GATE

DS & AI
CS & IT

Linear Algebra - I

From 04th Aug : 11:00 AM to 1:30 PM

Lecture No. 11

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

EIGEN VALUES – EIGEN VECTORS

(BASIC CONCEPTS)

Topics to be Covered

Topic

EIGEN VALUES-EIGEN VECTORS

→ Properties of Eigen Values

Def^n: Consider Sq. Mat $A_{n\times n}$ then Non Zero Vector X is called Eigen vector, corresponding to Eigen value λ (Real/Complex)/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda . X}$$

λ = Eigen value

X = Eigen vector.

LHS is the Multi of Two Matrices = RHS is the Scalar Multi in a Mat

(Tough job) (Easy job)

Here we are considering Homog. System as follows;

$$AX = \lambda X \Rightarrow AX - \lambda X = 0 \Rightarrow \boxed{(A - \lambda I)X = 0}$$

Hence this system will satisfy all the properties of Homo System.

C. Equ^n of A :- $Ax = \lambda x$

$A_{n\times n}$

$(A-\lambda I)X = 0$ — (1)

$\boxed{MX = 0}$

$\downarrow$

Non Zero E. Vector

Non Zero solution

∞ soln.

$\rho(M) < n$ or $|M| = 0$

$\Rightarrow \rho(A-\lambda I) < n$ or $\boxed{|A-\lambda I| = 0}$

is Necessary cond^n for the existence of Non Zero eigen vector is

$\rho(A-\lambda I) < n$ or $\boxed{|A-\lambda I| = 0}$ — (2)

(*) Equ^n (2) is called Characteristic equ^n of A & Roots of this equation i.e. Values of λ are called Eigen Values / Eigen Roots / Char Values / Char Roots / Latent Roots / Special Values.

Q: Find the E.Values of $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}_{4\times4}$

Sol: C.Equⁿ is $|A - \lambda I| = 0$

$$\begin{vmatrix} (0-\lambda) & 0 & 1 & 1 \\ -1 & (2-\lambda) & 0 & 1 \\ -1 & 0 & (2-\lambda) & 1 \\ 1 & 0 & -1 & (0-\lambda) \end{vmatrix} = 0$$

------------ HW

$$\boxed{\lambda(\lambda-2)(\lambda-1)^2 = 0} \Rightarrow \lambda = 0, 2, 1, 1$$

Note ①

1) Sum of E.Values $= 4 = Tr(A)$

2) Product of E.Values $= 0 = |A|$

3) $(\lambda = 0) \iff |A| = 0$

4) Total No. of E.Values = order

(a) × 1, 1, 1, 1 ∵ $|A| \neq 0$

(b) × 2, 2, 0, 0 ∵ A is Not U.T.M / L.T.M ....

(c) ✓ 0, 2, 1, 1

(d) × 1, −1, −1, 3, ∵ $|A| \neq 0$

∵ $Tr = 4$, $|A| = 0$

PROPERTIES of Values → Let $A_{n\times n}$ having Eigen Values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

① T. Number of E-Values of A = order of A (whether Different or Repeated)

② Sum of E-Values = Trace(A) i.e $\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n = Tr(A)$

③ Product of E Values = Det(A) i.e $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n = |A|$

④ (Zero is an E. Value of A) iff $\Longleftrightarrow$ (A is Singular) i.e $(\lambda = 0) \Longleftrightarrow (|A| = 0)$

⑤ Number of Non Zero E.Values of $A \leq \rho(A)$

eg if $\rho(A_{6\times 6}) = 4$ then A has at least two Eigen Values as $0, 0$.

⑥ If sum of all the elements in each Row (or each Column) is unique constant K then that Constant K will be one of the E.Value of A.

(7) Don't use E-Operations in a given Mat while calculating E-Values

But we can apply 3rd E-Operation in it's C.Eqn is in $|A-\lambda I|=0$

(8) E-Values of U.T.M, L.T.M, Diag Mat, scalar Mat, Identity Mat are just the diagonal elements.

eg $A=\begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\lambda=2,0,-3,-1$ $\lambda=2,-1,1$ $\lambda=2,-3,4$

(9) If λ is an Eigen Value of A then to find Eigen Value of any algebraic expression formed by A, we can Replace A with λ in that expression.

Conclusion i.e Equivalent Matrices have same RANK But may have different Determinant as well as different E-Values.

