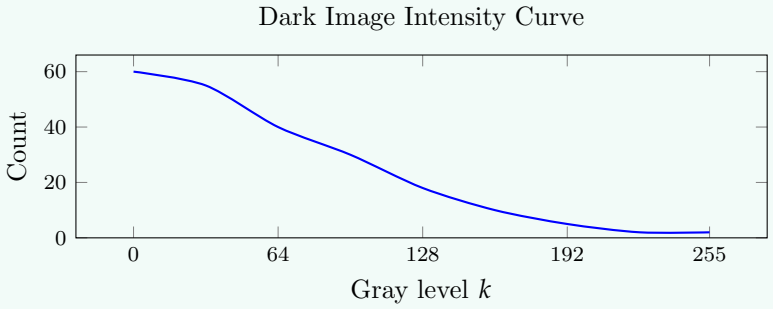
Example 1.4.2 (Histogram Interpretation)

- A **bright image** shows histogram concentrated on high intensity values.



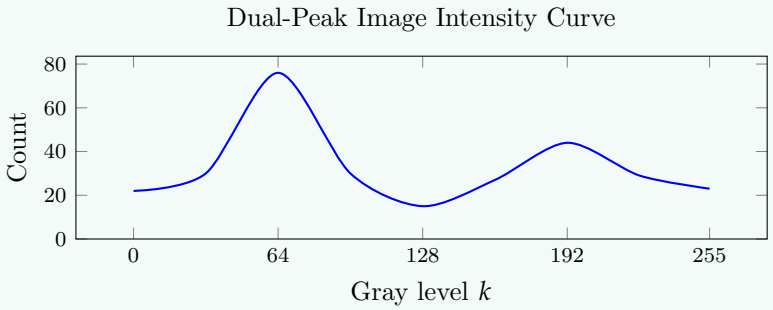
Intensity profile of a bright image: curve shifted towards high gray levels

- A **dark image** shows histogram concentrated on low values.



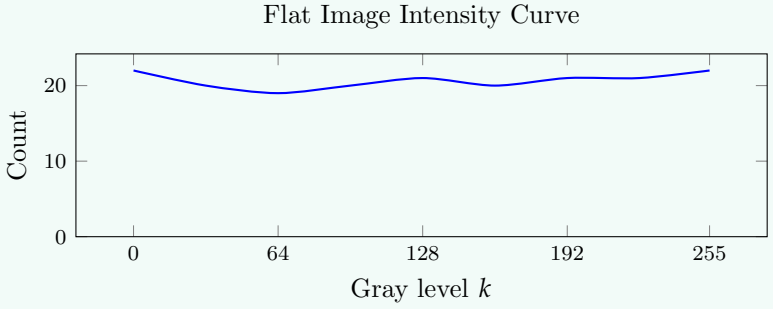
Intensity profile of a dark image: curve shifted towards low gray levels

- A **dual-peak image** indicates presence of both dark (background) and bright (foreground) regions.



Intensity profile of a dual-peak image: two distinct peaks at low and high gray levels

- A **flat histogram** corresponds to uniformly distributed intensities.



Intensity profile of a flat image: uniform distribution across all gray levels

At a Glance:

Histogram Type	Typical Scene	Segmentation Implication
Dark-skewed	Low-light or dark objects	Threshold near lower gray levels
Bright-skewed	Bright background/lighting	Threshold near higher gray levels
Bimodal	Foreground vs. background	Global threshold often effective
Flat/noisy	Low contrast/high noise	Consider contrast stretch or adaptive threshold

1.5 Binarization

Binarization is the process of converting a grayscale image into a binary image, where each pixel is assigned a value of either 0 (black) or 1 (white). This is typically done by applying a threshold to the image, such that pixels above the threshold are set to 1 and those below are set to 0.

$$B(x,y) = \begin{cases} 1 & I(x,y) \geqslant T \\ 0 & I(x,y) < T \end{cases}$$

where T is the threshold.

Choosing an appropriate threshold is crucial for effective binarization. we can use histogram-based methods to select the threshold, by first order statistics or second order statistics (Otsu's method).

1.5.1 Probability and Statistical Prerequisites

Basic Probability Concepts Consider an image histogram with L gray levels $(0, 1, ..., L-1)$. Let p_k denote the normalized probability for gray level k:

$$p_k = \frac{n_k}{N}$$

where n_k is the number of pixels with gray level k, and N is the total pixel count.

