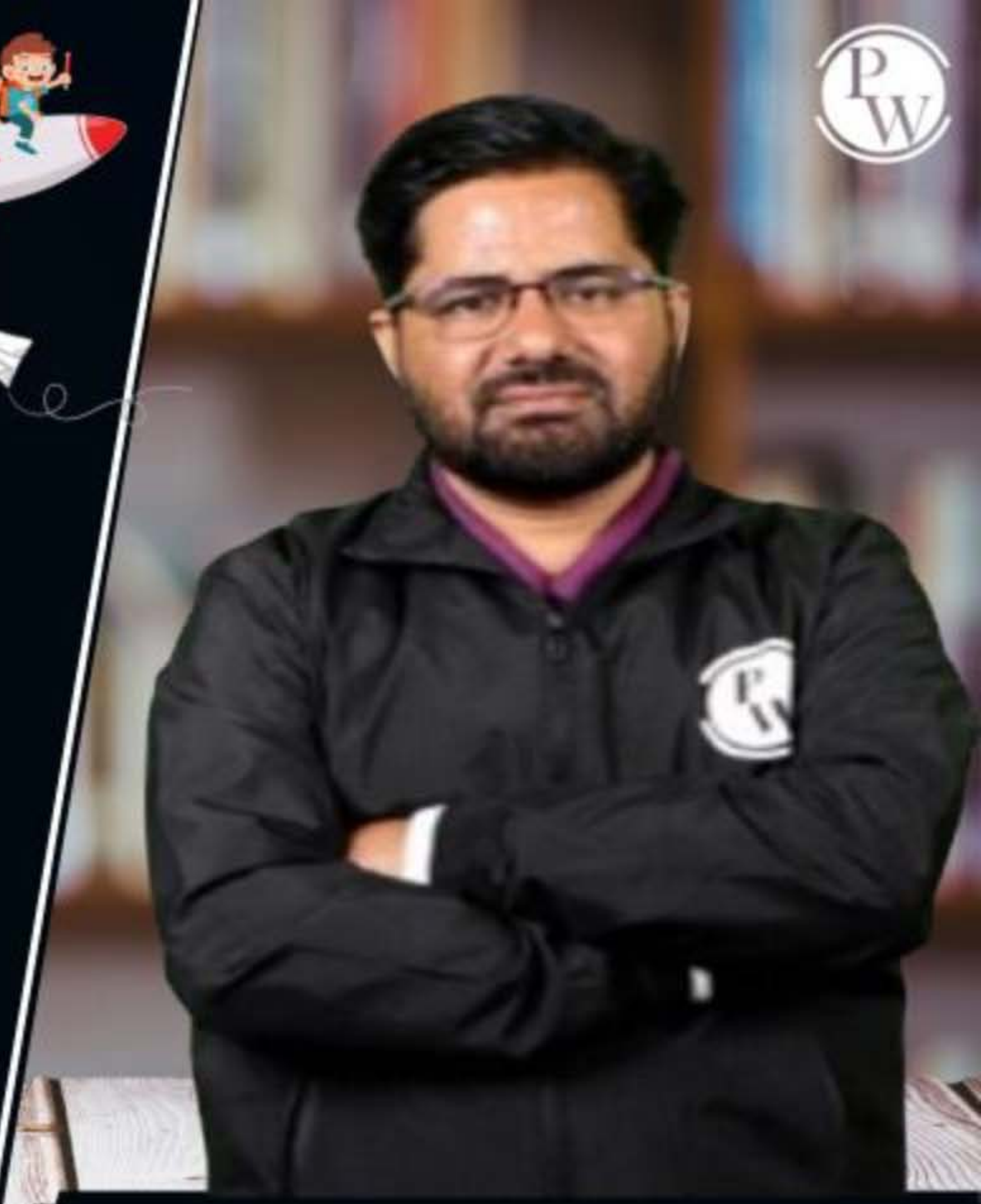
GATE

DS & AI
CS & IT

Linear Algebra

Lecture No. 02



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Recap of previous lecture

Topic BASICS of Determinants

Topics to be Covered

Topic

① Remaining Portion of Determinants

② BASICS of MATRICES

PROPERTIES of Det :-

(1) If in a Mat, Any two Rows (or any two columns) are identical then it's $\det = 0$

