DS & AI
CS & IT

Linear Algebra

Lecture No. 05

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Recap of previous lecture

Topic

Types of Matrices

Topics to be Covered

Topic

① Remaining Part of Types of Matrices

② Vectors & their Properties.

Some Confusions :-

RECAP

① $(A+B+C)^T = A^T + B^T + C^T$

② $(A+B+C)^\theta = A^\theta + B^\theta + C^\theta$

③ $Tr(A+B+C) = Tr(A) + Tr(B) + Tr(C)$

④ $|A+B+C| \leq |A| + |B| + |C|$

⑤ $(A+B+C)^{-1} = \dfrac{adj(A+B+C)}{|A+B+C|}$

⑥ $AB \neq BA$ But $Tr(AB) = Tr(BA)$

⑦ $|ABC| = |A| \cdot |B| \cdot |C|$

⑧ Reversal Law :-

(i) $(ABC)^T = C^T B^T A^T$

(ii) $(ABC)^\theta = C^\theta B^\theta A^\theta$

(iii) $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

QUICK RECAP:-

1. Singular Mat if $|A| = 0$
2. Non Sing Mat if $|A| \neq 0$
3. Invertible Mat if A^{-1} exist, $A^{-1} = \frac{\text{adj} A}{|A|}$
4. Real Mat if $\bar{A} = A$ or $(A^{\theta} = A^{T})$
5. Complex Mat if $\bar{A} \neq A$
6. Symm Mat if $A^{T} = A$
7. Skew Symm Mat if $A^{T} = -A$
8. Hermitian Mat if $A^{\theta} = A$
9. Skew Herm Mat if $A^{\theta} = -A$
10. Idempotent if $A^2 = A$
11. Involutary if $A^2 = I$
12. Nilpotent if $A^K = O$, K = least +ve int
13. U.T.M $A = [a_{ij}]_{n \times n} = 0 \ \forall \ i > j$
14. L.T.M $A = [a_{ij}]_{n \times n} = 0 \ \forall \ i < j$
15. Diag Mat:

$$A = [a_{ij}]_{n \times n} = \begin{cases} 0, & i \neq j \\ \text{atleast one element is Non Zero}, & i = j \end{cases}$$

Q. If A is an Idempotent Mat then $(I-A)$ will be?

(a) Null Mat

(b) Identity

(c) Involutary

(d) Idempotent ✓

Given $A^2 = A$ —— (1)

M-I $\cancel{AA = A}$ (wrong step)

$A = I$

So $(I-A) = I - I = O$

∵ for an Idempotent Mat

$|A| = 0$ or 1

is we are not sure that Mat A given in Question is Non Singular so Not possible to Cancellation law.

M-II $A^2 = A$ —— (1) (given)

Now, $(I-A)^2 = (I-A).(I-A)$

$= I^2 - IA - AI + A^2$

$= I - A - A + (A)$

$= (I-A)$

So $(I-A)$ is an Idempotent Mat.

If T is Idempdotent then $\boxed{T^k = T}$ for

(a) k = 2

(b) All integer k

(c) All positive integer k ≥ 2

(d) All of the above

$\Rightarrow k = 2, 3, 4, 5, 6$

ie $k \geq 2$

Last Quest:

$A^2 = A$ given.

$\cancel{(I-A) = A(I-A)}$??

$= AI - A^2$

$= A - (A)$

$= 0$

if T is an Idempotent $\Rightarrow T^2 = T$

So $T^3 = T^2 \cdot T = T \cdot T = T^2 = T$

& $T^4 = T^2 \cdot T^2 = T \cdot T = T^2 = T$

$T^5 = T, T^6 = T$ - - - - -

(*) Max Number of Different elements that are required to construct a Symm Mat $A_{n\times n} = \frac{n(n+1)}{2}$

eg $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}_{2\times 2}$ i.e Max elements Required $= 3 = (1+2)$

eg $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & k \end{bmatrix}_{3\times 3}$ " " " " $= 6 = (1+2+3)$

eg $A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}_{4\times 4}$ " " " " $= 10 = (1+2+3+4)$

i.e $A_{n\times n} = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

M-II $A = \begin{bmatrix} x_1 & - & - & \cdots & - \\ - & x_2 & & & - \\ - & - & x_3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ - & - & \cdots & & x_n \end{bmatrix}_{n\times n}$ = Symm. Mat / So Max Different elements

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

if A is any General Mat of $n\times n$ then

Total elements in $A = n^2$

Note:

E (5th) J (10th) O (15th) T (20th) Y (25th)

(*) Max Number of Ind. entries that are required to construct skew symm Mat $A_{n\times n} = \frac{n(n-1)}{2}$

Total elements in any General Mat of $n\times n = n^2$

So Total Ind entries $= \frac{n^2-n}{2}$

$$\begin{bmatrix} 0 & \alpha & \beta & \gamma & \cdots & \delta \\ -\alpha & 0 & a & b & \cdots & c \\ -\beta & -a & 0 & & & \vdots \\ -\gamma & -b & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ -\delta & -c & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{n\times n}$$

At diag Position, we have No liberty to put any value except 0 so 0 is also Dependent

Orthogonal Mat → If $\boxed{AA^T = I}$ (or $A^{-1} = A^T$) then A is called Orthogonal Mat.

