GATE

DS & AI
CS & IT

Linear Algebra - I

From 04^{th} Aug : 11:00 Am to 1:30 PM

Lecture No. 10

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

Homogeneous System of Linear Equations

Topics to be Covered

Topic

Eigen Values & Eigen Vectors

① Concept

② Properties of Eigen Values.

Monday onwards : 11:00 AM to 1:30 PM
(From 4th August 2025)

Methods of Solving Non Homog System → Consider $\boxed{A_{m\times n}\,X_{n\times 1} = B_{m\times 1}}$

RANK Method (always applicable)

(ie for $m>n$, $m=n$, $m<n$)

Matrix Method

(applicable only when $m=n$)

① if $\boxed{\rho(A)=\rho(A:B)}$ = No. of Variables ⇒ unique sol. ⇐ if $|A|\neq 0$

② if $\boxed{\rho(A)=\rho(A:B)}$ < No. of Variables ⇒ ∞ sol. ⇐ if $|A|=0$, $(\text{adj}A)B=0$

③ if $\rho(A)\neq\rho(A:B)$ ⟹ No sol. ⇐ if $|A|=0$, $(\text{adj}A)B\neq 0$

RECAP

(*) Consistent system → System is called Consistent if ∃ Solution.
RECAP.
(whether unique or ∞ Sol.)

In consistent system → System is called Inconsistent if we have No Sol.

(*) Necessary condition for a system $\boxed{AX = B}$ to be consistent is ?

$$\rho(A) = \rho(A : B).$$

Another Condition for Consistency : $\boxed{\text{B must be L-D on Columns vectors of A.}}$

RECAP $A = [X_1, X_2, X_3, X_4]$

→ $\rho(A) = 4 \Rightarrow$ vectors are LI

→ $\rho(A) < 4 \Rightarrow$ " " LD

Methods of Solving Homog. System

RECAP

$(A_{m\times n} X_{n\times 1} = O_{m\times 1})$

RANK Method (always applicable)

$(m>n, m=n, m<n)$

① if $\rho(A)$ = No. of Variables ⇒ unique sol.

② if $\rho(A) <$ " " ⇒ ∞ sol.

MATRIX Method

(only for $m=n$)

① if $|A| \neq 0$ ⇒ unique sol exist

② if $|A| = 0$ ⇒ ∞ sol exist

(*) In this Chapter, RECAP

(i) unique sol $\cong$ Trivial sol. $\cong$ ZERO solⁿ **always** exist.

(ii) ∞ solⁿ $\cong$ Non Trivial sol also exist $\cong$ Non Zero sol also exist.

(iii) $\boxed{\text{No sol} \neq \text{ZERO sol.}}$ $\qquad |A|=0$ or $\rho(A) < n$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \qquad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ANALYSIS (PODCAST) → Consider $\boxed{x+y+z=0} \Rightarrow \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\end{bmatrix} \Rightarrow \boxed{A_{1\times 3}X_{3\times 1}=O_{1\times 1}}$

∵ this is type of underdetermined system so it has ∞ solⁿ.

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\underbrace{\begin{bmatrix}0\\0\\0\end{bmatrix}}_{\text{Zero sol}},\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\-1\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix},\begin{bmatrix}5\\-3\\-2\end{bmatrix},\ldots\ldots\infty\text{ sol}}_{\text{Non Zero sol}}$$

Here Non Zero solⁿ's are also uncountable (i.e. ∞)

But out of these ∞ solⁿ, only Two are L.I & Rest are L.D on them.

Q. How many L.I solⁿ are there for $\boxed{x+y+z=0}$ at a time? Two Ans

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{cases} X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/4 \\ -1/4 \\ 0 \end{bmatrix} \cdots\cdots \infty \text{ sol exist.} \\ X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/5 \\ 0 \\ -2/5 \end{bmatrix} \cdots\cdots \infty \text{ sol exist} \\ X_3 = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}, X_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, X_5 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \cdots\cdots \infty \text{ sol exist.} \end{cases}$$

Null Space of $AX = 0$.

