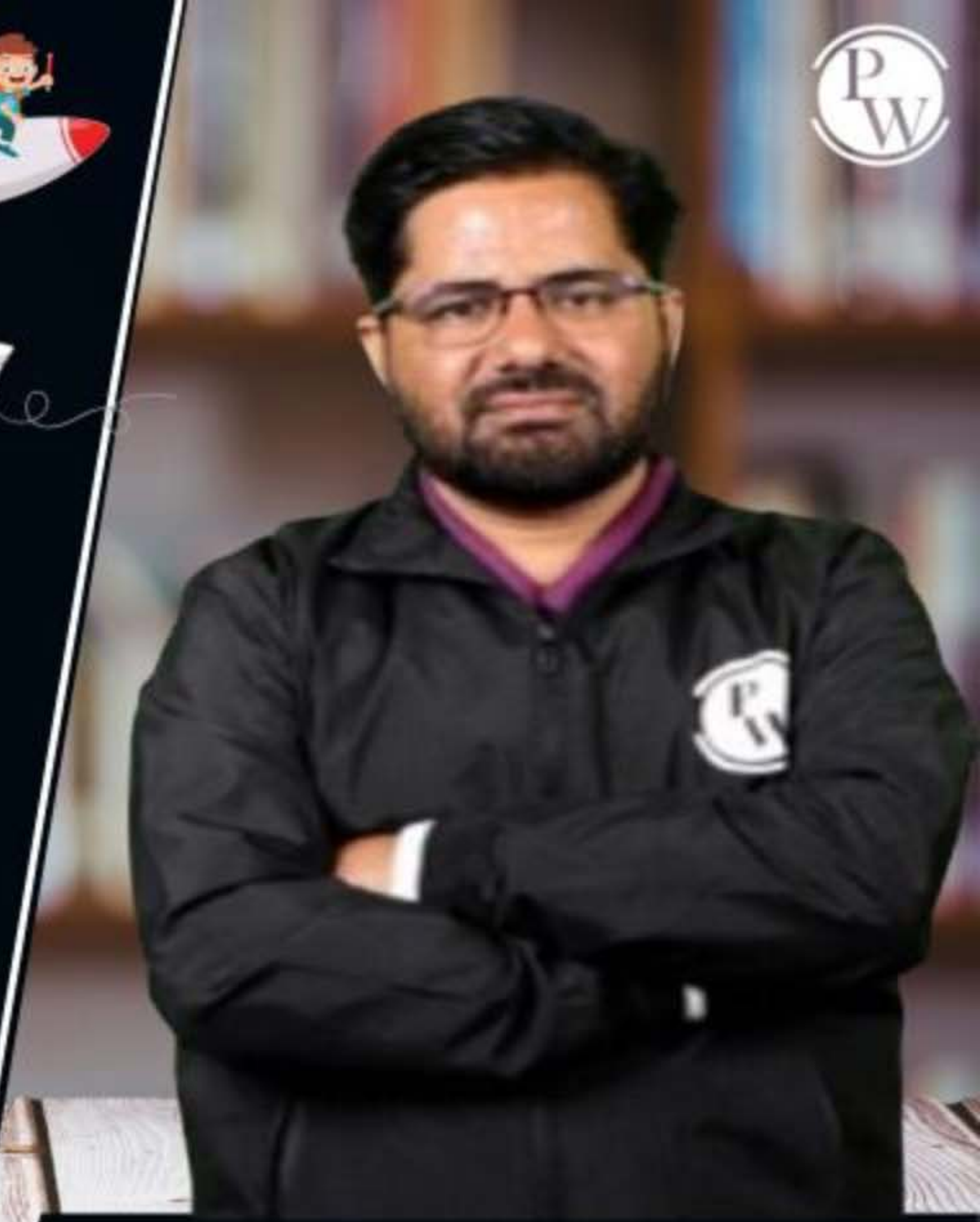
DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

(NOT for CS/IT)

Linear Algebra - II

Lecture No. 01



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Recap of previous lecture

Topic

EIGEN VALUES & EIGEN VECTORS

Topics to be Covered

Topic

QUADRATIC FORM

LINEAR ALGEBRA (PART-2)