Class Probabilities, Means, and Variances

For a given threshold T:
Class Probability

- Probability that the pixel belongs to class 0 (background)

$$\omega_0(T) = \sum_{k=0}^{T} p_k \quad \text{(Probability of class 0 - background - black pixels)}$$

- Probability that the pixel belongs to class 1 (foreground)

$$\omega_1(T) = \sum_{k=T+1}^{L-1} p_k \quad \text{(Probability of class 1 - foreground - white pixels)}$$

Class Means

- Probability that the pixel takes values k given that it belongs to class 0

$$\mu_0(T) = \sum_{k=0}^{T} k \cdot p(k|C_0) \quad \text{(Mean of class 0)}$$

$$\text{where} \quad p(k|C_0) = \frac{p(k, C_0)}{p(C_0)} = \frac{p_k p(C_0|k)}{p(C_0)}$$

$$\text{for} k \in \{0 \dots T\} \quad p(k|C_0) = \frac{p_k}{\omega_0(T)}$$

$$\mu_0(T) = \frac{1}{\omega_0(T)} \sum_{k=0}^{T} k p_k$$

$$\text{also, for} k \in \{T+1 \dots L-1\} \quad p(k|C_0) = 0$$

$$\mu_0(T) = \frac{1}{\omega_0(T)} \sum_{k=0}^{L-1} k p_k$$

- Probability that the pixel takes values k given that it belongs to class 1

$$\mu_1(T) = \sum_{k=T+1}^{L-1} k \cdot p(k|C_1) \quad \text{(Mean of class 1)}$$

$$\text{where} \quad p(k|C_1) = \frac{p(k, C_1)}{p(C_1)} = \frac{p_k p(C_1|k)}{p(C_1)}$$

$$\text{for} k \in \{T+1 \dots L-1\} \quad p(k|C_1) = \frac{p_k}{\omega_1(T)}$$

$$\mu_1(T) = \frac{1}{\omega_1(T)} \sum_{k=T+1}^{L-1} k p_k$$

$$\text{also, for} k \in \{0 \dots T\} \quad p(k|C_1) = 0$$

$$\mu_1(T) = \frac{1}{\omega_1(T)} \sum_{k=0}^{L-1} k p_k$$

- Overall Image Mean

$$\begin{aligned} \mu_T &= \sum_{k=0}^{L-1} k p_k \\ &= \sum_{k=0}^{T} k p_k + \sum_{k=T+1}^{L-1} k p_k \\ &= \mu_0(T)\omega_0(T) + \mu_1(T)\omega_1(T) \end{aligned}$$

Class Variances

- Variance of class 0

$$\begin{aligned} \sigma_0^2(T) &= \sum_{k=0}^{T} (k - \mu_0(T))^2 p(k|C_0) \\ &= \sum_{k=0}^{T} (k - \mu_0(T))^2 \frac{p_k}{\omega_0(T)} \end{aligned}$$

- Variance of class 1

$$\begin{aligned}\sigma_1^2(T) &= \sum_{k=T+1}^{L-1}(k-\mu_1(T))^2 p(k|C_1) \\ &= \sum_{k=T+1}^{L-1}(k-\mu_1(T))^2 \frac{p_k}{\omega_1(T)}\end{aligned}$$

- Total Image Variance

$$\begin{aligned}\sigma^2(T) &= \sum_{k=0}^{L-1}(k-\mu_T)^2 p_k \\ &= \sum_{k=0}^{T}(k-\mu_T)^2 p_k + \sum_{k=T+1}^{L-1}(k-\mu_T)^2 p_k \\ &= \sigma_0^2(T) + \sigma_1^2(T) + \omega_0(T)[\mu_0(T)-\mu_T]^2 + \omega_1(T)[\mu_1(T)-\mu_T]^2\end{aligned}$$

Definition 1.5.1: Within Class Variance $\sigma_w^2(T)$

The variance of pixel intensities within class is given by:

$$\sigma_w^2(T) = \omega_0(T)\sigma_0^2(T) + \omega_1(T)\sigma_1^2(T)$$

Also known as intra-class variance, $\sigma_w^2(T)$ measures the compactness of pixel intensities within each class. **maximizing** $\sigma_w^2(T)$ leads to better class separability.

