DATA SCIENCE & ARTIFICIAL INTELLIGENCE

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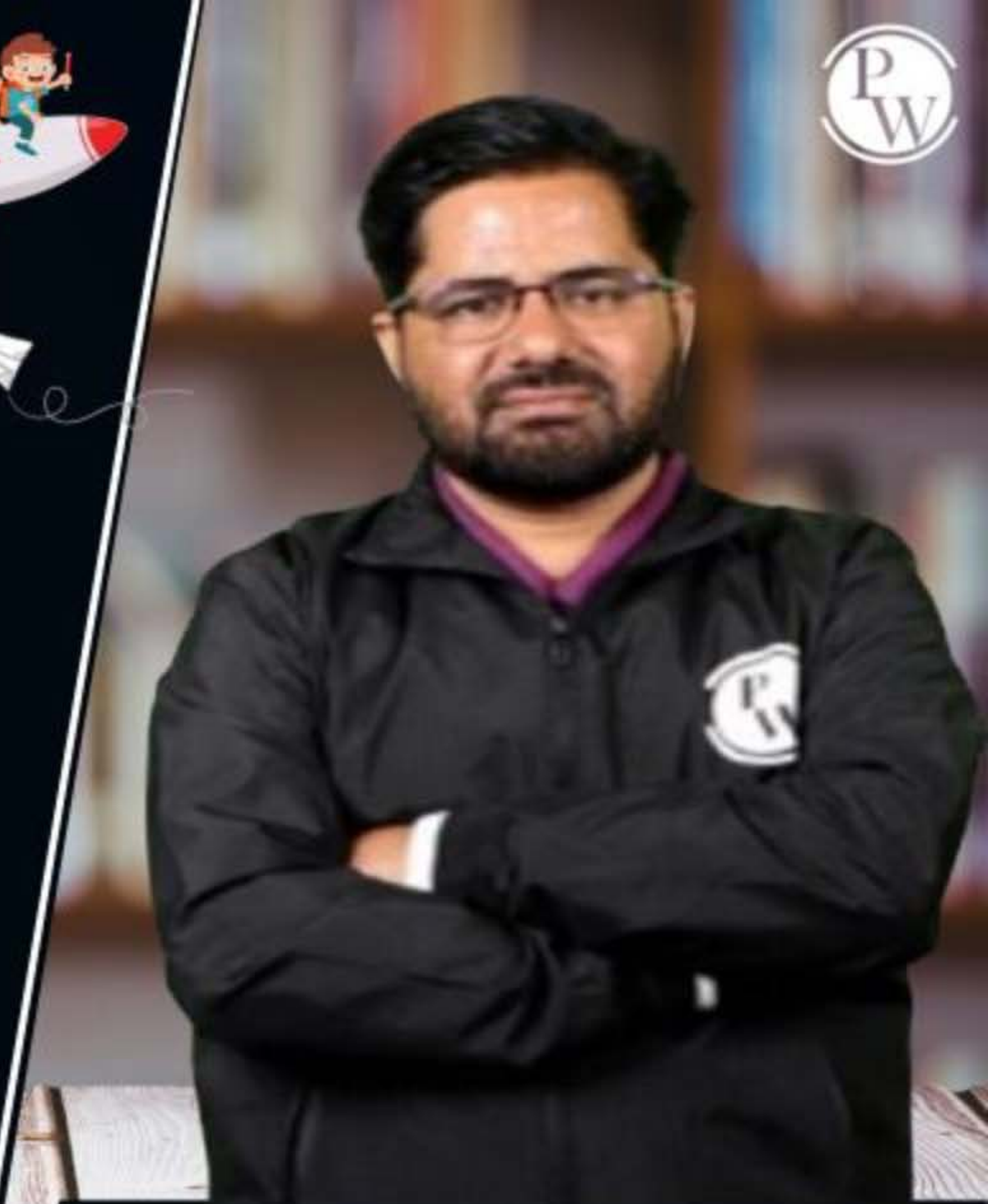
Linear Algebra- II

Lecture No. 02



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Recap of previous lecture

Topic QUADRATIC FORMS

Topics to be Covered

Topic SINGULAR VALUE DECOMPOSITION

(S.V.D)

SINGULAR VALUE DECOMPOSITION

① E.Values (given) $\rightarrow$ Trace, $|A|$, $A^{-1} \equiv$ ✓ using C.H.T

② E Values & E.Vectors (given) $\rightarrow$ then we can Calculate A using Diagonalisation

But this concept is applicable only for SQ. Matrices.