(10) Let $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots \lambda_n$ are the E Values of $A_{n\times n}$ then

(i) E Values of A^T are $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ is same as that of A.

(ii) E Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots\ldots, \lambda_n^m$; $(m \in N)$

(iii) E Values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \ldots\ldots \frac{1}{\lambda_n}$ (Provided $|A| \neq 0$)

(iv) E. Values of $(\text{adj} A)$ are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots\ldots, \frac{|A|}{\lambda_n}$

But take Care, this short cut of finding E Values of $(\text{adj} A)$ is applicable when $|A| \neq 0$

if, $|A| = 0$ then to find E Values of $(\text{adj} A)$, use conventional Approach.

Proof: 10(ii): if λ is an EValues of A

then $\boxed{AX = \lambda X}$ —— (1)

$$A(AX) = A(\lambda X)$$
$$A^2X = \lambda(AX)$$
$$= \lambda(\lambda X)$$
$$\boxed{A^2X = \lambda^2 X}$$

ie λ^2 is an EValue of A^2.

Proof 10(iii) →

$$AX = \lambda X$$

if $|A| \neq 0$, A^{-1} exist

$$A^{-1}(AX) = A^{-1}(\lambda X)$$
$$IX = \lambda(A^{-1}X)$$
$$\lambda(A^{-1}X) = IX$$
$$\boxed{A^{-1}X = \left(\frac{1}{\lambda}\right)X}$$

$\frac{1}{\lambda}$ is an EValue of A^{-1}

Proof 10(iv) if $|A| \neq 0$.

Consider $AX = \lambda X$

$$A^{-1}(AX) = \lambda(A^{-1}X)$$
$$\lambda(A^{-1}X) = IX$$
$$A^{-1}X = \left(\frac{1}{\lambda}\right)X$$
$$\frac{(\operatorname{adj} A)}{|A|}X = \left(\frac{1}{\lambda}\right)X$$
$$(\operatorname{adj} A)X = \left(\frac{|A|}{\lambda}\right)X$$

$\frac{|A|}{\lambda}$ is an EValue of adj A.

Q. If Evalues of $A=\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$ are 4 & 8 then $x+y=$?

(a) -4 (b) 6 (c) 10 (d) 14

$\lambda_1+\lambda_2=Tr(A)$ & $\lambda_1\cdot\lambda_2=|A|$

$4+8=2+y$ & $(4)(8)=2y-3x$

$y=10$ & $32=2(10)-3x$

$3x=-12$

$x=-4$

So $x+y=-4+10=6$

Q. If Trace & Det of $A_{2\times 2}$ are -2 & -35 resp then $\lambda_1+\lambda_2=$?

(a) 12 (b) -12 (c) 2 (d) -2

M-I $\lambda_1+\lambda_2=Tr(A)=-2$

M-II $\lambda_1+\lambda_2=-2$ — (1) & $\lambda_1\cdot\lambda_2=-35$ — (2)

$\Rightarrow \lambda_1=5$ & $\lambda_2=-7$

Consider a 2 × 2 square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

where x is unknown. If the eigen values of the matrix A are $\underbrace{(\sigma + j\omega)}_{\lambda_1 =}$ and $\underbrace{(\sigma - j\omega)}_{\lambda_2 =}$, then x is equal to

(a) $+j\omega$ (b) $-j\omega$

(c) $+\omega$ (d) $-\omega$

$$\boxed{j = \sqrt{-1} = i}$$

& $j^2 = -1 = i^2$

w.k.that $\lambda_1 \cdot \lambda_2 = |A|$

$(\sigma + j\omega)(\sigma - j\omega) = \sigma^2 - \omega x$

$\sigma^2 - j^2\omega^2 = \sigma^2 - \omega x$

$-j^2\omega^2 = -\omega x$

$j^2\omega = x$

$x = j^2\omega = i^2\omega = -\omega$

Q. If one of the EValue of $A = \begin{bmatrix} 40 & -16 & -24 \\ -11 & 30 & -19 \\ 26 & 24 & -50 \end{bmatrix}$ is λ then other EValues will be ?

