DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Not for CS/IT

Linear Algebra- II

Lecture No. 05

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Recap of previous lecture

Topic

VECTOR SPACE

&

PARTITION MATRIX

Topics to be Covered

Topic PROJECTION MATRIX

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \Rightarrow \vec{a} = |\vec{a}|\,\hat{a} = \boxed{(\text{Magnitude}).(\text{unit Vector})}$$

General points: Consider $A = \begin{bmatrix} - \\ - \\ - \\ \vdots \\ - \end{bmatrix}_{n\times 1}$, $A^T = \begin{bmatrix} - & - & \cdots & - \end{bmatrix}_{1\times n}$

$$M = A^T A \longrightarrow [\quad]_{1\times n} [\quad]_{n\times 1} = [\quad]_{1\times 1} = \text{SCALAR} = \|A\|^2$$

$$N = AA^T = [\quad]_{n\times 1} [\quad]_{1\times n} = [\quad]_{n\times n} = \text{Mat (Symm.)}$$

$$N^T = (AA^T)^T = (A^T)^T . A^T = AA^T = M \text{ is Symm.}$$

$A_{n\times 1}$ & $B_{n\times 1}$ then $A \cdot B = A^T B = [\quad]_{1\times 1} = \text{scalar}$

General Points: ① $\because \hat{a} = \frac{\vec{a}}{|\vec{a}|} \Rightarrow \boxed{\vec{a} = |\vec{a}|.\hat{a}}$

Any vector = (it's Magnitude) × (unit vector in it's direction)

② let $A_{n\times 1}$ then $A^T_{1\times n}$

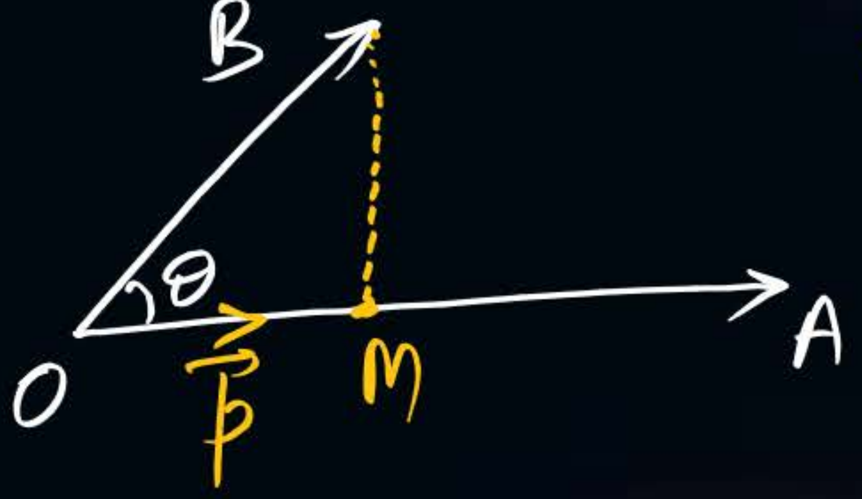
- $(A^TA)_{1\times 1}$ = symmetric = scalar &
- $(AA^T)_{n\times n}$ = symmetric Mat.

③ let $A_{n\times 1}$ & $B_{n\times 1}$

- $(A^TB)_{1\times 1}$ = Scalar Quantity.
- $A \cdot B = A^TB$ = scaler Quantity
- $\|A\| = \sqrt{A^TA} \Rightarrow \|A\|^2 = A^TA$

④ if $A \cdot B = 0$ then A & B are Called Orthogonal Vectors.

PROJECTION of VECTOR $\rightarrow$ If we want to find projection of $\vec{B}$ onto (along) the vector $\vec{A}$ then process will be as follows;



$\triangle OMB$: $\frac{OM}{OB} = \cos\theta$

$OM = |\vec{B}|\cos\theta$

where $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

Now Any vector $=$ (it's Magnitude) (Unit vector in it's dir^n)

$$\vec{p} = (OM)(\hat{A}) = |B|\cos\theta\left(\frac{\vec{A}}{|\vec{A}|}\right)$$

$$\vec{p} = |\vec{B}|\cdot\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}\left(\frac{\vec{A}}{|\vec{A}|}\right) = \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2}\right)\vec{A} = \left(\frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|^2}\right)\vec{A}$$

$$\boxed{\vec{p} = \left(\frac{A^TB}{A^TA}\right)_{1\times1}\vec{A}} = (\lambda)\vec{A}$$

i.e. $\vec{p}$ & $\vec{A}$ are collinear.