Here $X_3 = X_1 + 2X_2$, $X_4 = X_1 - X_2$, $X_5 = X_1 + X_2$ i.e. all are LD on X_1 & X_2

i.e. only X_1 & X_2 are LI & Rest are LD on them.

Actually we are getting only Two Ind families Solutions and Rest are LD on them

Shortcut :→ $\underbrace{\text{No. of L.I Solutions of}}_{\text{Nullity of } A \,/\, \text{Dimension of Null space.}}$ $AX=0$ $=$ $\boxed{\text{No. of Columns} - \rho(A)}$

Rank - Nullity Th : $\boxed{N(A) = \text{No. of Columns} - \rho(A)}$

Null space (Solution space / Solution set) ⟶

Set formed by various solutions of Homog System $AX=0$ is called Null space

& Dimension of Null space = Counting of LI Soln of $(AX=0)$ = Nullity(A)

Column Space → (Range space) if $A_{m\times n}$ then set formed by all the Linear combinations of Columns of A

Row space if $A_{m\times n}$ then " " " " " " " Rows of A.

Dimension of Column space = Dim of Row space = $\rho(A)$ i.e. No. of LI Colums / No of LI Rows.

Eg: Find the Nullity of $\boxed{x+y+z=0}$ —— (1)

Here $A_{1\times 3} \Rightarrow \rho(A) = 1$ So $N(A) =$ No. of Columns $- \rho(A) = 3-1 = 2$

i.e. (1) is two LI Solutions

Eg: If $A_{4\times 5}$ matrix s.t All the Solutions of $\boxed{AX=0}$ is Scalar Multiplication of $\begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}$ then $\rho(A) =$?

Sol: ATQ, above system has only one LI Sol.

So $N(A) = 1$. Hence By Rank-Nullity Th,

$N(A) =$ No. of Columns $- \rho(A)$

$1 = 5 - \rho(A) \Rightarrow \rho(A) = 4$ $A_{4\times 5}$

Sol. $X = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 \\ -2 \\ 6 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} - \\ - \\ - \\ - \\ - \end{bmatrix} \cdots\cdots \infty$. But all are LD on X.

→ Null Space

PODCAST:-

$$A = \begin{array}{c} \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \\ \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \end{array}$$

$C(A)$ (Range space) $= \{ C_1, C_2, C_3, C_4, (2C_1 + 4C_3 - C_4), (-C_1 + 2C_2 + C_3 + 2C_4), (4C_1 - C_2 + 2C_3 + 2C_4), \ldots \}$

