GATE

Linear Algebra

Vector Space & Quadratic Forms

DPP 04

Q1 Determine whether (3, -1) can be expressed as a unique linear combination of which one of the following.

(A) $V_1 = (2, 0)$ $V_2 = (1, 1)$

(B) $V_1 = (2, 2)$ $V_2 = (1, 1)$

(C) $V_1 = (9, -3)$ $V_2 = (-6, 2)$

(D) None of these

Q2 Write down the quadratic form corresponding to the matrix .

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

Q3 Express the quadratic form $x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$ as product of matrices.

Q4 Write down the matrix of the quadratic form.
$x_1^2 + 2x_2^2 - 7x_3^2 + x_4^2 - 4x_1x_2 + 8x_1x_3 .$
$- 6x_3x_4$

Q5 Find the matrix A for the quadratic form $10x^2 + 2y^2 + 5z^2 + 6yz - 10zx - 4xy$.

Q6 Find a real symmetric matrix C of the quadratic form .
$Q(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + 6x_3^2 + 2x_1x_2$
$+ x_2x_3 + 3x_1x_3$

Q7 Consider a vector $\vec{p} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ in the coordinate system $(\hat{i}, \hat{j}, \hat{k})$.

The vector is rotated anti – clockwise about the Y axis by an angle of 60° The vector $\vec{p}$ in the noted coordinates system $(\hat{i}, \hat{j}, \hat{k})$ is.

(A) $(1 - \sqrt{3})\hat{i} + 3\hat{j} + (1 + \sqrt{3})\hat{k}$

(B) $(1 + \sqrt{3})i + 3\hat{j} + (1 - \sqrt{3})\hat{k}$

(C) $(1 - \sqrt{3})\hat{i} + (3 + \sqrt{3})\hat{j} + 2\hat{k}$

(D) $(1 - \sqrt{3})\hat{i} + (3 - \sqrt{3})\hat{j} + 2\hat{k}$

Q8 Consider a vector $\vec{p} = \hat{i} + 2\hat{j} + \hat{k}$ in the coordinate system $(\hat{i}, \hat{j}, \hat{k})$

The vector is rotated clockwise about the z axis by an angle of 45° The vector $\vec{p}$ in the rotated co-ordinate system will be -

(A) $-\frac{1}{\sqrt{2}}\hat{i} + \frac{3}{\sqrt{2}}\hat{j} + \hat{k}$

(B) $\frac{1}{\sqrt{2}}\hat{i} - \frac{3}{\sqrt{2}}\hat{j} + \hat{k}$

(C) $\frac{3}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \hat{k}$

(D) $\frac{3}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}$

Q9 Find the singular value decomposition of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Q10 Find the singular value Decomposition of

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Q11 Find the singular value Decomposition of

$$A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$$

Q12 Using LU decomposition solve the equations -

$x + y + z = 1$

$4x + 3y - z = 6$

$3x + 5y + 3z = 4$.

Q13 Consider the system of equation and solve them by using LU decomposition –

$x_1 + x_2 - x_3 = 4$

$x_1 - 2x_2 + 3x_3 = -6$

$2x_1 + 3x_2 + x_3 = 7$

Q14 The Trace of the projection matrix that projects any n – dimensional vector on to the vector $(1, 1, 1 \ldots\ldots 1)^T$?

Q15 Find the projection matrix P on to the space spanned by $a_1 = (1, 0, 1)$ and $a_2 = (1, 1, -1)$?

Q16 Find the Projection of B = (4, 3, 1, 0) on to the column space of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

Q17

Let $A = \begin{bmatrix} 1 & 2 & \vdots & 1 \\ 3 & 4 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 2 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & 2 & 3 & \vdots & 1 \\ 4 & 3 & 6 & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$ Compute AB

using the indicated partitioning's.

