Numerical methods - 2075

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A 'Group':

1. What is non-linear equation? Derive the required expression to calculate the root of non-linear equation using secant method. Using this expression find a root of following

$$x^2 + \cos(x) - e^{-x} - 2 = 0$$

=> Equation whose graph does not form a straight line is called a non linear equation. The variables are either of degree less than one or greater than one but never equal to one.
General form: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$.

Secant method (Chord method)

Secant method is an improvement over the method of false position as it does not require $f(x_0) * f(x_1) < 0$ of that method.

It is given by $$\boxed{x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_1)}$$

It is derived from Newton Raphson method:

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Given a non-linear eqn $f(x)=0$
From newton's raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{—(i)}$$

Here, $f'(x_n)$ is calculated as.

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \quad \text{—(ii)}$$

Putting value in (i)

$$\cancel{x_{n+1} = x_n - f(x_n)(x_n - x_{n-1})}$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\therefore \boxed{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \times f(x_n)}$$

which is secant method.

For, initial approximation x_0 and x_1, x_2 can be calculated as

$$\boxed{x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_1)}$$

Note: $\boxed{\text{Use radian in calculator}}$

Given Eqn,

$$x^2 + \cos(x) - e^{-x} - 2 = 0$$

$$\Rightarrow f(x) = x^2 + \cos(x) - e^{-x} - 2$$

Let, Initial approximations be

$x_0 = 1.4$ ∴ $f(x_0) = -0.11662$

$x_1 = 1.5$ ∴ $f(x_1) = 0.09760$

So, the root lies between x_0 and x_1

Now, 1st Iteration,

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_1)$$

$$= 1.5 - \frac{1.5 - 1.4}{0.09760 + 0.11662} \times 0.09760$$

$$= 1.5 - 0.089326$$

$$\therefore x_2 = 1.41067$$

2nd Iteration

$x_0 = 1.5$ ∴ $f(x_0) = 0.09760$

$x_1 = 1.41067$ ∴ $f(x_1) = -0.09454$

$$x_2 = 1.41067 - \frac{1.41067 - 1.5}{-0.09454 - 0.09760} \times (-0.09454)$$

$$= 1.41067 - (-0.04395)$$

$$\therefore x_2 = 1.45462$$

<u>3rd Iteration</u>

$x_0 = 1.41067 \quad \therefore f(x_0) = -0.09454$

$x_1 = 1.45462 \quad \therefore f(x_1) = -0.00165$

$$x_2 = 1.45462 - \frac{1.45462 - 1.41067}{-0.00165 + 0.09454} \times (-0.00165)$$

$$= 1.45462 - \{(-0.00078) \times (0.00165)\}$$

$$x_2 = \underline{\underline{1.4554}}$$

<u>4th Iteration.</u>

$x_0 = 1.45462 \quad \therefore f(x_0) = -0.00165$

$x_1 = 1.4554 \quad \therefore f(x_1) = 0.00002$

$$x_2 = 1.4554 - \frac{1.4554 - 1.45462}{0.00002 + 0.00165} \times 0.00002$$

$$x_2 = \underline{\underline{1.4553}}$$

Here, we can see root is $\boxed{1.455}$ as the 3rd & 4th iteration have values equal to three decimal places.

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Q.2 What is matrix factorization? Factorize the given matrix A into LU dolittle algorithm and also solve $Ax=b$ for given b using L and U.

=) Matrix factorization is the process of decomposing a matrix into two triangular matrices (one lower triangular and another upper triangular).

Given,

$$A = \begin{bmatrix} 2 & 4 & -4 & 0 \\ 1 & 5 & -5 & -3 \\ 2 & 3 & 1 & 3 \\ 1 & 4 & -2 & 2 \end{bmatrix} \quad \& \quad b = \begin{bmatrix} 12 \\ 18 \\ 8 \\ 8 \end{bmatrix}$$

Here, we know $A = LU$

$$\begin{bmatrix} 2 & 4 & -4 & 0 \\ 1 & 5 & -5 & -3 \\ 2 & 3 & 1 & 3 \\ 1 & 4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -4 & 0 \\ 1 & 5 & -5 & -3 \\ 2 & 3 & 1 & 3 \\ 1 & 4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ l_{21}U_{11} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23} & l_{21}U_{14}+U_{24} \\ l_{31}U_{11} & l_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+U_{33} & l_{31}U_{14}+l_{32}U_{24}+U_{34} \\ l_{41}U_{11} & l_{41}U_{12}+l_{42}U_{22} & l_{41}U_{13}+l_{42}U_{23}+L_{43}U_{33} & L_{41}U_{14}+L_{42}U_{24}+L_{43}U_{34}+U_{44} \end{bmatrix}$$

Equating the corresponding elements, we get.