1. QUADRATIC FORM
2. Singular Value Decomposition (S.V.D)
3. VECTOR SPACE (BASIS, SUBSPACE, SPAN, DIMENSION)
4. PROJECTION MATRIX
5. PARTITION MATRIX

General Observation: Consider a vector $X = \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3\times 1}$ & let $A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$

Let us Calculate: $X^T A X = \begin{bmatrix} - & - & - \end{bmatrix}_{1\times 3} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3} \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3\times 1}$

$$Q(X) = \begin{bmatrix} - & - & - \end{bmatrix}_{1\times 3} \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3\times 1} = \begin{bmatrix} - \end{bmatrix}_{1\times 1} = \text{Scalar Quantity}$$

Again Considering $X = \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}_{4\times 1}$, $A = \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times 4}$ then

$$Q(X) = X^T A X = \begin{bmatrix} - & - & - & - \end{bmatrix}_{1\times 4} \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times 4} \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}_{4\times 1} = \begin{bmatrix} - & - & - & - \end{bmatrix}_{1\times 4} \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}_{4\times 1} = \begin{bmatrix} - \end{bmatrix}_{1\times 1} = \text{Scalar Q.}$$

this Concept of converting X into a value is possible when $A =$ Symm Mat.

Quadratic Form → "It is a process of assigning value to any Column Mat"

& it is denoted as $Q(X)$ and is defined as

$$\boxed{Q(X) = X^T A X} = [\quad]_{1\times 1} = \text{Constant/scalar Value.}$$

where A is called Matrix associated with Quad. Form.

Note ① Here A is always a Symmetric Mat.

② if $A = I$ then $Q(X) = ? = X^T A X = X^T I X = X^T X = \|X\|^2$ where $\|X\| = \sqrt{X^T X}$

i.e if $A = I$ then $\boxed{Q(X) = (\text{NORM } X)^2}$ or $\boxed{Q(X) = \|X\|^2} = x_1^2 + x_2^2 + \dots + x_n^2$

eg if $A = \begin{bmatrix} 7 & 4 \\ 4 & 5 \end{bmatrix}_{2\times 2}$ then find Quad Form of $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$

$$Q(X) = X^T A X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} (7x_1 + 4x_2) & (4x_1 + 5x_2) \end{bmatrix}_{1\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

$$= 7x_1^2 + 4x_2x_1 + 4x_1x_2 + 5x_2^2$$

$$= 7x_1^2 + 8x_1x_2 + 5x_2^2$$

(ii) if $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ then $Q(X) = ?$ $= 7(3)^2 + 8(3)(-1) + 5(-1)^2 = 63 - 24 + 5 = 44$ Ans

Type I

Q. Find the Quadratic Forms of (1) $A=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$, (2) $A=\begin{bmatrix}3 & -1\\ -1 & 4\end{bmatrix}$

Let $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ then

(1) $Q(X)=X^TAX=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$=\begin{bmatrix}x_1 & -2x_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$=\begin{bmatrix}(x_1^2-2x_2^2)\end{bmatrix}_{1\times 1}$

$Q(X) = x_1^2-2x_2^2$

(2) $Q(X)=X^TAX$

$=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}3 & -1\\ -1 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$=\begin{bmatrix}(3x_1-x_2) & (-x_1+4x_2)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$=\begin{bmatrix}(3x_1^2-x_2x_1-x_1x_2+4x_2^2)\end{bmatrix}_{1\times 1}$

$=3x_1^2-2x_1x_2+4x_2^2$

Q. Find the Quadratic Forms of ① $A=\begin{bmatrix}2 & 0 & -1\\ 0 & 4 & 3\\ -1 & 3 & 1\end{bmatrix}$, ② $A=\begin{bmatrix}1 & -3 & 0 & 1\\ -3 & 0 & 2 & 4\\ 0 & 2 & 3 & 5\\ 1 & 4 & 5 & -3\end{bmatrix}_{4\times 4}$

① Let $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ then $Q(X)=X^TAX$

is $Q(X)=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}2 & 0 & -1\\ 0 & 4 & 3\\ -1 & 3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

$=\begin{bmatrix}(2x_1-x_3) & (4x_2+3x_3) & (-x_1+3x_2+x_3)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

