Selling with Evidence

Frédéric Koessler and Vasiliki Skreta

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Motivating Example

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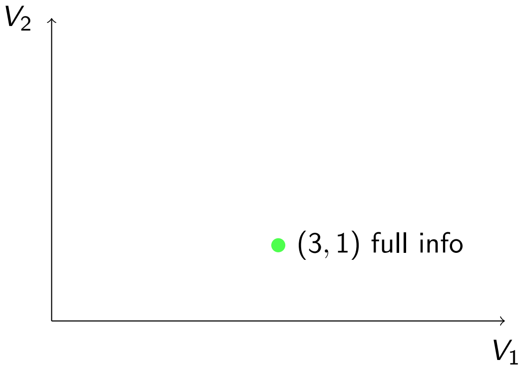

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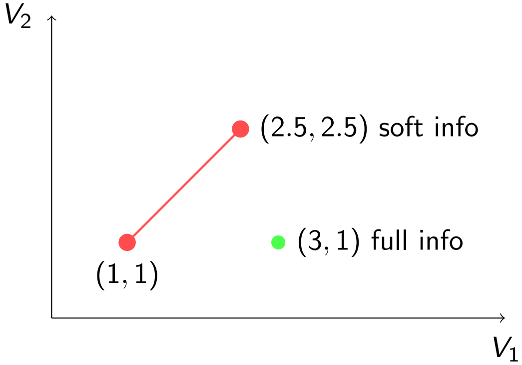

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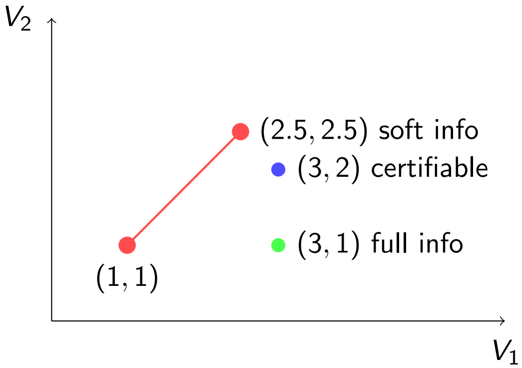

Myerson 1983: Informed Principal

- Inscrutable Principle: all principals choose the same mechanism
- Strong solution: a solution is strong if it is
  - Safe: IC for every type of principal when her type is known to the agent
  - Undominated: IC and not dominated by other IC mechanism.
- Neutral Optima: IC and not blocked by any justifiable blocking rule
- Expectational Equilibrium: for any deviation, there exists a belief and strategy forms a BNE and is worse for all principals
- Core Mechanisms: IC and no other mechanism such that if all that prefer the latter choose it, it is IC for these principals
- Thm: a strong solution, if exists, is neutral optima. Neutral Optima exist and are both expectational equilibria and core mechanisms.

Model Setting

1. Allocation $(p, x) : S \times T \to [0, 1] \times [-\mathcal{X}, \mathcal{X}]$
2. The buyer
    - Type $t \in T$, $\tau \in \Delta(T)$
    - Valuation $u(s, t)$
    - Utility $U(s, t) = p(s, t)u(s, t) - x(s, t)$
    - Interim Utility $U_\pi(s, t) = \sum_{s \in S} \pi(s)U(s, t), \forall \pi \in \Delta(S)$
3. The seller
    - Type $s \in S$, $\pi^0 \in \Delta(S)$
    - Valuation $v(s, t)$
    - Utility $V(s, t) = x(s, t) - p(s, t)v(s, t)$
    - Interim Utility $V(s) = \sum_{t \in T} \tau(t)V(s, t)$

Certification and Mechanisms

- Exogenous Certification Structure: $\mathcal{E} \subset 2^S$, closed under intersection.
  - Certifiable Set of Events for $s \in S$: $\mathcal{E}(s) = \{E \in \mathcal{E} : s \in E\}$
  - Smallest Certifiable Set $E^*(s) = \cap_{E \in \mathcal{E}(s)} E$
  - Own-Type Certifiability: $\{s\} \in \mathcal{E}, \forall s \in S$
- Reporting correspondence:

$$\begin{aligned} R : S &\rightrightarrows S \\ s &\mapsto R(s) = \{\tilde{s} \in S : E^*(\tilde{s}) \in \mathcal{E}(s)\} \end{aligned}$$

  Note: $R(s) = \{s\}$ under own-type certifiability.
- Mechanism:

$$\begin{aligned} \mathcal{M} : \mathcal{E} \times M_S \times M_T &\to [0,1] \times [-\mathcal{X}, \mathcal{X}] \\ (E, m_s, m_t) &\mapsto (p, x) \end{aligned}$$

- Default Allocation $(p, x) = (0, 0)$ if the buyer rejects the mechanism.

Feasible Allocations

Definition (Feasible Allocation)

An allocation (p, x) is *feasible* for belief π if

- $V_\pi(s) \geq 0$
- there exists a mechanism $\mathcal{M}$, reporting and participation strategies that implements (p, x) and forms BNE given $\mathcal{M}$ and π.

