Numerical Methods. 2077

-Ankit Pangeni

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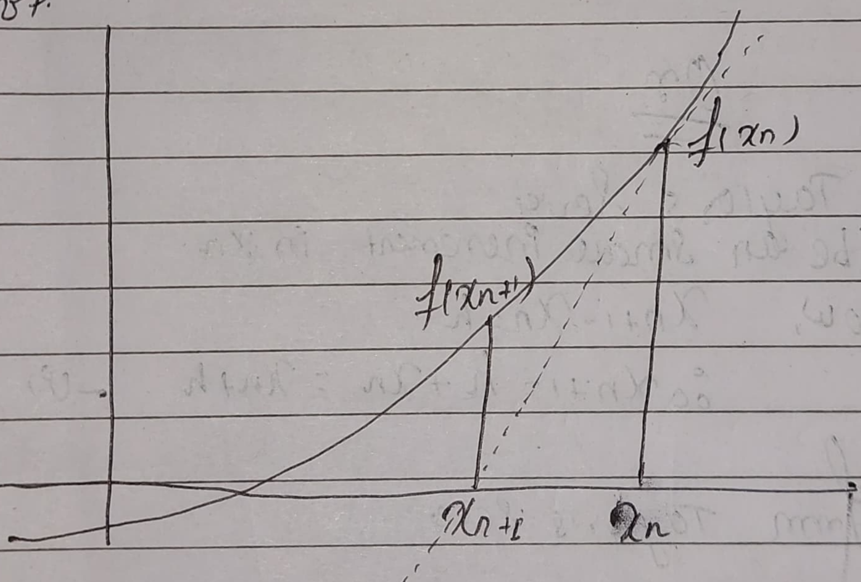
Group 'A'

Q.1. Derive the formula for Newton Raphson method. Solve the equation $x^2+4x-9=0$ using Newton Raphson method. Assume error precision is 0.01. Discuss drawbacks of the Newton Raphson Method

⇒ Newton-Raphson method is based on the principle that if the initial guess of the root of $f(x)=0$ is at x_n, & if one draws the tangent to the curve at $f(x_n)$, the point x_{n+1} where the tangent crosses the x-axis is an improved estimate of the root.



Using the definition of slope of function, at $x=x_n$

$$f'(x_n)=\frac{f(x_n)-f(x_{n+1})}{x_n-x_{n+1}}$$

If $f(x)$ have root at x_{n+1}, then $f(x_{n+1})=0$.

$\therefore f'(x_n)=\dfrac{f(x_n)}{x_n-x_{n+1}}$

This gives,

$$\boxed{x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}}$$

This is called Newton-Raphson formula for solving non linear eqns of form $f(x)=0$

Or

Using Taylor's Series.
Let h be an small increment in x_n
We know, $x_{n+1}-x_n=h$

$\therefore x_{n+1}=h+x_n=x_n+h$ — (i)

Here, from Taylor's series.

$$f(x+h)=f(x)+h\frac{f'(x)}{1!}+\frac{h^2}{2!}f''(x)+\cdots$$

Since, h is very small, so neglecting its higher order terms

$f(x+h) = f(x) + h f'(x)$

$\therefore f(x_n+h) = f(x_n) + h f'(x_n)$

from Eqn (i)

$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n)$

If x_{n+1} is root of given polynomial, then $f(x_{n+1}) = 0$

$\therefore f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0$

$f'(x_n)(x_{n+1} - x_n) = -f(x_n)$

$(x_{n+1} - x_n) = \dfrac{-f(x_n)}{f'(x_n)}$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Which is newton's raphsm formula

Given, Eqn $x^2+4x-9=0$

$\therefore f(x) = x^2 + 4x - 9$

$f'(x) = 2x + 4$

Here, Let initial guesses be 1 & 2

$\therefore f(1) = -4$

$f(2) = 3$

So, root lies between 1 and 2.

Let, $x_0 = 1.5$

<u>1st Iteration</u>

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.5 - \frac{(-0.75)}{7}$$

$$= 1.6071$$

$$\text{Error} = \left| \frac{1.6071 - 1.5}{1.6071} \right|$$

$$= 0.066$$

<u>2nd iteration</u>

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.6071 - \frac{0.011}{7.2142}$$

$$= 1.6055$$

$$\text{Error} = \left| \frac{1.6055 - 1.6071}{1.6055} \right| = 0.00099$$

Since Error $= 0.00099 < 0.01$

So, $\boxed{\text{root} = 1.605}$

Drawbacks of Newton-Raphson Method

1) Its convergence is not guaranteed.
2) Division by zero problem can occur

Ex: $f(x) = 1 - x^2$

$f'(x) = -2x$

At, $x_0 = 0$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{0} \text{ (undefined)}$$

3) It converges slowly in case of multiple roots.
4) Oscillations near local maximum & minimum
5) Symbolic derivative is required.