No. of LI Column vectors / Dimension of Column space $= \rho(A)$

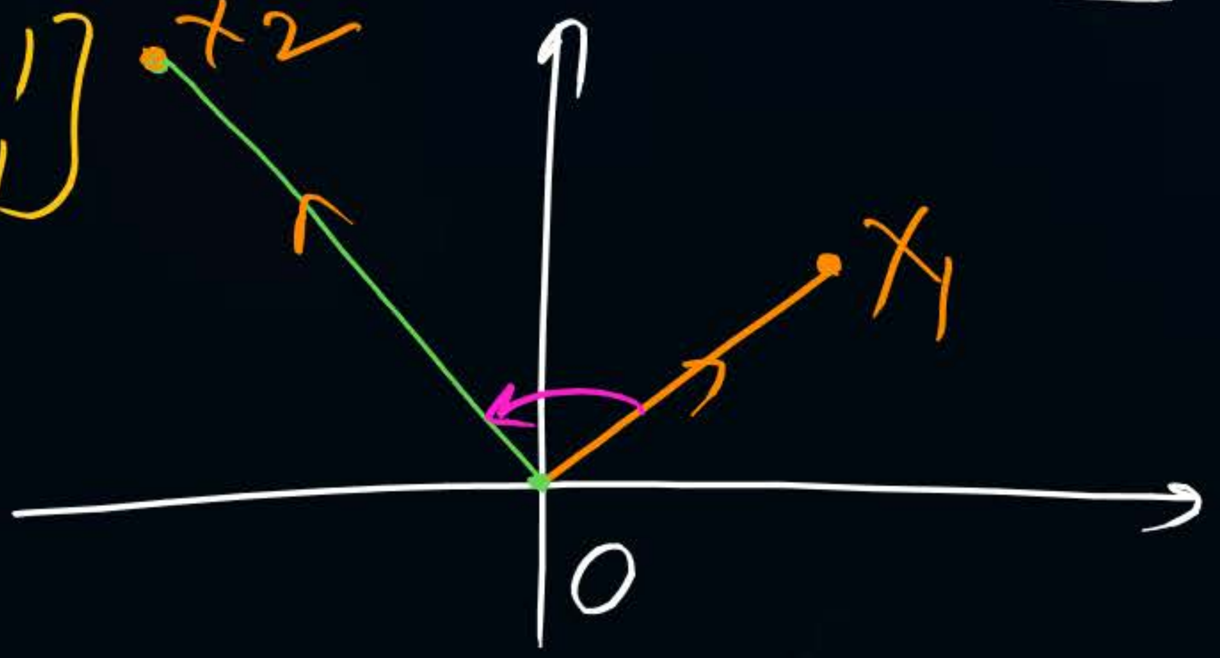
$R(A) = \{ R_1, R_2, R_3, (k_1R_1 + k_2R_2 + k_3R_3) ; k_i \in R \}$

No of LI Row vectors / Dimension of Row space $= \rho(A)$

EIGEN VALUES & EIGEN VECTORS

PODCAST: Consider $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$ & let $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$AX = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = X_2$$