Definition 1.5.2: Between Class Variance $\sigma_b^2(T)$

The variance of pixel intensities between class is given by:

$$\begin{aligned}\sigma_b^2(T) &= \sigma^2 - \sigma_w^2(T) \\ &= \sigma^2 - \big(\omega_0(T)\sigma_0^2(T) + \omega_1(T)\sigma_1^2(T)\big) \\ &= \omega_0(T)\omega_1(T)\left[\mu_0(T) - \mu_1(T)\right]^2\end{aligned}$$

Also known as inter-class variance, $\sigma_b^2(T)$ measures the separability of pixel intensities between classes. **minimizing** $\sigma_w^2(T)$ leads to better class compactness.

Derivation of $\sigma_b^2(T)$ from definition of $\sigma_w^2(T)$

Derivation of $\sigma_b^2(T)$.

$$\begin{aligned}\sigma^2 &= \sigma_w^2(T) + \sigma_b^2(T) \\ \sigma_b^2(T) &= \sigma^2 - \sigma_w^2(T) \\ &= \sigma^2 - \big(\omega_0(T)\sigma_0^2(T) + \omega_1(T)\sigma_1^2(T)\big)\end{aligned}$$

Now, expand the total variance σ^2. Let $\mu = \omega_0\mu_0 + \omega_1\mu_1$ be the global mean. let's also ignore (T) for simplicity. Then

$$\begin{aligned}\sigma^2 &= \sum_i \omega_i\big(\sigma_i^2 + \mu_i^2\big) - \mu^2 \\ &= \omega_0(\sigma_0^2 + \mu_0^2) + \omega_1(\sigma_1^2 + \mu_1^2) - \mu^2.\end{aligned}$$

Substitute this back:

$$\begin{aligned}\sigma_b^2(T) &= \left[\omega_0(\sigma_0^2+\mu_0^2)+\omega_1(\sigma_1^2+\mu_1^2)-\mu^2\right]-\left(\omega_0\sigma_0^2+\omega_1\sigma_1^2\right)\\ &= \omega_0\mu_0^2+\omega_1\mu_1^2-\mu^2.\end{aligned}$$

Since $\mu = \omega_0\mu_0 + \omega_1\mu_1$, we expand:

$$\begin{aligned}\sigma_b^2(T) &= \omega_0\mu_0^2+\omega_1\mu_1^2-\left(\omega_0\mu_0+\omega_1\mu_1\right)^2\\ &= \omega_0\mu_0^2+\omega_1\mu_1^2-\left(\omega_0^2\mu_0^2+\omega_1^2\mu_1^2+2\omega_0\omega_1\mu_0\mu_1\right)\\ &= \omega_0(1-\omega_0)\mu_0^2+\omega_1(1-\omega_1)\mu_1^2-2\omega_0\omega_1\mu_0\mu_1\\ &= \omega_0\omega_1\mu_0^2+\omega_0\omega_1\mu_1^2-2\omega_0\omega_1\mu_0\mu_1\\ &= \omega_0\omega_1(\mu_0-\mu_1)^2.\end{aligned}$$

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1.5.2 Thresholding using First Order statistics

Thresholding using first-order statistics involves selecting a threshold based on the mean or median intensity values of the image histogram. The basic idea is to compute a global threshold that separates the foreground from the background by analyzing the intensity distribution.

Mean Thresholding The threshold is set to the mean intensity value of the image.

$$T = \frac{1}{N}\sum_{x,y} I(x,y)$$

where N is the total number of pixels.

Median Thresholding The threshold is set to the median intensity value, which is more robust to outliers.

$$T = \text{median}(I(x,y))$$

These methods are simple and computationally efficient but may not perform well for images with complex backgrounds or varying illumination.

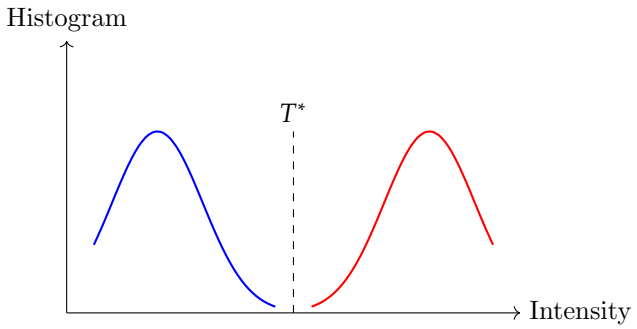
1.5.3 Otsu's Binarization

Otsu's Binarization method uses Second Order Statistics to find the optimal threshold for image binarization.

Definition 1.5.3: Otsu's Binarization

The process of determining the threshold T^* that maximizes the inter-class variance between foreground and background, thus optimally segmenting a bimodal histogram image.

