Solution to Binomial Trees in Weighted Quick-Union

Problem Statement

The problem asks us to:

1. Show that the number of nodes at each level in the worst-case trees for weighted quick-union are binomial coefficients.
2. Compute the average depth of a node in a worst-case tree with $N = 2^n$ nodes.

Solution

Part 1: Binomial Trees in Weighted Quick-Union

The worst-case trees in the *Weighted Quick-Union* algorithm are balanced trees. Each tree has a structure similar to a *binomial tree*, which arises naturally when performing unions on sets of different sizes. In the worst-case scenario, we keep unioning trees of equal size, which increases the height of the tree.

A binomial tree B_k of height k has the following structure:

- The root has two subtrees, each of which is a binomial tree of height $k-1$.
- The number of nodes at level i in a binomial tree of height k corresponds to the binomial coefficient $\binom{k}{i}$, where:

$$\binom{k}{i} = \frac{k!}{i!(k-i)!}$$

Thus, in the worst-case scenario, the number of nodes at level i in a tree of height k follows the binomial distribution $\binom{k}{i}$.

Part 2: Average Depth of a Node in a Worst-Case Tree with $N = 2^n$

Let's now compute the average depth of a node in a worst-case tree when there are $N = 2^n$ nodes.

Step 1: Total number of nodes The worst-case tree has $N = 2^n$ nodes, and its height is n, as it is a balanced tree and follows the structure of a binomial tree.

Step 2: Number of nodes at each level The number of nodes at level i in a tree of height n is given by the binomial coefficient $\binom{n}{i}$.

Step 3: Average depth calculation The depth of a node is its distance from the root. The total sum of depths is calculated by summing the product of the depth i and the number of nodes at that depth $\binom{n}{i}$. The formula for the total sum of depths is:

$$\text{Sum of depths} = \sum_{i=0}^{n} i \times \binom{n}{i}$$

The average depth is the total sum of depths divided by the total number of nodes $N = 2^n$:

$$\text{Average depth} = \frac{\sum_{i=0}^{n} i \times \binom{n}{i}}{2^n}$$

Step 4: Simplifying the sum Using the identity for binomial coefficients, we know that:

$$\sum_{i=0}^{n} i \times \binom{n}{i} = n \times 2^{n-1}$$

Thus, the sum of depths is $n \times 2^{n-1}$.

Step 5: Final average depth The average depth is then:

$$\text{Average depth} = \frac{n \times 2^{n-1}}{2^n} = \frac{n}{2}$$

Thus, the average depth of a node in a worst-case tree with $N = 2^n$ nodes is $\frac{n}{2}$.

Final Answer

1. The number of nodes at each level in the worst-case trees for weighted quick-union follows binomial coefficients.

2. The average depth of a node in a worst-case tree with $N = 2^n$ nodes is $\frac{n}{2}$.