(2) If in a Mat, All the elements in any Row (or in any Column) are all zero then value of it's $\det = 0$

(3) $|ABC| = |A| \cdot |B| \cdot |C|$

(4) $|A+B+C| \neq |A|+|B|+|C|$

ie $|A+B+C| \leq |A|+|B|+|C|$

(5) $|A^m| = |A|^m$, $m \in N$

(6) $|A^T| = |A|$

(7) $|A|^{-1} = \frac{1}{|A|}$ eg If $|A| = 5$ then $|A|^{-1} = 5^{-1} = \frac{1}{5}$

(8) $|A^{-1}| = \frac{1}{|A|}$ ie $\boxed{\det(A^{-1}) = \frac{1}{\det A}}$

where $\cancel{A^{-1} = \frac{1}{A}}$ PAAP ; $A^{-1} = \frac{\operatorname{adj} A}{|A|} = \frac{(\operatorname{Cof} A)^T}{|A|}$ ✓

(9) Max No. of terms in $|A|_{n \times n} = n!$

(10) Area of $\triangle$ formed by (x_1, y_1), $(x_2 y_2)$, $(x_3 y_3)$ is $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

eg Area of $\triangle$ formed by $(2,7)$, $(3,6)$, $(4,5)$ will be? $= \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 3 & 6 & 1 \\ 4 & 5 & 1 \end{vmatrix} = \cdots\cdots = 0$

(i.e these points are Collinear.)

eg Area of $\triangle$ formed by $(1,0)$, $(2\ 2)$, $(4,3)$ will be? $= \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{3}{2}$

$\because$ area can't be -ve so Ans $= 1.5$.

(11) $$\frac{d}{dx}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix}$$

(12) $\begin{vmatrix} a+l & b & c \\ d+m & e & f \\ g+n & h & i \end{vmatrix} = ? = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} l & b & c \\ m & e & f \\ n & h & i \end{vmatrix}$

or $\begin{vmatrix} a+l & b+m & c+n \\ d & e & f \\ g & h & i \end{vmatrix} = ? = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} l & m & n \\ d & e & f \\ g & h & i \end{vmatrix}$

eg $\begin{vmatrix} 1+1 & -3 & 0 \\ 2+3 & +1 & 2 \\ 2+2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -3 & 0 \\ 3 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix}$

$=27 \qquad\qquad =9 \qquad\qquad =18$

Which one of the following does NOT equal

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} ? = |A| = (x-y)(y-z)(z-x)$$

(a) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

(d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

(a) $\begin{vmatrix} 1 & x^2+x & x \\ 1 & y^2+y & y \\ 1 & z^2+z & z \end{vmatrix} + \begin{vmatrix} 1 & x^2+x & 1 \\ 1 & y^2+y & 1 \\ 1 & z^2+z & 1 \end{vmatrix} \Rightarrow 0$

$= \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + \underbrace{\begin{vmatrix} 1 & x & x \\ 1 & y & y \\ 1 & z & z \end{vmatrix}}_{0} + 0$

$C_2 \leftrightarrow C_3$

$= - \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -|A|$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix} = ? = (x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$

$= (x-y)(y-z)\left[y+z-x-y\right] = (x-y)(y-z)(z-x) = |A|$

(d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - 2R_3} \begin{vmatrix} 0 & (x+y-2z) & (x^2+y^2-2z^2) \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

Q. if $\Delta = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3x \\ 0 & 2 & 5x \end{vmatrix}$ then $\frac{d\Delta}{dx} = ?$

Sol: $\frac{d\Delta}{dx} = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2 & 3x \\ 0 & 2 & 5x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 0 & 3 \\ 0 & 2 & 5x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3x \\ 0 & 0 & 5 \end{vmatrix}$

$= \left[1[10x-6x]-1[10x^2-6x^2]\right] + (-3)[2x-0] + (5)[2x-x^2] = -9x^2+8x$

Q. (MSQ) if $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3x \\ 0 & 2 & 5x \end{vmatrix}$ s.t. $f'(x) = -1$ then $x = ?$

