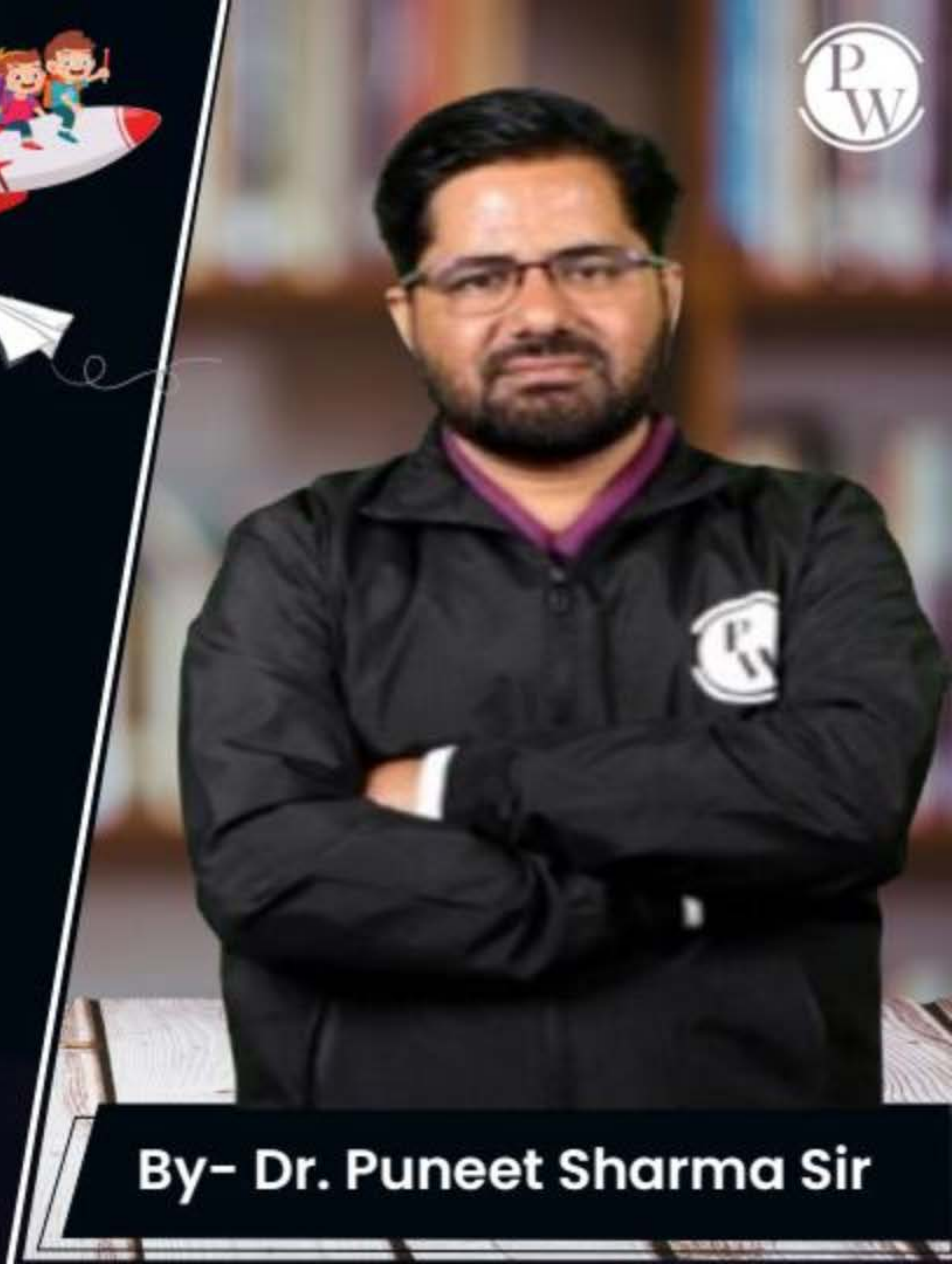
GATE

DS & AI
CS & IT

Linear Algebra

Lecture No. 09



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Recap of previous lecture

Topic

Non Homogeneous System

Topics to be Covered

Topic

(1) Remaining Part of Non Homog System of Equ^{n}s

(2) Homogeneous System of Equ^{n}s

Methods of Solving Non Homog System → Consider $\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$

RANK Method (always applicable)
(i.e. for $m > n$, $m = n$, $m < n$)

Matrix Method
(applicable only when $m = n$)

① if $\boxed{\rho(A) = \rho(A:B)}$ = No. of Variables ⇒ unique sol. ⟸ if $|A| \neq 0$

② if $\boxed{\rho(A) = \rho(A:B)}$ < No. of Variables ⇒ ∞ sol. ⟸ if $|A| = 0$, $(\text{adj}A)B = 0$

③ if $\rho(A) \neq \rho(A:B)$ ⟹ No sol. ⟸ if $|A| = 0$, $(\text{adj}A)B \neq 0$

(*) Consistent system → System is called Consistent if ∃ Solution.
(whether unique or ∞ Sol.)

