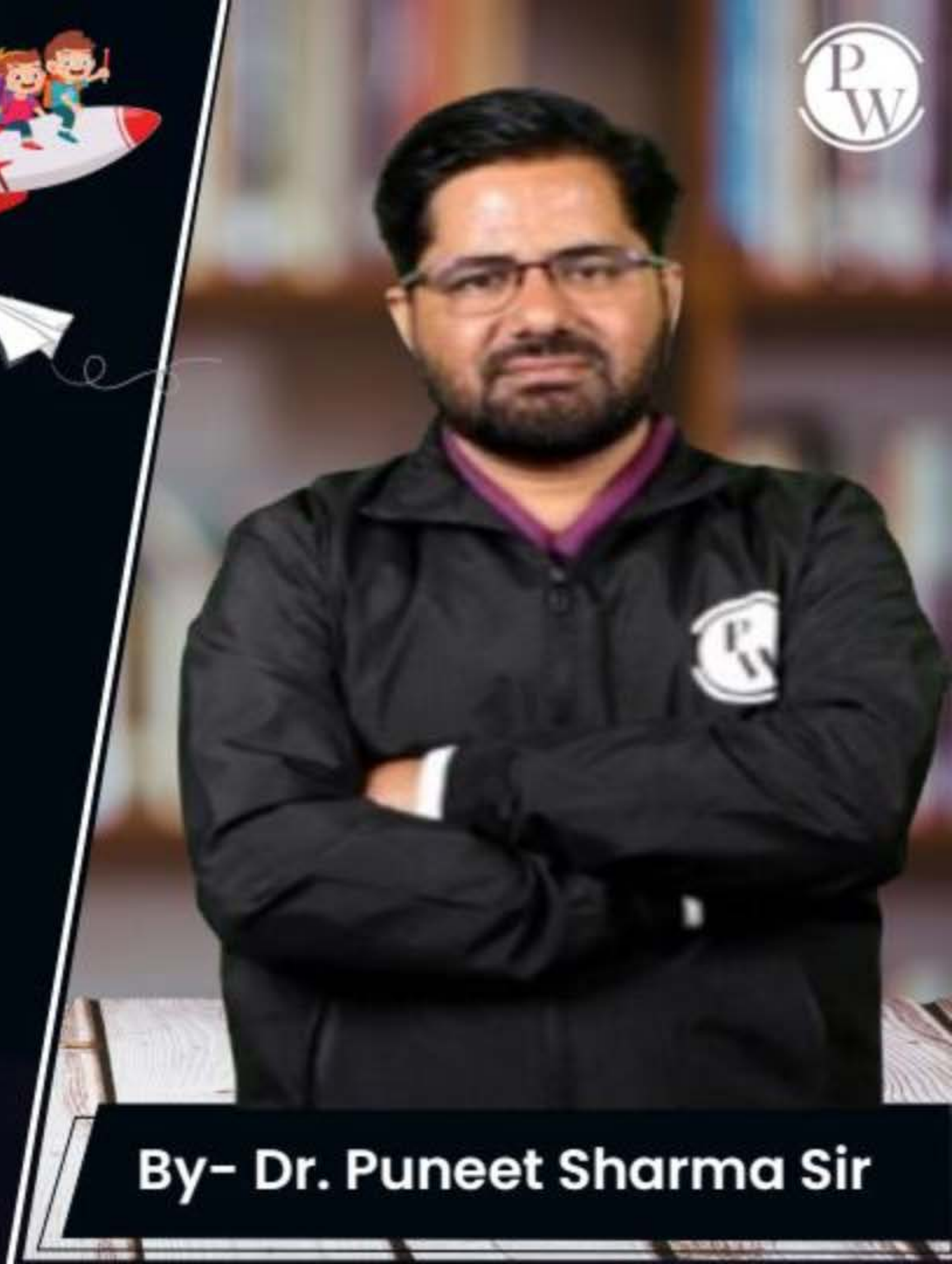
GATE

DS & AI
CS & IT

Linear Algebra

Lecture No. 03



Recap of previous lecture

Topic

→ BASICS of Determinants

→ BASICS of MATRICES

Topics to be Covered

Topic

ALGEBRA of MATRICES

MATRIX.

