GATE

DS & AI
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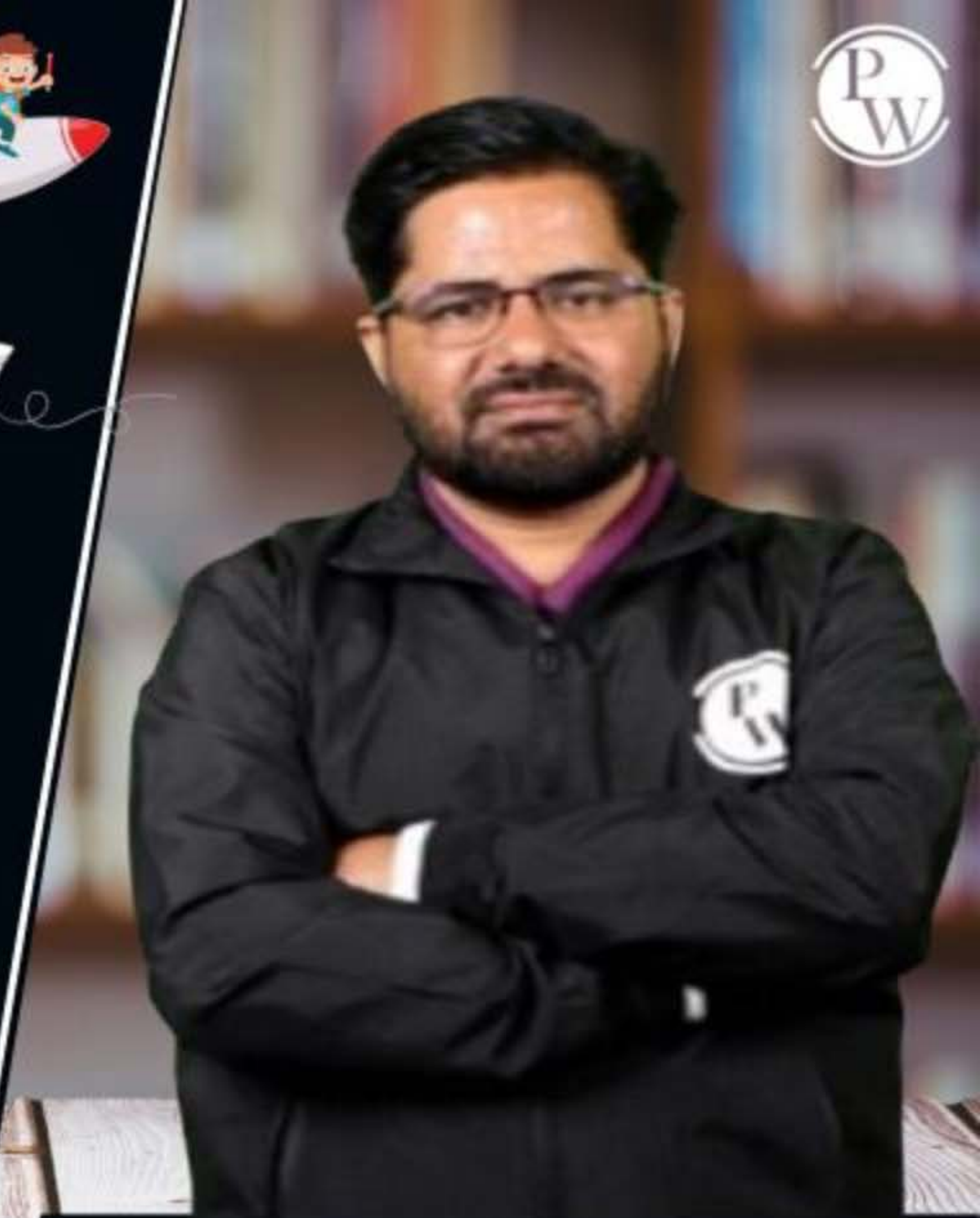

Linear Algebra

Lecture No. 08

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Recap of previous lecture

Topic

→ RANK of MATRIX

→ LD & LI VECTORS

Topics to be Covered

Topic

SYSTEM of LINEAR EQUATION

(Non Homogeneous System)

Properties of Rank –

(1) $\rho(A_{m\times n}) \le \min\{m,n\}$ eg $\rho(A_{6\times 4}) \le 4$

(2) $\rho(A_{m\times n}) \ge 1$ i.e $\boxed{1 \le \rho(A_{m\times n}) \le \min\{m,n\}}$