M-II we have shown that,

$$\vec{p} = \left(\frac{A^T B}{A^T A}\right)_{1\times 1} A$$

$$= A\left(\frac{A^T B}{A^T A}\right)$$

$$= \left(\frac{A A^T}{A^T A}\right)_{n\times n} B$$

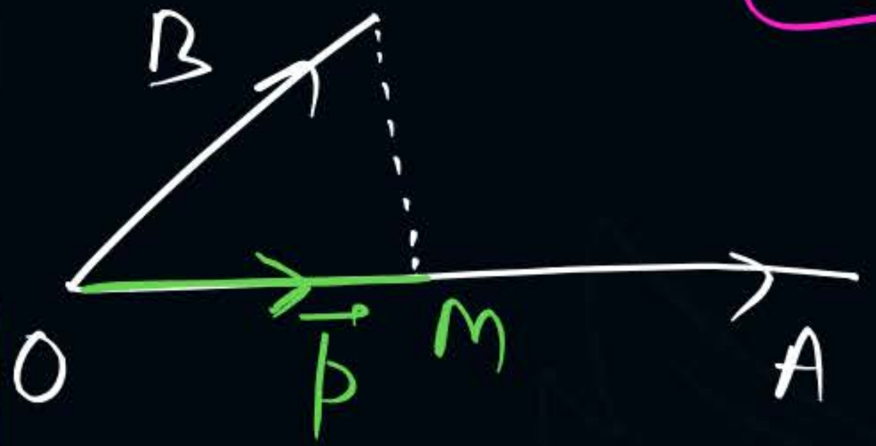
$$\boxed{\vec{p} = P B} \quad \text{where } P = \frac{A A^T}{A^T A} = \text{Projection Mat}$$

Projection along a line :→

Projection of Vector B along a line A is given as

$$\boxed{\vec{p} = P B} \quad \text{where } \boxed{P = \frac{A A^T}{A^T A}} = \left[\quad\quad\right]_{n\times n}$$

$= \text{Projection Mat.}$



Q: Find the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ onto the vector $\begin{bmatrix}-2\\1\end{bmatrix}$

Sol: Here $B=\begin{bmatrix}1\\1\end{bmatrix}$, $A=\begin{bmatrix}-2\\1\end{bmatrix}$

$$A^TB=\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\left[(-2+1)\right]=-1$$

$$A^TA=\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}=\left[(4+1)\right]=5$$

So $\boxed{\vec{p}=\left(\frac{A^TB}{A^TA}\right)\vec{A}}=\left(-\frac{1}{5}\right)\vec{A}=-\frac{1}{5}\begin{bmatrix}-2\\1\end{bmatrix}$