Q.3. Why partial pivoting is used with Naive Gauss Elimination method? Solve the following system of equations using Gauss Elimination method with partial pivoting? How Gauss Jordan method differs from Gauss elimination method?

$2x + 2y - z = 6$
$4x + 2y + 3z = 4$
$x + y + z = 0$

=) Partial pivoting in Gauss elimination method is the modified process of Gauss elimination method which helps reduce the rounding errors; you are less likely to add / subtract with very small (or very large) numbers

It basically avoids the round off errors that could be caused when dividing every entry of a row by a pivot value that is relatively small in comparision to its remaining row entries. So before continuing with the row reduction process, we switch the rows so that the pivot element with highest coefficient is always at top. The simplest strategy is to select an element in the same coumn that is below the diagonal and has the largest absowte vawe.

Given system of linear equations:

$$2x+2y-z=6$$
$$4x+2y+3z=4$$
$$x+y+z=0$$

It's matrix representation is

$$\left[\begin{array}{ccc:c} 2 & 2 & -1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Here, we look at the 1st coumn and compare the magnitude of each elements of the coumn.

$|4| > |2| > |1|$, So, exchange R_1 & R_2

Apply, $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 2 & 2 & -1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Apply forward elimination
$R_2 \to 2R_2 - R_1$
$R_3 \to 4R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 0 & 2 & -5 & 8 \\ 0 & 2 & 1 & -4 \end{array}\right]$$

Now, we look in the 2nd column, below the 1st row and compare the magnitude of each elements.

$|2| = |2|$

So, no row change is required

Apply forward elimination,
$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 0 & 2 & -5 & 8 \\ 0 & 0 & 6 & -12 \end{array}\right].$$

Now, ~~using~~ $R_3 \to \frac{1}{6} R_3$

$$\sim \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 0 & 2 & -5 & 8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Now, using backward substitution

$\Rightarrow \boxed{z = -2}$

$\Rightarrow 2y - 5z = 8 \quad \therefore 2y + 10 = 8 \quad , \boxed{y = -1}$

$\Rightarrow 4x + 2y + 3z = 4$

$4x - 2 - 6 = 4$

$\boxed{x = 3}$

$\therefore$ By using Gauss Elimination method with partial pivoting, we solved the given system of equations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Gauss Jordan method is the modification of Gauss elimination method.
Basically, in Gauss elimination method we reduce the matrix formed by the system of linear equations into the echelon form whereas in Gauss Jordan method, we reduce the matrix into row reduced echelon form

It means, an identity matrix is formed in case of Gauss Jordan matrix so that we do not need to perform the backward substitution and can obtain the solution directly from the identity matrix. Whereas, in Gauss elimination method, it forms an upper triangular matrix and we have to perform backward substitution to find the solution.

Gauss-Jordan is more complicated than Gauss elimination (required more computational work). For small systems it is convienient to use Gauss Jordan method.

Q.2. How Interpolation differs from regression? Write down algorithm and program for Lagrange interpolation

⇒ Interpolation is the process of finding the value of a function $y = f(x)$ at a given point that lies in between the data points given in table.
The given data points: $(x_0, y_0), (x_1, y_1) \ldots (x_{n-1}, y_{n-1}), (x_n, y_n)$

⇒ It is basically the process of finding a curve which fits the input/output relationship based on given data points.

⇒ Whereas, regression is the process of finding the curve/line of best fit based on given data points.
It finds a functional relationship between dependent and independent variable. A regression fit may not require the function to have exact values at given point; we just want a good approximation. ie. $y = f(x)$ might not satisfy relation $y_i = f(x_i)$ for any given data point.

⇒ Some examples of interpolation: Lagrange, newton's divided difference, newtons backward & backward difference. Some examples of regression:- linear regression, exponential regression, quadratic regression.

