DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

Not for CS/IT

Linear Algebra- II

Lecture No. 06

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Recap of previous lecture

Topic

PROJECTION MATRIX

(Part 1)

Topics to be Covered

Topic PROJECTION MATRIX

(Part 2)

Q: Find a Matrix that projects every point in 2D plane onto the line $x+2y=0$. Also find the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ onto that line.

Sol: vector form of line $\boxed{x+2y=0}$ is $A=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$ $\quad x=-2y$ $\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2y\\y\end{bmatrix}=y\begin{bmatrix}-2\\1\end{bmatrix}$

So Projection Mat $P=\dfrac{AA^T}{A^TA}=\text{------}=\boxed{\dfrac{1}{5}\begin{bmatrix}4&-2\\-2&1\end{bmatrix}}$

(ii) $B=\begin{bmatrix}1\\1\end{bmatrix}$ So $\vec{p}=P\vec{B}=\dfrac{1}{5}\begin{bmatrix}4&-2\\-2&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\dfrac{-1}{5}\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$

Note: $Tr(P)=1$, $|P|=0$, $\rho(A)=1$, $\lambda=0\ \&\ 1$, No. of LI Evectors = Two = order

P = Symm & Idempotent $\qquad$ Hence Diagonalizable.

PODCAST: Along the line $x+2y=0$, let us take $A=\begin{bmatrix}-4\\2\end{bmatrix}$

$$A^TA = \begin{bmatrix}-4 & 2\end{bmatrix}\begin{bmatrix}-4\\2\end{bmatrix} = [20]_{1\times 1} = 20$$

$$AA^T = \begin{bmatrix}-4\\2\end{bmatrix}\begin{bmatrix}-4 & 2\end{bmatrix} = \begin{bmatrix}16 & -8\\-8 & 4\end{bmatrix}$$

So $$P = \frac{AA^T}{A^TA} = \frac{1}{20}\begin{bmatrix}16 & -8\\-8 & 4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$$

& $$\vec{p} = PB = \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2/5\\-1/5\end{bmatrix}$$

i.e Projection Mat of A & (KA) would be SAME.

i.e Scaling will not effect Projection Mat.

Given a vector $V = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and a subspace W spanned by $W = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, what is the projection matrix P that projects any vector onto W, and what is the projection of V onto W?

(a) $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$

(b) $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$

(c) $P = \begin{pmatrix} 0.5 & 2.5 \\ 1.5 & 1.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$ $\because Tr = 2 \neq 1$

$\times$

(d) $P = \begin{pmatrix} 0.5 & 0.25 \\ 1.5 & 1.25 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ $\because Tr = 1.5 \neq 1$

$\times$

let $A = W = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow P = \frac{WW^T}{W^TW} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Here $B = V$ so $\vec{p} = PB = PV = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$

Also (c) & (d) are Not symmetric

11 In a 2-dimensional space R^2, consider the subspace W spanned by the vector $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let P be the projection matrix onto W. Whcih of the following vectors is the image of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ under projection P?

$= A$ $\quad B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$

$$P = \frac{AA^T}{A^TA} = \frac{WW^T}{W^TW} = \frac{V_1V_1^T}{V_1^TV_1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{So } \vec{p} = PB = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$$

For a 2×2 projection matrix P that projects $\boxed{\text{onto the line } y = x,}$ what is the matrix P?

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\neq$ Symm. ✗

(b) $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\neq$ Symm ✗

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Trace $\neq 1$ ✗

(d) $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ ✓

line $y - x = 0$

So $A = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

i.e. $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

0 Which of the following matrices represents a projection onto the line L with direction vector $d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ Trace $\neq 1$

(b) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ $Tr \neq 1$

$$P = \frac{AA^T}{A^TA} = \frac{dd^T}{d^Td} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$= (c)$

$\because d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, d^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$dd^T = \begin{bmatrix} \text{we will get} \\ \text{all the entries} \end{bmatrix}$ So (b) is not possible