(MSQ)

(a) $\lambda - 20$

(b) $20 - \lambda$

(c) $20 + \lambda$

(d) 0

$\because |A| = ? \xrightarrow{C_1 \to C_1 + C_2 + C_3} \begin{vmatrix} 0 & -16 & -24 \\ 0 & 30 & -19 \\ 0 & 24 & -50 \end{vmatrix} = 0$

$\because |A| = 0 \xrightarrow{\text{using prop (4)}} \lambda = 0$

Now, $\lambda_1 + \lambda_2 + \lambda_3 = Tr(A) = 20$

$0 + \lambda + \lambda_3 = 20 \Rightarrow \lambda_3 = (20 - \lambda)$

(M-II) By prop (6), one of the EValue is $\lambda = 0$ & other EValues are λ & $(20 - \lambda)$

Q: If P and Q are two sq. Matrices of same order s.t $\boxed{PQ = QP = I}$ then 0 is an eigen value of ?

(a) P But not Q

(b) Q But not P

(c) Both P & Q

(d) Neither P Nor Q

$$PQ = QP = I \Rightarrow \begin{cases} P^{-1} = Q \\ Q^{-1} = P \end{cases}$$

ie Both P^{-1} & Q^{-1} exist

$\Rightarrow$ ∴ P & Q are Non sing.

ie $|P| \neq 0$ & $|Q| \neq 0$.

So By prop (4), (d)

Q. one of the E-Value of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ will be? (a) 5 (b) 0 (c) 10 (d) −1

$= 10 \quad 10 \quad 10 \quad 10$

M-I using prop (6), one of the E Value is $\lambda = 10$. Ans

M-II C. Equn is $|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 & 4 \\ 2 & (3-\lambda) & 4 & 1 \\ 3 & 4 & (1-\lambda) & 2 \\ 4 & 1 & 2 & (3-\lambda) \end{vmatrix} = 0$$

$C_1 \to C_1 + (C_2 + C_3 + C_4)$;

$$\begin{vmatrix} (10-\lambda) & 2 & 3 & 4 \\ (10-\lambda) & (3-\lambda) & 4 & 1 \\ (10-\lambda) & 4 & (1-\lambda) & 2 \\ (10-\lambda) & 1 & 2 & (3-\lambda) \end{vmatrix} = 0$$

$$(10-\lambda) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & (3-\lambda) & 4 & 1 \\ 1 & 4 & (1-\lambda) & 2 \\ 1 & 2 & \ & (3-\lambda) \end{vmatrix} = 0$$

$(10-\lambda)[\text{cubic equn in } \lambda] = 0$

$\lambda = 10, \lambda_2, \lambda_3, \lambda_4$

i.e. Ans $= 10$ Ans

Q. The E.Values of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4}$ are ?

$= 4 \quad 4 \quad 4 \quad 4$

M-I

By property (6), one E.Value is $\lambda = 4$.

Now, $\rho(A) =$ one

i.e. A has at most one Non Zero E.V. which is $\lambda = 4$. So

other E.Values will be 0, 0, 0.

Hence $\lambda = 4, 0, 0, 0$.

M-II C.Eqn $|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 & 1 \\ 1 & 1 & (1-\lambda) & 1 \\ 1 & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$C_1 \longrightarrow C_1 + (C_2 + C_3 + C_4)$

$$\begin{vmatrix} (4-\lambda) & 1 & 1 & 1 \\ (4-\lambda) & (1-\lambda) & 1 & 1 \\ (4-\lambda) & 1 & (1-\lambda) & 1 \\ (4-\lambda) & 1 & 1 & (1-\lambda) \end{vmatrix}$$

$$(4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0 \xrightarrow{R_2 - R_1,\ R_3 - R_1,\ R_4 - R_1} (4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[-\lambda^3\right] = 0$$
$$\lambda^3(4-\lambda) = 0$$
$$\lambda = 0, 0, 0, 4$$

M-III

WRONG APP

$$\cancel{A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_1}$$

$= U.T.M.$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\lambda = 4, 0, 0, 0 \qquad\qquad \lambda = 1, 0, 0, 0$$

i.e we have justified that,

Equivalent Matrices have different E. Values

Q. $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \frac{-1+i\sqrt{3}}{2} = \omega & 0 \\ 3 & 4 & \frac{-1-i\sqrt{3}}{2} = \omega^2 \end{bmatrix}$ then $Tr(A^{102}) = \underline{?}$

$\omega = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

$\omega^2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$

$\overline{\omega} = \omega^2, \overline{\omega^2} = \omega$

$\omega^3 = 1, 1 + \omega + \omega^2 = 0$

$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \omega & 0 \\ 3 & 4 & \omega^2 \end{bmatrix}$

= L.T.M.

