GATE

DS & AI
CS & IT

Linear Algebra - I

Lecture No. 12

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Recap of previous lecture

Topic

EIGEN VALUES - EIGEN VECTORS

→ Basic Concepts

→ Properties of E-Values

Topics to be Covered

Topic

EIGEN VALUES-EIGEN VECTORS

→ Cayley Hamilton Theorem

→ Procedure of finding E. Vectors

PROPERTIES of Values → Let $A_{n\times n}$ having Eigen Values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

① T. Number of E-Values of A = order of A (whether Different or Repeated)

② Sum of E-Values = Trace(A) i.e $\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n = Tr(A)$

③ Product of E Values = Det(A) i.e $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n = |A|$

④ (Zero is an E. Value of A) iff (A is Singular) i.e $(\lambda=0) \Longleftrightarrow (|A|=0)$

⑤ Number of Non Zero E.Values of A $\leq \rho(A)$

eg if $\rho(A_{6\times 6})=4$ then A has at least two Eigen Values as 0, 0.

⑥ If sum of all the elements in each Row (or each Column) is unique constant K then that Constant K will be one of the E.Value of A.

(7) Don't use E-Operations in a given Mat while Calculating E-Values

But we can apply 3rd E-Operation in it's C.Eqn i.e. in $|A-\lambda I|=0$

(8) E-Values of U.T.M, L.T.M, Diag Mat, scalar Mat, Identity Mat are just the diagonal elements.

eg $A=\begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\lambda=2,0,-3,-1$ $\quad$ $\lambda=2,-1,1$ $\quad$ $\lambda=2,-3,4$

(9) If λ is an Eigen Value of A then to find Eigen Value of any algebraic expression formed by A, we can Replace A with λ in that expression.

Conclusion i.e Equivalent Matrices have Same RANK But may have different Determinant as well as different E.Values.

(10) Let $\lambda_1, \lambda_2, \lambda_3 ---- \lambda_n$ are the E Values of $A_{n\times n}$ then

(i) E Values of A^T are $\lambda_1, \lambda_2, \lambda_3, ----, \lambda_n$ is same as that of A.

(ii) E Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, ----, \lambda_n^m$; $(m \in N)$

(iii) E Values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} ---- \frac{1}{\lambda_n}$ (provided $|A| \neq 0$)