PROJECTION MATRIX (along a subspace, spanned by set of vectors) →

let x_1, x_2, x_3 are given vectors & we want to find the projection of $\vec{B}$ onto the subspace spanned by x_1, x_2, x_3 then process will be as follows;

Consider a Mat $A = [x_1\, x_2\, x_3]$ then

→ Projection Mat $\boxed{P = A(A^TA)^{-1}A^T}$

→ Projection Vector $\vec{p} = PB$

Verify: let $A = [x_1]_{n\times 1}$, $A^T = [\ \ ]_{1\times n}$,

then $A^TA = [\ \ ]_{1\times 1}$ = Scalar Quantity

So $P = A\underbrace{(A^TA)^{-1}}_{1\times 1}A^T$

$= \dfrac{AA^T_{\ n\times n}}{A^TA_{\ 1\times 1}} = \left(\dfrac{AA^T}{A^TA}\right)_{n\times n}$

Q. The projection of $B = \begin{bmatrix}4\\5\\6\end{bmatrix}$ onto the $\underbrace{\text{Column space}}$ of $A = \begin{bmatrix}1 & 2\\1 & 3\\0 & 0\end{bmatrix}$ will be ?

$C(A)$ = set of all Linear Combinations of Columns of A where $A = [X_1 \; X_2]$

$= \{(c_1X_1 + c_2X_2) ; c_1, c_2 \in \text{SCALARS}, X_1 \,\&\, X_2 \text{ are Column of } A\}$

Here X_1, X_2 lies in 2D as well as they are L.I. also (By observation)

So $C(A)$ is nothing but 2-D Space or $\boxed{C(A) = R^2}$.

$A^TA = \begin{bmatrix}1 & 1 & 0\\2 & 3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\\0 & 0\end{bmatrix} = \begin{bmatrix}2 & 5\\5 & 13\end{bmatrix}$, $|A^TA| = 1$ so $(A^TA)^{-1} = \frac{1}{(1)}\begin{bmatrix}13 & -5\\-5 & 2\end{bmatrix}$

Standard Form of Projection Mat onto the subspace SPANNED by A is given as

$\boxed{P = A(A^TA)^{-1}A^T} = \begin{bmatrix}1 & 2\\1 & 3\\0 & 0\end{bmatrix}\begin{bmatrix}13 & -5\\-5 & 2\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\2 & 3 & 0\end{bmatrix} = \cdots = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ = P. Mat.

ie Projection Mat $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Projected vector $\vec{p} = \underline{PB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

Analysis:

(1) $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

Consider $[A : B] = \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 0 & 0 & 6 \end{array}\right]$

$\rho(A) = 2, \quad \rho(A : B) = 3$

ie $AX = B$ is InConsistent

$\Rightarrow \quad AX = \vec{p}$ Now it is Consistent.

$\therefore [A : \vec{p}] = \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \rho(A) = \rho(A : \vec{p})$ ie Consistent.

KID'S EXAMPLE :→

Doctor (Heart Beat)

$x = 116$
$x = 73$
$x = 94$
$x = 124$
$x = 53$
$x = 144$

(Inconsistent)

Project (medicine) → $x = 98$

Consistent.

(2) Use $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$, $Tr(P) = 2$, $|P| = 0$, $\rho(P) = 2$

$\Downarrow$

$\lambda = 0, 1, 1$

$P^T = P \Rightarrow$ symm,

$P^2 = P \Rightarrow$ Idempotent

AM of '0' = one
AM of '1' = two

$\left.\right\}$ i.e. A is Diagonalizable.

$\therefore$ No. of LI E.Vectors for A = three = order

Q: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then Find projection of B onto A ?

$A = [x_1 \;\; x_2]$

Also find the orthogonal complement onto the columns of A.

OR

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then split B into $\vec{p} + \vec{q}$ where $\vec{p}$ is in the Column space and $\vec{q}$ will be $\perp^r$ to that space.