So to Calculate Rectangular Mat we will use the concept of S.V.D

SINGULAR VALUE DECOMPOSITION

If A is any sq. Mat (which is not known) and we know it's E.Values & E.vectors then by using the concept of Diagonalisation we can find A as follows,

Let $A_{3\times3}$ s.t it E. Values are $\lambda_1, \lambda_2, \lambda_3$ & Corresponding E Vectors are X_1, X_2, X_3

then Modal Mat $P = [X_1 \; X_2 \; X_3]$, $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ & $P^{-1}AP = D$

$\Rightarrow A = PDP^{-1} = \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}_{3\times3}$

$\boxed{\text{unfortunately}}$, this concept is applicable only for sq. Mat & if we want to find any Rectangular Mat then what should be the procedure ?

Don't worry, Now we will use the concept of S.V.D which will discussed after the understanding of some Basic Information.

Singular Values of A $\rightarrow$ $\boxed{\text{Sq. Roots of E. Values of } A^TA \text{ are called Sing. Values of } A.}$

if $A_{m\times n}$ then $(A^TA)_{n\times n}$ & let E Values of A^TA are $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$

then singular values of A are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \dots, \sqrt{\lambda_n}$

ie sing. values of A are $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n$ where $\boxed{\sigma_i = \sqrt{\lambda_i}}$ $\forall i$

eg let $A_{2\times 3}$ then $(A^T)_{3\times 2}$ so $(A^TA)_{3\times 3}$ & $(AA^T)_{2\times 2}$

let $\lambda_1, \lambda_2, \lambda_3$ are the E Values of (A^TA) then $\sigma_1, \sigma_2, \sigma_3$ are the Sing Values of A

where $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $\sigma_3 = \sqrt{\lambda_3}$ $\Rightarrow \boxed{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \lambda_1 + \lambda_2 + \lambda_3} = Tr(A^TA)$

Note (1) If $A_{m\times n}$ be any Rectangular Mat then $\underset{n\times n}{(A^TA)}$ & $\underset{m\times m}{(AA^T)}$ both are $\boxed{\text{Symmetric}}$

Let $P = A^TA$ then $P^T = (A^TA)^T = A^T(A^T)^T = A^TA = P \Rightarrow P$ is symmetric

& $Q = AA^T$ then $Q^T = (AA^T)^T = (A^T)^T A^T = AA^T = Q \Rightarrow Q$ is also symmetric

(2) Non Zero E Values of A^TA and AA^T are Same.

for eg $A_{3\times 5}$ then $\nearrow (A^TA)_{5\times 5} \longrightarrow 0, 0, \lambda_1, \lambda_2, \lambda_3$

$\searrow (AA^T)_{3\times 3} \longrightarrow \lambda_1, \lambda_2, \lambda_3$

that why $\boxed{\rho(AA^T) = \rho(A^TA)}$.