Optimization Criteria Otsu's method selects a threshold T^* to partition the image pixels into two classes (often foreground and background) so that the variance between these classes (**inter-class variance**) is maximized, or equivalently, the variance within each class (**intra-class variance**) is minimized. These criteria are mathematically dual—maximizing $\sigma_b^2(T)$ is identical to minimizing $\sigma_w^2(T)$ since their sum is the total variance σ^2.



A typical bimodal histogram: T^* chosen to best separate two peaks.

Claim 1.5.1 Equivalence of Maximization and Minimization Criteria

Maximizing the between-class variance $\sigma_b^2(T)$ is equivalent to minimizing within-class variance $\sigma_w^2(T)$, as

$$\arg\max_T \sigma_b^2(T) = \arg\min_T \sigma_w^2(T)$$

since σ^2 is fixed for a given image histogram.

Note:-

In practice, Otsu's method typically maximizes $\sigma_b^2(T)$, but formulations based on minimizing $\sigma_w^2(T)$ yield the same optimal threshold T^*.

1.5.4 Method 1: Maximization of Inter-Class Variance

Threshold T^* is selected to maximize the between-class variance $\sigma_b^2(T)$.

$$T^* = \arg\max_T \sigma_b^2(T) = \arg\max_T \left[\omega_0(T)\omega_1(T)\left(\mu_0(T) - \mu_1(T)\right)^2\right]$$

The algorithm for Otsu's binarization by maximizing $\sigma_b^2(T)$ is as follows:

1. Compute the histogram and probabilities $p_i = n_i/N$ for each gray level i.
2. For each possible threshold T (from gray level 1 to $L-2$):
   - Calculate class weights (probabilities): $\omega_0(T)$ and $\omega_1(T)$.
   - Calculate class means: $\mu_0(T)$ and $\mu_1(T)$.
   - Compute between-class variance:

$$\sigma_b^2(T) = \omega_0(T)\,\omega_1(T)\left[\mu_0(T) - \mu_1(T)\right]^2$$

3. Find the threshold T^* that maximizes $\sigma_b^2(T)$.

Example 1.5.1 (Numerical Example)

Suppose an image has normalized histogram values: $p_0 = 0.4$, $p_1 = 0.2$, $p_2 = 0.2$, $p_3 = 0.2$ Try threshold $T = 1$:

$$\omega_0(1) = p_0 + p_1 = 0.6$$
$$\omega_1(1) = 0.4$$
$$\mu_0(1) = \frac{0 \cdot 0.4 + 1 \cdot 0.2}{0.6} = 0.333$$
$$\mu_1(1) = \frac{2 \cdot 0.2 + 3 \cdot 0.2}{0.4} = 2.5$$

Now,

$$\sigma_b^2(1) = 0.6 \times 0.4 \times (0.333 - 2.5)^2 \approx 0.6 \times 0.4 \times 4.694 \approx 1.126$$

similarly for $T = 2$:

$$\begin{aligned}
\omega_0(2) &= p_0 + p_1 + p_2 = 0.8 \\
\omega_1(2) &= 0.2 \\
\mu_0(2) &= \frac{0 \cdot 0.4 + 1 \cdot 0.2 + 2 \cdot 0.2}{0.8} = 0.75 \\
\mu_1(2) &= \frac{3 \cdot 0.2}{0.2} = 3
\end{aligned}$$

Now,

$$\sigma_b^2(2) = 0.8 \times 0.2 \times (0.75 - 3)^2 \approx 0.8 \times 0.2 \times 5.0625 \approx 0.81$$

T	$\omega_0(T)$	$\omega_1(T)$	$\mu_0(T)$	$\mu_1(T)$	$\sigma_b^2(T)$
1	0.6	0.4	0.3333	2.5	**1.1267**
2	0.8	0.2	0.75	3	0.81

Maximum σ_b^2 occurs at $T = 1$, $\sigma_b^2(1) \approx$ **1.1267**.

1.5.5 Method 2: Minimization of Within-Class Variance

Threshold T^* is selected to minimize the within-class variance $\sigma_w^2(T)$.

$$T^* = \arg\min_T \sigma_w^2(T) = \arg\min_T \left[\omega_0(T)\sigma_0^2(T) + \omega_1(T)\sigma_1^2(T)\right]$$

This approach explicitly minimizes intra-class variance $\sigma_w^2(T)$.

1. For each threshold T, calculate:

$$\sigma_w^2(T) = \omega_0(T)\sigma_0^2(T) + \omega_1(T)\sigma_1^2(T)$$

   where $\sigma_k^2(T)$ are variances for classes $k = 0, 1$, based on histogram statistics within each class.