Sol: $A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$(A^T A)^{-1} = \frac{1}{(3)} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\boxed{P = A(A^T A)^{-1} A^T}$$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

ie for $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $P = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow \vec{p} = PB = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

(ii) w.k. that $\vec{p} + \vec{q} = \vec{B}$ where $\vec{q}$ = orthogonal complement of $\vec{p}$.

$$\Rightarrow \vec{q} = B - \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

(iii) Also find Projection Mat for orthogonal complement

Sol: w.k. that $P + Q = I \Rightarrow Q = I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix}$

PROPERTIES of Projection Mat →

① Trace of projection Mat on to the line = 1 (always)

② Trace of projection Mat onto the subspace is not Necessarily 1

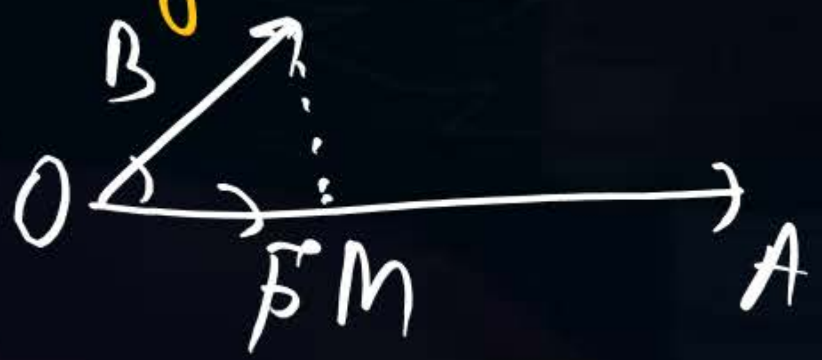
③ Determinant of Projection Mat is always Zero except identity Mat
doesn't matter whether onto the line or onto the subspace.

④ Projection Mat is also an Idempotent Matrix

if we take the projection of $\vec{p}$ onto A again then it will be vector itself

$P(\vec{p}) = P(PB) = PB$

$\Rightarrow P^2B = PB \Rightarrow P^2 = P$



(5) $\because P$ is an Idempotent Mat

i.e. $P^2 = P$

$\Rightarrow \lambda^2 = \lambda \Rightarrow \lambda(\lambda-1) = 0$

or $\boxed{\lambda = 0 \text{ or } 1}$

i.e. EValues of Projection Mat are either 0 or 1 or Both

(6) Projection Mat is always Symm

Proof: w.k.that $P = A(A^TA)^{-1}A^T$.

Now, $P^T = \left(A(A^TA)^{-1}A^T\right)^T$

$= \left[A\{A^{-1}(A^T)^{-1}\}A^T\right]^T$

$= \left[A\{A^{-1}(A^{-1})^T\}A^T\right]^T$

$= (A^T)^T \cdot ((A^{-1})^T)^T \cdot (A^{-1})^T \cdot A^T$

$= A \; A^{-1}(A^{-1})^T \; A^T$

$= A \; \underbrace{A^{-1}(A^T)^{-1}} \; A^T$

$= A \cdot (A^TA)^{-1} A^T = P$

i.e. $P^T = P$ Hence Proved.

(7) If A is sq. Matrix s.t A^{-1} exist then $\begin{cases} P = I \\ \boxed{\vec{p} = \vec{B}} \text{ (i.e vector itself.)} \end{cases}$

Proof: $P = \boxed{A(A^TA)^{-1}A^T} = A\{A^{-1}(A^T)^{-1}\}A^T = (AA^{-1})((A^T)^{-1}A^T) = I \cdot I = I$

& $\vec{p} = PB = IB = B$ i.e vector itself Hence proved.

(8) if B is orthogonal to the Columns of A then $\begin{cases} P = \text{have No Shortcut.} \\ \boxed{\vec{p} = \vec{0}} \end{cases}$

let $A = [X_1\ X_2\ X_3]$ s.t $X_1 \cdot B = X_2 \cdot B = X_3 \cdot B = 0$

i.e $[X_1\ X_2\ X_3] \cdot B = 0$

$A \cdot B = 0$

or $A^T B = 0$

so $\vec{p} = PB = (A(A^TA)^{-1}A^T)B$

$= (A(A^TA)^{-1})(A^TB)$

$= A(A^TA)^{-1} \cdot 0 = \vec{0}$

(a) If B is LD on columns of A then $\to P =$ No shortcut

$\boxed{\vec{p} = B}$ itself.