(4) Also we have $\boxed{Tr(AA^T) = Tr(A^TA)}$; In Fact $AB \neq BA$ But $Tr(AB) = Tr(BA)$

(5) Let E. Value of A^TA is λ_i then Singular Value of A is $\boxed{\sigma_i = \sqrt{\lambda_i}}$ $\forall i$

for eg $A_{2\times3}$ then $(A^T)_{3\times2}$ & $(A^TA)_{3\times3}$ = Symm. Mat, let E Values of A^TA are $\lambda_1, \lambda_2, \lambda_3$

then Singular Values of A are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}$

(6) If A is a Square Mat of $n\times n$ then $\boxed{|A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_n}$

(7) If $A_{n\times n}$ s.t A is Symmetric then $\boxed{\text{Singular Values of } A = \text{Eigen Values of } A}$

(8) if $A_{m\times n}$ then $(A^TA)_{n\times n}$

Sum of the squares of Sing Values of A = Sum of Eigen Values of A^TA = $Tr(A^TA)$

i.e $\boxed{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots + \sigma_n^2 = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = Tr(A^TA)}$

Q. Find the singular values of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times3}$

Sol: $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times3}$, $A^T = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}_{3\times2} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}_{3\times3}$$

$$\& \ AA^T = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}_{2\times2}$$

we know that,

Non zero E values of A^TA & AA^T are same

so instead of calculating E values of $(A^TA)_{3\times3}$

we will try to find the E values $(AA^T)_{2\times2}$

so C. Eqn of AA^T is

$$\lambda^2 - (\text{Trace})\lambda + (\text{Det}) = 0$$

$$\lambda^2 - (450)\lambda + (32400) = 0$$

$$(\lambda - 360)(\lambda - 90) = 0 \Rightarrow \lambda = 360, 90$$

i.e. E.Values of AA^T are 360 & 90

so E.Values of $\underset{3\times3}{(A^TA)}$ are $360, 90, 0$

Hence singular Values of A are $\sqrt{360}, \sqrt{90}, \sqrt{0}$

i.e. $\sigma_1 = \sqrt{360}$, $\sigma_2 = \sqrt{90}$, $\sigma_3 = 0$.

Q: Find the singular values of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

Sol: $A^T A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \begin{matrix} = 45 \\ = 45 \end{matrix}$

By property of E Value, $\lambda = 45$.

Now $\because Tr(A^T A) = 50$

$\lambda_1 + \lambda_2 = 50$

$45 + \lambda_2 = 50 \Rightarrow \lambda_2 = 5$

i.e. E Values of $A^T A$ are 45 & 5

So singular values of A are $\sqrt{45}$ & $\sqrt{5}$

i.e. $\sigma_1 = \sqrt{45}$, $\sigma_2 = \sqrt{5}$ are the sing. Values of A

Note:

(1) E Values of $A^T A = ? = 45$ & 5

& " " $AA^T = ? = 45$ & 5

(2) E. Values of $A = ? = 3$ & 5

Singular Values of $A = ? = \sqrt{45}$ & $\sqrt{5}$

(3) By observation, we have

$\boxed{\lambda_1 \cdot \lambda_2 = \sigma_1 \cdot \sigma_2 = |A|}$

$\left((3)(5) = \sqrt{45} \cdot \sqrt{5} = 15 \right)$

$\because$ A is Square Mat.

Right and Left Singular Vectors of A →

if $A_{m\times n}$ then Unit E-Vectors of A^TA are called Right Singular vectors of A

& " " of AA^T " " left " " of A

Consider $A_{2\times 3} \Rightarrow (A^TA)_{3\times 3}$ → λ_1, x_1 ; λ_2, x_2 ; λ_3, x_3 , then $V_i = \frac{x_i}{\|x_i\|} = [V_1\ V_2\ V_3]$ are called Right Sing vectors of A

$\because (A^TA)$ = Symm Mat so V_1, V_2, V_3 are orthonormal vectors

i.e. $V_i \cdot V_j = V_i^T V_j = 0$ & $V_i \cdot V_i = V_i^T V_i = 1$. $\forall\ i \neq j$

Similarly $(AA^T)_{2\times 2}$ $\begin{cases} \lambda_1, \gamma_1 \\ \lambda_2, \gamma_2 \end{cases}$ then $U_i = \frac{\gamma_i}{\|\gamma_i\|} = [U_1\ U_2]$

$\because (AA^T) =$ Symm so U_1 & U_2 are also orthonormal vectors

i.e $U_i \cdot U_j = U_i^T U_j = 0$ & $U_i \cdot U_i = U^T U = 1 \ \forall i \neq j$

In short; Right Sing Vectors $V = [V_1 V_2 V_3]$

& Left Sing Vectors $U = [U_1\ U_2]$

Relationship b/w V_i & U_i →

$$\boxed{AV_i = \sigma_i U_i}$$

i.e $AV_1 = \sigma_1 U_1$, $AV_2 = \sigma_2 V_2$, ...