$=\begin{bmatrix}(2x_1^2-x_3x_1)+(4x_2^2+3x_3x_2)+(-x_1x_3+3x_2x_3+x_3^2)\end{bmatrix}_{1\times 1}$

$=2x_1^2+4x_2^2+x_3^2-2x_1x_3+6x_2x_3$

② $X=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$ so $Q(X)=X^TAX$

$=\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}\begin{bmatrix}- & - & - & -\\ - & - & - & -\\ - & - & - & -\\ - & - & - & -\end{bmatrix}_{4\times 4}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

$=x_1^2+3x_3^2-3x_4^2-6x_1x_2+2x_1x_4$
$+4x_2x_3+8x_2x_4$
$+10x_3x_4$

PQ: if $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ -1 & 1 & 5 & 0 \\ 3 & 5 & 3 & 1.5 \\ 4 & 0 & 1.5 & -4 \end{bmatrix}_{4\times 4}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ then $Q(X) = ?$

Sol:

$$Q(X) = X^T A X$$

$$= 2x_1^2 + x_2^2 + 3x_3^2 - 4x_4^2 - 2x_1x_2 + 6x_1x_3 + 8x_1x_4$$

$$10x_2x_3 + 0.x_2x_4 + 3x_3x_4$$

Type 2: Q. Find the Symmetric Matrix that is associated with the following Quadratic forms;

① $x_1^2 - 5x_1x_2 + 3x_2^2$

② $x_1^2 + 4x_3^2 - 2x_1x_2 + 5x_2x_4$

③ $x^2 + 2y^2 - 3z^2 + 4xy - 5yz + 3zx$

① $Q(X) = 1.x_1^2 - \frac{5}{2}x_1x_2 - \frac{5}{2}x_2x_1 + 3x_2^2$

$= X^T A X$

where $A = \begin{bmatrix} 1 & -5/2 \\ -5/2 & 3 \end{bmatrix}$ = symm

② $Q(X) = X^T A X$

$= 1.x_1^2 + 0.x_2^2 + 4x_3^2 + 0x_4^2$

$-2x_1x_2 + 0.x_1x_3 + 0.x_1x_4$

$+ 0.x_2x_3 + 5x_2x_4 + 0.x_3x_4$

where $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5/2 \\ 0 & 0 & 4 & 0 \\ 0 & 5/2 & 0 & 0 \end{bmatrix}_{4\times 4}$

(3) $Q(X) = x^2 + 2y^2 - 3z^2 + 4xy - 5yz + 3zx$

$$= 1 \cdot x_1^2 + 2x_2^2 - 3x_3^2 + 4x_1x_2 - 5x_2x_3 + 3x_3x_1$$

$$= X^T A X$$

where $A = \begin{bmatrix} 1 & 2 & 3/2 \\ 2 & 2 & -5/2 \\ 3/2 & -5/2 & -3 \end{bmatrix}_{3\times 1}$

Q: if Quad. form of X is given as $Q(x) = a^2 - 2b^2 + 3c^2 + d^2 - ab + 4bc - 2cd + 2ad$ then find symm Mat associated with above Q. Form.

$$Q(x) = x_1^2 - 2x_2^2 + 3x_3^2 + x_4^2 - x_1x_2 + 4x_2x_3 - 2x_3x_4 + 2x_1x_4$$

$$A = \begin{bmatrix} 1 & -1/2 & 0 & 1 \\ -1/2 & -2 & 2 & 0 \\ 0 & 2 & 3 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

TYPE 3 Q: If Quadratic form of any vector x is given as;

(1) $x_1^2 - 4x_2^2 + 3x_1x_2$ then evaluate $Q\begin{bmatrix}1\\-3\end{bmatrix} = ?$ $= (1)^2 - 4(-3)^2 + 3(1)(-3)$

$= 1 - 36 - 9 = -44$

(2) $x^2 + 4y^2 - z^2 + 2zx + 4xy$ then evaluate $Q\begin{bmatrix}1\\2\\-3\end{bmatrix} = ?$

$$Q(x) = 1^2 + 4(2)^2 - (-3)^2 + 2(-3)(1) + 4(1)(2)$$

$$= 1 + 16 - 9 - 6 + 8$$

$$= 10$$

Relation b/w Quadratic form & Norm of X →

Let $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $\|X\| = \sqrt{X^T X} = \sqrt{(x_1^2 + x_2^2 + x_3^2)}$ i.e $\boxed{X^T X = \|X\|^2}$