$U_{11}=2$

$U_{12}=4$

$U_{13}=-4$

$U_{14}=0$

$L_{41}U_{12}+L_{42}U_{22}=4$

$0.5\times 4+L_{42}\times 3=4$

$L_{42}=\frac{2}{3}=0.666$

$L_{31}U_{13}+L_{32}U_{23}+U_{33}=1$

$-4+\left(-\frac{1}{3}\right)(-3)+U_{33}=1$

$U_{33}=4$

$L_{21}U_{11}=1$

$\therefore L_{21}=\frac{1}{2}=0.5$

$L_{31}U_{11}=2$

$\therefore L_{31}=\frac{2}{2}=1$

$L_{41}U_{11}=1$

$\therefore L_{41}=\frac{1}{2}=0.5$

$L_{21}U_{13}+U_{23}=-5$

$0.5\times(-4)+U_{23}=-5$

$U_{23}=-3$

$L_{21}U_{12}+U_{22}=5$

$0.5\times 4+U_{22}=5$

$\therefore U_{22}=3$

$L_{31}U_{12}+L_{32}U_{22}=3$

$1\times 4+L_{32}\times 3=3$

$L_{32}=-\frac{1}{3}=-0.333$

$L_{21}U_{14}+U_{24}=-3$

$0.5\times 0+U_{24}=-3$

$U_{24}=0+(-3)$

$U_{24}=-3$

$$L_{41}U_{13} + L_{42}U_{23} + L_{43}U_{33} = -2$$

$$\frac{1}{2} \times (-4) + \frac{2}{3} \times (-3) + L_{43} \times 4 = -2$$

$$-2 - 2 + L_{43} \times 4 = -2$$

$$4L_{43} = 2$$

$$L_{43} = \frac{1}{2}$$

$$L_{31}U_{14} + L_{32}U_{24} + U_{34} = 3$$

$$0 + -\frac{1}{3} \times (-3) + U_{34} = 3$$

$$U_{34} = 2$$

$$L_{41}U_{14} + L_{42}U_{24} + L_{43}U_{34} + U_{44} = 2$$

$$0.5 \times 0 + \frac{2}{3} \times (-3) + \frac{1}{2} \times 2 + U_{44} = 2$$

$$-2 + 1 + U_{44} = 2$$

$$U_{44} = 3$$

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$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1 & -0.333 & 1 & 0 \\ 0.5 & 0.666 & 0.5 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 2 & 4 & -4 & 0 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Now, we know, to solve $Ax = b$

$Ly = b$ where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1 & -0.333 & 1 & 0 \\ 0.5 & 0.666 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \\ 8 \\ 8 \end{bmatrix}$

From this,

$y_1 = 12$

$0.5 \times 12 + y_2 = 18 \quad \therefore y_2 = 12$

$y_1 + (-0.333) y_2 + y_3 = 8$

$\therefore 12 + -\frac{1}{3} \times 12 + y_3 = 8$

$y_3 = 0$

&

$0.5 y_1 + 0.666 y_2 + 0.5 y_3 + y_4 = 8$

$6 + \frac{2}{3} \times 12 + 0 + y_4 = 8$

$y_4 = -6$

$\therefore Y = \begin{bmatrix} 12 \\ 12 \\ 0 \\ -6 \end{bmatrix}$

Now,

$UX = Y$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 & -4 & 0 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 0 \\ -6 \end{bmatrix}$$

Using backward substitution,

$3x_4 = 10$

$\therefore x_4 = \frac{-6}{3} = -2$

$$4x_3 + 2x_4 = 0$$
$$4x_3 + 2\times(-2) = 0$$
$$4x_3 - 4 = 0$$
$$x_3 = 1$$

$$3x_2 - 3x_3 - 3x_4 = 12$$
$$3x_2 - 3 + 6 = 12$$
$$3x_2 + 3 = 12$$
$$x_2 = 3$$

$$2x_1 + 4x_2 - 4x_3 = 12$$
$$2x_1 + 12 - 4 = 12$$
$$2x_1 + 8 = 12$$
$$2x_1 = 4$$
$$x_1 = 2$$

$$\therefore \quad X = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$$

Q.4 Calculate the real negative root of following equation using Newton method.

$$x^4 + 2x^3 + 3x^2 + 4x = 5$$

=)

$$f(x) = x^4 + 2x^3 + 3x^2 + 4x - 5$$
$$f'(x) = 4x^3 + 6x^2 + 6x + 4$$

Here, we need to find the -ve root

$$f(-3) = 37 > 0$$
$$f(-2) = -1 < 0$$

So, root lies between -3 & -2.