(a) 1 (b) -1

(c) $-1/9$ (d) 0

Here $f'(x) = -9x^2+8x$

$-9x^2+8x = -1$

$9x^2-8x-1=0$

$(9x+1)(x-1)=0 \Rightarrow x = 1 \;\&\; -\frac{1}{9}$

HWQ (1) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = ? = (a-b)(b-c)(c-a)$ Learn

HWQ (2) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = ? = (a-b)(b-c)(c-a)(a+b+c)$

CW (3) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = ? = -(a+b+c)\left[a^2+b^2+c^2-ab-bc-ca\right]$

OR

$= -\left(a^3+b^3+c^3-3abc\right)$

OR

$= -\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

(3) $|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \xrightarrow{C_1 \to C_1 + C_2 + C_3} \begin{vmatrix} (a+b+c) & b & c \\ (a+b+c) & c & a \\ (a+b+c) & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$

$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = (a+b+c)\left[-(b-c)^2 - (a-b)(a-c)\right]$

$= -(a^3+b^3+c^3-3abc)$ Ans

$= -\left\{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\right\}$ Ans

$= -\frac{1}{2}\left[(a+b+c)\left\{2a^2+2b^2+2c^2-2ab-2bc-2ca\right\}\right]$

$= -\frac{1}{2}\left[(a+b+c)\left\{(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca)\right\}\right]$

$= -\frac{1}{2}\left[(a+b+c)\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}\right]$ Ans

(13) Special Property of Det. :- Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}_{3\times 3}$ & $B = \begin{bmatrix} 8 & -4 & 12 \\ 16 & 0 & 4 \\ 4 & 8 & -4 \end{bmatrix} = 4A$

Here, $|A| = 15$ & $|B| = 960$

$|4.A| = 64 \times 15 = 4^3 \cdot |A|$

M II $|B| = \begin{vmatrix} 8 & -4 & 12 \\ 16 & 0 & 4 \\ 4 & 8 & -4 \end{vmatrix} = 4 \times 4 \times 4 \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 4^3 \cdot |A|$

is in general, if $A_{n\times n}$ & K is any scalar then

$[KA] = K \cdot [A]$

$\boxed{|KA| = K^n \cdot |A|}$

Eg: if $|A|_{3\times3} = 6$ then

(1) $|(2A)^{-1}| = ? = \frac{1}{|2A|} = \frac{1}{2^3 \cdot |A|} = \frac{1}{8\times6}$

(2) $|2 \cdot A^{-1}| = ? = 2^3 \cdot |A^{-1}| = 2^3 \cdot \frac{1}{|A|} = \frac{8}{6}$

$\because A_{3\times3} \Rightarrow (A^{-1})_{3\times3}$

Eg: if $|A|_{3\times3} = -1$, $|B|_{3\times3} = 4$ then $|5AB| = ?$ (a) 0 (b) -4 (c) -20 (d) -500

Sol: $\because A_{3\times3}$ & $B_{3\times3} \Rightarrow (AB)_{3\times3}$

So $|5AB| = 5^3 \cdot |AB| = 125\,|A| \cdot |B|$

$= 125(-1)(4) = -500$

ANALYSIS (PODCAST) $\rightarrow$ w.k. that $A^{-1} = \dfrac{\text{adj}A}{|A|}$

$$A \cdot A^{-1} = \frac{A \cdot (\text{adj}A)}{|A|}$$

$$I = \frac{A(\text{adj}A)}{|A|}$$

i.e. $A(\text{adj}A) = |A| \cdot I$

if $A_{n\times n}$ then $\boxed{A(\text{adj}A) = |A| \cdot I_{n\times n}}$

i.e. $A(\text{adj}A) = |A| I_n = |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$

$$= \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}_{n\times n} = \text{Scalar Mat}$$

So $\text{Trace}(A\,\text{adj}A) = \underbrace{|A| + |A| + \cdots + |A|}_{} = n|A|$

$$|A\,\text{adj}A| = \underbrace{|A| \cdot |A| \cdots\cdots |A|}_{n\text{ times}} = |A|^n$$

(14) if $A_{n\times n}$ then evaluate $|A.\text{adj}A|$, $|\text{adj}A|$, $|\text{Cof}A|$, $|A^{-1}| = ?$