Let $A = [x_1\ x_2\ x_3]$

$\because$ B is LD on columns of A so $\exists$ linear combination of the type.

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = B$$

$$[x_1\ x_2\ x_3]\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = B$$

$$AX = B \quad \text{—} \ (1)$$

$$\vec{p} = P\vec{B} = \left[A(A^TA)^{-1}A^T\right]B$$

$$= \left[A(A^TA)^{-1}A^T\right]AX \quad \text{By (1)}$$

$$= \left[A(\cancel{A^TA})^{-1}(\cancel{A^TA})\right]X$$

$$= (A\,I)X = AX = B$$

Q: Find the projection Matrix & projection of B $(b_1\ b_2\ b_3)$ onto the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} = [x_1\ x_2\ x_3]$

Sol: $\because |A| = \ldots\ldots\ldots = -42$ i.e. $|A| \neq 0 \Rightarrow A^{-1}$ exist.

By prop (7), $P = I$ & $\vec{p} = B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Ans

Q: If V is a subspace spanned by (1,1,0,1) & (0,0,1,0) then find

① the projection Matrix P onto V

② The vector in V which is closest to the vector B = (0 1 0 -1)

(ii): $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ $\because X_1 \cdot B = 0$ & $X_2 \cdot B = 0$

ie B is $\perp^r$ to Columns of A so $\boxed{\vec{p} = \vec{0}}$ Ans.

(i) Do it by conventional Method:→

Let $A = [X_1 \; X_2] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A^T A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$; $(A^T A)^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$$\boxed{P = A(A^T A)^{-1} A^T} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

ANALYSIS.

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}_{4\times 4} = \text{Symm}, \text{Idempotent}, \rho(A) = 2$$

$(P^T = P)$ $(P^2 = P)$

$\Downarrow$

$\lambda = 0, 0, 1, 1$

$Tr(P) = 2$, $|P| = 0$,

GM of $(\lambda = 0)$ = order $- \rho(A - 0.I) = 4 - \rho(A) = 4 - 2 = 2$

GM of $(\lambda = 1)$ = order $- \rho(A - 1.I) = 4 - \rho\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = 4 - 2 = 2$

$\because$ No. of LI EVectors = order of A Hence Diagonalisable.

Q: find the projection Matrix & projection of B $(b_1\ b_2\ b_3)$ onto the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix}_{3\times3} = [x_1\ x_2\ x_3]$

we want to find the projection of $\vec{B}$ onto the subspace spanned by x_1, x_2, x_3

$\because |A| = \dots\dots = -42$

i.e $|A| \neq 0 \Rightarrow A^{-1}$ exist So By Property (7)

$P = I$

$\vec{p} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Q. Find the projection of $B(1\ 3\ 4)$ onto $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = [X_1\ X_2]$

M-I By observation, $\boxed{B = X_1 + 3X_2}$

i.e. B is LD on columns of A

So By Property (9), $\vec{p} = B = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

M-II Consider $M = [X_1\ X_2\ B]$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

$\because |M| = 0$ So X_1, X_2, B are LD $\Rightarrow$ By Prop (9), $\vec{p} = B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Q: Find the projection of $B(3\ -2\ 1)$ onto the columns of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -4 \end{bmatrix}$

(a) $\vec{B}$

(b) $\vec{0}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\because X_1 \cdot B = 0 = X_2 \cdot B$ $\qquad = [X_1\ X_2]$

ie B is orthogonal to the columns of A

Hence by Prop (8), $\vec{p} = \vec{0}$ Ans

Consider the vectors space R^3 and the subspace W spanned by vectors $W_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $W_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$. What

HW

$A = [W_1 \; W_2]_{3\times 2}$

is the projection matrix P that projects any vector onto W and what is the projection of $V = \begin{pmatrix}3\\4\\5\end{pmatrix}$ onto W?