Special Matrix Involved in S.V.D $\rightarrow$ Let $\sigma_1, \sigma_2, \sigma_3$ are the Non Zero Sing-Values

then $D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$ & $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix}_{m \times n}$.

S.V.D $\rightarrow$ $\boxed{A = U.\Sigma.V^T}$ is called S.V.D of A

i.e $\underset{m \times n}{A} = \underset{m \times m}{U}.\underset{m \times n}{\Sigma}.\underset{n \times n}{V}^T$ is S.V.D in proper order

Conclusions

(✱) let E. Value of A^TA is λ_i then Singular Value of A is $\boxed{\sigma_i = \sqrt{\lambda_i}}$ $\forall i$

for eg $A_{2\times3}$ then $(A^T)_{3\times2}$ & $(A^TA)_{3\times3}$ = Symm. Mat, let E Values are $\lambda_1, \lambda_2, \lambda_3$

then Singular Values of A are $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, $\sqrt{\lambda_3}$

(✱) Σ is special Mat, includes D, which is formed by Non Zero sing values of A

let Sing Values of $A_{2\times3}$ are $\sigma_1, \sigma_2, 0$ then $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$

$A_{2\times3}$, $(A^T)_{3\times2}$ $\begin{cases} (A^TA)_{3\times3} \rightarrow (V)_{3\times3} \\ (AA^T)_{2\times2} \rightarrow (U)_{2\times2} \end{cases}$ so $\boxed{A_{2\times3} = U_{2\times2} \Sigma_{2\times3} V^T_{3\times3}}$

Q: Find the S.V.D of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$?

Sol: $A^TA = \begin{bmatrix} 80 & 140 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$, $AA^T = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}$

for A^TA:

$\lambda_1 = 360$, $x_1 = [1 \ 2 \ 2]'$

$\lambda_2 = 90$, $x_2 = [-2 \ -1 \ 2]'$

$\lambda_3 = 0$, $x_3 = [2 \ -2 \ 1]'$

So $V_1 = \dfrac{X_1}{\|X_1\|} = \dfrac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $V_2 = \dfrac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$, $V_3 = \dfrac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

So set of Right Singular vectors

$V = [V_1 \ V_2 \ V_3] = \dfrac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

& we have $\sigma_1 = \sqrt{360}$, $\sigma_2 = \sqrt{90}$, $\sigma_3 = 0$

Now $AV_1 = \sigma_1 U_1 \Rightarrow U_1 = \dfrac{1}{\sigma_1} AV_1$

$U_1 = \dfrac{1}{\sqrt{360}}\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}\dfrac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \dfrac{1}{3\sqrt{360}}\begin{bmatrix} 54 \\ 18 \end{bmatrix} = \dfrac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$U_2 = \dfrac{1}{\sigma_2} AV_2 = \dfrac{1}{\sqrt{90}}\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}\dfrac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

$= \dfrac{1}{3\sqrt{90}}\begin{bmatrix} 9 \\ -27 \end{bmatrix} = \dfrac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

i.e. for $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3}$, Right Sing Vector $V = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

Sing Values of A are $\sqrt{360}, \sqrt{90}, 0$ & left Sing vector $U = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$

Now constructing a Diagonal Mat with the help of Non Zero Sing. Values.

$$D = \begin{bmatrix} \sqrt{360} & 0 \\ 0 & \sqrt{90} \end{bmatrix}$$ & Taking one special Mat $$\Sigma = \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}_{2\times 3}$$