Def^n: Matrix is a Rectangular arrangement of m.n numbers.

$$A=[a_{ij}]_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$1\le i\le m$

$1\le j\le n$

H. lines = Rows

V. lines = Columns

Square Mat: Def^n ①: if No. of Rows = No of Columns, then it is Sq. Mat

Def^n ②: If in a Matrix, Diag exist then it must be Sq. Mat

Def^n ③ if in a Mat, Corresponding element exist for every element.

Sq. Matrix:

$$A=[a_{ij}]_{n\times n}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

① for diag. elements, $i=j\ \forall\ i\ \&\ j$

② for upper diag elements, $i<j\ \forall\ i\ \&\ j$

③ for lower diag elements, $i>j$ " "

④ for off diag elements, $i\neq j$ " "
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⑤ Corresponding elements are a_{ij} & a_{ji}

Trace(A) = $\boxed{\text{Sum of diag elements}}$

or $Tr(A)=\sum_{i=1}^{n}(a_{ii})$

eg $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ = U.T.M

$a_{ij}=0\ \forall\ i>j$

Some Special Matrices :-

$$\begin{bmatrix} 2 & -4 & 0 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 2 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

U.T.M	L.T.M	Diag Mat	Scalar Mat	Identity Mat
$Tr = 8$	$Tr = 7$	$Tr = 4$	$Tr = 20$	$Tr = 4$
$Det = -24$	$Det = 0$	$Det = 0$	$Det = 5^4 = 625$	$Det = 1$

(*) To find Det of U.T.M, L.T.M, Diag Mat, Scalar Mat, Identity Mat, we can multiply diagonal elements i.e $\boxed{\text{Det} = \text{Product of Diag elements}}$

(*) if $A = [a_{ij}]_{n \times n}$ s.t $\underbrace{a_{ij} = 0 \ \forall \ i > j}_{\text{Lower Diag elements} = 0}$ then A is U.T.M

Singular Mat: if $|A|=0$ then A is Called Singular Mat

Non Singular Mat: → if $|A|\neq 0$ then A is Called Non singular Mat

Invertible Mat → if A^{-1} exist then A is Called Invertible Mat

Def^n: $A^{-1}=\dfrac{\text{adj}\,A}{|A|}=\dfrac{(\text{Cof}\,A)^T}{|A|}$

A Necessary Condition for a Mat to be invertible is $|A|\neq 0$

(*) if A & B are two Matrices s.t

$\boxed{AB=BA=I}$ then → Inverse of A = B i.e $A^{-1}=B$

→ Inverse of B = A $B^{-1}=A$

i.e Both are the Inverses of each other.

Q: If $M_{n\times n}$ s.t $M^4 = I$ then

$M^{-1} = ? = M^3$

$(M^2)^{-1} = ? = M^2$

$(M^3)^{-1} = ? = M$

Sol: ① $M^4 = I$

$\Rightarrow M \cdot M^3 = I \Rightarrow M^{-1} = M^3$

② $M^4 = I$

$\Rightarrow M^2 \cdot M^2 = I \Rightarrow (M^2)^{-1} = M^2$

③ $M^4 = I$

$\Rightarrow M^3 \cdot M = I \Rightarrow (M^3)^{-1} = M$

GATE if $M_{n\times n}$ s.t $M \neq I$, $M^2 \neq I$, $M^3 \neq I$ But $\boxed{M^4 = I}$ then $M^{-1} = ?$

(a) $M^{4K+1} = M^{4K} \cdot M = M$

(b) $M^{4K+2} = M^{4K} \cdot M^2 = M^2$

(c) $M^{4K+3} = M^{4K} \cdot M^3 = M^3$

(d) M^{9K}

$K \in N$.

Given $M^4 = I$

Now, $M^8 = M^4 \cdot M^4 = I \cdot I = I^2 = I$

Again $M^{12} = M^4 \cdot M^4 \cdot M^4 = I \cdot I \cdot I = I^3 = I$

Similarly $M^{16} = I$, $M^{20} = I$ ....., $M^{4K} = I$.