Let us take a symm Mat of the type $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I$

& we will try to calculate $Q(X) = ?$

$$Q(X) = X^T A X = X^T . I . X = X^T X = \|X\|^2$$

Conclusion: i.e If $A = I$ then $\boxed{Q(X) = \|X\|^2}$.

eg if $A = I$ & $X = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ then $Q(X) = ? = 3^2 + (-2)^2 + 1^2 = 14$ Ans

Maximum and Minimum Values of Quadratic form $Q(X)$ {when $\|X\|=1$}

let $A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times3}$ = Symm Mat

λ_{min}, X_1 let $H_1 = \frac{X_1}{\|X_1\|}$

λ, X_2

λ_{Max}, X_3 $H_3 = \frac{X_3}{\|X_3\|}$

Let Minimum eigen Value of A λ_{min}

& Max " " " λ_{max}

then Min Value of $Q(X) = \lambda_{min}$

& Max " of $Q(X) = \lambda_{max}$

i.e $\boxed{\lambda_{min} \le Q(X) \le \lambda_{max}}$

& these Min & Max Values of Quadratic form occurs at unit E.Vector of λ_{min} & λ_{max} Resp.

Verification let $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be any vector s.t. $\boxed{\|X\| = 1}$

then min & max value of Quad. form of $X = ?$

Sol: $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda_1 = 2$, $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $N_1 = \dfrac{X_1}{\|X_1\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 6$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $N_2 = \dfrac{X_2}{\|X_2\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(Already Calculated in Previous chapter)

$Q(X) = X^T A X = \cdots\cdots = 4x_1^2 + 4x_1x_2 + 4x_2^2$

Now, $Q(N_1) = N_1^T A N_1 = Q\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 4\left(\frac{1}{2}\right) + 4\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{2}\right) = 2 = \lambda_{min}$

Again $Q(N_2) = N_2^T A N_2 = Q\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{2}\right) = 6 = \lambda_{Max}$

ie $\boxed{2 \le Q(X) \le 6}$

Now let us take another Random vector say $X = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\|X\| = 1$

$$Q(X) = 4x_1^2 + 4x_1x_2 + 4x_2^2$$

$$\Rightarrow Q\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = 4\left(\frac{9}{25}\right) + 4\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + 4\left(\frac{16}{25}\right)$$

$$= \frac{36+48+64}{25} = \frac{148}{25} = 5.92$$

Hence Verified

TYPE-4

Q. for the Matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the Max and Minimum value of $X^T A X$ will be ? subject to the constraint $X^T X = 1$ $\Rightarrow \|X\| = 1$ given.

Already discussed.

Q: For the Matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, Max and minimum values of $\boxed{X^T A X}$ lies in the interval ? (Provided $\|X\| = 1$)

MSQ

$Q(X)$

(a) $[1 \; 6]$

(b) $[-5 \; 5]$

(c) $[1 \; 7]$

(d) $[2 \; 3]$

Sol: $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda = 2, \; Q_{min} = 2$

$\lambda = 6, \; Q_{Max} = 6$

Q: if for any vector X, Quadratic form associated with Matrix A is given as $Q(X) = \boxed{2x^2 - 2xy + 2y^2}$ then sum of the squares of minimum & Maximum value of Q(X) will be ________ subject to the constraints $X^T X = 1$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{matrix} = 1 \\ = 1 \end{matrix}$$

By property of E Value, one E. Value of A is $\lambda = 1$

Now $Tr(A) = 4$

$\lambda_1 + \lambda_2 = 4$

$1 + \lambda_2 = 4$

$\lambda_2 = 3$

i.e. $A \begin{cases} \lambda = 1, & Q_{min} = 1 \\ \lambda = 3, & Q_{Max} = 3 \end{cases}$

Hence Ans $= 1^2 + 3^2 = 10$ Ans

Q: If $Q(x) = \boxed{4x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2}$ is the Quadratic form of any unit vector x then sum of minimum & Maximum Value of $Q(x)$ is ?