i.e. $\vec{p}=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$

M-II

$$A^TA=\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}=5$$

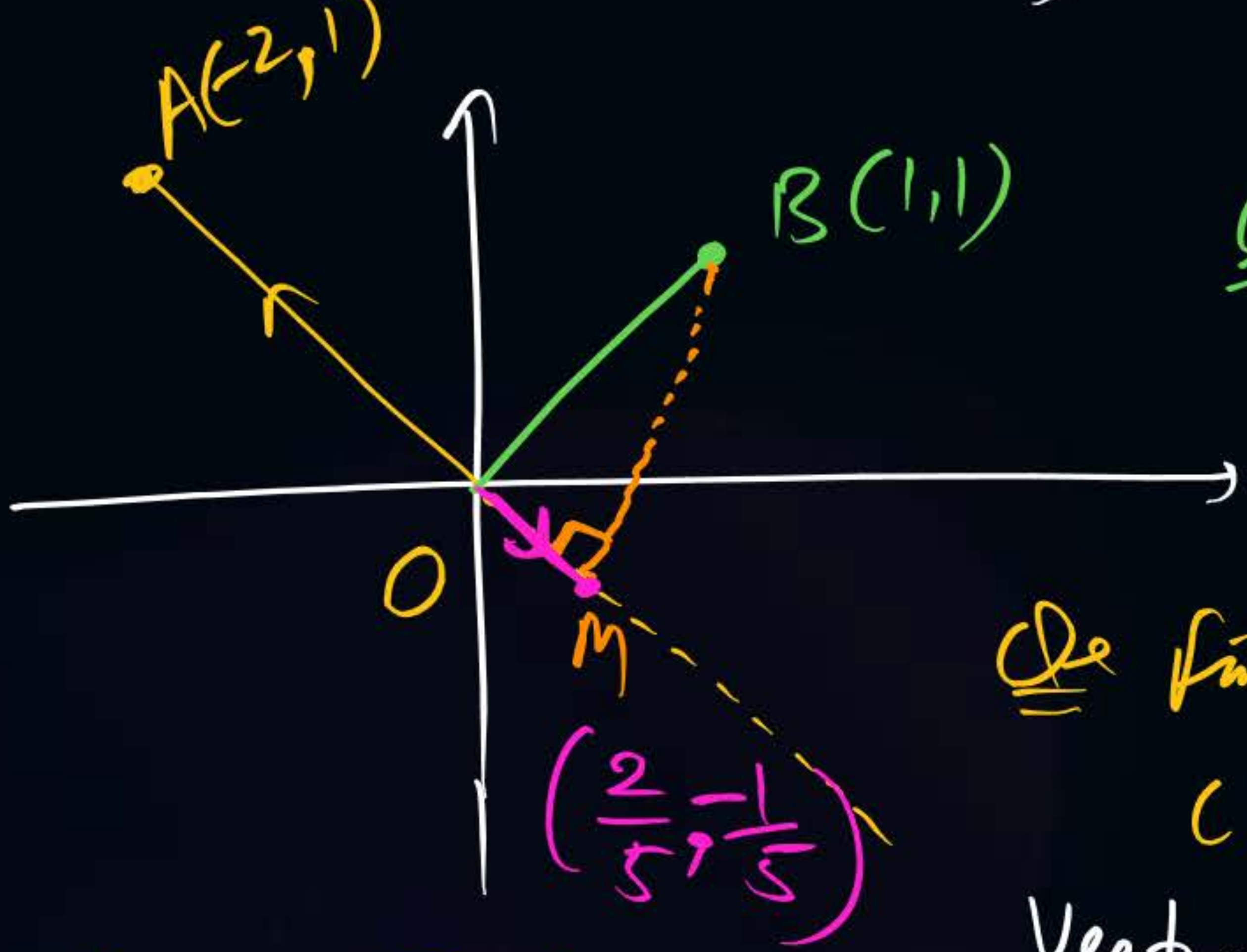
$$AA^T=\begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix}=\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$$

So Projection Mat $P=\frac{AA^T}{A^TA}=\frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

& Projected vector is $\vec{p}=PB$

$$\vec{p}=\frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$$

PODCAST: $A=\begin{bmatrix}-2\\1\end{bmatrix}$, $B=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{p}=\begin{bmatrix}2/5\\-1/5\end{bmatrix}=\frac{-1}{5}\begin{bmatrix}-2\\1\end{bmatrix}=\frac{-1}{5}A$



Q: what Multiple of $\begin{bmatrix}-2\\1\end{bmatrix}$ will be closest to $\begin{bmatrix}1\\1\end{bmatrix}$

(a) -1 (b) $\frac{1}{5}$ (c) $-\frac{1}{5}$ (d) $\frac{-2}{5}$

Q: find the point on the line $x+2y=0$ which is closest to the point $(1,1)$?

(a) $(-2\ 1)$ (b) $(2\ -1)$

(c) $\left(\frac{2}{5},-\frac{1}{5}\right)$ (d) $\left(\frac{-2}{5},\frac{1}{5}\right)$

Vector along the line $\boxed{x+2y=0}$ is $A=\begin{bmatrix}-2\\1\end{bmatrix}$

& given Point $B=\begin{bmatrix}1\\1\end{bmatrix}$ so closest point = P. Vector = $\vec{p}=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$

Q: Find a Matrix that projects every point in 2D plane $\boxed{\text{onto}}$ along the line $\boxed{x+2y=0}$

Also find the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ onto that line.