Let, $x_0 = -2$ $\therefore f(x_0) = -1$

1st Iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= -2 - \frac{(-1)}{(-16)} = -2.0625$$

$$\therefore x_1 = -2.0625$$

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2nd Iteration

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -2.0625 - \frac{0.06007}{19.9521}$$

$$= -2.0655$$

$\therefore x_2 = -2.0655$

3rd Iteration

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -2.0655 - \frac{0.11405}{-18.043}$$

$$= -2.0591$$

$\therefore x_3 = -2.0591$

4th Iteration

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= -2.0591 - \frac{0.00075}{-17.83}$$

$\therefore x_4 = -2.0591$

Here, we can see, the value of x_3 and x_4 are same upto four decimal places,

So, real negative root = -2.0591

Q.5 What is least squares approximation of fitting a function? How does it differ with polynomial interpolations? Explain with suitable example

=) Least squares approximation is the process of ~~approximating~~ estimating the value of parameters of a particular model, based on the observed pairs of values.
It finds the functional relationship between a dependent and independent variables.

Some examples of least square approximation are linear regression, exponential regression, quadratic/polynomial regression etc.

~~Inter~~ Polynomial interpolation is the process of finding a curve which fits the input/output relationship based on the given data points.
In case of interpolation, we don't have to worry about variance of the fitted curve. That means given the set of n data points (x_i, y_i) we look for the function $f(x)$ that satisfies the relation $y_i = f(x_i)$ for all given data points.
Some examples of polynomial interpolation are:- Lagrange interpolation, newton's divided interpolation, newton's forward and backward difference interpolation.

Q.6. Find the lowest degree polynomial, which passes through the following points

	x_0		x_1			
x	-2	-1	1	2	3	4
$f(x)$	-19	0	2	-3	-4	5

Using this polynomial, estimate $f(x)$ at $x=0$.

$\Rightarrow$ Note: It says lowest degree polynomial, so we use lagrange interpolation.

We know, lowest degree polynomial is the 1st order polynomial. so,

$$\therefore P_1(x) = \sum_{i=0}^{1} f_i l_i(x)$$

$$= f_0 l_0(x) + f_1 l_1(x)$$

$$= f_0 \frac{(x-x_1)}{(x_0-x_1)} + f_1 \frac{(x-x_0)}{(x_1-x_0)}$$

Here, 0 lies betn -1 & 1

So, $x_0 = -1$ $x_1 = 1$

$f_0 = 0$ $f_1 = 2$

$\therefore P_1(x) = \frac{0 \times (x-1)}{(-1-1)} + \frac{2(x-(-1))}{2}$

$= \frac{2(x+1)}{2}$

$\therefore P_1(x) = x+1$

Hence, the lowest degree polynomial is $x+1$

Now, at $x=0$

$f(x) = P(0) = 0+1 = 1$

$\therefore$ The value of $f(x)$ at $x=0$ is 1.

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Q.7. Fit function of type $y = a + bx$ for the following points using least square method

x	-1	1.2	2	2.7	3.6	4
f(x)	1	20	27	33	41	45

=)

X	Y	XY	X^2
-1	1	-1	1
1.2	20	24	1.44
2	27	54	4
2.7	33	89.1	7.29
3.6	41	147.6	12.96
4	45	180	16

$\Sigma X = 12.5$ $\Sigma Y = 167$ $\Sigma XY = 493.7$ $\Sigma X^2 = 42.69$

$n = 6$

Here, the regression function is

$y = a + bx$ - (i)

The regression coefficients can be found as

$$b = \frac{n\Sigma XY - \Sigma X \Sigma Y}{n\Sigma X^2 - (\Sigma X)^2} = \frac{6 \times 493.7 - 12.5 \times 167}{6 \times 42.69 - (12.5)^2}$$

$$= 8.756$$

$$a = \bar{y} - b\bar{x}$$

$$= \frac{\Sigma y}{n} - b\,\frac{\Sigma x}{n}$$

$$= \frac{167}{6} - \frac{8.756 \times 12.5}{6}$$

$$= 27.833 - 18.241$$

$$= 9.592$$

$\therefore$ The fitted regression function is.