Lemma (Revelation Principal)

An allocation (p,x) is feasible for belief π given $\mathcal{E}$ iff

- *S-IC:* $V(s) \geq V(s'|s), \forall s \in S, s' \in R(s)$
- *S-IR:* $V(s) \geq 0, \forall s \in S$
- *B-IC:* $U_\pi(t) \geq U_\pi(t'|t), \forall t, t' \in T$
- *B-IR:* $U_\pi(t) \geq 0, \forall t \in T$

Under own-type certifiability $R(s) = \{s\}$, S-IC always satisfied.

Ex-Ante Optimal and Full Information Allocation

Definition (Ex-ante Optimal)

An allocation (p,x) is *ex-ante optimal* if it solve $\max_{(p,x)} \sum_{s \in S} \pi^0(s) V(s)$ subject to S-IC, S-IR, B-IC, B-IR for interim belief $\pi = \pi^0$.

Definition (Full-Information Allocation)

An allocation (p,x) is a *full-information allocation* if it for every $s \in S$ solves $\max_{(p,x)} V(s)$ subject to the ex post IC and IR:

$$U(s,t) \geq p(s,t')u(s,t) - x(s,t'), \forall t, t' \in T$$

$$U(s,t) \geq 0, \forall t \in T$$

- Full-info allocation might not be feasible.(S-IC)
- Always feasible under own-type certifiability.
- If it is feasible, it is feasible regardless of π.

Mechanism Proposal Game

1. Nature choose s (resp. t) according to π^0 (resp. τ) and privately informs the seller (resp. buyer)
2. Seller certifies $F \in \mathcal{E}(s)$ and proposes $\mathcal{M}$. Buyer observes F and $\mathcal{M}$
3. They both observes a public signal drawn from $\text{Unif}[0,1]$ and play $\mathcal{M}$

Definition (Expectational Equilibrium)

An allocation (p, x) is an *expectational equilibrium* iff

- it is feasible for π^0
- for any other $F \in \mathcal{E}$ and $\tilde{\mathcal{M}}$, there exists a belief $\tilde{\pi} \in \Delta_F(S)$, reporting and participation strategies that form a continuation BNE given $\tilde{\mathcal{M}}$ and $\tilde{\pi}$, inducing a dominated profit vector $\tilde{V}(s) \leq V(s), \forall s \in F$

Strong Pareto Optimal Allocation

Definition (π-buyer-feasible)

An allocation (p, x) is π-buyer-feasible if it satisfy the B-IC and B-IR for interim belief π.
$\mathcal{V}^B(\pi) = \{(V^{(p,x)}(s))_{s \in S} \in \mathbb{R}^{|S|} : (p, x)\text{is } \pi\text{-buyer-feasible}\}$.
$\mathcal{V}^B = \cup_{\pi \in \Delta(S)} \mathcal{V}^B(\pi)$

Definition (SPO Allocation)

$$\mathcal{V}^{SPO} = \{V^* \in \mathcal{V}^B : \mathcal{V}^B(\pi) \cap \{V : V \geqslant_\pi V^*\} = \emptyset, \forall \pi \in \Delta(S)\}$$

where $V \geqslant_\pi V^*$ means $V(s) \geq V^*(s) \forall s$ and strict for some s with $\pi(s) > 0$. $\mathcal{V}^{SPO}(\pi) = \mathcal{V}^{SPO} \cap \mathcal{V}^B(\pi)$

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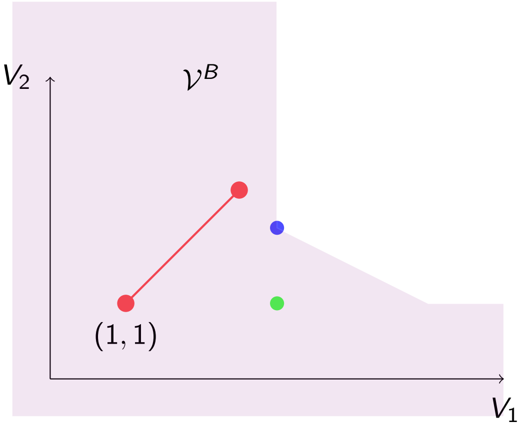

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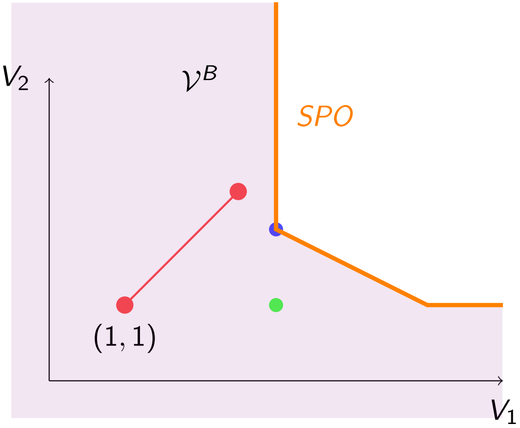

Main Result

Proposition (Existence)

$$\mathcal{V}^{SPO}(\pi) \neq \emptyset, \forall \pi \in \Delta(S)$$

Proposition

If an SPO allocation (p, x) is feasible for the prior π^0, then it is an expectational equilibrium of the mechanism proposal game.