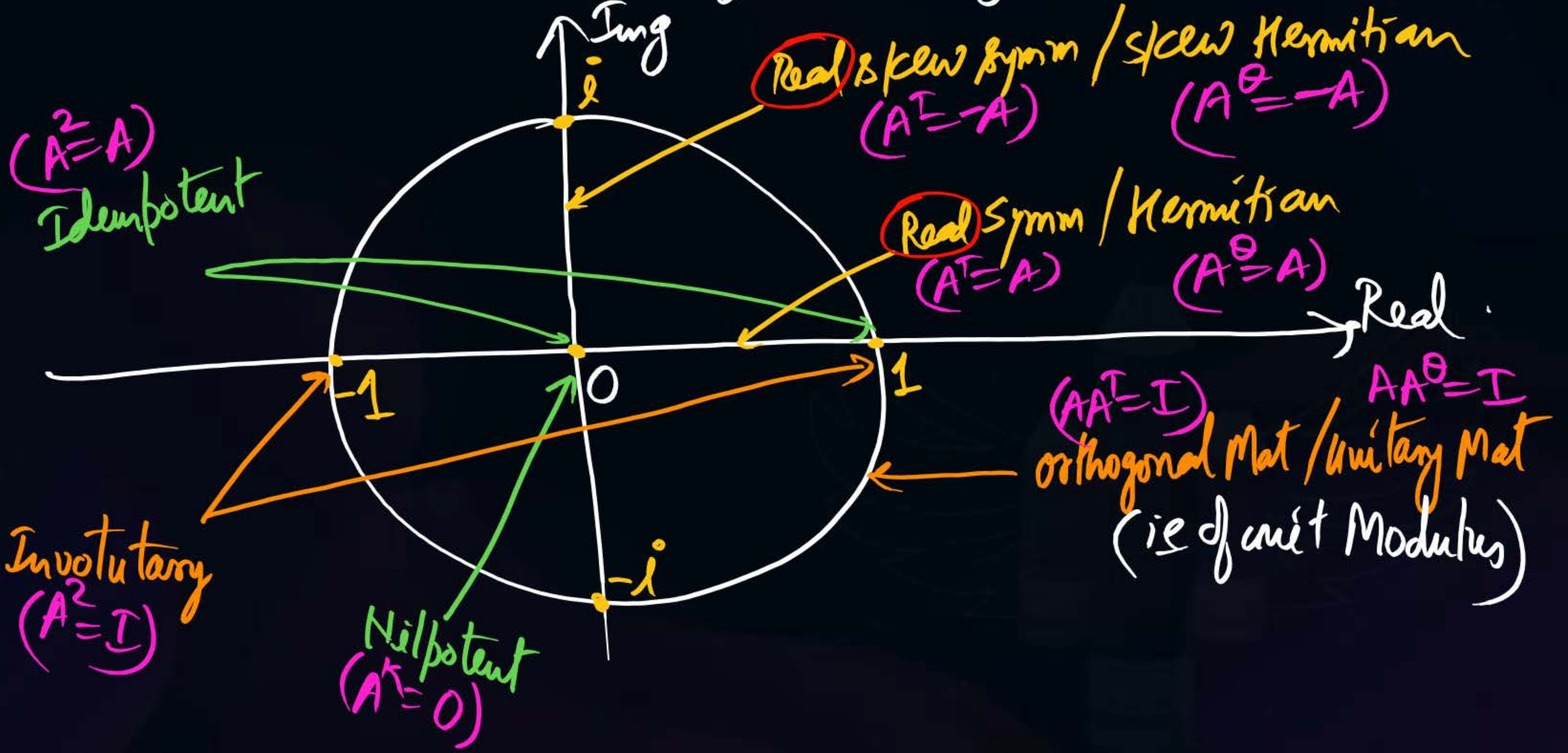
(iv) E. Values of (adj A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, ----, \frac{|A|}{\lambda_n}$

But take Case, this short cut of finding E Values of (adj A) is applicable when $|A| \neq 0$

if, $|A| = 0$ then to find E Values of (adj A), use conventional Approach.

(1) Shortcut Method to Learn Various Th based on E Values →

Consider a unit circle centered at origin i.e $x^2+y^2=1$



(*) If $\{\lambda_i, X_i\}$ is an E-pair of $A_{n\times n}$ then which is false?

(a) $\exists$ Non Zero E Vector of A if $\rho(A-\lambda I) < n$ (T)

(b) if $A^{\theta} = A$ then $\lambda_i \in R \ \forall i$ (T) $\because$ A is Hermitian

(c) if $\boxed{A^{-1} = A^T}$ then $|\lambda_i| = 1 \ \forall i$ (T) $\because$ A is orthogonal.

(d) $\{\lambda_i^m, X_i^m\}$ is an E-pair of A^m? (F) $<$ λ^m is an E Value of A^m (T); X^m is an E Vector of A^m (F)

(e) If $\boxed{A = A^{-1}}$ then the Eigen Value of A is 1 (T) $\because$ A is Involutory

$\because X = \begin{bmatrix} \ \end{bmatrix}_{n\times 1} \Rightarrow X^2 = X_{n\times 1} X_{n\times 1} = N.D$ ie Any Power of X is not defined

ie Mat A & A^m have <u>Same</u> E. Vectors

The eigen values of A = $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ are _____.

$|A| = 0-(i)(-i) = +i^2 = -1$

(a) Purely imaginary (b) Zero

(c) Real (d) None of the above

(M-I) $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

Complex skew symm. (Not necessarily have purely Img E Values).

Hermitian. (Purely Real)

(M-II) C. Eqn; $\lambda^2 - (Tr(A))\lambda + (|A|) = 0$

$\lambda^2 - (0)\lambda + (-1) = 0 \Rightarrow \lambda = \pm 1$ (Real)

C.H.T

$$A_{3\times3} \to \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \quad (\text{Equ}^n)$$

$$\lambda \to A \; ; \quad A^3 + a_1A^2 + a_2A + a_3I = O \quad (\text{Mat Equ}^n)$$

$$A_{4\times4} \to \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

$$\lambda \to A \; ; \quad A^4 + a_1A^3 + a_2A^2 + a_3A + a_4I = O$$

$$Tr(A) = -a_1, \quad |A| = -a_3, \; a_4$$

Cayley-Hamilton Theorem → (C.H.T) → $\boxed{\text{"Every Sq. Mat satisfies it's own CEq"}}$

i.e we can Replace $\lambda \to A$ in CEq^n.