$\lambda = 1, \omega, \omega^2$

E. Values of A are $\lambda = 1, \omega, \omega^2$

" " A^{102} are $= 1^{102}, \omega^{102}, (\omega^2)^{102}$

$= 1, \omega^{102}, \omega^{204}$

$= 1, (\omega^3)^{34}, (\omega^3)^{68}$

$= 1, 1^{34}, 1^{68} = 1, 1, 1$

i.e. E Values of A^{102} are 1, 1, 1

So $Tr(A^{102}) = 1 + 1 + 1 = 3$

Qs

2012

If $A_{2\times 2}$ s.t. $a_{11} = a_{12} = a_{21} = 1$ & $a_{22} = -1$ then

E. Values of A^{19} are ?

(a) $\pm\sqrt{2}$

(b) ± 2

(c) ± 1

(d) $\pm 512\sqrt{2}$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\lambda_1 + \lambda_2 = Tr(A) = 0 \Rightarrow \lambda_1 = -\lambda_2$

$\lambda_1 \cdot \lambda_2 = |A| = -2$

$(-\lambda_2)(\lambda_2) = -2$

$\lambda_2^2 = 2 \Rightarrow \lambda_2 = \pm\sqrt{2}$

i.e. $\lambda_1 = \sqrt{2}$ & $\lambda_2 = -\sqrt{2}$

C. Equⁿ is $|A - \lambda I| = 0$

$\lambda^2 - 2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$

So E Values of A are $\sqrt{2}$ & $-\sqrt{2}$

" " A^{19} are $(\sqrt{2})^{19}$ & $(-\sqrt{2})^{19}$

$= (\sqrt{2})^{18}(\sqrt{2})$ & $(-\sqrt{2})^{18}(-\sqrt{2})$

E Values of A^{19} are $2^9\sqrt{2}$ & $-2^9\sqrt{2}$

$= 512\sqrt{2}$ & $-512\sqrt{2}$

Q. If $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ $\begin{matrix} = 6 \\ = 6 \end{matrix}$ & $B = A^2 - 2A + 3I$ then Find the Product of the Eigen Values of B?

(a) 81
(b) 30
(c) 12
(d) 144

By prop (6), one E-Value is $\lambda_A = 6$

$\because Tr(A) = 8 \Rightarrow 6 + \lambda = 8$

$\lambda_A = 2$

ie E-Values of A are 2 & 6.

Now using prop (9).

$$B = (A^2 - 2A + 3I) \begin{cases} \lambda_B = (2)^2 - 2(2) + 3(1) = 3 \\ \lambda_B = (6)^2 - 2(6) + 3(1) = 27 \end{cases}$$

So Req Product $= 3 \times 27 = 81$

Shortcut Method to write C. Equ^n

of 2×2 Mat A →

$$\boxed{\lambda^2 - (Tr A)\lambda + |A| = 0}$$

eg $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \lambda^2 - (8)\lambda + (12) = 0$

ie $\lambda = 6$ & 2

eg $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \lambda^2 - (0)\lambda + (-2) = 0$

$\lambda^2 - 2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$

eg $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - (0)\lambda + (1) = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

MII (Verification) $\rightarrow$ $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ & $B = A^2 - 2A + 3I$.

C. Equ^n of A is $\lambda^2 - 8\lambda + 12 = 0 \Rightarrow (\lambda-2)(\lambda-6) = 0 \Rightarrow \lambda = 2, 6$

Now $B = A^2 - 2A + 3I$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - 2\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 12 & 15 \end{bmatrix}$$

C Eq of B is $\lambda^2 - (\text{Tr}B)\lambda + (|B|) = 0$

$\lambda^2 - 30\lambda + 81 = 0 \Rightarrow (\lambda-3)(\lambda-27) = 0$

$\lambda = 3$ & 27 Hence Verified

Qs If Eigen Value of $X_{2\times2}$ are -2 & -3 then Trace of $(X+I)^{-1}(X+5I)$ will be?

(a) -4

(b) 4

(c) -5

(d) 12

ATQ, $X_{2\times2} \begin{cases} \lambda_X = -2 \\ \lambda_X = -3 \end{cases}$

Now we will use prop (9).