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

$\lambda = 1 = Q_{min}$

$\lambda = 3$

$\lambda = 5 = Q_{max}$

So Ans $= 1 + 5 = 6$

(a) 6

(b) 4

(c) 5

(d) 9

(M-II) using the concept of Partition Matr.

$$A = \left[\begin{array}{cc|c} 4 & -1 & 0 \\ -1 & 4 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

$|A| = |A_1| \cdot |A_2|$

$= (16-1)(1) = 15$

E.Values of A = E.Values of $A_1 = 3 \,\&\, 5$

& " of $A_2 = 1$

$A_1 = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{matrix} = 3 \\ = 3 \end{matrix}$, $\lambda = 3 \,\&\, 5$

Q. Min & Max values of $Q(x)$ for $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ lies in b/w ?

(a) (-1 1)

(b) (1 5)

(c) [1 2]

(d) (3 5)

(e) None

~~$\because A$ is U.T.M so $\lambda = 1$, $Q_{min} = 1$~~

~~$\lambda = 2$, $Q_{max} = 2$~~

WRONG DATA GIVEN in QUESTION.

$A =$ Not Symm.

Relation b/n Quadratic form and Eigen Values $\rightarrow$

Positive Definite Mat $\rightarrow$ If (all the E. Values are +ve) $\Longleftrightarrow$ then (A is called +ve definite.)

Negative Definite Mat $\rightarrow$ If all the E Values are -ve then $\Longleftrightarrow$ A is called -ve definite.

INDEFINITE Mat $\rightarrow$ If E Values are +ve as well as -ve $\Longleftrightarrow$ then A is Indefinite.

Semi +ve Definite $\rightarrow$ If E Values are 0 or +ve then $\Longleftrightarrow$ A is called semi +ve Def.

Now, if A is Mat associated with Quad. Form then $Q(X)$ is called.

(i) +ve definite (i.e $Q(X) > 0$) if All the E Values are +ve

(ii) -ve definite ($Q(X) < 0$) if " " " " " -ve

(iii) Indefinite if all the " " are +ve as well as -ve.

Type 5 Q. The nature of $Q(X) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ will be?

(a) +ve Definite

(b) −ve Definite ×

(c) Indefinite ✓

(d) Can't say anything

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}_{3\times 3}$$

M-I

$\lambda_1 + \lambda_2 + \lambda_3 = 6 = +ve$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = -10$ (−ve)

Case I: $Tr = +ve \Rightarrow$ All 3 can't be −ve

$|A| = -ve$
- one −ve & two +ve ✓
- two −ve & one +ve NP ($\because |A| = -ve$)
- All three −ve N.P ($\because Tr = +ve$)

ie A has −ve as well as +ve E Values so $Q(X)$ = Indefinite.

M-II $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$,

C.Equ^n is $|A - \lambda I| = 0$

- - - - - - - - - - -

$(\lambda+1)(\lambda-5)(\lambda-2) = 0$

$\lambda = -1, 2, 5$

i.e. $Q(X) =$ Indefinite

Verification:

$Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

let $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ then

$Q(X_1) = 3(1)^2 + 0 + (-1)^2 + 0 + 0 = 4$ (+ve)

$Q(X_2) = 0 + 2(1)^2 + (-1)^2 + 0 + 4(1)(-1)$

$= 2 + 1 - 4 = -1$ (-ve)

i.e. $Q(X)$ is sometimes +ve & sometimes -ve so $Q(X)$ is Indefinite

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$. Which of the following statements is true regarding the quadratic form $Q(x) = X^T AX$?

(a) $Q(x)$ is positive definite.

(b) $Q(x)$ is negative definite

(c) $Q(x)$ is indefinite.

(d) $Q(x)$ is positive, semi-definite.

M-I

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 8$ (+ve) → Either All three are +ve
→ or two -ve & one +ve.