Now Calculating $$U \Sigma V^T = \frac{1}{\sqrt{10}}\underbrace{\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}}_{2\times 3} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}_{3\times 3}$$

$$= \cdots\cdots = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3} = A$$

Q: Obtain the S.V.D of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$ Ans: $A = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Sol: $A^TA = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \begin{matrix} = 45 \\ = 45 \end{matrix}$

ie one EValue is 45 & Tr = 50

So other EValue is $\lambda = 5$

ie Hence $\sigma_1 = \sqrt{45}$ & $\sigma_2 = \sqrt{5}$

$A^TA = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$ → $\lambda_1 = 45$, $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; → $\lambda_2 = 5$, $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$V_1 = \frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $V_2 = \frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Right Sing Vectors $V = [V_1, V_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$AV_1 = \sigma_1 U_1 \Rightarrow U_1 = \frac{1}{\sigma_1}AV_1 = \frac{1}{\sqrt{45}}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ie $U_1 = \frac{1}{\sqrt{90}}\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$AV_2 = \sigma_2 U_2 \Rightarrow U_2 = \frac{1}{\sigma_2}AV_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

ie $U_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

So left Sing Vectors $U = [U_1, U_2] = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

$A=\begin{bmatrix}3&0\\4&5\end{bmatrix}$, $U=\frac{1}{\sqrt{10}}\begin{bmatrix}1&-3\\3&1\end{bmatrix}$, $V=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}$, $D=\begin{bmatrix}\sqrt{45}&0\\0&\sqrt{5}\end{bmatrix}_{2\times 2}$

$(\sigma_1=\sqrt{45},\ \sigma_2=5)$

So S.V.D of $A_{2\times 2} = U_{2\times 2}\Sigma_{2\times 2}V^T_{2\times 2}$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix}1&-3\\3&1\end{bmatrix}\cdot\begin{bmatrix}\sqrt{45}&0\\0&\sqrt{5}\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$$

$= \underline{\text{Ans}}$

Q. If $A_{2\times2}$ s.t. $M = A^T A$ and $N = AA^T$ and also we have,

$M\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}45\\45\end{bmatrix}$, $M\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-5\\5\end{bmatrix}$, $N\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}45\\135\end{bmatrix}$, $N\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}-15\\5\end{bmatrix}$ then find $A = ?$

Sol: $M\begin{bmatrix}1\\1\end{bmatrix} = 45\begin{bmatrix}1\\1\end{bmatrix}$, $M\begin{bmatrix}-1\\1\end{bmatrix} = 5\begin{bmatrix}-1\\1\end{bmatrix}$, $N\begin{bmatrix}1\\3\end{bmatrix} = 45\begin{bmatrix}1\\3\end{bmatrix}$, $N\begin{bmatrix}-3\\1\end{bmatrix} = 5\begin{bmatrix}-3\\1\end{bmatrix}$

$\lambda_1 = 45$, $\lambda_2 = 5$ & $\lambda_1 = 45$, $Y_1 = \begin{bmatrix}1\\3\end{bmatrix}$, $\lambda_2 = 5$, $Y_2 = \begin{bmatrix}-3\\1\end{bmatrix}$

$X_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $X_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

$U = \frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}$

$V = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$ $\sigma_1 = \sqrt{45}$, $\sigma_2 = \sqrt{5}$ $\Sigma = D = \begin{bmatrix}\sqrt{45} & 0\\0 & \sqrt{5}\end{bmatrix}$,

$$\underset{2\times2}{A} = U\underset{2\times2}{\Sigma}V^T = \frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}\cdot\begin{bmatrix}\sqrt{45} & 0\\0 & \sqrt{5}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$$

Ans. $= \begin{bmatrix}3 & 0\\4 & 5\end{bmatrix}$

Q. If $A_{2\times 2}$ s.t $M = A^T A$ and $N = AA^T$ and also we have,

$M\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}45\\45\end{bmatrix}$, $M\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-5\\5\end{bmatrix}$, $N\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}45\\135\end{bmatrix}$, $N\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}-15\\5\end{bmatrix}$ then Find $A = ?$