(3) $\rho(\text{Null Mat}) = 0$ (defined / Assumption)

(4) if $A_{n\times n}$ s.t A is Non Singular then $\rho(A) = n$ & $\rho(A^{-1}) = n$

i.e $\boxed{\rho(A) = \rho(A^{-1})}$

(5) $\rho(A) = \rho(A^T) = \rho(A^\theta) = \rho(AA^T) = \rho(AA^\theta)$

(if $AA^T = I$ then only A is called O-Mat)

~~i.e $\rho(A)$ & ρ(orthogonal Mat) are same~~

(BLUNDER) / PAAP

(6) If A & B are two Matrices s.t AB is defined then

$\boxed{\rho(AB) \le \min\{\rho(A), \rho(B)\}}$ i.e Rank of the product can never exceeds their individual Rank.

eg if $\rho(A_{6\times 7}) = 5$ & $\rho(B_{7\times 5}) = 3$ then $\rho(AB)_{6\times 5} \le 3$

(7) $\rho(A+B) = \rho(A) + \rho(B)$ Not always True

i.e $\rho(A+B) \le \rho(A) + \rho(B)$ (True)

(8) ρ(Row Mat $= \rho(A_{1\times n}) = 1$

ρ(Column Mat$) = \rho(B_{n\times 1}) = 1$

$\Rightarrow \rho(\text{Row} \times \text{Column}) = \rho(AB)_{1\times 1} = 1$ or 0

$\boxed{\rho(\text{Column} \times \text{Row}) = \rho(BA)_{n\times n} = 1 \text{ or } 0}$

Methods of Checking the Nature of Vectors →

Consider the given vectors are $x_1, x_2, x_3, \ldots, x_r$ then Construct a Matrix A as follows; $A = [x_1\, x_2\, x_3 \ldots x_r] \neq$ Row Mat

M-I General Method (always applicable) →

(i) if $\rho(A) =$ No. of vectors $\Rightarrow$ Vectors are LI

(ii) if $\rho(A) <$ " " " $\Rightarrow$ " " LD

M-II Tricky Method (applicable only when A is Sq Mat) →

(i) if $|A| \neq 0 \Rightarrow$ Vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

VARIOUS Defⁿ of RANK:-

Defⁿ $\rho(A_{6\times7}) = 4$

→ Mat A will have at most **4** LI Row vectors.

→ Mat A " " at most **4** LI Column vectors.

Defⁿ of Rank: "It is the order of Non singular Submatrix of Highest order that can exist in a given Mat"

Defⁿ In Books:

if $\boxed{\rho(A_{6\times7}) = 4}$ then

→ ∃ at least one Non Singular submatrix of order 4×4

→ Every square submatrix of order 5×5 & 6×6 are Singular eg $A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 3 & 6 \\ 2 & 1 & -2 & 3 & 1 \\ 1 & -4 & 1 & -3 & 2 \end{bmatrix}_{3\times 5} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \end{array}$

$E = [c_1\ c_4\ c_5]_{3\times 3} \Rightarrow \rho(E) = 2 < 3$

i.e. c_1, c_4, c_5 are LD

$\rho(A) = 3$ → A will have at Most $\underline{3}$ LI Row vectors.

→ " " " " $\underline{3}$ LI Column vectors

$B = [c_1\ c_2\ c_3\ c_4]_{3\times 4}$

$\rho(B) = 3$

$\rho(B) < 4$ (definitely)

So c_1, c_2, c_3, c_4 are LD

$C = [c_1\ c_2\ c_3]_{3\times 3}$

$\rho(C) = 3$ = No. of Vectors $\Rightarrow c_1, c_2, c_3$ are LI

$D = [c_1\ c_2\ c_4]_{3\times 3}$

$\rho(D) = 2 <$ No. of Vectors (3)

$\Rightarrow c_1, c_2, c_4$ are LD

System of Equations

Non Homogeneous system ($AX=B$)