$M^4 = I$

$M \cdot M^3 = I \Rightarrow M^{-1} = M^3 = M^3 \cdot I = M^3 \cdot M^{4K} = M^{4K+3}$

Q if $A_{n\times n}$ s.t $A^2 = I$ then $A^{-1} = ?$ (a) I ~~(b) A~~ (c) A^2 (d) O

$$A^2 = I$$

$$A \cdot A = I \longrightarrow (1)$$

w.k.that $A A^{-1} = I \longrightarrow (2)$

$\Rightarrow A^{-1} = A$ i.e A is Inverse of Itself.

or A is self Invertible.

Shortcut Method of finding Inverse of 2×2 Mat →

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{\text{adj} A}{|A|} = \dfrac{(\text{cof } A)^T}{|A|} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Eg $A = \begin{bmatrix} (3+2i) & -i \\ i & (3-2i) \end{bmatrix}$ then $A^{-1} = ?$ $= \dfrac{1}{12}\begin{bmatrix} 3-2i & +i \\ -i & 3+2i \end{bmatrix}$

PYQ

$|A| = (3+2i)(3-2i) - (i)(-i)$

$= (3)^2 - (2i)^2 + (i^2)$

$= 9 - 4i^2 + i^2$

$= 9 - 3i^2 = 9 - 3(-1) = 12$

Q. If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ & $A^{-1} = \begin{bmatrix} 0.5 & a \\ 0 & b \end{bmatrix}$ then $a+b=?$

(a) 60

(b) $1/60$

(c) $7/20$

(d) 6

$\because |A| = 6$ i.e. A^{-1} exist.

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & 1/60 \\ 0 & 1/3 \end{bmatrix}$$

$a = \frac{1}{60}$, $b = \frac{1}{3}$ so $a+b = \frac{7}{20}$

Shortcut Method of finding Inverse of 3×3 Mat →

eg $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & 1 & 2 \end{bmatrix}_{3\times 3}$ then $A^{-1} = ? = \dfrac{\text{adj}A}{|A|} = \dfrac{(\text{Cof}A)^T}{|A|} = \dfrac{1}{34}\begin{bmatrix} -4 & 7 & -8 \\ -16 & 11 & 2 \\ 2 & 5 & 4 \end{bmatrix}$

Sol: $\because |A| = 34$ (i.e $\neq 0$)

So A is Non singular $\Rightarrow A^{-1}$ exist.

1	-2	3	1	-2
2	0	4	2	0
-3	1	2	-3	1
1	-2	3	1	-2
2	0	4	2	0

Top Row of $A^{-1} = ?$ $\begin{bmatrix} -4 & 7 & -8 \end{bmatrix}$ ✗

$\begin{bmatrix} \frac{-4}{34} & \frac{7}{34} & \frac{-8}{34} \end{bmatrix}$ ✓

$\text{Cof}A = ? = (\text{adj}A)^T = \begin{bmatrix} -4 & -16 & 2 \\ 7 & 11 & 5 \\ -8 & 2 & 4 \end{bmatrix}$

(PQ) if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix}$ then $A^{-1} = ?$ also find $\operatorname{adj} A = ?$

Sol: $\because |A| = \text{.........} = 0$

i.e. A is singular $\Rightarrow A^{-1} = \text{DNE}$

1	2	3	1	2
4	5	6	4	5
3	4	5	3	4
1	2	3	1	2
4	5	6	4	5

$\operatorname{adj} A = \begin{bmatrix} \text{Do yourself.} \end{bmatrix}$

If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$. Then top row of R^{-1} is

(a) [5 6 4]

(b) [5 –3 1]

(c) [2 0 –1]

(d) $\left[2 \quad -1 \quad \frac{1}{2}\right]$

$|R| = +1[2+3] - 0 + (-1)[6-2]$

$= 5 - (4) = 1$

ie $|R| \neq 0 \Rightarrow R^{-1}$ exist.