(1)

(a) $\begin{bmatrix}45 & 0\\0 & 5\end{bmatrix}$

(b) $\begin{bmatrix}3 & 0\\4 & 5\end{bmatrix}$

(c) $\begin{bmatrix}3 & 4\\0 & 5\end{bmatrix}$

(d) $\begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}$

(2)

(a) $\frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}\begin{bmatrix}45 & 0\\0 & 5\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$

(b) $\frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}\begin{bmatrix}\sqrt{45} & 0\\0 & \sqrt{5}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

(c) $\frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$

(d) $\frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}\begin{bmatrix}\sqrt{45} & 0\\0 & \sqrt{5}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$

Given a 3 × 3 matrix E with $\boxed{\text{SVD} = U\Sigma V^T}$, if E has exactly one zero singular value, what can be said about determinant of E?

(a) The determinant of E is zero.

(b) The determinant of E is the product of its non-zero singular values.

(c) The determinant of E is equal to the sum of its singular values.

(d) det $(E) = 1$

$\because E_{3\times 3}$ i.e Sq. Mat $\Rightarrow |E| =$ Product of E Values

$=$ Product of Singular Values

$= \sigma_1 \cdot \sigma_2 \cdot (0)$

$= 0$

Consider a matrix A where $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}_{3\times 3}$. The matrix $A^T A$ is :

$$A^T A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}^T \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Which of the following is correct about the eigen values of $A^T A$?

(a) They are the squares of the singular values of A.

(b) They are the sums of the squares of the singular values of A.

(c) They are the absolute values of singular values of A.

(d) They are one less than the singular values of A.

(2) which of the following is correct about the sum of the E.Values of $A^T A = ?$

$= \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n$

$= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \dots + \sigma_n^2$

= sum of the sq. of the sing. Values

= (b).

w.k.that sing. Values of A = Sq. Roots of E Values of $A^T A$.

or $(\text{Sing Values of } A)^2 = \text{E Values of } A^T A \Rightarrow (\sigma)^2 = \lambda$

PODCAST: $A=\begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$ = Symm., $|A-\lambda I|=0 \dashrightarrow$ EValues of A are $\boxed{4.87, 2.65, 1.46}$

$(\lambda^3-9\lambda^2+24\lambda-19=0)$

Now, $A^TA=\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} & & \end{bmatrix}=\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$

EValues of A^TA are $(4.87)^2, (2.65)^2, (1.46)^2$

So Sing. Values of A are $\sqrt{(4.87)^2}, \sqrt{(2.65)^2}, \sqrt{(1.46)^2}$

$= \boxed{4.86, 2.65, 1.46}$ Ans

ie when A is Symm, Sing. Values of A = EValues of A.

Let A be a 2×2 matrix given by :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Patience

The singular value decomposition (SVD) of A is given by $\boxed{A = U\Sigma V^T}$, where U and V are orthogonal matrices and Σ is a diagonal matrix with non-negative entries. If the singular values of A are σ_1 and σ_2, where $\sigma_1 \geq \sigma_2 \geq 0$.

What is the sum of the square of singular values of A? $= \underset{(\text{for } A)}{\sigma_1^2 + \sigma_2^2} = \underset{(\text{for } A^TA)}{\lambda_1 + \lambda_2} = Tr(A^TA) = 30$

(a) 5.24
(b) 6.37
(c) ✓ 30
(d) 5

$$A^TA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

M-II C. Equ of A^TA is $\lambda^2 - (30)\lambda + (4) = 0$

$\lambda = 15 \pm \sqrt{221}$

so $\sigma_1^2 + \sigma_2^2 = \lambda_1 + \lambda_2$

$= (15 + \sqrt{221}) + (15 - \sqrt{221})$

$= 30$

PODCAST:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, E-Values of A are $\frac{5+\sqrt{17}}{2}$ & $\frac{5-\sqrt{17}}{2}$