$$\boxed{\hat{y} = 9.592 + 8.756x.}$$

Q.8. Calculate the integral value of the function given below from $x = 1.8$ to $x = 3.4$ using Simpson's $\frac{1}{3}$ rule.

x	1.8	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4
F(x)	0.003	0.778	1.632	2.566	3.579	4.672	7.097	7.097	8.429

$\Rightarrow$ Here, $x_0 = 1.8$, $x_n = 3.4$

No. of segments $(k) = 8$

$$\therefore h = \frac{3.4 - 1.8}{8} = 0.2$$

$x_0 = 1.8$, $x_1 = 2.0$, $x_2 = 2.2$, $x_3 = 2.4$, $x_4 = 2.6$,
$x_5 = 2.8$, $x_6 = 3.0$, $x_7 = 3.2$, $x_8 = 3.4$

Now, using Simpson $\frac{1}{3}$ rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[\cancel{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) +} \; 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)\right]$$

$$= \frac{0.2}{3}\left[0.003 + 4 \times 0.778 + 2 \times 1.632 + 4 \times 2.566 + 2 \times 3.579 + 4 \times 4.672 + 2 \times 7.097 + \cancel{8.429}\; 4 \times 7.097 + 8.429\right]$$

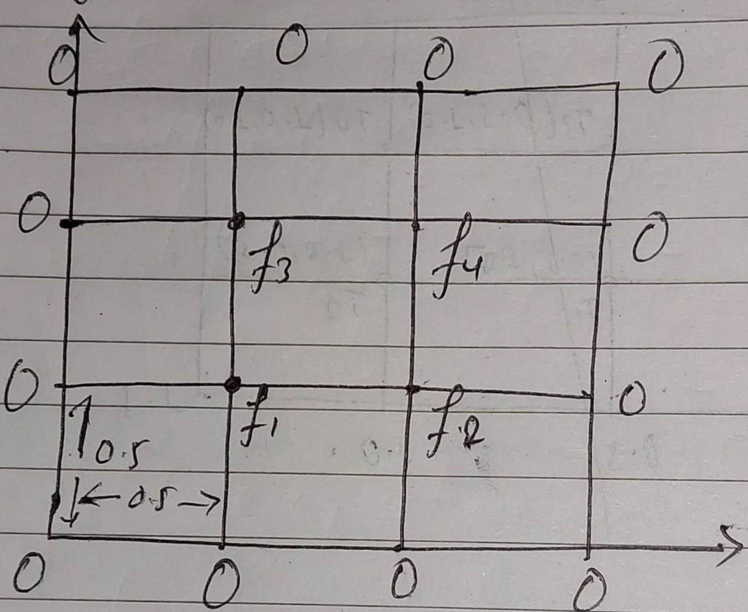
$= \frac{0.2}{3} \times \cancel{65.412}\ 93.5$

$= \cancel{4.948}\ 6.233$

Hence, the integral value of the function is ~~4.948~~ 6.233

12. Solve the poisson's equation $\nabla^2 f = 2xy$ over the square domain $0 \le x \le 1.5$, $0 \le y \le 1.5$ with $f = 0$ on the boundary & $h = 0.5$

⇒ The given grid can be represented as



For f_1

$$f_3 + f_2 + 0 + 0 - 4f_1 = 2 \times 0.5 \times 0.5$$
$$f_3 + f_2 - 4f_1 = 0.5 \quad -(i)$$

For f_2

$$f_1 + f_4 + 0 + 0 - 4f_2 = 2 \times 1.0 \times 0.5$$
$$f_1 + f_4 - 4f_2 = 1 \quad -(ii)$$

For f_3

$$f_1 + f_4 + 0 + 0 - 4f_3 = 2 \times 0.5 \times 1.0$$
$$f_1 + f_4 - 4f_3 = 1 \quad -(iii)$$

For f_4

$$f_3 + f_2 + 0 + 0 - 4f_4 = 2 \times 1 \times 1$$

$$f_2 + f_3 - 4f_4 = 2 \quad -(iv)$$

Hence the obtained eqns are:

$$-4f_1 + f_2 + f_3 + 0f_4 = 0.5$$
$$f_1 - 4f_2 + 0f_3 + f_4 = 1$$
$$f_1 + 0f_2 + 4f_3 + f_4 = 1$$
$$0f_1 - f_2 + f_3 - 4f_4 = 2$$

Solving these equations using calculator

$f_1 = -0.3958$, $f_2 = -0.5416$, $f_3 = -0.5416$,
$f_4 = -0.7708$