Unitary Mat → If $\boxed{AA^\theta = I}$ (or $A^{-1} = A^\theta$) then A is called Unitary Mat.

(*) Unitary Mat of Real Nos is orthogonal also. (Learn).

Let A is $\underbrace{\text{unitary Mat}}_{AA^\theta = I \;\longrightarrow\; (1)}$ formed by Real Nos then it is $\underbrace{\text{Real Mat also}}_{(A^\theta = A^T) \;\longrightarrow\; (2)}$.

Using (1) & (2) we can conclude that $AA^T = I$ which is condition of Ortho. Mat

(*) if A is an orthogonal Mat then $\overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}}$ $|A| = 1$ or -1. (Learn).

w.k.that $AA^T = I \Rightarrow |AA^T| = |I| \Rightarrow |A|\cdot|A^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

Q. Check the nature of following Matrices; (1) $A=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$, (2) $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix}$

Sol: Let us Calculate $AA^T = ?$

$$AA^T=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

So A is orthogonal Mat.

Note: $\boxed{\omega^3=1,\ 1+\omega+\omega^2=0}$

$\bar{\omega}=\omega^2,\ \overline{\omega^2}=\omega$

(2) $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix}$, $\bar{A}=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \bar{\omega} & \overline{\omega^2}\\ 1 & \overline{\omega^2} & \bar{\omega}\end{bmatrix}$

$$=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{bmatrix}$$

Now $A^\theta=(\bar{A})^T=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{bmatrix}$

Now $AA^\theta=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}=I$

$\because AA^\theta=I$ So A is unitary Mat.

Q: if $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}_{4\times 4}$ s.t. A is an orthogonal mat then $(AA')^{-1} = ?$

MCQ

(a) I_2

(b) I_3

(c) I_4

(d) I

$$AA^T = I$$

$$\Downarrow$$

$$AA' = I \text{ given}$$

Now $(AA')^{-1} = (I)^{-1} = I = I_{4\times 4} = I_4$

Qs (MSQ) if $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ then A is

By observation, $A =$ Herm. Mat ie $A^\theta = A$ — (1)

(a) Idempotent

(b) Involutary

(c) unitary

(d) Hermitian

Now, $A^2 = A.A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \ldots\ldots$

$\ldots\ldots = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

ie A is Involutary also.

Now $AA^\theta = ? = A.A = A^2 = I$ ie unitary also.

ordered pair $= (x_1, x_2) = x_1\hat{i} + x_2\hat{j} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$ 2 dim vector

(Point in 2D)

ordered Triplet $= (x_1, x_2, x_3) = x_1\hat{i} + x_2\hat{j} + x_3\hat{k} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$ 3 dim vector

(Point in 3D)

ordered Quadruple $= (x_1 x_2 x_3 x_4) = ? = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$ 4- dim vector

$\vdots$

ordered n-tuple $= \quad ? \quad = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} =$ n-dim vector

VECTORS & their Properties

Ordered n-tuple (n-dim vector) → Any ordered set of n numbers is called Ordered n-tuple
Generally it is represented in the form of Column Matrix
(But we can also represent it in the form of Row Mat).

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \text{ or } \begin{bmatrix} x_1, & x_2 & x_3 & \cdots & x_n \end{bmatrix}_{1\times n}$$

ie $A_{3\times 4} = \begin{array}{c} \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \\ C_1 \;\; C_2 \;\; C_3 \;\; C_4 \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$ has 3 Row vectors & 4 Column vectors

Consider $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$ then

① **Dot product** → $\boxed{X \cdot Y = X^T Y} = (x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n)$

Proof: $X \cdot Y = [x_1\ x_2\ x_3 \dots x_n]_{1\times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} = [(x_1 y_1 + x_2 y_2 + \dots + x_n y_n)]_{1\times 1}$

Note - Try to calculate $XY = ? = X_{n\times 1} Y_{n\times 1} = \text{ND}$

② **Norm of Vector** - $\boxed{\|X\| = \sqrt{X^T X}} = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2}$
(Length)

Note :- $X \cdot Y = Y \cdot X = (x_1 y_1 + x_2 y_2 + \dots + x_n y_n)$ i.e Dot product is Commutative

(3) Normalised Vector → if $\boxed{\|X\| = 1}$ then X is Called Normalised vector
(Unit vector)

(4) orthogonal vectors → if $\boxed{X \cdot Y = 0}$ then [X & Y are called orthogonal vectors]

(5) orthonormal vectors → if [$X \cdot Y = 0$, $\|X\| = 1$, $\|Y\| = 1$] then [Vectors are orthonormal Vectors.]

