DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Not for CS/IT

Linear Algebra - II

Lecture No. 03

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Recap of previous lecture

Topic

S.V.D

Topics to be Covered

Topic

$\boxed{\text{VECTOR SPACE}}$

- → SUBSPACE
- → BASIS
- → DIMENSION of VECTOR SPACE
- → Concept of SPANNIG in Vector Space.

VECTOR SPACE

Def^n: $\rightarrow$ Any set of vectors V is called V.space if it satisfy following properties;

① Let $X, Y \in V \Rightarrow X+Y \in V$ i.e Closure property of vector addition holds.

② $\forall X, Y \in V \Rightarrow X+Y = Y+X$ i.e Commutative prop of V.addition holds.

③ $\forall X, Y, Z \in V \Rightarrow (X+Y)+Z = X+(Y+Z)$ i.e Associative prop of V.addition holds.

④ $\forall X \in V \ \exists \ 0 \in V$ s.t $X+0 = X$ i.e additive Identity exist

⑤ $\forall X \in V \ \exists -X \in V$ s.t $X+(-X) = 0$ i.e Additive inverse exist.

⑥ $\forall X \in V \ \exists$ scalar c s.t $c.X \in V$ i.e Closure prop for scalar multiplication holds.

⑦ $\forall X \in V \ \exists$ scalars λ & μ s.t $\lambda(\mu X) = (\lambda\mu)X$ i.e Asso. Prop for Scalar Multi holds.

⑧ $\forall X \in V \ \exists$ scalar 1 s.t $1.X \in V$ i.e Multiplicative identity exist.

(9) $\forall X, Y \in V \; \exists$ scalar λ s.t. $\lambda(X+Y) = \lambda X + \lambda Y$ i.e. distributive prop holds.

(10) $\forall X \in V \; \exists$ scalars λ & μ s.t. $(\lambda+\mu)X = \lambda X + \mu X$ " " " ".

Any set of vectors `V' satisfying all 10 properties is called V-space.

EXAMPLE:-

$V = R^2 = $ 2-D vector space, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$V = R^3 = $ 3-D " " , $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$V = R^4 = $ 4-D " " "

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$V = R^n = $ n-Dim " " $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Verification :- Consider $V = \mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x, y \in \mathbb{R} \right\}$

① Let $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, Y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ then $X+Y = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \in \mathbb{R}^2$ i.e. Closure prop for V. Addition Holds.

② $\because X+Y = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = Y+X$ i.e. Commutative prop for V. Addition Holds.

③ Let $Z = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ then $(X+Y)+Z = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ i.e. Associative prop for

& $X+(Y+Z) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ V. addition Holds.

④ $\because 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ & $X+0 = X$ i.e. Additive Identity exist.

⑤ $\because X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, -X = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \in \mathbb{R}^2$ & $X+(-X) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \in \mathbb{R}^2$ i.e. Additive inverse exist

(6) let $c=5$, $x=\begin{bmatrix}2\\3\end{bmatrix}$ then $cx=5x=\begin{bmatrix}10\\15\end{bmatrix}\in\mathbb{R}^2$ i.e Closure prop for Scalar Multi Holds.

(7) let $\lambda=3, \mu=-5$ then $\lambda(\mu x)=3(-5x)=3\begin{bmatrix}-10\\-15\end{bmatrix}=\begin{bmatrix}-30\\-45\end{bmatrix}\in\mathbb{R}^2$

$$\& \ (\lambda\mu)x=(3(-5))x=-15\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}-30\\-45\end{bmatrix}\in\mathbb{R}^2$$

i.e $\lambda(\mu x)=(\lambda\mu)x$ so Associative prop for scalar Multi Holds.

(8) let $c=1 \Rightarrow cx=1.x=1.\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}=x\in\mathbb{R}^2$ i.e Multi Identity exist.

(9) let $\lambda=3, x=\begin{bmatrix}2\\3\end{bmatrix}, y=\begin{bmatrix}-1\\2\end{bmatrix}$ then $\lambda(x+y)=3\left(\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}\right)=3\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}3\\15\end{bmatrix}$

$$\& \ \lambda x+\lambda y=3\begin{bmatrix}2\\3\end{bmatrix}+3\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix}+\begin{bmatrix}-3\\6\end{bmatrix}=\begin{bmatrix}3\\15\end{bmatrix}$$

(10) $\lambda=3, \mu=-5, x=\begin{bmatrix}2\\3\end{bmatrix}$

$(\lambda+\mu)x=(3-5)x=-2x$

$\lambda x+\mu x=3x-5x=-2x$

i.e Both types of Distributive Prop Holds.