Sol: we have already shown that

(i) $|A.\text{adj}A| = |A|^n$

(ii) $|A|.|\text{adj}A| = |A|^n$

$$|\text{adj}A| = \frac{|A|^n}{|A|} = |A|^{(n-1)}$$

(iii) we know that,

Det of any Mat = Det of it's Transpose

So $|\text{Cof}A| = |(\text{Cof}A)^T| = |\text{adj}A| = |A|^{n-1}$

(iv) $A^{-1} = \frac{\text{adj}A}{|A|} = \frac{\text{Matrix}}{\text{Number}}$

$$|A^{-1}| = \left|\frac{\text{adj}A}{|A|}\right| = \frac{1}{|A|^n}|\text{adj}A|$$

$$= \frac{|A|^{n-1}}{|A|^n} = \frac{1}{|A|}$$

ie $\det(A^{-1}) = \frac{1}{\det(A)}$

(15) if $A_{n\times n}$ then $|\underbrace{\text{adj adj adj adj} \ldots \text{adj}}_{r \text{ times}} A| = |A|^{(n-1)^r}$

eg $|\text{adj} A| = |A|^{(n-1)^1}$, $|\text{adj adj} A| = |A|^{(n-1)^2}$ & so on - - - -.

Qs if $|A|_{4\times 4} = 5$ then $|\text{adj adj adj} A| = ? = |A|^{(n-1)^3} = (5)^{(4-1)^3} = (5)^{3^3}$

(a) 5^3 (b) 5^9

(c) 5^{27} (d) 0

Here $n=4$, $r=3$

Note: $(a^b)^c = a^{bc}$

$(a)^{b^c} = a^{\underbrace{b\times b\times b\times \ldots \times b}_{c \text{ times}}}$

$= (5)^{3\times 3\times 3}$

$= 5^{27}$

MATRIX.

Defⁿ: Matrix is a Rectangular arrangement of $m \cdot n$ numbers.

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$1 \le i \le m$

$1 \le j \le n$

H. Lines = Rows

V. Lines = Columns

Square Mat: Defⁿ ①: if No. of Rows = No. of Columns. then it is Sq. Mat

Defⁿ ②: If in a Matrix, Diag exist exist then it must be Sq. Mat

Defⁿ ③ if in a Mat, Corresponding element exist for every element.

Sq. Matrix :

$$A = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

1. for diag. elements, $i = j \ \forall i \,\&\, j$
2. for upper diag elements, $i < j \ \forall i \,\&\, j$
3. for lower diag elements, $i > j$ " "
4. for off diag elements, $i \neq j$ " "

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5. Corresponding elements are a_{ij} & a_{ji}

Trace(A) = sum of diag elements

or $Tr(A) = \sum_{i=1}^{n} (a_{ii})$

eg $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ = U.T.M

$a_{ij} = 0 \ \forall i > j$

Some Special Matrices :-

$$\begin{bmatrix} 2 & -4 & 0 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 2 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

U.T.M	L.T.M	Diag Mat	Scalar Mat	Identity Mat
Tr = 2	Tr = 7	Tr = 4	Tr = 20	Tr = 4
Det = −24	Det = 0	Det = 0	Det = 5^4 = 625	Det = 1

(*) To Find Det of U.T.M, L.T.M, Diag Mat, Scalar Mat, Identity Mat, we can multiply diagonal elements i.e $\boxed{\text{Det} = \text{Product of Diag elements}}$

(*) if $A = [a_{ij}]_{n\times n}$ s.t $\underbrace{a_{ij} = 0 \ \forall\, i > j}_{\text{Lower Diag elements} = 0}$ then A is U.T.M

Q. Evaluate, $|A| = \begin{vmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 4 & 0 \\ -2 & -4 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{vmatrix}_{4\times 4} = ?$

(a) 0 (b) 4

(c) 16 (d) 36 ✓

(M-I) $|A| = \cancel{0\times 0\times 0\times 0} = 0$ ☹

∵ A is Neither U.T.M, Nor L.T.M & not a Diag Mat, So we can not use the shortcut method defined in previous slide.

So we will find $|A|$ by conventional approach.

(M-II) Expanding along C_4.

$$|A| = -6 \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & 4 \\ 0 & 0 & -6 \end{vmatrix} = -6\left[(-6)(0+1)\right] = 36$$

Thank
you

Keep Hustling!