Sol: vector along the line $\boxed{x+2y=0}$ is $A=\begin{bmatrix}-2\\1\end{bmatrix}$

(1) So $P=\dfrac{AA^T}{A^TA}=\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$ already Calculated

(2) $\vec{p}=PB=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$ already Calculated

Q: Find a Matrix that projects every point in 2D plane onto the line $x+2y=0$. Also find the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ onto that line.

Sol: vector form of line $\boxed{x+2y=0}$ is $A=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$

So Projection Mat $P=\dfrac{AA^T}{A^TA}= \cdots\cdots = \boxed{\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}}$

(ii) $B=\begin{bmatrix}1\\1\end{bmatrix}$ so $\vec{p}=P\vec{B}=\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\dfrac{-1}{5}\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$

Note: $Tr(P)=1$, $|P|=0$, $\rho(A)=1$, $\lambda=0$ & 1, No. of LI Evectors = Two = order

P = Symm & Idempotent

Hence Diagonalizable.

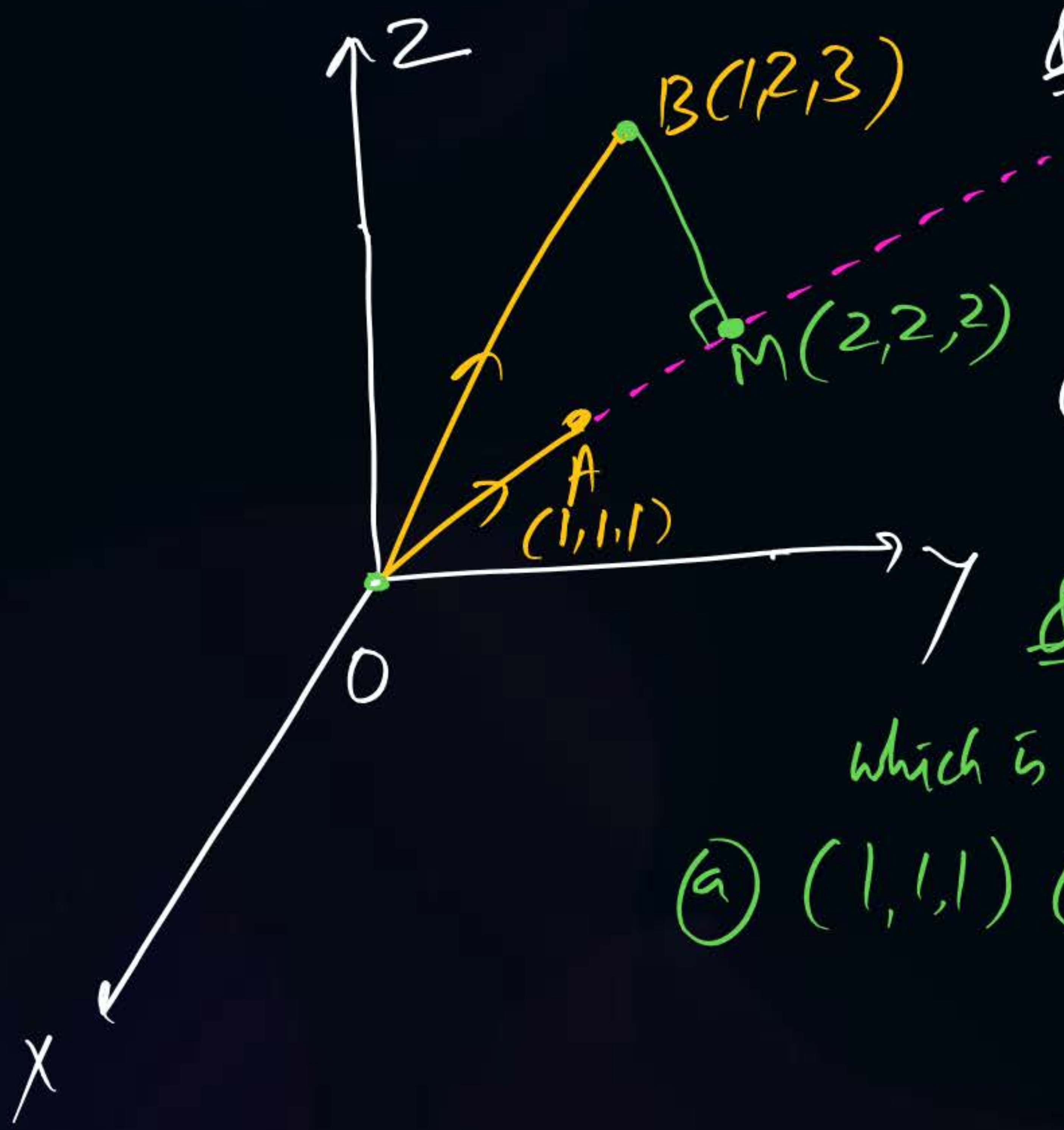
Q. The projection of $B=\begin{bmatrix}1\\2\\3\end{bmatrix}$ onto the line $A=\begin{bmatrix}1\\1\\1\end{bmatrix}$ will be?