Homogeneous System ($AX=0$)

eg
$$\left.\begin{aligned} 2x-y+4z&=0\\ x-2-3z&=0\\ 3x+2y-z&=0\\ -4x+4y+2z&=0\end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2x-y+4z&=0\\ x+0y-3z&=2\\ 3x+2y-z&=0\\ -4x+4y+2z&=0\end{aligned}\right\} \Rightarrow \begin{bmatrix} 2 & -1 & 4\\ 1 & 0 & -3\\ 3 & 2 & -1\\ -4 & 4 & 2\end{bmatrix}_{4\times 3}\begin{bmatrix} x\\ y\\ z\end{bmatrix}_{3\times 1} = \begin{bmatrix} 0\\ 2\\ 0\\ 0\end{bmatrix}_{4\times 1}$$

$$A_{4\times 3}\,X_{3\times 1} = B_{4\times 1}$$

$$[A \vdots B] = \left[\begin{array}{ccc|c} 2 & -1 & 4 & 0\\ 1 & 0 & -3 & 2\\ 3 & 2 & -1 & 0\\ -4 & 4 & 2 & 0\end{array}\right]_{4\times 4} = \text{Augmented Mat.}$$

① Consider

$$\boxed{\underbrace{A}_{\text{Coefficient Mat}}{}_{m\times n}\ \underbrace{X}_{\text{solution of system / Variable Mat / unknown vector}}{}_{n\times 1} = \underbrace{B}_{\text{Constant Mat}}{}_{m\times 1}}$$

m: No. of equations

n: No. of Variables

$[A \vdots B]$ = Aug. Mat.

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} \text{ Ans}$$

② if $m > n$ then system is overdetermined (Tough)

if $m = n$ " " is equally determined

if $m < n$ " " is underdetermined (Easy)

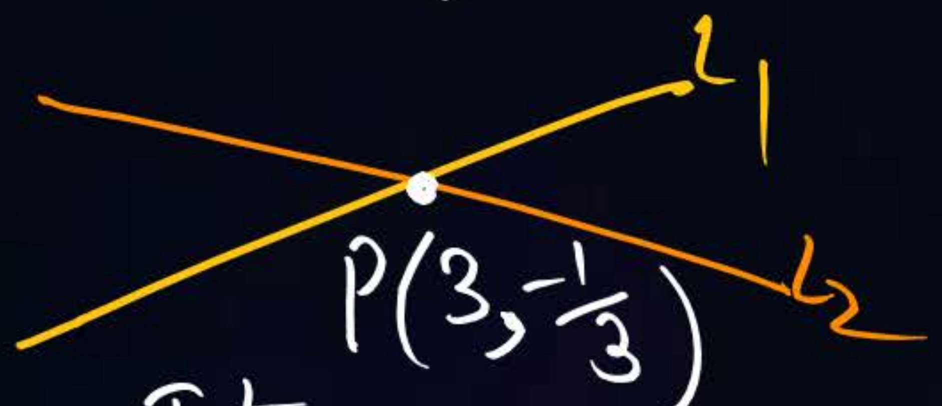
(3) Nature of Solutions →

eg
$$2x+3y=5$$
$$x-3y=4$$

$$X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\-1/3\end{bmatrix}$$

= unique vector exist

= unique sol^n exist.



Intersecting lines

eg
$$2x+3y=5$$
$$4x+6y=10$$

$$X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1/3\end{bmatrix}, \begin{bmatrix}1/2\\4/3\end{bmatrix}, \ldots\ldots$$

= Multiple vectors exist

= ∞ sol^n exist

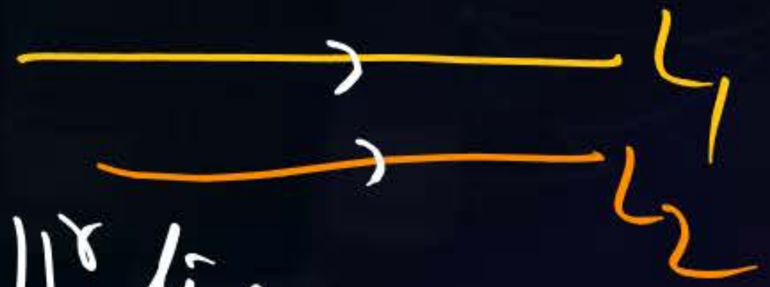
overlapping lines

eg
$$2x+3y=5$$
$$4x+6y=9 \Rightarrow 2x+3y=4.5$$

$$X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}?\\?\end{bmatrix}$$ = No vector exist

= No sol^n exist

$\because$ Mathematically Both the equations are contradicting each other.



$\parallel^r$ lines.