Subspace → let V is any Vector space & W is any subset of V ie $\boxed{W \subset V}$

if W is itself a vector space then W is called subspace of V.

OR

if W satisfies all the 10 properties of V.space then W is called Subspace of V.Space (V)

Shortcut: In order to CROSS check whether W is subspace or Not, only check following two conditions

Subspace

① $\forall X, Y \in W \Rightarrow \boxed{X+Y \in W}$ ie closure prop for V. Addition

② $\forall x \in W, \exists$ scalar c st $\boxed{cX \in W}$ ie closure prop for scalar Multi.

Q) Let M_n denote the vector space of all $n \times n$ real matrices. Consider the following subsets of M_n

MSQ

1. $W_1 = \{A \in M_n \mid A^2 = I\}$, where I is the identity matrix.
2. $W_2 = \{A \in M_n \mid \text{rank}(A) = 1\}$
3. $W_3 = \{A \in M_n \mid \text{trace}(A) = 0\}$
4. $W_4 = \{A + B \mid A \in M_n, B \text{ is a fixed matrix in } M_n\}$

Which of the following statements is correct?

(a) W_1 is a linear subspace of M_n.

(b) W_2 is a linear subspace of M_n.

c) W_3 is a linear subspace of M_n (T)

d) W_4 is a linear subspace of M_n only if B is a zero matrix. (T)

i.e. W_1 is Not Closed under Vector addition

Hence W_1 is not Subspace

(1) $W_1 = \{A \in M_{n\times n} : A^2 = I\}$ Let $A, B \in W_1$ then $A^2 = I$, $B^2 = I$ (given)

Now, $(A+B)^2 = A^2 + B^2 + AB + BA = I + I + AB + BA \neq I$.

i.e. $(A+B)$ is Not Necessarily Involutary Mat i.e. $(A+B)^2 \neq I \Rightarrow \boxed{A+B \notin W_1}$

$W_2 = \{A : \rho(A) = 1\}$, let $A, B \in W_2$ then $\rho(A) = 1$, $\rho(B) = 1$ (given)

Now $\rho(A+B) \leq \rho(A) + \rho(B)$ i.e. $\rho(A+B) \leq 1+1$ i.e. $\rho(A+B) \leq 2$

$\Rightarrow$ it is not necessary that $\rho(A+B) = 1 \Rightarrow \boxed{A+B \notin W_2}$

i.e. we are not sure that W_2 is closed under V. addition

Hence W_2 is not a subspace.

$W_3 = \{A : Tr(A) = 0\}$, Let $A, B \in W_3 \Rightarrow Tr(A) = 0$ & $Tr(B) = 0$

Now $Tr(A+B) = Tr(A) + Tr(B) = 0 + 0 = 0$ i.e. $\boxed{A+B \in W_3}$ Closure Prop Holds.

Again $Tr(CA) = C\,Tr(A) = C(0) = 0$ i.e. $\boxed{CA \in W_3}$ Closure Prop Holds.

Hence W_3 is a subspace of $M_{n \times n}$

$W_4 = \{ A+B : B \text{ is a fixed Mat} \}$.

we have two possibilities for B: either $B = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

or $B \neq O = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$

if we take $B \neq O$ then

$A+B \neq A$ ie additive identity DNE in W_4.

& Hence W_4 is not a Vector Space.

if we take $B = O$ then

$A+B = A+O = A$ ie Additive identity exist

& Hence W_4 can be a Vector Space

only when we take $B = O$ = fixed Matrix

Q) Let S_1, S_2, S_3 be sets of real-valued functions defined as:

MCQ

$S_1 = \{ f \mid f(3) = 0 \}$

$S_2 = \{ g \mid g(x) = x+1 \text{ for all } x \in R \}$

$S_3 = \{ h \mid h(x) = c \text{ for some constant } c \in K \}$

Which of the following statements is correct?

a) S_1 is a vector space & S_2 is not a vector space.

b) ~~S_2 is a vector space & S_3 is not a vector space.~~

c) S_1 & S_3 are vector spaces, but S_2 is not a vector space

d) ~~S_2 & S_3 are vector spaces but S_1 is not a vector space.~~

Ans = (c)

But in case of MSQ

Ans = (a) & (c) both.

(1) $S_1 : \{ f(x) : f(3) = 0 \}$

Let $f_1(x), f_2(x) \in S_1$ then

$f_1(3) = 0$ & $f_2(3) = 0$

Now $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$= 0 + 0 = 0$

ie $f_1 + f_2 \in S_1$

$\boxed{\forall f_1, f_2 \in S_1 \Rightarrow (f_1 + f_2) \in S_1}$

Now $(cf)(3) = c\, f(3) = c(0) = 0$

ie $cf \in S_1$

$\boxed{\forall f \in S_1 \ \exists \text{ scalar } c \text{ st } cf \in S_1}$

Hence S_1 is a V. Space.