Inconsistent system → System is called Inconsistent if we have No Sol.

(*) Necessary condition for a system $\boxed{AX = B}$ to be consistent is ?
$\rho(A) = \rho(A \vdots B)$.

Another Condition for Consistency : $\boxed{\text{B must be L-D on Columns vectors of A.}}$

RECAP $A = [X_1, X_2, X_3, X_4]$
→ $\rho(A) = 4 \Rightarrow$ vectors are L I
→ $\rho(A) < 4 \Rightarrow$ " " LD

(*) $\boxed{\rho(A) \not> \rho(A:B)}$ i.e. Rank of Coeff Mat can never exceed Rank of Augumented Mat.

e.g. $$[A:B] = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & B \end{array} \\ \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 14 \end{array}\right] \end{array}$$

$\rho(A) = 3$, $\rho(A:B) = 3$

unique sol.

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{0} & 10 \end{array}\right]$$

$\rho(A) = 2$, $\rho(A:B) = 3$

(No sol.)

$$[A:B] = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & B \end{array} \\ \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right] \end{array}$$

$\rho(A) = 2 = \rho(A:B)$

(∞ sol.)

$$[A:B] = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & B \end{array} \\ \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 0 \end{array}\right] \end{array}$$

$\rho(A) = 3 = \rho(A:B)$ unique sol.

e.g (1): $\rho(A:B) = 3 <$ No. of Column vectors.

i.e C_1, C_2, C_3 & B are LD

e.g (3) $\rho(A:B) = 2 <$ No of Column vectors.

i.e C_1, C_2, C_3, B are LD

Q. If $A_{3\times 4}$ matrix s.t. $AX = B$ is an Inconsistent system then $\boxed{\text{highest possible}}$ Rank of A will be?

(a) 0

(b) 1

(c) 2

(d) 3

M-I $\because A_{3\times 4} \Rightarrow [A : B]_{3\times 5}$

& for Inconsistency $\rho(A) \neq \rho(A:B) \Rightarrow \underbrace{\rho(A) < \rho(A:B)}$

$\because \text{Max}\,\rho(A:B)_{3\times 5} = 3$ $\Downarrow$

so, $\text{Max}\,\rho(A) = 2$

M-II for $\boxed{\rho(A) < \rho(A:B)}$ we have following cases:

Case (i) $\rho(A) = 1$, $\rho(A:B) = 2$, Case (iii) $\rho(A) = 2$ & $\rho(A:B) = 3$

Case (ii) $\rho(A) = 1$, $\rho(A:B) = 3$ so $\text{Max}\,\rho(A) = 2$

(x) underdetermined system can not have unique solution

Note: (1) underdetermined, Non Homog system if consistent

always consist Infinite sol.

or $\rho(A)=\rho(A:B)$

(2) In case of overdetermined system, we may have all types of answers

if $\rho(A) < \rho(A:B) \Longrightarrow$ No sol.

if $\rho(A)=\rho(A:B)$ → unique sol.

OR

→ ∞ sol.

Eg the Nature of the Sol of following system:

① $x_1 - 2x_2 + 4x_3 = 5$

$2x_1 - 4x_2 + 8x_3 = 7$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 2 & -4 & 8 & 7 \end{array}\right]_{2\times 4}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$\rho(A) = 1$, $\rho(A:B) = 2$

So No Sol.

② $x_1 - 2x_2 + 4x_3 = 5$

$2x_1 - 4x_2 + 8x_3 = 10$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 2 & -4 & 8 & 10 \end{array}\right]_{2\times 4}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 1 = \rho(A:B) < (3)$

∞ solⁿ exist.

Given a system of equations :

$x + 2y + 2z = b_1$

$5x + y + 3z = b_2$

Which of the following is true regarding its solution ?

(a) The system has a unique solution for any given b_1 and b_2

(b) The system will have infinitely many solutions for any given b_1 and b_2

(c) Whether or not a solution exists depends on the given b_1 and b_2

(d) The system would have no solution for any values of b_1 and b_2

$$[A \vdots B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -9 & -7 & (b_2 - 5b_1) \end{array}\right]_{2\times 4}$$

$$\rho(A) = 2 = \rho(A \vdots B)$$

i.e. Consistent $\Rightarrow \infty$ soln.