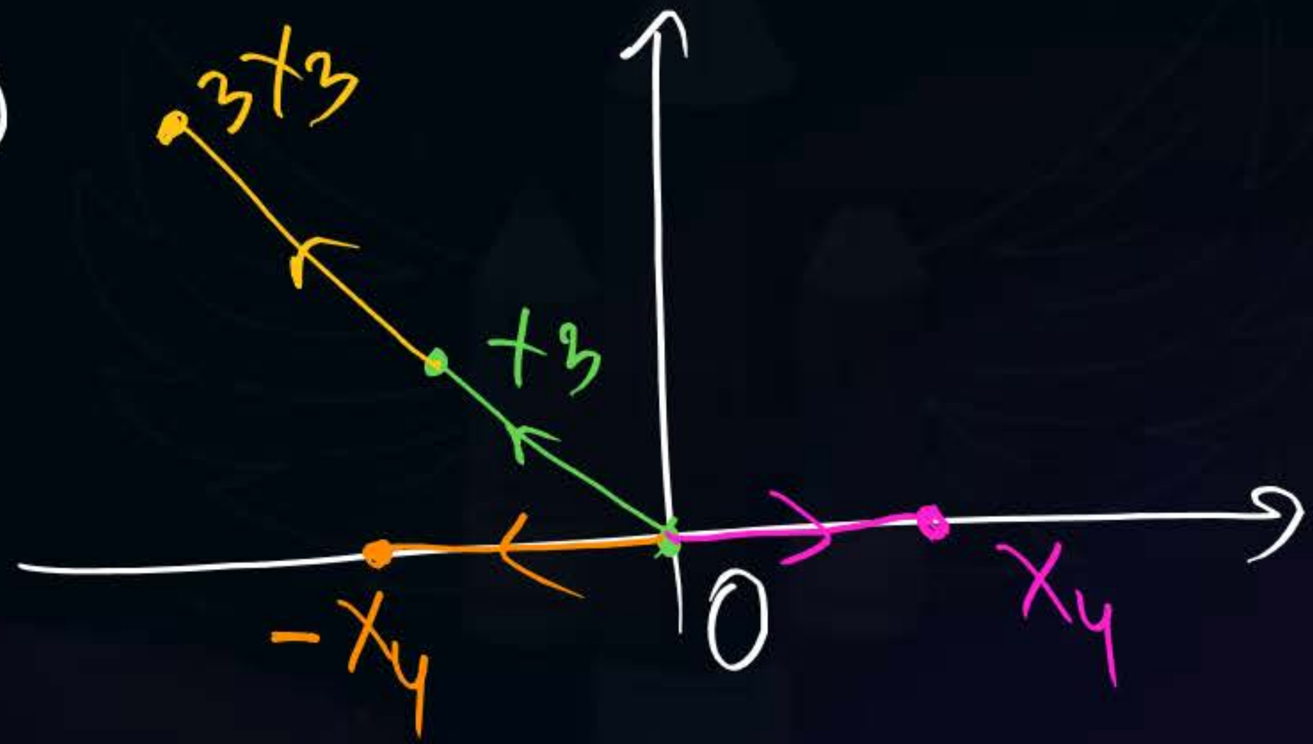


Let us consider $X_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ then

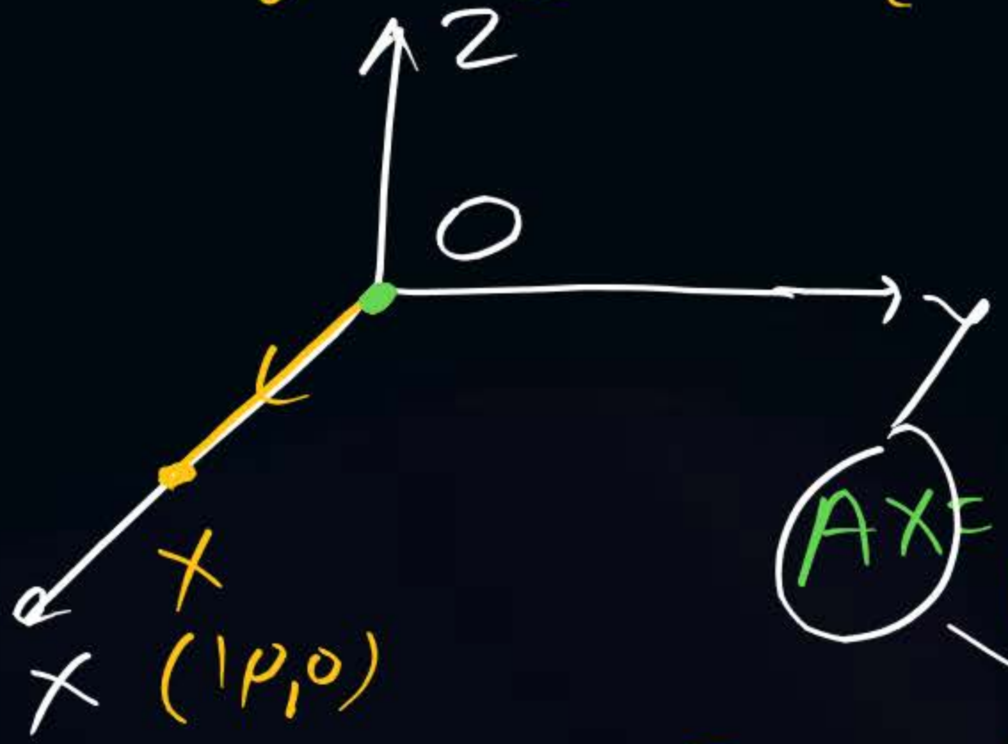
$$AX_3 = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 3X_3$$

Let us consider $X_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$AX_4 = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -X_4$$



Eg: $A=\begin{bmatrix}0&0&5\\0&0&0\\0&0&0\end{bmatrix}$, $X=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $AX=\begin{bmatrix}0&0&5\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}=0.X$ = Point vector



let $X=\begin{bmatrix}0\\0\\0\end{bmatrix}$ then

$AX=\begin{bmatrix}0&0&5\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0.\begin{bmatrix}0\\0\\0\end{bmatrix}=0.X$

i.e. Eigen Value can be Zero but Eigen Vector can't

i.e. Eigen Vector is always Non Zero Vector.