$S_3 = \{ h(x) : h(x) = c \}$

Let $h_1(x), h_2(x) \in S_3$ then $h_1(x) = c$, $h_2(x) = c$

Now $(h_1 + h_2)(x) = h_1(x) + h_2(x) = c + c = 2c = c_1 \Rightarrow (h_1 + h_2) \in S_3$

i.e $\boxed{\forall h_1, h_2 \in S_3 \Rightarrow h_1 + h_2 \in S_3}$ i.e closure holds.

Let $h(x) \in S_3$ then $h(x) = c$

Let λ is any scalar then $(\lambda h)(x) = \lambda[h(x)] = \lambda(c) = c_2 \Rightarrow (\lambda h) \in S_3$

i.e $\boxed{\forall h \in S_3 \ \exists \text{ scalar } \lambda \text{ s.t } \lambda h \in S_3}$ i.e closure holds.

Hence S_3 is also a V. Space.

$S_2 = \{ g(x) : g(x) = x+1 \}$

Let $g_1(x)$ & $g_2(x) \in S_2$ then $g_1(x) = x+1$ & $g_2(x) = x+1$

Now $(g_1+g_2)(x) = g_1(x) + g_2(x) = (x+1)+(x+1) = 2(x+1) \neq x+1$

i.e. $g_1+g_2 \notin S_2 \Rightarrow$ Closure prop for V. Addition Not Holds.

So S_2 is not a V. Space.

Linear span → Let S is any set of vectors then

"set of all linear combinations of vectors in S is called Linear SPAN of S"
and it is denoted by $L\{S\}$

Let $S = \{X_1, X_2, X_3, X_4\}$ then all the linear combination of these vectors can be written as $\boxed{K_1X_1 + K_2X_2 + K_3X_3 + K_4X_4}$ where $K_i \in R$.

then $L\{S\} = \{(K_1X_1 + K_2X_2 + K_3X_3 + K_4X_4) ; K_i \in R\}$ = Linear SPAN of S.

SPANNING in VECTOR SPACE → Let W is any subspace & S is any subset of W

then W is called SPANNED by S if $\boxed{W = L\{S\}}$
generated

OR

S SPANS W ie $\boxed{L(S) = W}$
generates

ie Any Random vector of W can be written as Linear comb of vectors in S.

for eg Consider $W = R^2$ & $S = \{X_1, X_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ then

$\boxed{W = L\{S\}}$ ie W is SPANNED by vectors X_1 & X_2

ie Any Random Vector in 2D can be generated with the Linear comb of X_1 & X_2

VERIFICATION :

$$S = \{X_1, X_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

then S spans R^2 i.e $L(S) = R^2$

Let $X = \begin{bmatrix}2\\-3\end{bmatrix} \in R^2$ then $\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\-3\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\end{bmatrix} = 2X_1 - 3X_2$

i.e $\boxed{X = 2X_1 - 3X_2}$

Let $Y = \begin{bmatrix}\lambda\\\mu\end{bmatrix} \in R^2$ then $Y = \begin{bmatrix}\lambda\\\mu\end{bmatrix} = \begin{bmatrix}\lambda\\0\end{bmatrix} + \begin{bmatrix}0\\\mu\end{bmatrix} = \lambda\begin{bmatrix}1\\0\end{bmatrix} + \mu\begin{bmatrix}0\\1\end{bmatrix} = \lambda X_1 + \mu X_2$

i.e we have justified that, Any Random Member of R^2 can be expressed as a Linear Combination of X_1 & X_2 i.e $L\{S\} = R^2 \Rightarrow S$ generates R^2 or S spans R^2

Similarly if we take $S = \{X_1\ X_2\ X_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

then S will generate R^3 i.e $\boxed{S \text{ spans } R^3}$ or $L\{S\} = R^3$

Cross check, let $X = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in R^3$ then $X = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}$

$$X = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{X = \alpha X_1 + \beta X_2 + \gamma X_3}$$

i.e any Random Member X of R^3 can be expressed as a L-comb of X_1, X_2, X_3

$\Longrightarrow S$ Spans $R^3 \Longrightarrow L\{S\} = R^3$

BASIS of Vector space → let W is any subspace & S is any given set of vectors.

then S is called $\boxed{\text{BASIS}}$ for W if
① S $\boxed{\text{SPANS}}$ W
② S contains L.I set of vectors.