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Q.4. Define the terms true error and relative error? Use Horner's method to evaluate polynomial $2x^3 - 3x^2 + 5x - 2$ at $x = 3$ and write down it's algorithm

=) True Error, denoted by Et is defined as the difference between the true value and approximate value of a function at any given point.

ie. True Error = True value - Approximate val

where,

True value = direct derivative $f'(x)$

Approximate value =) $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

Relative Error also known as relative true error denoted by Er is defined as the ratio between the true Error & true value

ie. $Er = \dfrac{\text{true error}}{\text{true value}}$

It is expressed in percentage (%)

Given polynomial,

$p(x) = 2x^3 - 3x^2 + 5x - 2$ at $x = 3$

Comparing the polynomial with

$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

we get, $a_3 = 2$, $a_2 = -3$, $a_1 = 5$, $a_0 = -2$

Now, new sequence of constants can be determined by recursive formula as below: $b_{n-1} = a_{n-1} + b_n x$

$b_3 = a_3 = 2$

$b_2 = a_2 + b_3 x = -3 + 2 \times 3 = 3$

$b_1 = a_1 + b_2 x = 5 + 3 \times 3 = 14$

$b_0 = a_0 + b_1 x = -2 + 14 \times 3 = 40$

$\therefore$ At $x = 3$, the value of polynomial is 40

Horner's method is used for either of two things, as an algorithm for evaluating polynomials efficiently, or as a method for approximating roots of a polynomial.

The algorithm for Horner's method is.

1. Start
2. Enter degree of polynomial, say n.
3. Enter the value at which polynomial is to be evaluated, x.
4. Initially set: $b_n = a_n$
5. While $n > 0$

   $b_{n-1} = a_{n-1} + b_n * x$
6. End while.
7. Display value of b_0, which is value of polynomial at x
8. Stop.

Q.5. Construct Newton's forward difference table for the given data points and approximate the value of $f(x)$ at $x=15$

x	10	20	30	40	50
$f(x)$	0.173	0.342	0.5	0.643	0.766

$\Rightarrow$ Here, $x = x_0 + sh$

$$s = \frac{x - x_0}{h} = \frac{15 - 10}{10} = 0.5$$

=) Let's construct the table.

x	$f(x)$	$\Delta f(x_0)$	$\Delta^2 f(x_0)$	$\Delta^3 f(x_0)$	$\Delta^4 f(x_0)$
10	0.173				
		0.169			
20	0.342		-0.011		
		0.158		-0.004	
30	0.5		-0.015		0.001
		0.143		-0.005	
40	0.643		-0.02		
		0.123			
50	0.766				

From table, we can see the polynomial is of 4th order

$$P_4(x) = f(x_0) + \sum_{k=1}^{4} \binom{s}{k} \Delta^k f(x_0)$$

$$= f(x_0) + \binom{s}{1}\Delta f(x_0) + \binom{s}{2}\Delta^2 f(x_0) + \binom{s}{3}\Delta^3 f(x_0) + \binom{s}{4}\Delta^4 f(x_0)$$

$$= f(x_0) + \frac{s}{1!}\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f(x_0)$$

$$= 0.173 + 0.5 \times 0.169 + \frac{0.5(0.5-1)}{2}(-0.011)$$

$$+ \frac{0.5(0.5-1)(0.5-2)}{6}(-0.004) +$$

$$\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(-0.001)$$

$$= 0.173 + 0.0845 + 0.001375 + (-0.00025) + (0.0000390625)$$

$$= 0.258$$

$$\therefore P_4(x) = f(x) = 0.258$$

Hence, the value of $f(x)$ at $x = 25$ is 0.258

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Q.6 Fit the curve $y = ae^{bx}$ through the following points.

x	1	2	3	4
y	1.65	2.70	4.50	7.35

Note: use (ln) on calculator

⇒ x	y	$\log(y)$	$x\log y$	x^2
1	1.65	0.5007	0.5007	1
2	2.70	0.9932	1.9864	4
3	4.50	1.5040	4.512	9
4	7.35	1.9947	7.9788	16

$\Sigma x = 10$ $\Sigma y = 16.2$ $\Sigma \log(y) = 4.9926$ $\Sigma x^2 = 30$

$\Sigma x \log y = 14.9779$

$n = 4$

We know, Regression line for exponential model is

$$y = ae^{bx}$$
$$\log y = \log a + bx$$

This equation is similar to linear Eqn $y = a + bx$. Now, we can evaluate the regression coefficients a and b as:

$$b = \frac{n\Sigma x \log y - \Sigma x \cdot \Sigma \log y}{n\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{4 \times 14.9779 - 10 \times 4.9926}{4 \times 30 - (10)^2}$$

$$= 0.49928$$

And,

$$\log a = \overline{\log y} - b\bar{x}$$

$$= \frac{\Sigma \log y}{n} - b\frac{\Sigma x}{n}$$

$$= \frac{4.9926}{4} - 0.49928 \times \frac{10}{4}$$

$$= 1.24815 - 1.2482$$
$$= -5 \times 10^{-5}$$

$$\log a = -5 \times 10^{-5}$$
$$a = e^{-5 \times 10^{-5}}$$
$$a = 0.99995$$

Hence, the curve is

$$y = 0.99995 \times e^{0.49928x}$$

Q.7. Discuss the Dolittle LU decomposition method for matrix factorization

⇒ Matrix factorization is the process of decomposing a matrix into two triangular matrices (one lower triangular 'L' and another upper triangular 'U').

