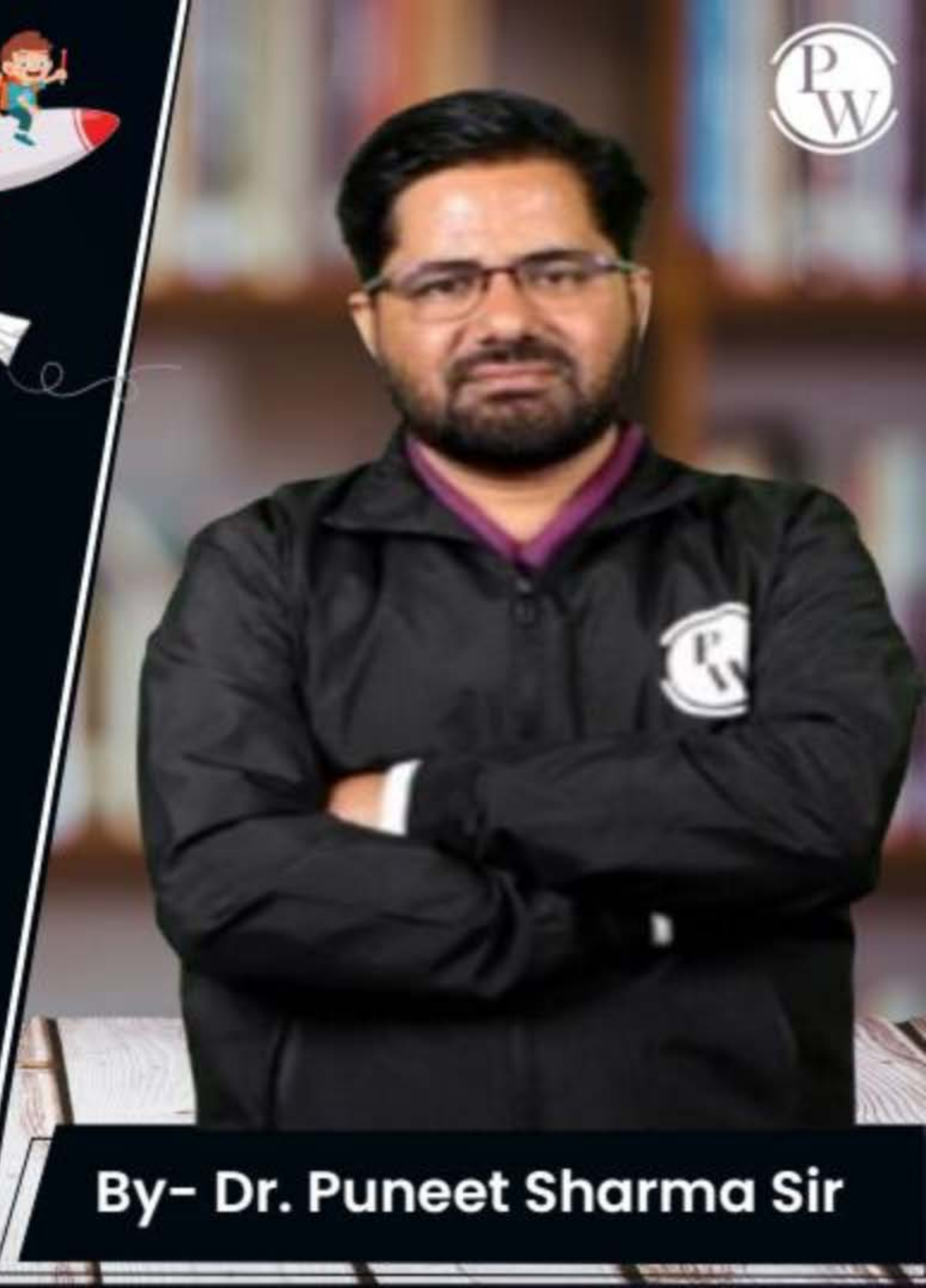
GATE

DS & AI
CS & IT

Linear Algebra

Lecture No. 04



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Recap of previous lecture

Topic

Algebra of Matrices

Topics to be Covered

Topic

Types of matrices

(*) In Matrix Algebra, Concept of Division is not defined b/w two Matrices

In Number System $ab = c \Rightarrow b = \frac{c}{a}$, $a \neq 0$. ✓

while In Matrix Algebra, $AB = C \Rightarrow B = \cancel{\frac{C}{A}}$ Blunder.