for eg Basis for $W = R^2$ is $S_1 = \{X_1 \; X_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\because$ $L\{S_1\} = R^2$ & S_1 contains LI vectors (By observation)

again, Basis for $W = R^3$ is $S_2 = \{X_1 \; X_2 \; X_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\because$ $L\{S_2\} = R^3$ & S_2 contains LI vectors (By Tricky Method)

$\because |S_2| \neq 0$

RECAP: Let $x_1, x_2, x_3 \ldots x_r$ are the given set of vectors

& consider $A = [x_1, x_2\ x_3 \ldots x_r]$

① **General Method**

(i) if $\rho(A) =$ No. of vectors $\Rightarrow$ LI

(ii) if $\rho(A) <$ " " $\Rightarrow$ LD

② **Tricky Method** (if A is Sq. Mat)

(i) if $|A| \neq 0 \Rightarrow$ vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

Note: In case of two vectors x_1 & x_2, there is no need to follow G. Method or Tricky Method, only follow observation Method.

Dimension of vector space $\rightarrow$ = $\boxed{\text{Number of vectors in it's BASIS}}$

for eg $\dim(R^2) = 2$ $\because$ Basis for $R^2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \{X_1, X_2\}$

" $\dim(R^3) = 3$ $\because$ Basis for $R^3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{X_1, X_2, X_3\}$

Let W is subspace of R^3 then $\dim W = ?$ = Not Necessarily 3

it will depend on the Basis of W (will be discussed Later)

if V is any Vector space s.t it's Basis $= \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\}$

then $\dim(V)$ = No. of vectors in Basis = 5

Q: Consider the set of column vectors defined by $X = \{(x_1\ x_2\ x_3)^T ; \boxed{x_1 + x_2 + x_3 = 0}\}$

then $S = \{(1\ {-1}\ 0)', (1\ 0\ {-1})'\}$ is Basis for subspace X ? YES

Sol: $\because X \subset R^3$ so X is subspace of R^3 defined as.

$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; x_1 + x_2 + x_3 = 0 \right\}$$

Now $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} = \{X_1, X_2\}$

will be Basis for X if
- S will SPAN X &
- S contains LI vectors.

Clearly X_1 & X_2 are LI (by observation) Hence 2^{nd} condn holds.

Now Let us take any Random vector of subspace X say it is $Y = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ then

$\boxed{Y = 3X_1 - X_2}$ so Y can be generated by X_1 & X_2 Hence $\boxed{\text{S SPAN X}}$

$\Rightarrow$ S is Basis for subspace X & $\dim(X) = 2$

Doubt: let $Z = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$, $S = \{X_1, X_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

$\therefore \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\boxed{Z = -3X_1 - 2X_2}$ i.e. Z is also generated by X_1 & X_2

Hence S $\boxed{\text{SPANS}}$ subspace X

Hence S is Basis for subspace X & $\dim(X) = $ No. of vectors in S

$= 2$

Consider the set of (column) vectors defined by $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0$, where $x^T = [x_1, x_2, x_3]^T\}$. Which of the following is TRUE?

(a) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a basis for the subspace X. T

(b) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X

(c) X is not a subspace for R^3

(d) None of the above

If the vectors, $e_1 = (1,0,2), e_2 = (0,1,0)$ and $e_3 = (-2,0,1)$ from an orthogonal basis of the three dimensional real space $\mathbb{R}^3$, then the vector $u = (4,3-3) \in \mathbb{R}^3$ can be expressed as

(a) $u = -\frac{2}{5}e_1 - 3e_2 - \frac{11}{5}e_3$

(b) $u = -\frac{2}{5}e_1 - 3e_2 + \frac{11}{5}e_3$

(c) $u = -\frac{2}{5}e_1 + 3e_2 + \frac{11}{5}e_3$

(d) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

$S = \{e_1\ e_2\ e_3\}$

$= \left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}$

= Basis of R^3 (given)

∵ S is Basis for R^3 So $\boxed{\text{S will SPAN } R^3}$

⇒ Any Random Vector of R^3 (say $u = \begin{bmatrix}4\\3\\-3\end{bmatrix}$)

Can be expressed as a Linear Combination of Vector in S

ie $u = k_1e_1 + k_2e_2 + k_3e_3$

..... M-II HW

M-I let us take (d)

$$\begin{bmatrix}4\\3\\-3\end{bmatrix} = -\frac{2}{5}\begin{bmatrix}1\\0\\2\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{11}{5}\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

$$-3 = -\frac{4}{5} + 0 - \frac{11}{5} = -\frac{15}{5} = -3$$

THANK - YOU