$B =$

(a) $P = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\-1 & 1 & 1\end{bmatrix}$, Projection of $V = \begin{bmatrix}5\\4\\3\end{bmatrix}$ ✗ $\because |P| \neq 0$ & Not symm

(b) $P = \frac{1}{3}\begin{bmatrix}2 & 1 & -1\\1 & 2 & 1\\-1 & 1 & 2\end{bmatrix}$, Projection of $V = \frac{1}{3}\begin{bmatrix}5\\16\\11\end{bmatrix}$ ✓

(c) $P = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix}$, Projection of $V = \begin{bmatrix}2\\3\\4\end{bmatrix}$ ✗ $\because |P| \neq 0$ & Not symm

(d) None of the above

$P = A(A^TA)^{-1}A^T = \ldots = \frac{1}{3}\begin{bmatrix}2 & 1 & -1\\1 & 2 & 1\\-1 & 1 & 2\end{bmatrix}$

& $\vec{p} = PV = \frac{1}{3}\begin{bmatrix}5\\16\\11\end{bmatrix}$

Given a 3×3 projection matrix P that projects onto the plane spanned by the vectors $[1, 0, 1]^T$ and

HW

$[0, 1, 1]^T$. What will be the $\boxed{\text{norm}}$ of $P\begin{bmatrix}1\\1\\1\end{bmatrix}$?

$$A = [x_1\ x_2] = \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T = ? = \frac{1}{3}\begin{bmatrix}2 & -1 & 1\\-1 & 2 & 1\\1 & 1 & 2\end{bmatrix}$$

(a) 1

(b) $\sqrt{2}$

(c) $\sqrt{3}$

(d) $2\sqrt{\frac{2}{3}}$ ✓

$$\vec{p} = P\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\2\\4\end{bmatrix}$$

$$\|\vec{p}\| = ? = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{16}{9}}$$

$$= \sqrt{\frac{24}{9}} = 2\sqrt{\frac{2}{3}}.$$

QUICK RECAP

PROJECTION MATRIX (along a subspace spanned by set of vectors) →

let W is the subspace spanned by set of vectors in A

& we want to to find projection of B onto the subspace spanned by A

then Projection Mat is given as.

$$\boxed{P = A(A^T A)^{-1} A^T}$$ & Projection vector is $\boxed{\bar{p} = PB}$

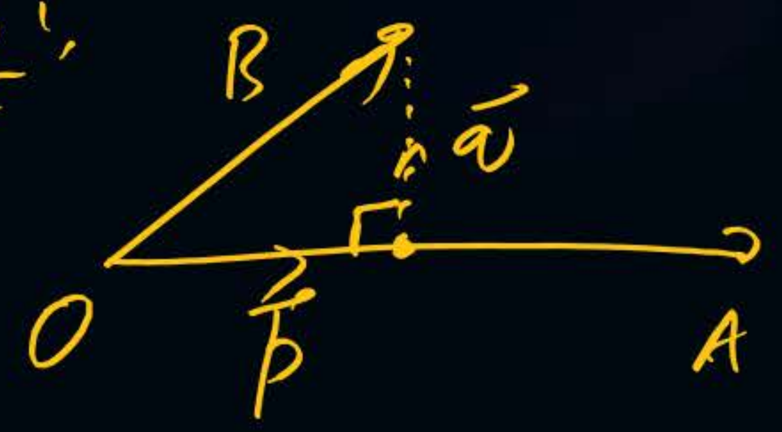
RECAP

PROJECTION MATRIX (along a vector A or ONTO the line A) →

$$P = \left(\frac{AA^T}{A^T A}\right)_{n\times n}$$ and Projection Vector of B onto A is $\boxed{\bar{p} = PB}$

RECAP

Note:



$\vec{p}$ = projection vector of B onto A

& it is given as $\boxed{\vec{p} = PB}$

Similarly $\vec{q}$ = orthogonal complement of B onto A

& it is given as $\boxed{\vec{q} = B - \vec{p}}$ Learn.