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Answer Key

Q1 (A)

Q2 $x^2 + 4z^2 + 4xy + 6yz + 10xz$

Q3 $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Q4 $\mathrm{A} = \begin{bmatrix} 1 & -2 & 4 & 0 \\ -2 & 2 & 0 & 0 \\ 4 & 0 & -7 & -3 \\ 0 & 0 & -3 & 1 \end{bmatrix}$

Q5 $\mathrm{A} = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

Q6 $\mathrm{A} = \begin{bmatrix} 1 & 1 & \frac{3}{2} \\ 1 & 4 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & 6 \end{bmatrix}$ is the real symmetric matrix.

Q7 (B)

Q8 (C)

Q9 See Solution

Q10 $\mathrm{A} = U \sum \mathrm{V}^{\mathrm{T}}$

Q11 See solution

Q12 x = 1,y=1/2,z=-1/2

Q13 $x_1 = 1, x_2 = 2, x_3 = -1$

Q14 (1)

Q15 See solution

Q16 $\frac{1}{35}\begin{bmatrix} 133 \\ 95 \\ 61 \\ -11 \end{bmatrix}$

Q17 $\mathrm{AB} = \begin{bmatrix} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

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Hints & Solutions

Q1 Text Solution:

$(3,-1)=\alpha(2,0)+\beta(1,1)$
$2\alpha+\beta=3$
$\beta=-1$
$2\alpha-1=3$
$2\alpha=4$
$\alpha=2$
Thus $\alpha=2,\ \beta=-1$

Thus it is expressed as the unique combination.

$(3,-1)=\alpha(2,2)+\beta(1,1)$
$2\alpha+\beta=3$
$2\alpha+\beta=-1$

Thus it can not be expressed as unique linear combination as no solution is there.

$(3,-1)=\alpha v_1+\beta v_2$
$=\alpha\ (9,-3)+\beta\ (-6,2)$
$9\alpha-6\beta=3$
$3\alpha-2\beta=1$

Thus also cannot be expressed as a unique linear combination.

Q2 Text Solution:

Q = X^T AX

here $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Q = X^T AX

$=[x\ \ y\ \ z]\begin{bmatrix}1&2&5\\2&0&3\\5&3&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$

$=[\ x+2y+5z\qquad 2x+3z\qquad 5x+3y+4z]$
$\begin{bmatrix}x\\y\\z\end{bmatrix}$

$=x^2+2xy+5xz+2xy+3yz+5xz+3yz+4z^2$

$=x^2+4z^2+4xy+6yz+10\,xz$

Q3 Text Solution:

Quadratic form is $x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3$

Q = X^T AX

$A=\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$

coefficient of

$x_1^2=1$
$x_2^2=2$
$x_3^2=2$

Now, coefficient of $x_1\,.\,x_2=-2$, thus they will be alloted as $-1,\ and\ -1$ to the elements of order 12 and 21,and the same goes for the element of order 23 and 32.

Q4 Text Solution:

$x_1^2+2x_2^2-7x_3^2+x_4^2-4x_1x_2+8x_1x_3-6x_3x_4$

Q = X^T AX

$X^T=[x_1\ \ x_2\ \ x_3\ \ x_4]$

$X=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ thus, $A=\begin{bmatrix}1&-2&4&0\\-2&2&0&0\\4&0&-7&-3\\0&0&-3&1\end{bmatrix}$

Q5 Text Solution:

10x^2 + 2y^2 + 5z^2 + 6yz −10zx − 4xy

Q = X^T AX

here X = $\begin{bmatrix}x\\y\\z\end{bmatrix}$

$A=\begin{bmatrix}10&-2&-5\\-2&2&3\\-5&3&5\end{bmatrix}$

$Q=X^T\begin{bmatrix}10&-2&-5\\-2&2&3\\-5&3&5\end{bmatrix}X.$

thus the matrix A is equal to

$A=\begin{bmatrix}10&-2&-5\\-2&2&3\\-5&3&5\end{bmatrix}$

The coeffcient of :

x^2 = 10

y^2 = 2

$z^2 = 5$

and thus 10, 2, 5 are diagonal elements for the extra diagonal elements we will consider the coefficients and will split them as we solved in question number 3

Q6 Text Solution:

$Q\ (x_1, x_2, x_3) = x_1^2\ + 4x_2^2\ + 6x_3^2\ + 2x_1x_2 + x_2x_3\ + 3x_1x_3$

Now $Q = X^T AX$

Here coefficient of $x_1^2 = 1$

$x_2^2 = 4$

$x_3^2 = 6$

thus diagonal elements are 1, 4, 6.