Defn: Consider Sq. Mat $A_{n\times n}$ then Non Zero Vector X is called Eigen vector, corresponding to Eigen value λ (Real/Complex)/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda . X}$$

λ = Eigen value

X = Eigen vector.

LHS is the Multi of Two Matrices = RHS is the Scalar Multi in a Mat

(Tough job) (Easy job)

Here we are considering ___Homog.___ System as follows;

$$AX = \lambda X \implies AX - \lambda X = 0 \implies \boxed{(A - \lambda I)X = 0}$$

Hence this system will satisfy all the properties of Homo System.

C. Equ^n of A :- $Ax = \lambda X$

$A_{n\times n}$ $(A-\lambda I)X = 0$ — (1)

$\boxed{MX = 0}$

$\downarrow$

Non Zero E. Vector

Non Zero Solution

∞ Sol.

$\rho(M) < n$ or $|M| = 0$

$\Rightarrow \rho(A-\lambda I) < n$ or $\boxed{|A-\lambda I| = 0}$

is Necessary cond^n for the existence of Non Zero eigen vector is

$\rho(A-\lambda I) < n$ or $\boxed{|A-\lambda I| = 0}$ — (2)

(*) Equ^n (2) is called Characteristic equ^n of A & Roots of this equation i.e. Values of λ are called

Eigen Values / Eigen Roots / Char Values / Char Roots / Latent Roots / Special Values.

eg: Find the E-Values of (1) $A=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}$ (2) $A=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ (3) $A=\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$

Sol: $A-\lambda I=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$=\begin{bmatrix}(2-\lambda)&1&1\\1&(2-\lambda)&1\\0&0&(1-\lambda)\end{bmatrix}$$

= Char Mat.

Now, Char Equ^n is $|A-\lambda I|=0$

$$\begin{vmatrix}(2-\lambda)&1&1\\1&(2-\lambda)&1\\0&0&(1-\lambda)\end{vmatrix}=0$$

- - HW - - - -

$\boxed{(\lambda-3)(\lambda-1)^2=0} \Rightarrow \lambda=3,1,1$

(ii) C.Equ is $|A-\lambda I|=0$

$$\begin{vmatrix}(0-\lambda)&-1\\1&(0-\lambda)\end{vmatrix}=0 \Rightarrow \boxed{\lambda^2+1=0} \Rightarrow \lambda=i,-i$$

(iii) C.Equ^n is $|A-\lambda I|=0$.

$$\begin{vmatrix}(8-\lambda)&-6&2\\-6&(7-\lambda)&-4\\2&-4&(3-\lambda)\end{vmatrix}=0 \xrightarrow{HW} \boxed{\lambda(\lambda-3)(\lambda-15)=0}$$

is $\lambda=0,3,15$

Q:- Find the E.Values of $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}_{4\times 4}$

Sol: C.Equ^n is $|A-\lambda I| = 0$

$$\begin{vmatrix} (0-\lambda) & 0 & 1 & 1 \\ -1 & (2-\lambda) & 0 & 1 \\ -1 & 0 & (2-\lambda) & 1 \\ 1 & 0 & -1 & (0-\lambda) \end{vmatrix} = 0$$

- - - - - - - - - - HW

$$\boxed{\lambda(\lambda-2)(\lambda-1)^2 = 0} \Rightarrow \lambda = 0, 2, 1, 1$$

Note (1)