Note :- (1) three vectors are orthogonal if they are pairwise orthogonal.

(2) if $AA^T = I$ then A is called orthogonal ~~Vector~~ Matrix

??

Q: Check the nature of following vectors given in following set

$$\{(1\ 2\ 1)', (2\ 1\ {-4})', (3\ {-2}\ 1)'\}$$

Sol: $X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$, $X_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$$X_1 \cdot X_2 = 1\times 2 + 2\times 1 + 1\times(-4) = 0$$

$$X_2 \cdot X_3 = 2\times 3 + 1\times(-2) + (-4)\times 1 = 0$$

$$X_3 \cdot X_1 = 3\times 1 + (-2)\times 2 + 1\times 1 = 0$$

i.e. X_1, X_2, X_3 are orthogonal $\Rightarrow$ LI also

$$\|X_1\| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\|X_2\| = \sqrt{2^2+1^2+(-4)^2} = \sqrt{21}$$

$$\|X_3\| = \sqrt{3^2+(-2)^2+1} = \sqrt{14}$$

$\because$ these vectors are not of unit norm so they are not orthonormal.

TWO GAJAB Ki PROPERTIES →

(1) "Column vectors of an orthogonal Matrix are Orthonormal Vectors"

eg $A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ is an orthogonal Mat ($\because AA^T = I$)

$\underset{C_1}{\shortparallel} \quad \underset{C_2}{\shortparallel} \quad \underset{C_3}{\shortparallel}$

$\because C_1 \cdot C_2 = C_2 \cdot C_3 = C_3 \cdot C_1 = 0$

& $\|C_1\| = \|C_2\| = \|C_3\| = 1$ Hence verified.

(2) If [vectors are orthogonal] $\underset{\not\Leftarrow}{\Rightarrow}$ then [these are L.I also.] (T)

Q $A = \begin{bmatrix} 1/9 & -4/9 & 8/9 \\ 8/9 & 4/9 & 1/9 \\ \alpha/9 & -7/9 & \beta/9 \end{bmatrix}$ is an orthogonal Matrix then $\alpha + \beta =$?

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$

$\Downarrow$

$C_1 \cdot C_2 = C_2 \cdot C_3 = C_3 \cdot C_1 = 0$

& $\|C_1\| = \|C_2\| = \|C_3\| = 1$

(a) 0

(b) 4

(c) -4

(d) 8

$\because C_1 \cdot C_2 = 0$

$\frac{1}{81}\left[-4+32-7\alpha\right] = 0$

$28 - 7\alpha = 0$

$\alpha = 4$

$\because C_2 \cdot C_3 = 0$

$\frac{1}{81}\left[-32+4-7\beta\right] = 0$

$-28 - 7\beta = 0$

$\beta = -4$

So $\alpha + \beta = 0$

Analysis :- $\|c_1\| = 1$ & $\|c_3\| = 1$

(PODCAST)

$$\sqrt{\frac{1}{81}+\frac{64}{81}+\frac{\alpha^2}{81}} = 1$$

$$1+64+\alpha^2 = 81$$

$$\alpha^2 = 16$$

$$\alpha = \pm 4$$

So $\alpha = 4$

$$\sqrt{\frac{64}{81}+\frac{1}{81}+\frac{\beta^2}{81}} = 1$$

$$64+1+\beta^2 = 81$$

$$\beta^2 = 16$$

$$\beta = \pm 4$$

$$\beta = -4$$

But $c_1 \cdot c_2 = 0 \Rightarrow \alpha = 4$

& $c_2 \cdot c_3 = 0 \Rightarrow \beta = -4$

$$\frac{1}{9}\sqrt{1+64+\alpha^2} = 1 \Rightarrow \sqrt{1+64+\alpha^2} = 9 \Rightarrow 1+64+\alpha^2 = 81 \Rightarrow \alpha^2 = 16$$

2009 / CS

(MSQ)

The vectors $(1,1,1)$ & $(1,a,a^2)$ (where $a = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega$.) are ?

(a) Linearly Ind.

(b) L. Dep.

(c) orthogonal.

(d) × orthonormal.

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$

$X \cdot Y = 1 \times 1 + 1 \times \omega + 1 \times \omega^2 = 1 + \omega + \omega^2 = 0$

i.e. X & Y are orthogonal $\Rightarrow$ LI also.

Now, $\|X\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

& $\|Y\| = \sqrt{1 + \omega^2 + (\omega^2)^2} = \sqrt{1 + \omega^2 + \omega^3 \cdot \omega} = \sqrt{1 + \omega^2 + \omega}$

$= 0$

Thank

you

Keep Hustling!