2. Find T^* that minimizes $\sigma_w^2(T)$.

Claim 1.5.2 Proof of Equivalence

Since $\sigma^2 = \sigma_w^2(T) + \sigma_b^2(T)$ for all T, maximizing $\sigma_b^2(T)$ is the same as minimizing $\sigma_w^2(T)$.
Now, because σ^2 is independent of T, for two thresholds T_1, T_2 we have

$$\sigma_b^2(T_1) > \sigma_b^2(T_2) \quad \Longleftrightarrow \quad \sigma_w^2(T_1) < \sigma_w^2(T_2),$$

since subtracting the same constant σ^2 from both sides reverses neither inequalities nor signs but simply gives

$$\sigma^2 - \sigma_b^2(T_1) < \sigma^2 - \sigma_b^2(T_2) \quad \Longleftrightarrow \quad \sigma_w^2(T_1) < \sigma_w^2(T_2).$$

Therefore

$$\arg\max_T \sigma_b^2(T) \;=\; \arg\min_T \sigma_w^2(T)$$

Equivalently, using the explicit form

$$\sigma_b^2(T) = \omega_0(T)\,\omega_1(T)\big(\mu_0(T) - \mu_1(T)\big)^2$$

maximizing this expression over T yields the same T that minimizes $\sigma_w^2(T)$ because

$$\sigma_w^2(T) = \sigma^2 - \sigma_b^2(T).$$

Hence any T that increases the between-class separation $\omega_0\omega_1(\mu_0-\mu_1)^2$ necessarily reduces the within-class scatter and vice versa.
Thus,

$$\text{Maximize } \sigma_b^2(T) \Leftrightarrow \text{Minimize } \sigma_w^2(T)$$

Note:-

- The identity relies on σ^2 being fixed (global histogram is fixed). If the data used to compute σ^2 changed with T, the equivalence would not hold.
- Since $\omega_0, \omega_1 \geq 0$ and μ_0, μ_1 are real, $\sigma_b^2(T) \geq 0$. Maximizing σ_b^2 therefore finds the threshold giving the largest possible separation between class means (weighted by class sizes), which is precisely the threshold that minimizes the residual within-class variance.

1.5.6 Algorithm: Otsu's Binarization (most used)

1. Compute image histogram.
2. For each threshold T:
   - Partition pixels into two classes.
   - Compute $\omega_k(T)$, $\mu_k(T)$, $\sigma_k^2(T)$ as needed.
   - Compute either $\sigma_b^2(T)$ or $\sigma_w^2(T)$.
3. Determine T^* by maximizing $\sigma_b^2(T)$ or minimizing $\sigma_w^2(T)$.
4. Binarize the image: pixels $\leqslant T^*$ are assigned to class 0; others to class 1.

Note:-

Otsu's method is most effective for bimodal histograms, but may be suboptimal for multi-modal or unimodal histograms. Variations and generalizations (multi-level, local thresholding) exist for more complex cases.

1.6 Connected Component Analysis

A *connected component* in a binary image is a maximal group of adjacent foreground pixels, where adjacency is defined by connectivity (typically, 4-connected or 8-connected).

Definition 1.6.1: Connected Component

A subset of foreground pixels (p, q) such that for any pixel p in the subset, there exists a sequence of pixels in the subset linking p to q via connected neighbors.

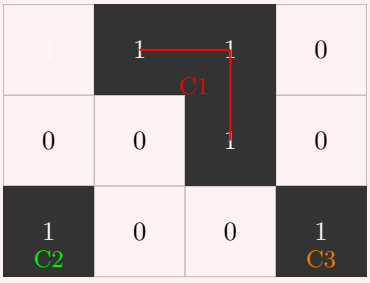
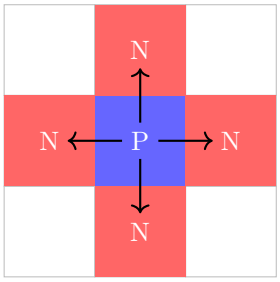


Illustration of 4-connected components in a binary image

Connectivity

- **4-Connectivity:** Each pixel is connected to its immediate horizontal and vertical neighbors.
  Neighbors for a pixel (x, y) are

$$\{(x-1, y), (x+1, y), (x, y-1), (x, y+1)\}$$



Plus-shaped 4-neighborhood of pixel P

- **8-Connectivity:** Includes diagonal neighbors as well.
  Neighbors for a pixel (x, y) are

$$\{(x-1, y), (x+1, y), (x, y-1), (x, y+1), (x-1, y-1), (x-1, y+1), (x+1, y-1), (x+1, y+1)\}$$

Definition 1.6.2: Path

Path from (x, y) to (s, t) is a sequence of pixels $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$ where $(x_0, y_0) = (x, y)$, $(x_n, y_n) = (s, t)$, and each consecutive pair of pixels are connected according to the chosen connectivity (4 or 8).