Now coefficient of $x_1 x_2 = 2$

$x_2 x_3 = 1$

$x_1 x_3 = 3$, thus they will be splitted accordingly.

Thus, $A = \begin{bmatrix} 1 & 1 & \frac{3}{2} \\ 1 & 4 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & 6 \end{bmatrix}$ is the real symmetric matrix.Now as transpose of A=A, thus it is a real symmetric matrix.

Q7 Text Solution:

$\overrightarrow{p} = 2\hat{i} + 3\hat{j} + 2\widehat{k}$

The vector can be written in column matrix as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Now, after Rotation we get -

$$\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = R\phi\,(y) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Here $\phi = 60^\circ$, thus $\cos\phi = \frac{1}{2}$ & $\sin\phi = \frac{\sqrt{3}}{2}$

$$\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = \begin{bmatrix} 1 + 0 + \sqrt{3} \\ 0 + 3 + 0 \\ -\sqrt{3} + 0 + 1 \end{bmatrix}$$

$$\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{3} \\ 3 \\ -\sqrt{3} + 1 \end{bmatrix}$$

thus the vector is $(1 + \sqrt{3})\,\widehat{i} + 3\,\widehat{j} + (-\sqrt{3} + 1)\widehat{k}$.

Q8 Text Solution:

$\overrightarrow{p} = \hat{i} + 2\hat{j} + \widehat{k}$

thus the vector can be written in column matrix as

$$\overrightarrow{A} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Now it is moved 45° in clockwise direction

thus $\theta = -45^\circ$

Now, $\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = R\,(\theta) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x^1 \\ y^1 \\ z^1 \end{bmatrix} = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) & 0 \\ \sin(-45^\circ) & \cos(-45^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} + \sqrt{2} \\ -\frac{1}{\sqrt{2}} + \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

Thus (c) is the correct option.

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Q9 Text Solution:

$$A=\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{matrix}\nearrow A^TA=\begin{bmatrix}5&4\\4&5\end{bmatrix}\\ \searrow AA^T=\begin{bmatrix}5&4\\4&5\end{bmatrix}\end{matrix}$$

E. Values and E. vector are as follows

$$\lambda_1=9, X_1=\begin{bmatrix}1\\1\end{bmatrix}, \lambda_2=1,\ X_2=\begin{bmatrix}1\\-1\end{bmatrix}$$

Hence right singular vector are

$$V_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, V_2=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Rightarrow V=[V_1V_2]$$

$$i.e.\ V=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{bmatrix}=\text{u also}$$

(as A^TA and AA^T are same)

Now singular vlaues of A are

$$\sigma_1=\sqrt{9}=3\ \&\ \ \sigma_2=\sqrt{1}=1$$

So, $\sum=\begin{bmatrix}3&0\\0&1\end{bmatrix}$, Hence S.V.D. is

S.V.D. = $u\sum V^T$

$$=\frac{1}{\sqrt{2}}\begin{bmatrix}1&+1\\1&-1\end{bmatrix}\begin{bmatrix}3&0\\0&1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&+1\\1&-1\end{bmatrix}$$

Q10 Text Solution:

$$A=\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\Rightarrow AA^T\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}$$

$$=\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix};$$

$$A^TA=\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}\Rightarrow$$

Eigenvalues and eigenvectors for AA^T :

$$(1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right]-(1-\lambda)$$

$$=(3-\lambda)(1-\lambda)\lambda=0\Rightarrow$$

$$\lambda_1=3;\ u_1=\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix};\lambda_2=1;\ u_2$$

$$=\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix};\lambda_3=0;\ u_3=\begin{bmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix};$$