Consider $A_{n\times n}$ then it's C.Eq is $|A-\lambda I|=0$

i.e $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots + a_{n-1}\lambda + a_n = 0$

By C.H.Th, $\lambda \to A$

$1.A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots + a_{n-1}A + a_n I = O$

Application:- using this Theorem, we can find $Tr(A)$, $|A|$ and A^{-1} as follows;

(1) $\boxed{Tr(A) = -a_1}$, (2) $|A| = (-1)^n a_n$, (3) A^{-1} = will be discussed in Quest.

Take care:- Coeff of A^n should be unity.

Note ① Constant term in the C.E of $\underline{A_{n\times n}} = a_n = \begin{cases} -|A| & , n = \text{odd} \\ +|A| & , n = \text{even.} \end{cases}$

Proof: w.k. that $|A| = (-1)^n a_n$

or $(-1)^n a_n = |A|$

$(-1)^n \cdot (-1)^n a_n = (-1)^n |A|$

$(-1)^{2n} a_n = (-1)^n \cdot |A|$

$a_n = (-1)^n |A| = \begin{cases} -|A|, & n = \text{odd} \\ |A|, & n = \text{even} \end{cases}$

② Shortcut Method of finding C Eq of $A_{2\times 2}$ Mat : $\longrightarrow$

C.Equ^n is $|A - \lambda I| = 0$

$\lambda^2 + a_1\lambda + a_2 = 0$

$\lambda^2 - (-a_1)\lambda + [(-1)^2 a_2] = 0$

$$\boxed{\lambda^2 - (\text{Tr}A)\lambda + (|A|) = 0}$$

Q: using C.H.T for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ evaluate $Tr(A)$, $|A|$, $A^{-1} = ?$

Sol: C. Equⁿ is $|A - \lambda I| = 0$

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$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By C.H.T, $\lambda \to A$

$$\boxed{A^3 - 5A^2 + 7A - 3I = 0} \quad \text{(1)}$$

$$\left(A^3 + a_1 A^2 + a_2 A + a_3 I = 0\right)$$

(1) $Tr(A) = -(a_1) = -(-5) = +5$

(2) $|A| = (-1)^3 a_3 = -(-3) = +3$

(3) $\because |A| \neq 0 \Rightarrow A^{-1}$ exist

Pre Multiply by A^{-1} in (1).

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = A^{-1} \cdot 0$$

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

$$(-3A^{-1}) = -(A^2 - 5A + 7I)$$

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$$ //

Qs using C.H.T, evaluate A^{-1} for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) $\frac{1}{3}(A^2 - 5A + 7I)$

(b) $A^2 - 5A + 7I$

Already Solved.

(c) $\frac{1}{3}(A^2 + 5A - 3I)$

(d) $\frac{1}{3}(A^2 - 5A + 3I)$

(4) For $A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ Evaluate $B=\underbrace{A^8-5A^7+7A^6-3A^5}+\underbrace{A^4-5A^3+7A^2-4A}+2I$

$$=A^5(A^3-5A^2+7A-3I)+A(A^3-5A^2+7A-3I)$$

$$-A+2I$$

By C.H.Th,

$$\boxed{A^3-5A^2+7A-3I=0}\quad\text{---(1)}$$

$$=A^5(0)+A(0)-A+2I$$

(a) $A-2I$ (b) $A^3-5A^2+7A-3I$

$$B=-A+2I=\text{(c)}$$

(c) $-A+2I$ (d) $2I$

(ii) $|B|=?$

$$B=-\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}+2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $|B|=-1$