Definition 1.6.3: Connectedness

the pixels (x, y) and (s, t) are connected if there exists a path between them consisting entirely of same polarity pixels (all foreground or all background).

A **4-Connected Component** is a set of pixels that are connected to each other through 4-connectivity paths.

1.6.1 Extraction of Connected Components

Claim 1.6.1 Labeling Algorithms

Every pixel belonging to a connected component can be assigned a unique label using the connected component labeling algorithm.

Definition 1.6.4: Region index array

Region index array is a matrix of the same size as the binary image, where each pixel is assigned a label (integer) corresponding to the connected component it belongs to. Background pixels are typically labeled as 0.

The goal is to assign distinct labels to all connected components in a binary image.

Labeling Algorithm

1. Scan the image pixel by pixel in a defined order (e.g., left-to-right, top-to-bottom).
2. For each foreground pixel, examine its neighbors (depending on the connectivity rule).
3. Assign a new label if no neighbor is labeled; otherwise assign the neighbor's label.

4. If multiple labeled neighbors, assign one and record equivalences.
5. After complete pass, resolve label equivalences: assign unique labels to all connected components.

Example 1.6.1 (Connected Component Labeling)
Say, Given the binary image (rows indexed from 1):

$$B = \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

We use *4-connectivity* and the standard two-pass algorithm (scan row-major). Denote left neighbour by N_L and up neighbour by N_U. Maintain a provisional label counter L (start $L \leftarrow 1$) and an equivalence table (for union-find).

First pass (scan, assign provisional labels and record equivalences).

Scan order (only foreground pixels shown). For each foreground pixel we inspect N_L, N_U and apply the standard rules:

- if both neighbours background, $\Rightarrow$ assign new label L, increment L;
  if $I(i-1,j) = I(i,j-1) = 0$ then, $R(i,j) = L$ and $L++$.
- if only one neighbour is labelled $\Rightarrow$ copy that label;
  if $I(i-1,j) = M$ or $I(i,j-1) = M$ then, $R(i,j) = M$.
- if both neighbours labelled with different labels $\Rightarrow$ assign one of them and record their equivalence.
  if $I(i-1,j) = M$ and $I(i,j-1) = N$ then, $R(i,j) = M$ and record equivalence $M \sim N$.

Step	Pixel	N_L	N_U	Action	Result / equivalences
1	$(1,2)$	0	none	both background $\Rightarrow$ new label	assign 1. $L \leftarrow 2$.
2	$(1,3)$	1	none	left labelled	assign 1.
3	$(2,3)$	0	1	up labelled	assign 1.
4	$(3,1)$	none	0	both background	assign 2. $L \leftarrow 3$.
5	$(3,4)$	0	0	both background	assign 3. $L \leftarrow 4$.

Note: no step produced conflicting labels from N_L and N_U, hence no equivalences were recorded (equivalence sets remain $\{1\},\{2\},\{3\}$).
Provisional labelled image (Region) after first pass:

$$R^{(1)} = \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{matrix}$$

Equivalence resolution (union-find / flattening).
Equivalence table (pairs recorded during first pass): none in this example. Therefore canonical representatives are

$$\text{repr}(1) = 1, \quad \text{repr}(2) = 2, \quad \text{repr}(3) = 3.$$

Second pass (replace provisional labels by their canonical representatives).
Apply

$$R_{\text{final}}(x,y) \leftarrow \text{repr}\big(R^{(1)}(x,y)\big)$$

Since representatives are identity, the final labelled image is

$$R_{\text{final}} = \begin{matrix} 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ \mathbf{2} & 0 & 0 & \mathbf{3} \end{matrix}$$

Components (explicit pixel sets).

$$C_1 = \{(1,2),(1,3),(2,3)\}, \quad C_2 = \{(3,1)\}, \quad C_3 = \{(3,4)\}.$$

Note:-

Correctly extracting connected components is crucial for counting objects, segmenting regions, and preparing for further analysis such as shape feature extraction.

Bibliography