(1) Sum of E.Values $= 4 = Tr(A)$

(2) Product of E.Values $= 0 = |A|$

(3) $(\lambda = 0) \Longleftrightarrow |A| = 0$

(4) Total No. of E.Values = order

PROPERTIES of Values → Let $A_{n\times n}$ having Eigen Values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

① T. Number of E-Values of A = order of A (whether Different or Repeated)

② Sum of E-Values = Trace(A) i.e $\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n = Tr(A)$

③ Product of E Values = Det(A) i.e $\lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\ldots\cdot\lambda_n = |A|$

④ (Zero is an E. Value of A) iff $\Longleftrightarrow$ (A is Singular) i.e $(\lambda=0) \Longleftrightarrow (|A|=0)$

⑤ Number of Non Zero E.Values of A $\leq \rho(A)$

eg if $\rho(A_{6\times 6})=4$ then A has at least two Eigen Values as 0, 0.

⑥ If sum of all the elements of each Row (or each Column) is unique constant K then that Constant K will be one of the E.Value of A.

(7) Don't use E-Operations in a given Mat while calculating E-Values

But

(8) E-Values of U.T.M, L.T.M, Diag Mat, Scalar Mat, Identity Mat are just the diagonal elements.

eg $A = \begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\lambda = 2, 0, -3, -1$ $\quad\quad$ $\lambda = 2, -1, 1$ $\quad\quad$ $\lambda = 2, -3, 4$

(9) If λ is an Eigen Value of A then to find Eigen Value of any algebraic expression formed by A, we can Replace A with λ in that expression.

(10) Let $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the E Values of $A_{n\times n}$ then

(i) E Values of A^T are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ is same as that of A.

(ii) E Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots, \lambda_n^m$; $(m \in N)$

(iii) E Values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \ldots \frac{1}{\lambda_n}$ (Provided $|A| \neq 0$)

(iv) E. Values of $(\text{adj} A)$ are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots, \frac{|A|}{\lambda_n}$

But take Case, this short cut of finding E Values of $(\text{adj} A)$ is applicable when $|A| \neq 0$

if $|A| = 0$ then to find E Values of $(\text{adj} A)$, use conventional Approach.

eg $A = \begin{bmatrix} -2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(U.T.M)

$|A| = -24.$

then

- E values of A are $-2, 1, 4, 3$
- " " A^T are $-2, 1, 4, 3$
- " " A^3 are $(-2)^3, (1)^3, (4)^3, (3)^3$
  $= -8, 1, 64, 27$
- " " A^{-1} are $\frac{1}{-2}, 1, \frac{1}{4}, \frac{1}{3}$
- " " adj A are $\frac{-24}{-2}, \frac{-24}{1}, \frac{-24}{4}, \frac{-24}{3}$
  $= 12, -24, -6, -8$

eg $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Diag Mat.

- E values of A are $1, -2, 0, 4$
- " of A^T are $1, -2, 0, 4$
- " of A^2 are $1, 4, 0, 16$
- " of A^{-1} are ~~$1, \frac{-1}{2}$, ND, $\frac{1}{4}$~~ Blunder.

$\because A^{-1} = \text{DNE}$

Q: If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then E Values of A^{-1} are ? → DNE

(ii) " of (adj A) are ?

Sol: E-Values of A are 0, 3, 15 (Already Calculated)

i.e $|A| = (0)(3)(15) = 0 \Rightarrow A^{-1} = \text{DNE}$

8	−6	2	8	−6
−6	7	−4	−6	7
2	−4	3	2	−4
8	−6	2	8	−6
−6	7	−4	−6	7

$$\text{adj}A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} \Rightarrow |\text{adj}A| = 0$$

i.e $\lambda = 0$

Now Calculate Remaining eigen Values of adj A by making its C-Equn. $\lambda = 0, 0, 45$

FROM MONDAY onwards : 11:00 AM to 1:30 PM
(4th Aug onwards)

THANK - YOU

Tel : $\boxed{\text{dr puneet sir pw}}$