Q. The system has ?

$3x+3y+2z=1$

$x+2y=4$

$10y+3z=-2$

$2x-3y-z=5$

(a) No sol.

(b) ∞ sol^n

(c) unique sol.

(d) finite No. of solutions

$$[A:B]=\left[\begin{array}{ccc|c}3&3&2&1\\1&2&0&4\\0&10&3&-2\\2&-3&-1&5\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{ccc|c}1&2&0&4\\3&3&2&1\\0&10&3&-2\\2&-3&-1&5\end{array}\right]$$

$$\xrightarrow[R_4-2R_1]{R_2-3R_1}\left[\begin{array}{ccc|c}1&2&0&4\\0&-3&2&-11\\0&10&3&-2\\0&-7&-1&-3\end{array}\right]$$

$$\xrightarrow{R_2\to 3R_2+R_3}\left[\begin{array}{ccc|c}1&2&0&4\\0&1&9&-35\\0&10&3&-2\\0&-7&-1&-3\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&2&0&4\\0&1&9&-35\\0&0&-87&-352\\0&0&62&?\end{array}\right]$$

$$\dashrightarrow\left[\begin{array}{ccc|c}1&2&0&4\\0&1&9&-35\\0&0&a_{23}&a_{24}\\0&0&0&0\end{array}\right]$$

$\rho(A)=3=\rho(A:B)$

$=$ No. of variables

$\Rightarrow$ unique sol.

$\because A_{4\times3}\Rightarrow$ Mat Method is not applicable

Q. Find the Nature of soln of

(MSQ) $x+y+z=1$

$2x+y+4z=1$

$4x+y+10z=1$

(a) No soln.

(b) unique soln.

(c) ∞ solns

(d) Finite No. of Ind soln.

$$[A:B]=\left[\begin{array}{ccc|c}1&1&1&1\\2&1&4&1\\4&1&10&1\end{array}\right]\xrightarrow[R_3-4R_1]{R_2-2R_1}\left[\begin{array}{ccc|c}1&1&1&1\\0&-1&2&-1\\0&-3&6&-3\end{array}\right]$$

$$\xrightarrow{R_3-3R_2}\left[\begin{array}{ccc|c}1&1&1&1\\0&-1&2&-1\\0&0&0&0\end{array}\right]$$

$\rho(A)=2=\rho(A:B)<3$

$\Rightarrow \infty$ soln exist

Procedure of finding soln →

$$AX=B\Rightarrow\begin{bmatrix}1&1&1\\2&1&4\\4&1&10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1\\0&-1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}\Rightarrow\begin{bmatrix}(x+y+z)\\(-y+2z)\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$x+y+z=1 \Rightarrow x+(2K+1)+K=1 \Rightarrow x=-3K$

$-y+2z=-1 \Rightarrow -y+2K=-1 \Rightarrow y=2K+1$

so $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -3K \\ 2K+1 \\ K \end{bmatrix}$

let $z=K$, $K \in$ Arbitrary Constant.

$$X=\begin{bmatrix} 0-3K \\ 1+2K \\ 0+K \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+\begin{bmatrix} -3K \\ 2K \\ K \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+K\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Put $K=0$, Put $K=1$, Put $K=-1$, Put $K=2$ .... ∞ solⁿ exist.

$X_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $X_2=\begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$, $X_3=\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$, $X_4=\begin{bmatrix} -6 \\ 5 \\ 2 \end{bmatrix}$

so out of these ∞ solutions only X_1 & X_2 are L.I and Rest are LD on X_1 & X_2

$X_3=2X_1-X_2$, $X_4=-X_1+2X_2$

Homogeneous System ($AX=0$)

Eg: $x-4y+2z=0$
$3x+2y-2z=0$
$-4x-y+4z=0$
$7x+2y+3z=0$

$\Rightarrow X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ is the sol of this system (T)

& this solution always exist. that's why

Homog System NEVER Inconsistent or Homog System Always Consistent

* Another Reason of Consistency :→ $\because \rho(A)=\rho(A:0)$ always Hence consistent.

* $\underbrace{\text{underdetermined}}_{\Downarrow}$, $\underbrace{\text{Homog System}}_{\Downarrow}$ always consist ∞ solutions

(Unique sol Not possible) (No sol. Not possible)

(*) In this Chapter,

(i) unique sol $\cong$ Trivial sol. $\cong$ ZERO solⁿ always exist.

(ii) ∞ solⁿ $\cong$ Non Trivial sol also exist $\cong$ Non Zero sol also exist.