Sol: $A^TA=\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}(1+1+1)\end{bmatrix}=3$

$$AA^T=\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

So $\boxed{P=\dfrac{AA^T}{A^TA}=\dfrac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}}$

& Hence $\vec{p}=PB=\dfrac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$



Q: what multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is closest to the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$? $\vec{p} = 2\begin{pmatrix}1\\1\\1\end{pmatrix}$

(a) $\begin{bmatrix}2\\2\\2\end{bmatrix}$, (b) $\begin{bmatrix}1\\1\\1\end{bmatrix}$, (c) 2 (d) 6

Q: the point along the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ which is closest to the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ will be?

(a) (1,1,1) (b) 2 (c) (2 2 2) (d) (1 2 3)

Q: What Multiple of $A(1,1,1)$ is closest to the point $B(2,4,4)$?

OR

Find the projection of $B(2,4,4)$ onto the vector $A(1,1,1)$

(a) 3

(b) 10

(c) 10/3

(d) 3/10

$$A=\begin{bmatrix}1\\1\\1\end{bmatrix},\ A^T=[1\ 1\ 1] \Rightarrow A^TA=[1\ 1\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix}=3$$

$$AA^T=\begin{bmatrix}1\\1\\1\end{bmatrix}[1\ 1\ 1]=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$$\boxed{P=\frac{AA^T}{A^TA}=\frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}_{3\times 3}}$$

$$\vec{p}=PB=\frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}2\\4\\4\end{bmatrix}=\begin{bmatrix}10/3\\10/3\\10/3\end{bmatrix}=\frac{10}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Note: $Tr(P)=1$, $|P|=0$

$\rho(A)=$ one, $\lambda=0,0,1$.

$P=$ Symm & Idempotent

GM of '0' = two

GM of '1' = one

No. of LI E.Vectors = three

Hence $P=$ Diagonalizable.

Q. Find the projection Matrix that projects every point in R^3 onto the line of Intersection of the planes $\boxed{x+y+z=0 \ \& \ x-z=0.}$

& Hence find the projection of $B(1\,2\,3)$?

Solving (1) & (2), $x+y+x=0 \Rightarrow 2x+y=0 \Rightarrow \boxed{2x+y+0z=0}$ — (3)

Along vector along line (3), is $A=\begin{bmatrix}1\\-2\\0\end{bmatrix}$

Now, $A^TA=\begin{bmatrix}1 & -2 & 0\end{bmatrix}\begin{bmatrix}1\\-2\\0\end{bmatrix}=\begin{bmatrix}(1+4+0)\end{bmatrix}=5$

$AA^T=\begin{bmatrix}1\\-2\\0\end{bmatrix}\begin{bmatrix}1 & -2 & 0\end{bmatrix}=\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 0\\0 & 0 & 0\end{bmatrix}$

So $P=\dfrac{AA^T}{A^TA}=\dfrac{1}{5}\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 0\\0 & 0 & 0\end{bmatrix}$

& $\vec{p}=PB=\frac{1}{5}\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-3\\6\\0\end{bmatrix}$

$\vec{p}=\begin{bmatrix}-\frac{3}{5} & \frac{6}{5} & 0\end{bmatrix}'$

Analysis: Line of Intersection is $\boxed{2x+y+0z=0}$ — lies in XY plane.

(1) if we take $A=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-2\\1\end{bmatrix}$ then after putting the values we are getting

$2(1)+(-2)+0(1)=0$ i.e $\boxed{0=0}$

But on XY plane, $z=0$ (always) so we should not take $z=1$.