Methods of Solving Non Homog System $\rightarrow$ Consider $\boxed{A_{m\times n}\, X_{n\times 1} = B_{m\times 1}}$

RANK Method (always applicable)
(i.e for $m>n$, $m=n$, $m<n$)

Matrix Method
(applicable only when $m=n$)

① if $\rho(A)=\rho(A:B)$ = No. of Variables $\Rightarrow$ unique sol. $\Leftarrow$ if $|A|\neq 0$

② if $\rho(A)=\rho(A:B)$ < No. of Variables $\Rightarrow$ ∞ sol. $\Leftarrow$ if $|A|=0$, $(\text{adj}A)B=0$

③ if $\rho(A)\neq\rho(A:B)$ $\Longrightarrow$ No sol. $\Leftarrow$ if $|A|=0$, $(\text{adj}A)B\neq 0$

Note: if $A_{n\times n}$ i.e. we have equally determined system.

then $\boxed{AX = B}$ — (1)

$$A^{-1}(AX) = A^{-1}.B$$

$$X = \frac{(\text{adj}\,A)}{|A|}.B$$

$$X = \frac{(\text{adj}\,A)\,B}{|A|} = \text{soln}$$

if $|A| = 0$, $(\text{adj}\,A)B \neq 0$ then $X = \frac{\text{Something}}{0} = \text{N.D.}$

(*) Consistent system → System is called consistent if ∃ solution.
(whether unique or ∞ sol.)

Inconsistent system → System is called Inconsistent if we have No sol.

(*) Necessary condition for a system $\boxed{AX = B}$ to be consistent is ___?___

$\rho(A) = \rho(A \vdots B)$.

Gate write the condition for the existence of $\underbrace{\text{at least one solution}}_{\text{consistency}}$ of an $\underbrace{\text{unknown vector}}$ in $PY = Q$

$\Downarrow$

$$\rho(P) = \rho(P \,\vdots\, Q)$$

The system of equations :

$$2x + y = 5$$

$$x - 3y = -1$$

$$3x + 4y = k$$

is consistent when k is ______

(a) 1 (b) 2

(c) 5 (d) 10

$$[A:B] = \left[\begin{array}{cc:c} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc:c} 1 & -3 & -1 \\ 2 & 1 & 5 \\ 3 & 4 & k \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{cc:c} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 13 & k+3 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{7}R_2} \left[\begin{array}{cc:c} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 13 & (k+3) \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 13R_2} \left[\begin{array}{cc:c} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & (k-10) \end{array}\right]_{3\times 3}$$

$\because \rho(A) = 2$ & for consistency $\rho(A:B) = 2$

$$k = 10$$

M-II

$$2x+y=5 \quad \text{—} \quad (1)$$

$$x-3y=-1 \quad \text{—} \quad (2)$$

$$\Rightarrow x=2,\ y=1$$

$$3x+4y=K \quad \text{—} \quad (3)$$

$$3(2)+4(1)=K$$

$$\boxed{K=10}$$

(*) $\boxed{\rho(A) \not> \rho(A:B)}$ i.e Rank of Coeff Mat can never exceed Rank of Augumented Mat.

eg $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 14 \end{array}\right]$, $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{0} & 10 \end{array}\right]$, $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$

$\rho(A) = 3$, $\rho(A:B) = 3$ unique sol.

$\rho(A) = 2$, $\rho(A:B) = 3$ (No sol.)

$\rho(A) = 2 = \rho(A:B)$ (∞ sol.)

$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 0 \end{array}\right]$

$\rho(A) = 3 = \rho(A:B)$ unique sol.

Q. For the system $Ax = B$ which of the following can be taken as condition for no soln. $\Rightarrow$ $\boxed{\rho(A) \neq \rho(A:B)}$

$\rho(A) > \rho(A:B)$ ✗

$\rho(A) < \rho(A:B)$ ✓

(a) $\rho(A) = \rho(A:B)$

(b) $\rho(A) < \rho(A:B)$ ✓

(c) $\rho(A) > \rho(A:B)$

(d) Both (b) & (c)

Q. The solution of the system:

$$\begin{cases} x+y+z=6 \\ x+2y+3z=10 \\ x+2y+5z=14 \end{cases}$$ will be.

(a) $\begin{bmatrix}4\\2\\0\end{bmatrix}$ (b) $\begin{bmatrix}2\\0\\4\end{bmatrix}$ ✓(c) $\begin{bmatrix}4\\0\\2\end{bmatrix}$ (d) $\begin{bmatrix}-4\\0\\-2\end{bmatrix}$

M-I using options, Ans = (c)

M-II w/o using options.