A coefficient matrix 'A' is decomposed into two triangular matrices L and U. If the diagonal elements of L contain 1 then it is called LU Decomposition method. i.e. DoLittle method assumes $l_{11} = l_{22} = l_{33} = \cdots l_{nn} = 1$

Let, coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

is decomposed into two triangular matrices

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$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$$

such that, $[A] = [L][U]$

Algorithm for Matrix factorization using Dolittle Decomposition

→ see from Book

Q.9. How Simpson's 1/3 rule differs from trapezoidal rule? Derive the formula for simpson's $\frac{1}{3}$ rule.

=) The trapezoidal rule ~~was~~ is based on approximating the integrand by a first order polynomial and then integrating the polynomial over interval of integration.

Simpson's $\frac{1}{3}$ rule is an extension of trapezoidal rule where the integrand is approximated by a second order polynomial. The simpson's $\frac{1}{3}$ rule assumes $n=2$

whereas the trapezoidal rule assumes $n=1$

General quadratic formula for integration is

$$\int_a^b f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx = nh\Big[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2-3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3-4n^2+4n)\Delta^3 f(x_0) + \cdots\Big] \quad \text{— (1)}$$

Here, $h = \frac{b-a}{2}$

By putting $n=2$ in above Eqn

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$$\int_{x_0}^{x_0+2h} f(x)dx = \int_{x_0}^{x_2} f(x)dx = 2h\left[f(x_0) + \Delta f(x_0) + \frac{1}{6}\Delta^2 f(x_0)\right]$$

$$= 2h\left[f(x_0) + (f(x_1) - f(x_0)) + \frac{1}{6}(f(x_0) - 2f(x_1) + f(x_2))\right]$$

$$= h\left[2f(x_0) + 2\{f(x_1) - f(x_0)\} + \frac{1}{3}(f(x_2) - 2f(x_1) + f(x_0))\right]$$

$$\therefore I = \int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

This eqn is called Simpsons $\frac{1}{3}$ rule

Algorithm

1. Start
2. Read value of lower and upper limit, say x_0 & x_2
3. Set $n = 2$
4. $h = (x_2 - x_0)/n$
5. $x_1 = x_0 + h$, $x_2 = x_0 + 2h$
6. Calculate values $f(x_0)$, $f(x_1)$ & $f(x_2)$
7. Calculate integration using
$$I = \int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$
8. Display I
9. Stop

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Q.10. Approximate the solution of $y' = 2x + y$, $y(0) = 1$ using Eulers method using step size 0.1. Approximate the value of $y(0.4)$.

$\Rightarrow$ Here, given, $f(x,y) = 2x + y$

$y(0) = 1$

$\therefore x_0 = 0,\ y_0 = 1$

Step-size $(h) = 0.1$

$x_1 = x_0 + h = 0.1$

$x_2 = 0.2$

$x_3 = 0.3$

$x_4 = 0.4$

We know, by Eulers method.

$$y(x_{i+1}) = y(x_i) + h\,(f(x_i, y_i))$$

Step 1.

$$y(x_1) = y(x_0) + h\, f(x_0, y_0)$$
$$y(0.1) = y(0) + h\, f(0,1)$$
$$= 1 + 0.1 \times 1$$
$$= 1.1$$

$\therefore (x_1, y_1) = (0.1, 1.1)$

Step 2

$y(x_2) = y(x_1) + hf(x_1, y_1)$

$y(0.2) = y(0.1) + 0.1\ f(0.1, 1.1)$

$= 1.1 + 0.1 \times 1.3$

$= 1.1 + 0.13$

$= 1.23$

$\therefore (x_2, y_2) = (0.2, 1.23)$

Step 3

$y(x_3) = y(x_2) + hf(x_2, y_2)$

$y(0.3) = y(0.2) + 0.1\ f(0.2, 1.23)$

$= 1.23 + 0.1 \times (2 \times 0.2 + 1.23)$

$= 1.393$

$\therefore (x_3, y_3) = (0.3, 1.393)$

Step 4

$y(x_4) = y(x_3) + hf(x_3, y_3)$

$y(0.4) = y(0.3) + 0.1 \times (2 \times 0.3 + 1.393)$

$= 1.393 + 0.1\ (2 \times 0.3 + 1.393)$

$= 1.5923$

Thus, $y(0.4) = 1.5923$

Q.11. A plate of dimension 18 cm × 18 cm is subjected to temperatures as follows. Left side at 100°c, right side at 200°c. Upper part at 50°c, and lower at 150°c. If a square grid length of 6cm × 6cm is assumed, what will be the temperature at the interior nodes?