Eigenvalues and eigenvectors for A^TA:

$$(2-\lambda)^2-1=(3-\lambda)(1-\lambda)=0\Rightarrow$$

$$\lambda_1=3;\ v_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix};\lambda_2=1;$$

$$v_2\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix};\Rightarrow$$

$$Av_i=\sigma_iu_i;\ \ i=1{,}2\Rightarrow\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$=\sigma_1\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1\sqrt{6}\end{bmatrix}$$

$$\Rightarrow\sigma_1=\sqrt{3};\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$=\sigma_2\begin{bmatrix}1/\sqrt{2}\\0\\-1\sqrt{2}\end{bmatrix}\Rightarrow\sigma_2=1$$

$$\Rightarrow A=\underbrace{[u_1\ \ u_2\ \ u_3]}_{U}\underbrace{\text{diag }\{\sigma_1,\sigma_2\}}_{D}\underbrace{\begin{bmatrix}e_1^T\\e_2^T\end{bmatrix}}_{V^T}=$$

$$\underbrace{\begin{bmatrix}1/\sqrt{6}&1/\sqrt{2}&1/\sqrt{3}\\2/\sqrt{6}&0&-1/\sqrt{3}\\1/\sqrt{6}&-1/\sqrt{2}&1/\sqrt{3}\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}\sqrt{3}&0\\0&1\\0&0\end{bmatrix}}_{D}$$

$$\underbrace{\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}}_{V^T}$$

Q11 Text Solution:

$$A=\begin{bmatrix}-4&-7\\1&4\end{bmatrix}, A^T$$

$$=\begin{bmatrix}-4&1\\-7&4\end{bmatrix}$$

$$\begin{matrix}\nearrow A^TA=\begin{bmatrix}17&32\\32&65\end{bmatrix}\Rightarrow\lambda=81,1\\ \searrow AA^T=\begin{bmatrix}65&-32\\-32&17\end{bmatrix}\Rightarrow\lambda=81,1\end{matrix}$$

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for A^TA

$$= \begin{bmatrix} 17 & 32 \\ 32 & 65 \end{bmatrix} \begin{matrix} \nearrow \lambda_1 = 81, X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \searrow \lambda_2 = 1, X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{matrix}$$

Hence, right singular vectors are

$V_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $V_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now, $A_1V_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sigma_1}A_1V_1$

$$= \frac{1}{\sqrt{81}}\begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and $A_2V_2 = \sigma_2 u_2 \Rightarrow u_2 = \frac{1}{\sigma_2}A_2V_2$

$$= \frac{1}{\sqrt{1}}\begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence $u = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, $\sum \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$, V

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

and S.V.D =

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Q12 Text Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = L.u$$

$$A= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

AX = B ⇒ (Lu) X = B⇒LY = B............(1)

where uX = Y ....(2)

Using (1) LY = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\Rightarrow Y = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

using 2, ux = y

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix}$$

Q13 Text Solution:

A = (unit LTM) (UTM)

$$A= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 13/3 \end{bmatrix}$$

Now, AX = B ⇒(Lu) X = B⇒ L (uX) = B ⇒LY = B ....(1)

Where uX = Y .....(2)

Now, LY = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$$\Rightarrow y = \begin{bmatrix} 4 \\ -10 \\ -13/3 \end{bmatrix}$$

using (2) uX = Y

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 13/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -13/3 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

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Q14 Text Solution:

Let $A=\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}_{n\times1}$ then $AA^T=\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}_{n\times1}$

$\begin{bmatrix}1 & 1 & 1 & \dots & \dots & 1\end{bmatrix}_{1\times n}$

$$\begin{bmatrix}1 & 1 & 1 & \dots & \dots & 1\\1 & 1 & 1 & \dots & \dots & 1\\1 & 1 & 1 & \dots & \dots & 1\\\vdots & & & & & \vdots\\1 & 1 & 1 & \dots & \dots & 1\end{bmatrix}_{n\times n}$$

and A^TA =

$$\begin{bmatrix}1 & 1 & 1 & \dots & \dots & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}=[n]_{1\times1}$$

$=n$

So,

$$P=\frac{AA^T}{A^TA}=\frac{1}{n}\begin{bmatrix}1 & 1 & 1 & \dots & 1\\1 & 1 & 1 & \dots & 1\\1 & 1 & 1 & \dots & 1\end{bmatrix}_{n\times n}$$

Hence Tr (P) = $\frac{1}{n}(1+1+1+\dots+1)$ = 1.