Solve above three eqn's simultaneously as discussed in class 8th ⟶ Do yourself.

M-III (RANK Method) ⟶

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 5 & 14 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{array}\right]_{3\times 4}$$

$\rho(A) = 3 = \rho(A:B)$

$\because \rho(A) = \rho(A:B) = $ No. of variables (3)

$\Rightarrow$ unique soln exist.

(ii) Procedure of finding Solution :

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 14 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} (x+y+z) \\ (y+2z) \\ 2z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$$

$$x+y+z=6 \Rightarrow x+0+2=6 \Rightarrow x=4$$

$$y+2z=4 \Rightarrow y+0=4 \Rightarrow y=0$$

$$2z=4 \Rightarrow z=2$$

So $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ Ans

this method is called Rank Method,
Echelon Form Method,
Gauss Ellimination method.
Backward Substitution method.

Q. Find the values of λ & μ for which following system has

① ∞ sol. ② No sol ③ unique sol.

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

$$[A:B]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \dashrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{(\lambda-3)} & (\mu-10) \end{array}\right]_{3\times 4}$$

3×3

For ∞ solⁿs :- $\rho(A)=\rho(A:B)<3$

Let $\rho(A)=2$ & $\rho(A:B)=2$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\lambda=3$ & $\mu=10$

Ans

For No sol :- $\rho(A)\neq\rho(A:B)$

$\Rightarrow \rho(A)<\rho(A:B)$

Let $\rho(A)=2$ & $\rho(A:B)=3$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\lambda=3$ & $\mu\neq 10$

(iii) for unique solⁿ $\rightarrow \rho(A) = \rho(A:B) = $ No. of variables (3)

i.e $\rho(A) = 3 \Rightarrow \lambda \neq 3$

So $\boxed{\lambda \neq 3, \mu \in R}$

& $\rho(A:B) = 3 \Rightarrow \mu$ can take any value

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{(\lambda-3)} & (\mu-10) \end{array}\right]$$

eg if $\lambda = 5, \mu = 14 \Rightarrow$ unique sol.

eg if $\lambda = 5, \mu = 0 \Rightarrow$ " "

eg if $\lambda = 7, \mu = 10 \Rightarrow$ " "

Q. Consider the S.O.Eq. $\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$

Notice that, 2nd & 3rd Column of Coeff. Mat are L.D then for how many values of α does the above system has ∞ solutions?

(a) 1
(b) 2
(c) 3
(d) 1/5
(e) ,

$$[A:B] = \begin{bmatrix} 2 & 1 & -4 & : & \alpha \\ 4 & 3 & -12 & : & 5 \\ 1 & 2 & -8 & : & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -8 & : & 7 \\ 4 & 3 & -12 & : & 5 \\ 2 & 1 & -4 & : & \alpha \end{bmatrix}$$

$$\ldots\ldots \begin{bmatrix} 1 & 2 & -8 & : & 7 \\ 0 & a_{22} & a_{23} & : & a_{24} \\ 0 & 0 & 0 & : & \frac{5\alpha-1}{2} \end{bmatrix}$$

for ∞ sol^n, $\rho(A) = \rho(A:B) < 3$

But $\rho(A) = 2 \Rightarrow \rho(A:B) = 2 \Rightarrow \alpha = \frac{1}{5}$,

Q. if $A_{n\times n}$ s.t. $A^2 = I$ then $AX = B$ has ______ sol.

(a) unique sol.

(b) ∞ sol.

(c) No sol.

(d) More than one but finite No of sol.

M-I $A^2 = I$

$|A^2| = |I|$

$|A|^2 = 1$

$|A| = \pm 1$

ie $|A| \neq 0$

$\Downarrow$

By Matrix Method unique sol exist.

M-II $A^2 = I \Rightarrow A = A^{-1}$

ie A^{-1} exist $\Rightarrow |A| \neq 0$

$\Downarrow$

By Matrix Method, unique sol exist.

M-II / GEN 'Z' → $A^2 = I$ (given)

Now $AX = B$

$$\Rightarrow A(AX) = A \cdot B$$

$$A^2 X = A_{n\times n} B_{n\times 1}$$

$$IX = [AB]_{n\times 1}$$

$$X = \begin{bmatrix} - \\ \vdots \\ \vdots \\ - \end{bmatrix}_{n\times 1}$$ ie unique vector exist ⇒ unique sol exist.

THANK - YOU

Tel:

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