Common sense.

(2) $\because B=\begin{bmatrix}1\\2\\3\end{bmatrix}$ lies in 3-D But line of intersection lies in 2D so projected vector will also lies in 2D. & it is given as $\vec{p}=\begin{bmatrix}-3/5\\6/5\\0\end{bmatrix}$ Hence justified

(3) $P = \frac{1}{5}\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow Tr(P) = 1,\ |P| = 0,\ \rho(A) = 1 \Rightarrow \boxed{\lambda = 0, 0, 1}$

$P^T = P \Rightarrow P = $ Symm.

$P^2 = P \Rightarrow P = $ Idempotent

GM of $(\lambda = 0) = $ order $- \rho(A - \lambda I)$

$= 3 - \rho(A - 0 \cdot I)$

$= 3 - \rho(A) = 3 - 1 = $ two

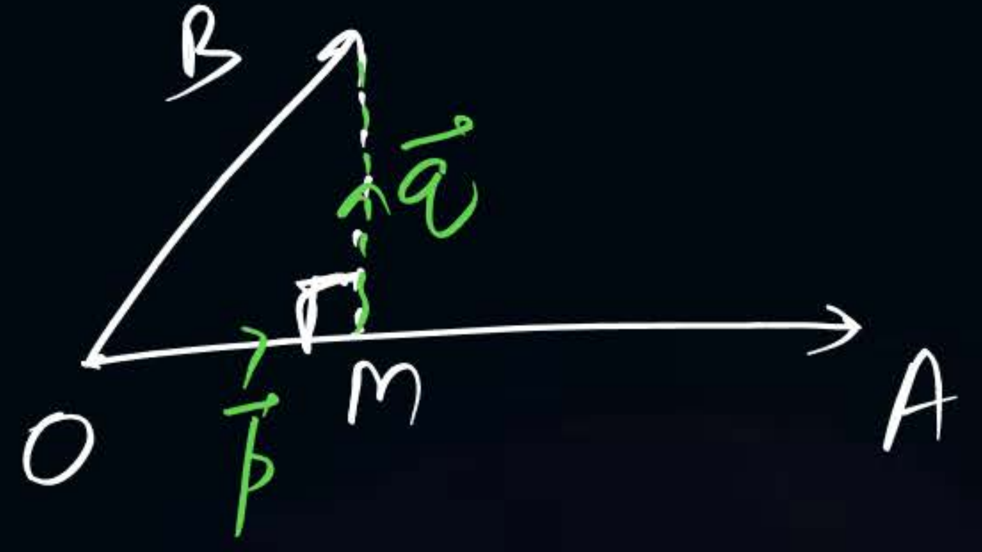
& GM of $(\lambda = 1) = $ one

Hence No. of LI Evectors = three = order

Hence P is Diagonalisable.

Orthogonal Complement of B → $\vec{p}$ = Projection Vector

$\vec{q}$ = orthogonal complement of $\vec{p}$



By vector Law, $\vec{p}+\vec{q}=\vec{B}$

$$\Rightarrow \boxed{\vec{q}=\vec{B}-\vec{p}}$$

$$PB + QB = B$$

$$(P+Q)B = B \Rightarrow \boxed{P+Q=I}$$

where Q = Projection Mat for orthogonal Complement of $\vec{p}$.

(*) What is the Nature of PQ? $PQ = P(I-P) = PI - P^2 = P - P = 0$

(Ans: NULL MAT)

Q: Find the projection Matrix P onto the line through $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ & hence find the projection of $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ onto A ?

(ii) Also find the Matrix Q that projects onto the line $\perp^r$ to A & hence find orthogonal Complement of B onto A.