Q15 Text Solution:

$A=\begin{bmatrix}1 & 1\\0 & 1\\1 & -1\end{bmatrix}$, $A^T=\begin{bmatrix}1 & 0 & 1\\1 & 1 & -1\end{bmatrix}$

$A^TA=\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\Rightarrow\left(A^TA\right)^{-1}=\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$

So prozection matrix is

$P = A\,(A^TA)^{-1}\,A^T$

$$=\begin{bmatrix}1 & 1\\0 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\1 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3\\0 & 3\\2 & -3\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\1 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}5 & 3 & -1\\3 & 3 & -3\\-1 & -3 & 5\end{bmatrix}$$

Q16 Text Solution:

$A=\begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 2\end{bmatrix}_{4x2}$, $A^T=\begin{bmatrix}1 & 1 & 1 & 1\\-2 & -1 & 0 & 2\end{bmatrix}_{zx4}$

$A^TA=\begin{bmatrix}4 & -1\\-1 & 9\end{bmatrix}\Rightarrow\left(A^TA\right)^{-1}$

$=\frac{1}{35}\begin{bmatrix}9 & 1\\1 & 4\end{bmatrix}$

Now prozection matrix is

$P = A(A^TA)^{-1}A^T$

$$=\begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 2\end{bmatrix}\frac{1}{35}\begin{bmatrix}9 & 1\\1 & 4\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\-2 & -1 & 0 & 2\end{bmatrix}$$

$$=\frac{1}{35}\begin{bmatrix}7 & -7\\+8 & -3\\9 & 1\\11 & 9\end{bmatrix}_{4\times2}\begin{bmatrix}1 & 1 & 1 & 1\\-2 & -1 & 0 & 2\end{bmatrix}_{2\times4}$$

$$P=\frac{1}{35}\begin{bmatrix}21 & 14 & 7 & -7\\14 & 11 & 8 & 2\\7 & 8 & 9 & 11\\-7 & 2 & 11 & 29\end{bmatrix}$$

So prozection is p= $PB=\frac{1}{35}\begin{bmatrix}133\\95\\61\\-11\end{bmatrix}$

Q17 Text Solution:

$A=\begin{bmatrix}A_1 & A_2\\O_{12} & A_3\end{bmatrix}$

$A_1=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, A_2=\begin{bmatrix}1\\0\end{bmatrix}, O_{12}=\begin{bmatrix}0 & 0\end{bmatrix}$

$A_3 = [2]$

and $B=\begin{bmatrix}B_1 & B_2\\O_{13} & B_3\end{bmatrix}$

$B_1=\begin{bmatrix}1 & 2 & 3\\4 & 3 & 6\end{bmatrix}\ B_2=\begin{bmatrix}1\\1\end{bmatrix}\ O_{13}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}$

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$B_3 = [1]$

$$AB = \begin{bmatrix} A_1 & A_2 \\ O_{12} & A_3 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ O_{13} & B_3 \end{bmatrix}$$

$$= \begin{bmatrix} A_1B_1 + A_2O_{13} & A_1B_2 + A_2B_3 \\ O_{12}B_1 + A_3O_{13} & O_{12}B_2 + A_3B_3 \end{bmatrix}$$

$$A_1B_1 + A_2O_{13} = \begin{bmatrix} 9 & 8 & 15 \\ 19 & 18 & 33 \end{bmatrix}$$

$$A_1B_2 + A_2B_3 = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$O_{12}B_1 + A_3O_{13} = [0 \quad 0] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$+ [2] [0 \quad 0 \quad 0]$

$= [0 \quad 0 \quad 0]$

$O_{12}B_2 + A_3B_3 = [2]$

$$AB = \begin{bmatrix} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$