Note If A is Non Sing Mat s.t

$$AB = C$$

$$A^{-1}(AB) = A^{-1}\cdot C$$

$$(A^{-1}A)B = A^{-1}C$$

$$IB = A^{-1}C \Rightarrow B = A^{-1}C$$

(*) In Matrix Algebra,

"Cancellation Law Holds only when Matrix is Non Singular."

eg if A is a Non Sing Mat s.t

$$\cancel{A}B = \cancel{A} \Rightarrow B = I \ (T)$$

Proof:

$$AB = A$$

$$A^{-1}(AB) = A^{-1} \cdot A$$

$$(A^{-1}A)B = I$$

$$\boxed{B = I}$$

eg if $\cancel{A}B = \cancel{A}$ then $B = I$ (False)

$\because$ we have No idea about the Non Singularity of A

Doubt: If $(ABC = AC) \not\Rightarrow (B = I)$

Que If $\boxed{XY=Y}$ — (1) & $\boxed{YX=X}$ — (2) then $X^2+Y^2=?$

MSQ

(a) $X+Y$

(b) $2I$

(c) O

(d) I

Case I: (If X & Y are Non Singular)

$\Rightarrow X^{-1}$ & Y^{-1} exist

So $XY=Y \Rightarrow X=I$

& $YX=X \Rightarrow Y=I$

So $X^2+Y^2 = I^2+I^2$

$= I+I = 2I$

(b)

Case II: (If X & Y are Singular)

$XY=Y$ — (1) & $YX=X$ — (2)

Now $X^2+Y^2 = X.X+Y.Y$

$= X(YX)+Y(XY)$

$= (XY)X+(YX)Y$

$= (Y)X+(X)Y$

$= X+Y$

(a)

If X and Y are two singular matrices such that XY = Y and YX = X then $X^2 + Y^2$ equals

(a) X + Y (b) XY

(c) YX (d) 2(X + Y)

$XY = Y$ — (1) & $YX = X$ — (2)

$X^2 + Y^2 = XX + YY$

$= X(YX) + Y(XY)$

$= (XY)X + (YX)Y$

$= (Y)X + (X)Y$

$= X + Y$

(a)

Conjugate of Complex no. $\rightarrow$ $z = x + iy$, $\bar{z} = x - iy$

$z_1 = 2 - 3i$, $\bar{z}_1 = 2 + 3i$

$z_2 = -5$, $\bar{z}_2 = -5$

$z_3 = 4i$, $\bar{z}_3 = -4i$

Tranjugate Mat $\longrightarrow$ if A is given Mat then $A^{\theta} = \overline{(A^T)}$ or $(\bar{A})^T$

(Transposed Conj of Mat)

eg $A = \begin{bmatrix} 2 & 4+i & -7i \\ 2-3i & 0 & 4 \end{bmatrix}_{2\times 3}$ then $A^{\theta} = ?$ $= \overline{(A^T)} = \begin{bmatrix} 2 & 2+3i \\ 4-i & 0 \\ 7i & 4 \end{bmatrix}_{3\times 2}$

Sol: $A^T = \begin{bmatrix} 2 & 2-3i \\ 4+i & 0 \\ -7i & 4 \end{bmatrix}_{3\times 2}$, $\bar{A} = \begin{bmatrix} 2 & 4-i & 7i \\ 2+3i & 0 & 4 \end{bmatrix}_{2\times 3}$