$\lambda_1 + \lambda_2 + \lambda_3 = 8$ (+ve) → All +ve ✓
→ two +ve, one -ve ✗
→ one +ve, two -ve ✓

Not feasible method this time

M-II $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

C.Eq is $|A - \lambda I| = 0$

$$\begin{vmatrix} (3-\lambda) & 1 & 2 \\ 1 & (2-\lambda) & 1 \\ 2 & 1 & (3-\lambda) \end{vmatrix} = 0$$

$$\boxed{\lambda^3 + (-\mathrm{Tr}\,A)\lambda^2 + (\,?\,)\lambda + (-|A|) = 0}$$

$$\lambda^3 + (-8)\lambda^2 + (?)\lambda + (-8) = 0$$

$$\lambda^3 - 8\lambda^2 + 15\lambda - 8 = 0$$

$$(\lambda - 1)[\lambda^2 - 7\lambda + 8] = 0$$

$$\lambda = 1, \; \lambda = \frac{7 \pm \sqrt{49 - 32}}{2}$$

$$\lambda = 1, \; \lambda = 3.5 \pm \frac{1}{2}\sqrt{17}$$

$$\lambda = 1, \; \lambda = 3.5 \pm (2.\_\_) = +ve, +ve$$

i.e All λ are +ve

$\Rightarrow f(X) = +$ve Definite.

PRINCIPAL AXES THEOREM $\rightarrow$ Consider $A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$ = Symm Mat.

let $A \begin{cases} \lambda_1, X_1 \\ \lambda_2, X_2 \\ \lambda_3, X_3 \end{cases} \Rightarrow \begin{aligned} N_1 &= X_1/\|X_1\| \\ N_2 &= X_2/\|X_2\| \\ N_3 &= X_3/\|X_3\| \end{aligned}$ i.e we are considering unit E.vectors.

$\because$ A is symm & $\lambda_1, \lambda_2, \lambda_3$ are diff so X_1, X_2, X_3 are orthogonal

so considering orthonormal Modal Mat $P = [N_1 \; N_2 \; N_3]$ i.e $P^{-1} = P^T$

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$,

if we put $X = PY$ then $Q(X) = X^T A X = \ldots\ldots = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = Y^T D Y = Q(Y)$

Proof: $X = PY$, $P^{-1} = P^T$ then

Note: $X = PY \Rightarrow \boxed{Y = P^{-1}X}$

$$Q(X) = X^T A X = (PY)^T A (PY)$$

$$= (Y^T P^T) A (PY) \quad \text{(By R. Law.)}$$

$$= Y^T (P^T A P) Y \quad \text{(By Associative law.)}$$

$$= Y^T (P^{-1} A P) Y \quad \text{(By Prop of orthogonal Mat)}$$

$$= Y^T . D . Y = Q(Y)$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \boxed{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2}$$

Use of P.A.Th i.e.

this Th converts one Quad. form (which includes CROSS PRODUCT terms) into another Quad form which is free from C.P terms.

FINAL conclusion of (P.A.Th) $\Rightarrow$

Let $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ & $Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ & $A=\begin{bmatrix}- & - & -\\- & - & -\\- & - & -\end{bmatrix}_{3\times3}$ = Symm.

λ_1, X_1

λ_2, X_2

λ_3, X_3

Let P = orthonormal Modal Mat $=\begin{bmatrix}\frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|}\end{bmatrix}_{3\times3}=\begin{bmatrix}N_1 & N_2 & N_3\end{bmatrix}$

$\Rightarrow$ $P^{-1}=P^T$. If we take $X=PY$ then

$$Q(X)=X^TAX=\ldots\ldots=\boxed{\lambda_1y_1^2+\lambda_2y_2^2+\lambda_3y_3^2}=Y^TDY=Q(Y)$$

which is free from Cross Product term. where $D=\begin{bmatrix}\lambda_1 & 0 & 0\\0 & \lambda_2 & 0\\0 & 0 & \lambda_3\end{bmatrix}$

Type 6 Qs if $Q(X) = \boxed{2x_1^2 - 2x_1x_2 + 2x_2^2}$ then write it's another Quadratic form which is free from cross product term?