⇒ Given, dimension of plate: 18 cm × 18 cm

$\therefore L = 18, \quad W = 18$

Square grid: 6cm × 6cm

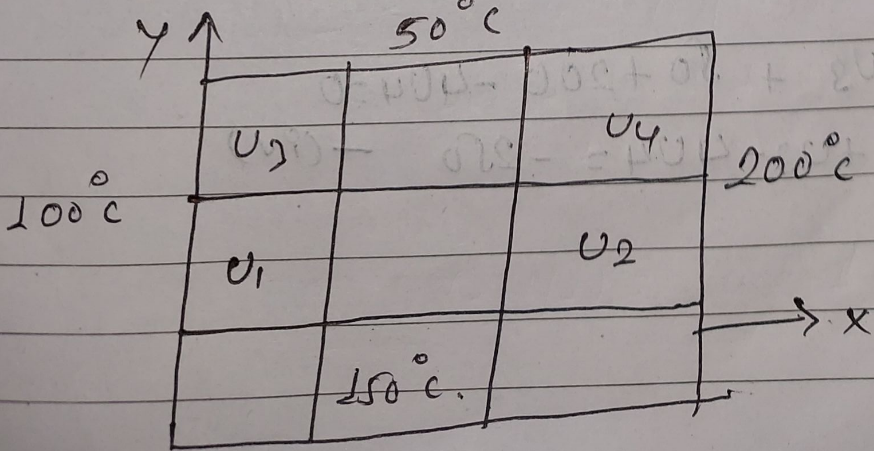
$\therefore h = 6, \; k = 6$

Now, $m = \frac{L}{h}$, $n = \frac{W}{k}$

$= 3 \qquad = 3$

$\therefore (m, n) = (3, 3)$

Let's divide into 3 by 3 grid. Let u_i represent the temperature where $i = 1, 2, 3, 4$



Now, to get the temperature at the interior nodes, we have to write Eqns for u_1, u_2, u_3, u_4

For u_1

$$u_3 + u_2 + 100 + 150 - 4u_1 = 0$$

$$\Rightarrow -4u_1 + u_2 + u_3 = -250 \quad -(i)$$

For u_2

$$u_4 + u_1 + 200 + 150 - 4u_2 = 0$$

$$u_1 - 4u_2 + u_4 = -350 \quad -(ii)$$

For u_3

$$u_1 + u_4 + 100 + 50 - 4u_3 = 0$$

$$u_1 - 4u_3 + u_4 = -150 \quad -(iii)$$

For u_4

$$u_2 + u_3 + 50 + 200 - 4u_4 = 0$$

$$u_2 + u_3 - 4u_4 = -250 \quad -(iv)$$

Eqn (i) to (iv) represents four simultaneous Eqns.

from (i)

$$4U_1 - U_2 - U_3 = 250$$

$$U_1 = \frac{250 + U_2 + U_3}{4}$$

from (ii)

$$U_2 = \frac{350 + U_1 + U_4}{4}$$

from (iii) $$U_3 = \frac{150 + U_1 + U_4}{4}$$

from (iv), $$U_4 = \frac{250 + U_2 + U_3}{4}$$

Using Gauss seidal method.
Initially suppose

$U_1 = 0$

$U_2 = 0$

$U_3 = 0$

$U_4 = 0$

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Iteration	U_1	U_2	U_3	U_4
1	62.5	103.125	53.125	101.562
2	101.562	138.281	88.281	119.140
3	119.140	147.07	97.07	123.535
4	123.535	149.267	99.267	124.633
5	124.633	149.816	99.816	124.908
6	124.908	149.954	99.954	124.977
7	124.977	149.988	99.988	124.994
8	124.994	149.997	99.997	124.998
9	124.998	149.999	99.999	124.999

From this

$U_1 = 124.99 \approx 125$

$U_2 = 149.99 \approx 150$

$U_3 = \cancel{199}99.99 \approx 100$

$U_4 = 124.999 \approx 125$

Hence, the temperature at interior nodes are 125, 150, 100, 125