M-II $A^{\theta} = (\bar{A})^T$ //

Real Mat → If all the elements in a Mat are Real Nos then A is called Real Mat

if $\bar{A} = A$ (or $A^\theta = A^T$) then A is Real Mat

Complex Mat → If at least one element in a Mat is Complex No then Mat is called Comp. Mat.

if $\bar{A} \neq A$ then A is Comp. Mat

eg $A = \begin{bmatrix} 2 & 4 & -1 \\ 2 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 2 & 4i \end{bmatrix}$

= Real Mat = Comp. Mat

(*) the Concept of Trace, Det, Inverse, Eigen Values are defined Square Matrices

While " of Transpose, Conjugate, Tranjugate, Rank are also defined for Rectangular Matrices

Some Confusions:-

(1) $(A+B+C)^T = A^T + B^T + C^T$

(2) $(A+B+C)^\theta = A^\theta + B^\theta + C^\theta$

(3) $Tr(A+B+C) = Tr(A) + Tr(B) + Tr(C)$

(4) $|A+B+C| \le |A| + |B| + |C|$

(5) $(A+B+C)^{-1} = \dfrac{adj(A+B+C)}{|A+B+C|}$

(6) $AB \neq BA$ But $Tr(AB) = Tr(BA)$

(7) $|ABC| = |A|\cdot|B|\cdot|C|$

(8) Reversal Law →

(i) $(ABC)^T = C^T B^T A^T$

(ii) $(ABC)^\theta = C^\theta B^\theta A^\theta$

(iii) $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

Q. If A, B, C, D are $\underbrace{\text{Non Singular}}$ Matrices of $\underbrace{\text{same order}}$ s.t

$\boxed{ABCD = I}$ then B & $B^{-1} = ?$

Sol: (M-I) (WRONG App)

$B = \dfrac{I}{\cancel{A}CD} = I \cdot A^{-1} \cdot C^{-1} \cdot D^{-1}$

$B = A^{-1}C^{-1}D^{-1}$ Blunder

Note: $A^{-1}\underbrace{C^{-1}D^{-1}} \neq A^{-1}\underbrace{D^{-1}C^{-1}}$

(M-II) $\boxed{ABCD = I}$

$A^{-1}(ABCD)D^{-1} = A^{-1} \cdot I \cdot D^{-1}$

$(A^{-1}A)BC(DD^{-1}) = A^{-1}D^{-1}$

$BC = A^{-1}D^{-1}$

$(BC)C^{-1} = A^{-1}D^{-1}C^{-1}$

$B = A^{-1}D^{-1}C^{-1}$ Ans

(ii) $B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$

$= (C^{-1})^{-1} \cdot (D^{-1})^{-1} \cdot (A^{-1})^{-1}$ (By R. Law)

$= CDA$ Ans

Some Special Types of Matrices $\rightarrow$

$$\begin{bmatrix} a & b & -c \\ b & d & \alpha \\ -c & \alpha & h \end{bmatrix}, \quad \begin{bmatrix} 0 & b & -c \\ -b & 0 & \alpha \\ c & -\alpha & 0 \end{bmatrix}_{3\times 3}, \quad \begin{bmatrix} 2 & 4+i & 2i \\ 4-i & 0 & -5 \\ -2i & -5 & -3 \end{bmatrix}, \quad \begin{bmatrix} 2i & 4+i & 2i \\ -4+i & 0 & -5 \\ 2i & 5 & -3i \end{bmatrix}$$

Symmetric Mat ($\because A^T = A$)

Skew Symm (Anti Symm) ($\because A^T = -A$) $\rightarrow |A| = 0$

Hermitian Mat ($\because A^\theta = A$)

Skew Hermitian ($\because A^\theta = -A$)

Note ① Symm Mat is symmetrical about leading diagonal
i.e. Corresponding elements are same & $A^T = A$

② Skew symm Mat : Diagonal elements are all zero and corresponding elements are of opposite sign & $A^T = -A$

③ Hermitian Mat : Diagonal elements are either zero or purely Real Nos & Corresponding elements are conjugates of each other & $A^\theta = A$.