(a) $2y_1^2 + 2y_2^2$

(b) $y_1^2 + 3y_2^2$ ✓

(c) $y_1^2 + 2y_2^2$

(d) $-y_1^2 + 5y_2^2$

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $=1$, $=1$ $\Rightarrow \lambda = 1$ & $Tr(A) = 4 \Rightarrow \lambda = 3$

ie EValues of A are 1 & 3

Consider $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$Q(Y) = Y^T D Y = [y_1 \; y_2] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \cdots = 1 \cdot y_1^2 + 3y_2^2$$

where $Q(X) = X^T A X = \underset{(\text{Put } X = PY)}{\cdots\cdots} = Y^T D Y = Q(Y)$

M-II $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $Tr(A) = 4 = \lambda_1 + \lambda_2$ & $|A| = \lambda_1 \lambda_2 = 3$

$$Q(X) = Q(Y) = \lambda_1 y_1^2 + \lambda_2 y_2^2$$
$$= 1 \cdot y_1^2 + 3 \cdot y_2^2$$

PODCAST: $A=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$

$\lambda_1=1,\ X_1=\begin{bmatrix}1\\1\end{bmatrix},\ N_1=\frac{X_1}{\|X_1\|}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$\lambda_2=3,\ X_2=\begin{bmatrix}-1\\1\end{bmatrix},\ N_2=\frac{X_2}{\|X_2\|}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$

$P=[N_1\ N_2]=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}$ where $P^{-1}=P^T$, let $X=\begin{bmatrix}x_1\\x_2\end{bmatrix},\ Y=\begin{bmatrix}y_1\\y_2\end{bmatrix}$

Here $X=PY \Rightarrow Y=P^{-1}X=P^TX=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\boxed{\frac{1}{\sqrt{2}}\begin{bmatrix}(x_1+x_2)\\(-x_1+x_2)\end{bmatrix}}$

Here $Q(X)=X^TAX=2x_1^2+2x_2^2-2x_1x_2=1\cdot y_1^2+3\cdot y_2^2=Y^TDY=Q(Y)$

Let $X=\begin{bmatrix}3\\4\end{bmatrix}\Rightarrow Q(X)=2(3)^2+2(4)^2-2(3)(4)=26$

& $Y=\frac{1}{\sqrt{2}}\begin{bmatrix}(x_1+x_2)\\(-x_1+x_2)\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}(3+4)\\(-3+4)\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}7\\1\end{bmatrix}$ Now $Q(Y)=1\left(\frac{49}{2}\right)+3\left(\frac{1}{2}\right)=26$

Q: The orthogonal Quadratic form of $Q(x) = 4x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2$ will be?

(2021)

(a) $-3y_1^2 + 2y_2^2 + 4y_3^2$ $\quad Tr \neq 9$

(b) $6y_1^2 + 5y_2^2 - 2y_3^2$ $\quad Tr = 9$, $|A| \neq 15$

(c) $3y_1^2 + 5y_2^2 + 1y_3^2$ $\quad Tr = 9$, $|A| = 15$

(d) $y_1^2 - y_2^2 + 6y_4^2$ $\quad Tr \neq 9$

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Tr = \lambda_1 + \lambda_2 + \lambda_3 = 9$$

$$|A| = \lambda_1 \lambda_2 \lambda_3 = 15$$

M-II $\quad A = [\ \ ]$ $\quad \lambda = 1,\ \lambda = 3,\ \lambda = 5$

$$Q(X) = X^T A X = \cdots\cdots = Y^T D Y = q(y)$$

$$= 1\cdot y_1^2 + 3y_2^2 + 5y_3^2 \quad \text{i.e. } (c)$$

The equation for the ellipsoid of inertia of a solid body is

$$P(x) \equiv \boxed{4x_1^2 + 4x_2^2 + 1.x_3^2 - 2x_1x_2.} \Rightarrow A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} < \begin{matrix} 1 \\ 3 \\ 5 \end{matrix}$$

What is the standard from in terms of a new orthogonal set of axes $O\{y_1, y_2, y_3\}$?

(a) $y_1^2 - 3y_2^2 + 3y_3^3$ ✓(b) $1.y_1^2 + 5y_2^2 + 3y_3^2$

(c) $y_1^2 - 5y_2^2 + 3y_3^2$ ✗ (d) $y_1^2 - 5y_2^2 - 3y_3^2$ ✗

Already Solved.

THANK - YOU