$A^T A = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$, E Values of $A^T A$ are $\lambda_1 = (15+\sqrt{221})$ & $\lambda_2 = (15-\sqrt{221})$

Sing Values of A are $\sigma_1 = \sqrt{15+\sqrt{221}}$ & $\sigma_2 = \sqrt{15-\sqrt{221}}$

ie $\sigma_1^2 + \sigma_2^2 = \lambda_1 + \lambda_2 = (15+\sqrt{221}) + (15-\sqrt{221})$

$= 30$

Consider a 3 × 3 matrix B given by :

Note: If A is symm Mat; $\boxed{\text{Sing Values of } A = \text{E Values of } A}$

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow B^T B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The singular value decomposition (SVD) of B is given by $B = U\Sigma V^T$, where Σ is a diagonal matrix with singular values on the diagonal. What is the sum of squares of the singular values of B?

$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = ?$

(a) 29 (b) 30

(c) 31 (d) 32

$$\underbrace{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}_{\text{(Sing Values of } B)} = \underbrace{\lambda_1 + \lambda_2 + \lambda_3}_{\text{(E-Values of } B^T B)} = \underbrace{16 + 9 + 4 = 29}_{\text{M-I}} = \underbrace{Tr(B^T B)}_{\text{M-II}}$$

Sp. Note: E Values of $B^T B$ are 16, 9, 4

So $\boxed{\text{Sing Values of } B}$ are $\sqrt{16}, \sqrt{9}, \sqrt{4} = 4, 3, 2$ which are the $\boxed{\text{E Values of } B}$

MSQ

Given a 5 × 5 matrix B with SVD $U\Sigma V^T$ which of the following statements is correct regarding the relationship between B and its SVD components.

(a) The matrix $U\Sigma V^T$ has the same rank as B.

(b) The product U^TU equals the identity matrix of size 5 × 5.

(c) The matrix Σ contains the eigen values of B on its diagonal. ×

(d) The columns of V are the right singular vectors of B and the rows of V are the left singular vector of B.

True False

ATQ, $\boxed{B = U\Sigma V^T}$

$\rho(B) = \rho(U\Sigma V^T)$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \end{bmatrix}$$

$B_{5\times5} \to (B^TB)_{5\times5} \to V = [v_1\ v_2\ v_3\ v_4\ v_5]_{5\times5}$

$B_{5\times5} \to (BB^T)_{5\times5} \to U = [u_1\ u_2\ u_3\ u_4\ u_5]_{5\times5}$

$\because U$ is O-Mat so $UU^T = I$

or $U^TU = I$

Let A be a 4×3 matrix. The singular value decomposition (SVD) of A is given by $A = U\Sigma V^T$. If U is a 4×4 orthogonal matrix, Σ is a 4×3 diagonal matrix and V is a 3×3 orthogonal matrix. Which of the following statements is true?

(a) Σ has exactly 3 non-zero singular values. ×

(b) The columns of U form an orthonormal basis for R^4.

(c) The columns of V form an orthonormal basis for R^4. ×

(d) The rows of U form an orthonormal basis for R^3. ×

??

$$\Sigma = \text{(i)} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}_{4\times 3}$$

$$\text{(ii)} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{4\times 3} \begin{cases} (A^TA)_{3\times 3} \longrightarrow V = [v_1 v_2 v_3]_{3\times 3}, \\ (AA^T)_{4\times 4} \longrightarrow U = [u_1 u_2 u_3 u_4]_{4\times 4} \end{cases}$$

$\underbrace{\sigma_1, \sigma_2, \sigma_3}_{\text{Not Necessarily Non Zero.}}$

$$\boxed{A_{4\times 3} = \underset{4\times 4}{U}\ \underset{4\times 3}{\Sigma}\ \underset{3\times 3}{V^T}}$$

THANK - YOU