④ Skew Hermitian Mat : Diagonal elements are either zero or purely Imaginary numbers and corresponding elements are −ve conjugates of each other. & $A^\theta = -A$

(*) If $A_{n \times n}$ s.t. A is skew symm then $|A| = \begin{cases} 0 & , n = \text{odd} \\ \text{Perfect square} & , n = \text{even} \end{cases}$

i.e. Det of skew symm Mat of odd order $= 0$

eg: $A = \begin{bmatrix} 0 & -1 & -2 & 3 & 4 \\ 1 & 0 & 1 & -2 & 0 \\ 2 & -1 & 0 & 4 & 1 \\ -3 & 2 & -4 & 0 & 2 \\ -4 & 0 & -1 & -2 & 0 \end{bmatrix}_{5 \times 5}$ = skew symm.

then $|A| = ? = 0$

eg $A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 4 & 0 \\ -2 & -4 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{bmatrix}_{4 \times 4}$ = skew symm.

$|A| = \cdots\cdots = 36 = (6)^2$ = Perfect sq.

Q If $A = \begin{bmatrix} 0 & - & - & - \\ - & 0 & - & - \\ - & - & 0 & - \\ - & - & - & 0 \end{bmatrix}_{4\times 4}$ is any skew symm Mat then Possible Value of $|A| = ?$

MCQ

(a) 3 (b) 17 (c) 81 (d) 31

Q if $A = \begin{bmatrix} 0 & - & - & - \\ - & 0 & - & - \\ - & - & 0 & - \\ - & - & - & 0 \end{bmatrix}_{4\times 4}$ is any skew symm. Mat then Possible values of $|A| = ?$

MCQ

(a) 3 (b) 0 (c) 81 (d) C.N.D.

$\because$ $81 = (9)^2$ & $0 = (0)^2$ i.e 0 is also treated as perfect square.

Q $A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{4\times 4}$ HW $|A| = \ldots\ldots\ldots\ldots = 0$

Q. If $A=[a_{ij}]_{n\times n}$ s.t $a_{ij}=i^2-j^2$ $\forall$ i & j then for $n\geq 3$, n = odd

Possible values of $n = 3, 5, 7, 9, \ldots$ $A^{-1}=?$

Let $n=3$, $A=[a_{ij}]_{3\times 3}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}_{3\times 3}$ = S. Symm Mat

$\because$ A is of odd order Skew symm Mat So $|A|=0$

$\Rightarrow A^{-1}=$ DNE.

(ii) Sum of all the elements in Skew symm Mat $A_{n\times n}=?=0$

Q. If $A = [a_{ij}]_{n\times n}$ s.t. $\forall\, i \,\&\, j$; $a_{ij} = -a_{ji}$ then the value of

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}) = \underline{\quad ? \quad}$$

$\Downarrow$

Corresponding elements are of opposite sign

Now For diag elements i.e. we can take $i=j$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & - & - \\ - & 0 & - \\ - & & 0 \end{bmatrix}$$

i.e. $a_{ij} = -a_{ji}$

$\Rightarrow a_{ii} = -a_{ii}$

$\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0 \Rightarrow$ Diag elements $= 0$.

i.e. A is skew symmt

So $\boxed{\text{Ans} = 0}$

(5) Every Sq Mat$^{A}_{n}$ can be expressed as the sum of symm & skew symm Mat.

By Common Sense;

Proof:

$$\boxed{A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right)}$$

$$\boxed{A = P + Q}$$

$P = \frac{A+A^T}{2}$ = symm. Mat

$Q = \frac{A-A^T}{2}$ = skew symm

Sq. Mat = symm + skew symm

where $P^T = \left(\frac{A+A^T}{2}\right)^T = \frac{A^T+(A^T)^T}{2} = \frac{A^T+A}{2} = P$ i.e. P is symm.