$R^{-1} = \frac{\text{adj } R}{|R|} = \frac{1}{(1)} \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$

$$\begin{array}{c|cccc} 1 & 0 & -1 & 1 & 0 \\ \hline 2 & 1 & -1 & 2 & 1 \\ 2 & 3 & 2 & 2 & 3 \\ 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & -1 & 2 & 1 \end{array}$$

HW

If $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ and Adj. $A = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{bmatrix}$

then k = ______

(a) –5 (b) 3

(c) –3 (d) 5

Inverse of Matrices of 4×4, 5×5, 6×6 .... →

(Not in a syllabus using conventional Approach)

Procedure: write $A = I \cdot A$

use E-Operations

$I = BA \Rightarrow A^{-1} = B$ Ans

Addition & Subtraction in a Matrix → is possible only when they are of <u>same</u> order.

eg $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & 0 \end{bmatrix}_{2\times 3}$, $C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}_{3\times 3}$

$A \pm B$ = defined But $A \pm C$ = N.D & $B \pm C$ = N.D.

$$A+B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & -1 & 2 \end{bmatrix} = B+A$$ i.e Mat addition is Commutative

$$A-B = \begin{bmatrix} 3 & 1 & -4 \\ -1 & 3 & 2 \end{bmatrix} \text{ \& } B-A = \begin{bmatrix} -3 & -1 & 4 \\ 1 & -3 & -2 \end{bmatrix}$$ Subtraction is Not Commutative

(*) **Matrix Multiplication** → is defined when

"Number of Columns in 1^{st} Mat = Number of Rows in 2^{nd} Matrix"

for eg, Consider $A_{m\times n}$ & $B_{n\times p}$ then
- $AB = (\text{Defined})_{m\times p}$
- $BA = N.D$ $\because B_{n\times p} A_{m\times n}$??

eg: $A = \begin{bmatrix} 1 & 2 & -4 \end{bmatrix}_{1\times 3}$ & $B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times 1}$ then $AB = ?$ & $BA = ?$

$$AB = \begin{bmatrix} 1 & 2 & -4 \end{bmatrix}_{1\times 3} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times 1} = \left[(1\times 2 + 2\times 1 + (-4)\times 3)\right]_{1\times 1} = [-8]_{1\times 1} = -8$$

- M = 3
- A = 2

$$BA = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times 1} \begin{bmatrix} 1 & 2 & -4 \end{bmatrix}_{1\times 3} = \begin{bmatrix} 2 & 4 & -8 \\ 1 & 2 & -4 \\ 3 & 6 & -12 \end{bmatrix}_{3\times 3}$$

- No. of Multi = 9 times
- No. of Addition = 0 time

The value of $\underset{3\times 1}{\overset{=A}{\begin{bmatrix}1\\2\\3\end{bmatrix}}} \times \underset{1\times 3}{\overset{=B}{\begin{bmatrix}3 & 2 & 4\end{bmatrix}}} \times \underset{3\times 1}{\overset{=C}{\begin{bmatrix}4\\6\\7\end{bmatrix}}}$ equals

(a) $\begin{bmatrix}52\\-104\\156\end{bmatrix}$

(b) $[52 \quad -104 \quad 15]$

(c) $\begin{bmatrix}52\\104\\156\end{bmatrix}$ ✓

(d) None of these

$$BC = \underset{1\times 3}{\begin{bmatrix}3 & 2 & 4\end{bmatrix}} \underset{3\times 1}{\begin{bmatrix}4\\6\\7\end{bmatrix}} = [(12+12+28)]_{1\times 1}$$

$$= [52]_{1\times 1}$$

$$A_{3\times 1}(BC)_{1\times 1} = \underset{3\times 1}{\begin{bmatrix}1\\2\\3\end{bmatrix}} [52]_{1\times 1}$$

$$= \begin{bmatrix}52\\104\\156\end{bmatrix}_{3\times 1}$$

M-II $$A_{3\times 1} B_{1\times 3} = \begin{bmatrix}3 & 2 & 4\\6 & 4 & 8\\9 & 6 & 12\end{bmatrix}_{3\times 3}$$

$$(AB)_{3\times 3} C_{3\times 1} = \begin{bmatrix}52\\104\\156\end{bmatrix}_{3\times 1}$$

Eg: $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 2 \end{bmatrix}_{2\times 3}$ & $B = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -2 \end{bmatrix}_{3\times 3}$ then $\rightarrow AB = ?$, $\rightarrow BA = ? = N.D.$