(iii) $\boxed{\text{No sol} \neq \text{ZERO sol.}}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Methods of Solving Homog. System

$(A_{m\times n} X_{n\times 1} = O_{m\times 1})$

RANK Method (always applicable)

$(m>n, m=n, m<n)$

① if $\rho(A)$ = No. of Variables $\Rightarrow$ unique sol.

② if $\rho(A) <$ " " $\Rightarrow$ ∞ sol.

MATRIX Method

(only for $m=n$)

① if $|A| \neq 0$ $\Rightarrow$ unique sol exist

② if $|A| = 0$ $\Rightarrow$ ∞ sol exist

Q: Find k for which
MSQ
$(3k-8)x + 3y + 3z = 0$
$3x + (3k-8)y + 3z = 0$
$3x + 3y + (3k-8)z = 0$
has Non Trivial Soln.

(a) $\frac{2}{3}$ (b) $\frac{11}{3}$ (c) $\frac{4}{3}$ (d) $\frac{8}{3}$

$$A = \begin{bmatrix} (3k-8) & 3 & 3 \\ 3 & (3k-8) & 3 \\ 3 & 3 & (3k-8) \end{bmatrix}_{3\times 3}$$

For Non Trivial Soln (∞ Soln); $|A| = 0$

$$\begin{vmatrix} (3k-8) & 3 & 3 \\ 3 & (3k-8) & 3 \\ 3 & 3 & (3k-8) \end{vmatrix} = 0$$

$C_1 \rightarrow C_1 + (C_2 + C_3)$

$$\begin{vmatrix} (3k-2) & 3 & 3 \\ (3k-2) & (3k-8) & 3 \\ (3k-2) & 3 & (3k-8) \end{vmatrix} = 0$$

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0 \Rightarrow (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & (3k-11) & 0 \\ 0 & 0 & (3k-11) \end{vmatrix} = 0$$

$\Uparrow$

$(3k-2)(3k-11)^2 = 0$

$k = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}$

HW Q / MSQ For the system to have Infinite sol, which of the following is/are True

$$px+qy+rz=0$$
$$qx+ry+pz=0$$
$$rx+py+qz=0$$

(a) $p=q=r$

(b) $p+q+r=0$

(c) $p^3+q^3+r^3-3pqr=0$

(d) None

$\Downarrow$

$|A|=0 \Rightarrow \begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix}=0$

At $p=q=r$, $|A|=0$ is justified so (a) ✓

$\rightarrow -\left[p^3+q^3+r^3-3pqr\right]=0$

or $p^3+q^3+r^3-3pqr=0$ i.e. (c) ✓

$C_1 \rightarrow C_1+C_2+C_3$, $\begin{vmatrix} (p+q+r) & q & r \\ (p+q+r) & r & p \\ (p+q+r) & p & q \end{vmatrix}=0$

At $p+q+r=0$
$|A|=0$ is justified
so (b) ✓

Q. Consider $\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$ then which one is false? m = No. of eqn's, n = No. of Variables.

(a) if $m > n$, $B \neq 0$, $\rho(A) < \rho(A:B)$ then System has NO Sol. (T)

(b) if $m = n$, $B = 0$, $|A| \neq 0$ then System has only Trivial Sol. (T)

(c) if $m = 3, n = 5$, $B = 0$ then System has also Non Zero Sol. (T)

(d) if $m = 5$, $n = 3$, $B = 0$ s.t. $\rho(A) = 3$ then System has only zero Sol. (T)

(e) if $m = n$, $B = 0$, then System has Sol. (T) it is obvious

(f) if $m > n$, $B = 0$ then System has Multiple Sol. (F)

(g) if $m > n$, $B = 0$, $\boxed{\rho(A) < n}$ then System has Multiple Sol. (T)

(h) Sun Rises from East (T) $\because$ it is obvious

The value of α for which the system of equation

HW

$$x + y + z = 0$$

$$y + 2z = 0$$

$$\alpha x + z = 0$$

has more than one solution is

∞ soln.

(a) –1 (b) 0

(c) $\frac{1}{2}$ (d) 1

for ∞ soln, $|A| = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ \alpha & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1[1-0] - 0 + \alpha[2-1] = 0$$

$$1 + \alpha = 0$$

$$\alpha = -1$$

Qe If $A_{3\times4}$ then $AX = 0$ has ______ soln's.

(a) No soln. (c) Multiple soln.

(b) unique soln (d) Data Inadequate.

$$A_{3\times4} X_{4\times1} = O_{3\times1}$$

underdetermined Homog System ⇒ ∞ soln

THANK - YOU