Q: $P_{3\times3}$ st it's C.Eq is $a(\lambda)=|\lambda I-P|=\lambda^3+\lambda^2+\lambda+1=0$ then Find Trace(P), $|P|$, P^{-1} and $|B|=?$ where $B=P^8+P^7+P^6+P^5+P^4+P^3+P^2+P+4I$

(a) 4
(b) 1
(c) 0
(d) 64 ✓

C.Eq of P is $f(\lambda)=0$

$$\boxed{|P-\lambda I|=0}$$

$$(-1)^3|\lambda I-P|=0$$

$$\boxed{|\lambda I-P|=0}$$

$$\lambda^3+1\lambda^2+\lambda+1=0$$

$$\lambda\to P,\quad \boxed{P^3+1P^2+P+1I=0} \longrightarrow (1)$$

$a_1=1$ & $a_3=1$

$Tr(P)=-(a_1)=-(1)=-1$

$|P|=(-1)^3a_3=-(1)=-1$

By (1), $P^{-1}(P^3+P^2+P+I)=P^{-1}\cdot O$

$$P^2+P+I+P^{-1}=0$$

$$P^{-1}=-(P^2+P+I)$$

(iv) $\boxed{P^3+P^2+P+I=0}$ —— (1)

$$B = \underbrace{P^8+P^7+P^6+P^5}+\underbrace{P^4+P^3+P^2+P}+4I$$

$$= P^5(P^3+P^2+P+I)+P(P^3+P^2+P+I)+4I$$

$$= P^5(0)+P(0)+4I$$

ie $B_{3\times3} = 4I_{3\times3} \Rightarrow |B| = |4I_{3\times3}| = 4^3|I_{3\times3}|$

$$= 64(1) = 64$$

Q. If E-Values of $A_{4\times4}$ are ± 1 & $\pm i$ then $A^4 = ?$

(a) A

(b) O

(c) I_3

(d) I_4

ATQ, $\lambda = 1, -1, i, -i$

so C.Equ^n is $(\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i)=0$

$$(\lambda^2-1)(\lambda^2+1)=0$$

$$\lambda^4-1=0$$

$$A^4-I=0$$

$$A^4=I=I_4=I_{4\times4}$$

TAG DAA QUEST :-

Q. if $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$ then which is/are True

(MSQ)

(a) $A^2 - 2A - 3I$ (Mat Poly)

(b) $(A - 3I)(A + I) = 0$

(c) $A - 2I - 3A^{-1}$ (Expression)

(d) $A^2 - 3A - 2I$

C. Equ^n is $\lambda^2 - (Tr(A))\lambda + (|A|) = 0$

$\lambda^2 - (2)\lambda + (-3) = 0$

$\lambda \to A$, $\boxed{A^2 - 2A - 3I = 0}$ Ans

multi by A^{-1}: $\boxed{A - 2I - 3A^{-1} = 0}$ Ans Equ^n

Factor: $\boxed{(A - 3I)(A + I) = 0}$ Ans

Q: if CE of $A_{4\times4}$ is $2\lambda^4-6\lambda^3+4\lambda^2-8\lambda+12=0$ then $|A|=? = (-1)^4 a_4$

Sol: C.Eqn is $1.\lambda^4-3\lambda^3+2\lambda^2-4\lambda+6=0$ $\qquad = (+)(6) = 6$

$(a_1=-3,\ a_2=2,\ a_3=-4,\ a_4=6)$

Q: constant term in the CEq of $A=\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}_{4\times4}$ will be ?

w.k.that constant term $= a_4 = +|A| = \ldots\ldots = 88$ Ans

PYQ

In matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $\underbrace{a + d}_{\text{Trace}} = \underbrace{ad - bc}_{\text{Det}} = 1$, then $A^3 =$ ______.

(a) $A - I$ (b) $A + I$

(c) $-I$ (d) 0

C. Equⁿ, $\lambda^2 - (Tr(A))\lambda + (|A|) = 0$

$\lambda^2 - (1)\lambda + (1) = 0$

By C.H.Th, $\lambda \to A$, $A^2 - A + I = 0$

$A^2 = A - I$ — (1)

$A^3 = A \cdot A^2$

$= A(A - I)$

$= A^2 - AI$

$= (A - I) - AI$

$= A - I - A$

$= -I$ (c)

THANK - YOU

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