(i) $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \end{bmatrix}$, $A^TA = 10$, $AA^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

so $P = \dfrac{AA^T}{A^TA} = \dfrac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ & projected vector $\vec{p} = PB = \dfrac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/10 \\ 12/10 \end{bmatrix}$

(2) w.k. that $P + Q = I$

$Q = I - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix} = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$

& orthogonal complement of B onto A is

$\vec{q} = \vec{B} - \vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/10 \\ 12/10 \end{bmatrix} = \begin{bmatrix} 6/10 \\ -2/10 \end{bmatrix}$

PROJECTION MATRIX (along a subspace, spanned by set of vectors) →

let x_1, x_2, x_3 are given vectors & we want to find the projection of $\vec{B}$ onto the subspace spanned by x_1, x_2, x_3 then process will be as follows;

Consider a Mat $A = [x_1\ x_2\ x_3]$ then

→ Projection Mat $\boxed{P = A(A^TA)^{-1}A^T}$

→ Projection Vector $\vec{p} = PB$

Verify: let $A = [x_1]_{n\times 1}$, $A^T = [\ \ ]_{1\times n}$,

then $A^TA = [\ \ \ ]_{1\times 1}$ = Scalar Quantity

so $P = A\underbrace{(A^TA)^{-1}}_{1\times 1}A^T$

$$= \frac{AA^T_{n\times n}}{A^TA_{1\times 1}} = \left(\frac{AA^T}{A^TA}\right)_{n\times n}$$

Q. The projection of $B=\begin{bmatrix}4\\5\\6\end{bmatrix}$ onto the Column space of $A=\begin{bmatrix}1&2\\1&3\\0&0\end{bmatrix}$ will be?

$C(A)$ = set of all Linear Combinations of Columns of A where $A=[X_1\ X_2]$

$=\{(c_1X_1+c_2X_2)\,;\ c_1,c_2\in \text{SCALARS},\ X_1 \,\&\, X_2 \text{ are Column of } A\}$

Here X_1, X_2 lies in 2D as well as they are L-I also (By observation)

So $C(A)$ is nothing but 2-D Space or $\boxed{C(A)=R^2}$

$A^TA=\begin{bmatrix}1&1&0\\2&3&0\end{bmatrix}\begin{bmatrix}1&2\\1&3\\0&0\end{bmatrix}=\begin{bmatrix}2&5\\5&13\end{bmatrix}$, $|A^TA|=1$ so $(A^TA)^{-1}=\frac{1}{(1)}\begin{bmatrix}13&-5\\-5&2\end{bmatrix}$

Standard Form of Projection Mat onto the subspace SPANNED by A is given as

$\boxed{P=A(A^TA)^{-1}A^T}=\begin{bmatrix}1&2\\1&3\\0&0\end{bmatrix}\begin{bmatrix}13&-5\\-5&2\end{bmatrix}\begin{bmatrix}1&1&0\\2&3&0\end{bmatrix}=\cdots=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ = P. Mat.

i.e. Projection Mat $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Projected vector $\vec{p} = PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

Analysis:

(1) $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{p} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

Consider $[A \vdots B] = \left[\begin{array}{cc:c} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 0 & 0 & 6 \end{array}\right]$

$\rho(A) = 2, \rho(A \vdots B) = 3$

i.e. $AX = B$ is InConsistent

$\Rightarrow AX = \vec{p}$ Now it is Consistent.

$\therefore [A \vdots \vec{p}] = \left[\begin{array}{cc:c} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \rho(A) = \rho(A \vdots \vec{p})$ i.e. Consistent.

KID'S EXAMPLE: $\rightarrow$

Doctor (Heart Beat)

$x = 116$
$x = 73$
$x = 94$
$x = 124$
$x = 53$
$x = 144$

(Inconsistent)

$\xrightarrow{\text{Project (medicine)}}$ $x = 98$

Consistent.

(2) Use $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$, $Tr(P) = 2$, $|P| = 0$, $\rho(P) = 2$

$\Downarrow$

$\lambda = 0, 1, 1$

$P^T = P \Rightarrow$ symm,

$P^2 = P \Rightarrow$ Idempotent

$\left.\begin{array}{l} \text{AM of '0'} = \text{one} \\ \text{AM of '1'} = \text{two} \end{array}\right\}$ i.e. A is Diagonalizable.

$\therefore$ No. of LI E.Vectors for A = three = order

Sp. Note: let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(Doubt by student)

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\rho(AB) = 0 \quad \& \quad \rho(BA) = 1.$$

i.e. if AB & BA Both exist, we are not sure that their Ranks are also equal. i.e. they may be

$$\rho(AB) \neq \rho(BA)$$

But $\boxed{\rho(AA^T) = \rho(A^TA)}$ always.

we are sure about above Result.

THANK - YOU