Let $\boxed{B = (X+I)^{-1}(X+5I)}$

$\lambda_B = (-2+1)^{-1}(-2+5) = (-1)^{-1}(3) = \frac{3}{(-1)^1} = -3$

$\lambda_B = (-3+1)^{-1}(-3+5) = (-2)^{-1}(2) = \frac{2}{(-2)} = -1$

So $Tr(B) = \lambda_B + \lambda_B = -3 + (-1) = -4$

M-II Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{cases} a+d = (-2)+(-3) = -5 \\ ad-bc = (-2)(-3) = 6 \end{cases}$

underdetermined system

So No unique Values of a, b, c, d exist

ie Not able to find X, Hence Not able to find B, Hence Not able to find Ans.

M-(II) $\because$ E Values of X are -2 & -3 so we can take, $X = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

$$X + I = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow (X+I)^{-1} = \frac{1}{2} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$X + 5I = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

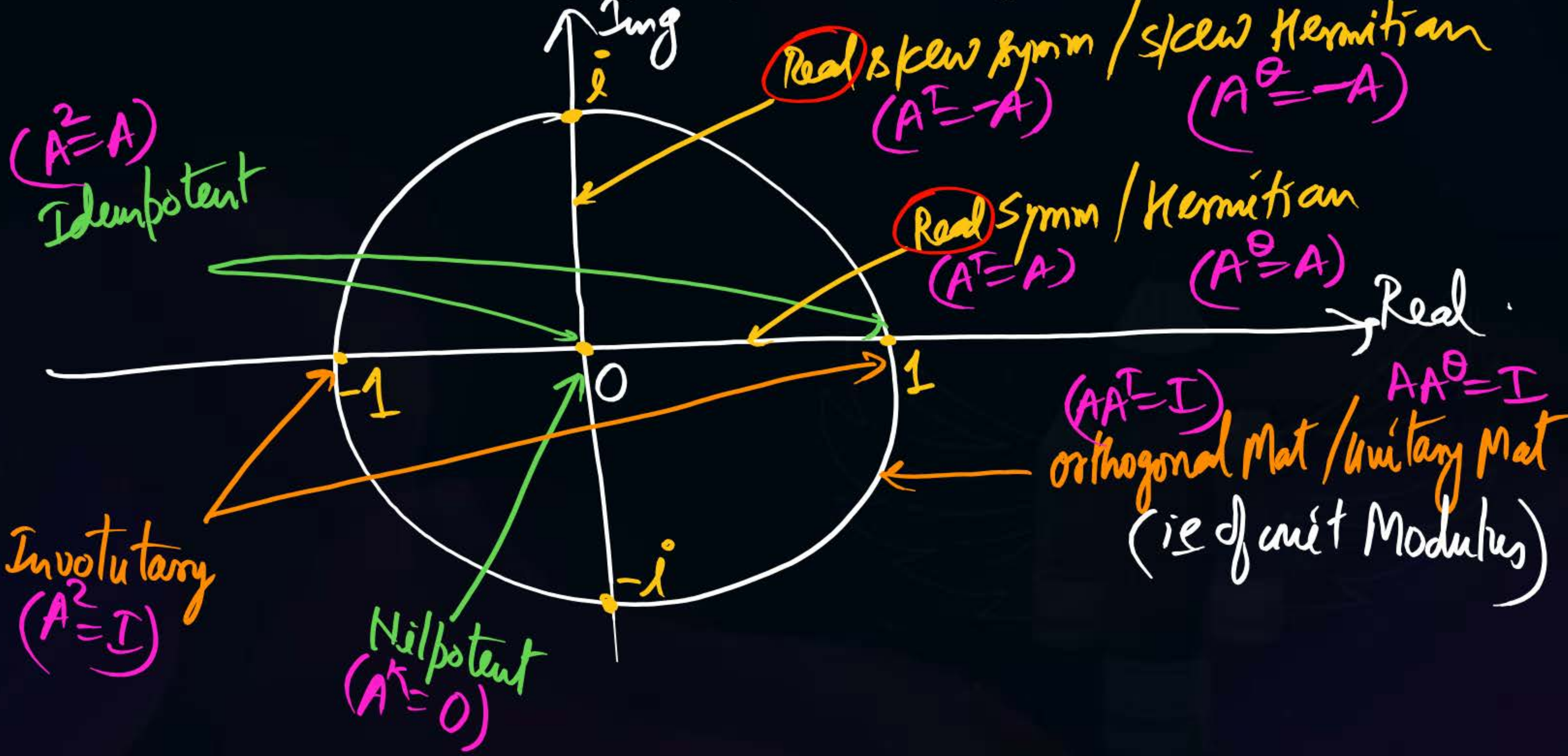
so $B = (X+I)^{-1} \cdot (X+5I) = \frac{1}{2} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$

Hence E Values of B are -3 & -1.