& $Q^T = \left(\frac{A-A^T}{2}\right)^T = \frac{A^T-(A^T)^T}{2} = \frac{A^T-A}{2} = -\left(\frac{A-A^T}{2}\right) = -Q$ i.e. Q is skew symm

(5) Every sq. Mat can be expressed as the sum of Hermitian & Skew Herm Mat. as follows; Consider sq Mat $A_{n\times n}$ then by common sense we can write.

$$\boxed{A = \left(\frac{A+A^{\theta}}{2}\right) + \left(\frac{A-A^{\theta}}{2}\right)}$$

$$A = R + S$$

$R = \frac{A+A^{\theta}}{2}$ = Herm. Mat

$S = \frac{A-A^{\theta}}{2}$ = Skew Herm. Mat.

for, $R^{\theta} = \left(\frac{A+A^{\theta}}{2}\right)^{\theta} = \dots\dots = R$ Hence Proved

$S^{\theta} = \left(\frac{A-A^{\theta}}{2}\right)^{\theta} = \dots\dots = -S$ " "

If A and B are two symmetric matrices. Then consider the following staements.

(i) A + B is symmetric

(ii) AB is symmetric

(iii) AB + BA is symmetric

(a) Only (i) is true

(b) (i) and (ii) are true

(c) (i) and (iii) are true

(d) (i), (ii) and (iii) are true.

Given $A^T = A$ & $B^T = B$

$(A+B)^T = A^T + B^T = (A+B)$

$(AB)^T = B^T A^T = BA \neq AB$

$(AB+BA)^T = (AB)^T + (BA)^T$

$= B^T A^T + A^T B^T$

$= BA + AB$

$= (AB+BA)$

Q. If A & B are two Hermitian Matrices then (AB − BA) will be ?

(a) Symm.

(b) Skew Symm.

(c) Hermitian

(d) Skew Hermitian

Given $A^{\theta} = A$ —— (1) & $B^{\theta} = B$ —— (2)

Now, $(AB - BA)^{\theta} = (AB)^{\theta} - (BA)^{\theta}$

$= B^{\theta}A^{\theta} - A^{\theta}B^{\theta}$

$= BA - AB$

$= -(AB - BA)$

Idempotent Mat → if $A^2 = A$ then A is called Idempotent Mat.

& for Idempotent Mat, $A^2 = A^3 = A^4 = \dots\dots = A$

i.e. [if A is an Idempotent Mat] $\underset{\not\Leftarrow}{\overset{\text{then}}{\Longrightarrow}}$ [$|A| = 0$ or 1.]

Involutary Mat → if $A^2 = I$ then A is called Involutary Mat.

& for Involutary Mat, $A^2 = I$, $A^4 = I$, $A^6 = I$ ......

i.e. if [A is an involutary Mat] $\underset{\not\Leftarrow}{\overset{\text{then}}{\Longrightarrow}}$ [$|A| = 1$ or -1]

Nilpotent Mat → if $\boxed{A^K = O}$ then A is called Nilpotent Mat of Power K. where K is least +ve integer.

eg if A is s.t. $A^2 \neq O$, $A^3 \neq O$, But $A^4 = O$ then $A^5 = A^6 = A^7 = \dots = O$

So A is Nilpotent Mat of Power 4

(*) If $A^2 = A$ then

$|A^2| = |A|$

$|A|^2 - |A| = 0$

$|A|(|A| - 1) = 0$

$|A| = 0$ or 1.

(*) if $A^2 = I$ then

$|A^2| = |I|$

$|A|^2 = 1$

$|A| = 1$ or -1

(*) $A^k = O$

$|A^k| = |O|$

$|A|^k = 0$

$|A| = 0$

Q. if $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ then A is ________ Mat

$$A^2 = AA = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\because A^2 = O$ so A is Nilpotent Mat of power 2.

eg $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ then A is ? Nilpotent of index 2

$(\because a = 2, b = 3)$

$(\because A^2 = O)$

Q. If A is an Idempotent Mat then (I-A) will be ?

(a) Null Mat

(b) Identity

(c) Involutary

(d) Idempotent

Thank you

Keep Hustling!