$\because \boxed{\vec{p} + \vec{q} = \vec{B}}$

or $P\vec{B} + Q\vec{B} = \vec{B}$

i.e. $\boxed{P + Q = I}$

i.e. orthogonal projection Mat of B onto A is given as

$\boxed{Q = I - P}$ Learn.

Properties of Projection Matrix → $\boxed{P = A(A^TA)^{-1}A^T}$ & $\boxed{\hat{p} = PB}$

(1) $\boxed{\text{Trace of Proj. Mat (onto the line)} = 1}$

RECAP & Trace of Proj Mat (onto the subspace) is Not Necessarily one

(2) Projection Mat is always Symmetric as well as Idempotent

(whether onto the line or onto the subspace)

(3) Det of Projection Mat (except identity Matrix) is always = 0

(Doesn't matter, whether onto the line or onto the subspace)

eg $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → $I^2 = I$ idempotent But $|I| \neq 0$

→ $I^T = I$ symm

(4) If A is Invertible (i.e. $|A| \neq 0$ & A^{-1} exist) then $P = I$ & $\vec{p} = B$ itself

(5) If B is L.D on Columns of A then $P =$ No idea & $\vec{p} = B$ itself

(6) If B is orthogonal to the Columns of A then $P =$ No idea & $\vec{p} = \vec{O}$.

RECAP

For a projection matrix P that projects onto a subspace spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in R^2, what will be det (P)?

(a) 0

(b) 0.5

(c) 1

(d) 2

By Prop: $|P| = 0$

If P is a 2×2 projection matrix such that $P\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$ and $P\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\0.5\end{bmatrix}$, what is det (P)?

(a) 1.5

(b) 0.25

(c) 0.5

(d) No valid Projection Matrix P

w.k. that $|P| = 0$ always except identity Mat

& for Identity Mat, $|P| = 1$

Let P be a projection matrix. Which of the following properties does not necessarily hold?

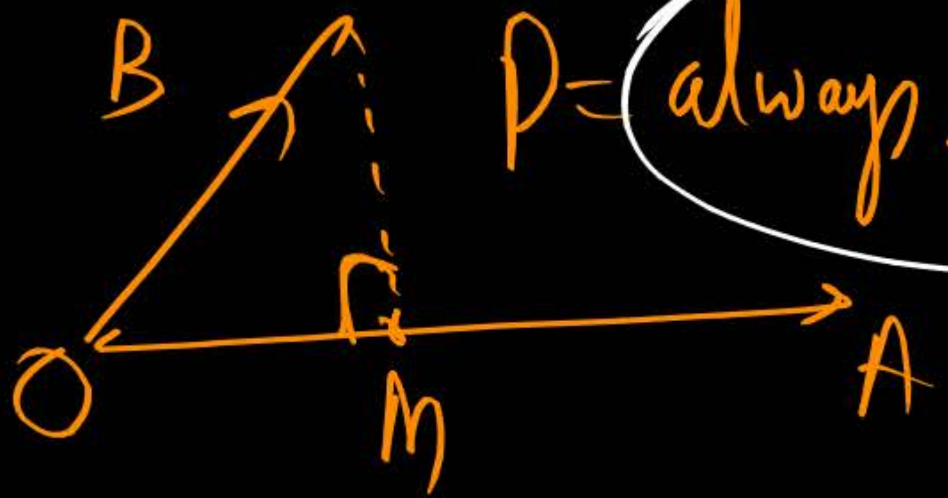
(a) P is idempotent. (T)

(b) The eigen values of P are either 0 or 1. (T)

(c) P is always diagonalizable. (T)

(d) P is always symmetric. ✓

Case I By default we take orthogonal Projection.



P = always symm.

Case II Sometimes projection may be of following type

this is not Orthogonal type of Projection

so P = Not necessarily symm.

Given a projection matrix P in R^n, how can you determine the dimension of the subspace onto which P projects?

(a) It is the number of non-zero columns in P.

(b) It is the rank of P.

(c) It is the number of zero eigen values of P.

(d) It is the trace of P.

Imp Point

Rank of Proj. Mat = {Dimension of Subspace formed by Columns of A}

$\rho(P)$ = dim(column space of A)

THANK - YOU