Note: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{\operatorname{adj} A}{|A|} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(1) Shortcut Method to Learn Various Th based on E Values →

Consider a unit circle centered at origin i.e $x^2+y^2=1$



Proof: Idempotent: $A^2 = A \Rightarrow |A^2| = |A| \Rightarrow |A|^2 - |A| = 0 \Rightarrow |A|(|A|-1) = 0 \Rightarrow |A| = 0$ or 1

$\Rightarrow \lambda^2 = \lambda \Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0$ or 1.

if (A is Idempotent) $\underset{\not\Leftarrow}{\Longrightarrow}$ Either $|A| = 0$ or $|A| = 1$.

OR

if (A is idempotent) $\underset{\not\Leftarrow}{\Longrightarrow}$ $\boxed{\lambda = 0 \text{ or } \lambda = 1 \text{ or Both}}$.

eg $A_{3\times3}$ s.t $A^2 = A \Rightarrow$

- $\lambda = 0, 0, 0$, $|A| = 0$
- $\lambda = 1, 1, 1$, $|A| = 1$
- $\lambda = 0, 1, 1$, $|A| = 0$
- $\lambda = 0, 0, 1$, $|A| = 0$

Prop: Involutary $A^2 = I \Rightarrow |A|^2 = 1 \Rightarrow |A| = 1$ or -1

" $A^2 = I \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$ ie $\boxed{\lambda = 1 \text{ or } -1 \text{ or both}}$

Analysis:

$$A = \begin{bmatrix} 2 & -4 & 5 \\ -4 & -1 & 3 \\ 5 & 3 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -4 & 5i \\ -4 & -1 & 3 \\ 5i & 3 & 0 \end{bmatrix}$$

= Real symm. = Complex symm. $\neq$ Hermitian Mat

$\lambda \in R$ $\qquad$ λ = Not Necessarily Real

eg: EValues of Real symm Mat are Real & +ve (False)

T F

Q. If E Values of $A = \begin{bmatrix} 2 & 5+i & -3 \\ x & -1 & 4 \\ -3 & 4 & 3 \end{bmatrix}$ are $\underbrace{\text{all Real Nos}}$ then $x = ?$

(a) $5+i$

(b) $5-i$

(c) 0

(d) Both a & b

Real E-Values $\Rightarrow$ Either A is Real Symm ✗ ($\because a_{12} = \text{Comp. No}$)

or A is Hermitian ✓

$\Downarrow$

Corresponding elements are conjugates of each other.

$\Downarrow$

$x = 5 - i$

Q. if $A = \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times 4}$ is an O-Mat then $(AA^T)^{-1} = ? = (I)^{-1} = I = I_{4\times 4}$

(ii) which of the following Can not be the Evalue of A ?

(a) $-\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow |\lambda| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$ (possible)

(b) $-1 \Rightarrow |\lambda| = |-1| = 1$ is possible

(c) $\frac{\sqrt{3}}{2} + \frac{i}{2} \Rightarrow |\lambda| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$ (possible)

(d) $\frac{1}{4} + i\frac{\sqrt{3}}{4} \Rightarrow |\lambda| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \neq 1$

Not possible

$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

(*) If $\{\lambda_i, X_i\}$ is an E-pair of $A_{n\times n}$ then which is false?

(a) $\exists$ Non Zero E Vector of A if $\rho(A-\lambda I) < n$

(b) if $A^\theta = A$ then $\lambda_i \in R \ \forall i$

(c) if $A^{-1} = A^T$ then $|\lambda_i| = 1 \ \forall i$

(d) $\{\lambda_i^m, X_i^m\}$ is an E-pair of A^m.

(e) If $A = A^{-1}$ then the Eigen Value of A is 1

THANK - YOU

Tel: $\boxed{\text{dr puneet sir pw}}$