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 2 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -2 \end{bmatrix}_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} = \begin{bmatrix} 7 & 5 & -11 \\ 9 & 0 & 12 \end{bmatrix}_{2\times 3}$$

$$a_{11} = (1).(2) + (-2).(-1) + (3).(1) = 7$$
$$a_{12} = (1).(-1) + (-2).(0) + (3).(2) = 5$$
$$a_{13} = (1).(3) + (-2).(4) + (3).(-2) = -11$$
$$a_{21} = (4).(2) + (1).(-1) + (2).(1) = 9$$
$$a_{22} = (4).(-1) + (1).(0) + (2).(2) = 0$$
$$a_{23} = (4).(3) + (1).(4) + (2).(-2) = 12$$

Number of times, symbol of Multiplication is used to find AB = 18 times

Number of times, symbol of addition is used to find AB = 12 times

Shortcut: Consider $A_{m\times n}$ & $B_{n\times p}$ then to find $(AB)_{m\times p}$

Number of Multiplications Required $= m.n.p$ & Number of Additions Req $= m.(n-1).p$

eg if $A_{2\times3}$ & $B_{3\times3}$ then to find AB → No. of Multi Req $= 2\times3\times3 = 18$

→ No. of Additions Req $= 2(3-1)3 = 12$

$m=2,\ n=3,\ p=3$

eg if $A_{3\times4}$ & $B_{4\times3}$ then to find AB → No. of Multi Req $= 3\times4\times3 = 36$

→ No. of Additions Req $= 3(4-1)3 = 27$

$m=3,\ n=4,\ p=3$

& to find BA → No. of Multi Req $= 4\times3\times4 = 48$

→ No. of Additions Req $= 4(3-1)4 = 32$

$B_{4\times3}A_{3\times4}$ i.e. $m=4$, $n=3,\ p=4$

MAJEDAAR QUESTION → Consider $A_{2\times3}$, $B_{3\times4}$ & $C_{4\times2}$ then find the minimum number of Multiplications & Additions that will be Required to find Matrix Product (ABC) ?

PYQ

(a) 40, 28

(b) 36, 26

(c) 40, 36

(d) 48, 28

we can find (ABC) either by using
- Case I: $(AB)C$ → $M = 40$, $A = 28$
- Case II: $A(BC)$ → $M = 36$, $A = 26$

Case I: $A_{2\times3} B_{3\times4} = (AB)_{2\times4}$
- Multi $= 2\times3\times4 = 24$
- Addition $= 2(3-1)4 = \boxed{16}$

$m=2, n=3, p=4$

Now $(AB)_{2\times4} C_{4\times2} = (ABC)_{2\times2}$
- Multi $= 2\times4\times2 = 16$
- Addition $= 2(4-1)2 = \boxed{12}$

$m=2, n=4, p=2$

Overall Multi $= 40$ & Addition $= 28$

Case II: A (B C) ;→ HW

Multi = 36

Add = 26 Ans

Commutative Law

$A+B=B+A$

$AB \neq BA$ in general.

Associative Law

$A+(B+C)=(A+B)+C$

$A(BC)=(AB)C$

i.e. Matrix Multiplication is Associative but not Commutative in general.

Q $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 0 & 1\end{bmatrix}_{2\times3}$ $B=\begin{bmatrix}2 & 1\\ 1 & 2\\ -1 & 0\end{bmatrix}_{3\times2}$

$AB=\begin{bmatrix}-3 & -3\\ 3 & 2\end{bmatrix}_{2\times2}$

& $BA=\begin{bmatrix}4 & -4 & 7\\ 5 & -2 & 5\\ 0 & 2 & -3\end{bmatrix}_{3\times3}$

Here $AB \neq BA$ in general

But $\boxed{Tr(AB)=Tr(BA)}$ always. Learn.

If A & B are two matrices of same order then which of the following is true.

(a) $(A + B)^2 = A^2 + 2AB + B^2$

(b) $(A - B)^2 = A^2 - 2AB + B^2$

(c) $(A + B)^2 + (A - B)^2 = 2A^2 + 2B^2$

(d) $(A + B)(A - B) = A^2 - B^2$

(a) $(A+B)^2 = (A+B)(A+B)$

$= A^2 + \underbrace{AB + BA}_{?} + B^2$

(b) $(A-B)^2 = (A-B)(A-B)$

$= A^2 - \underbrace{AB - BA}_{?} + B^2$

(d) $(A+B)(A-B) = A^2 - AB + BA - B^2$

$= A^2 - (\underbrace{AB - BA}_{?}) - B^2$

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Thank

you

Keep Hustling!