Proposition

If an SPO allocation (p, x) is feasible for the prior π^0, then it is ex-ante optimal.

Proof of Existence

- Step 1: define an exchange economy where a bundle of goods for s consists of a vector of slacks $c = (c(s,t), c(s,t,t'))_{t,t'}$ and the indirect utility of trader s given c is

$$V_I(s|c) = \max_{x(s,\cdot),p(s,\cdot)} \sum_{t \in T} \tau(t)(x(s,t) - p(s,t)v(s,t)$$

subject to the constraint

$$p(s,t)u(s,t) - x(s,t) \geq p(s,t')u(s,t) - x(s,t') - c(s,t,t'), \forall t,t' \in T$$

$$p(s,t)u(s,t) - x(s,t) \geq -c(s,t), \forall t \in T$$

Let $C(s)$ be the non-empty closed and convex set of slacks such that the maximization problem is nonempty.

Proof of Existence

- Step 2: define the demand function $D(s|\gamma)$ given the price $\gamma(t)$ and $\gamma(t, t')$ for the slacks $c(s, t)$ and $c(s, t, t')$

$$D(s|\gamma) := \arg\max_{c \in C(s)} V_I(s|c)$$

subject to the budget constraint

$$\sum_{t \in T} \gamma(t)c(s, t) + \sum_{t, t'} \gamma(t, t')c(s, t, t') \leq 0$$

Definition

A Walrasium equilibrium relative to π is a non-negative price vector γ and slack vectors c such that

- $c \in D(s|\gamma)$
- $\sum_{s \in S} \pi(s)c(s, t) \leq 0, \forall t \in T; \sum_{s \in S} \pi(s)c(s, t, t') \leq 0, \forall t, t' \in T;$

Proof of Existence

- Step 3: show the Walras' Law is satisfied: if $c \in D(s|\gamma)$, then

$$\sum_{t \in T} \gamma(t) c(s,t) + \sum_{t,t'} \gamma(t,t') c(s,t,t') = 0$$

- Step 4: show that a Walrasian equilibrium exists.
- Step 5: show any Walrasian Equilibrium profit vector $V_I(s|c)$ relative to π is an SPO profit vector with belief π.

Proof of Expectational Equilibria

1. Given an SPO profit vector $\hat{V}$, define $\mathcal{V}(\pi)$ the convex hull of eqm profits with off path belief π. And $\mathcal{V} = conv \cup_{\pi \in \Delta_F(S)} \mathcal{V}(\pi)$
2. The correspondence on $\Delta_F(S) \times \mathcal{V}$:

$$(\pi, V) \to \left(\arg \max_{\pi' \in \Delta_F(S)} \sum_{s \in S} \pi'(s)(V(s) - \hat{V}(s) \right) \times \mathcal{V}(\pi)$$

   has a fixed point (π^*, V^*)
3. Suppose $I = \{s \in F : V^*(s) > \hat{V}\} \neq \emptyset$, then $\pi^*(s) = 0, \forall s \notin I$.
4. $\tilde{V}(s) = V^*(s)$ if $s \in I$ and $\tilde{V}(s) = \hat{V}(s)$ if $s \notin I$ is π^* feasible which implies $\hat{V}$ not SPO. Contradiction!

Proof of Ex-Ante Optimality

Let $\hat{V}$ be an SPO allocation and V^* be an ex-ante optimal allocation. SBCT $\hat{V}$ not ex ante optimal.

- $S_2 = \{s \in S : \hat{V}(s) < V^*(s)\} \neq \emptyset$ and $\pi^0(S_2) \neq 0$
- Define allocation $\tilde{p}(s,t) = p^*(s,t)$ and

$$\tilde{x}(s,t) = p^*(s,t)v(s,t) + \underbrace{\hat{V}(s) + \frac{\mathbb{I}_{s\in S_2}}{\pi^0(S_2)}\left[\mathbb{E}_S V^*(s,t) - \mathbb{E}_{S,T}\hat{V}(s,t)\right]}_{\tilde{V}(s,t)}$$

- $\tilde{V}$ dominates $\hat{V}$: $\mathbb{E}_{S,T}V^*(s,t) > \mathbb{E}_{S,T}\hat{V}(s,t) \Rightarrow \tilde{V}(s) > \hat{V}, \forall s \in S_2$
- $(\tilde{p},\tilde{x})$ is π^0-feasible: $\mathbb{E}_S[\tilde{x}(s,t)] =$ $\mathbb{E}_S[p^*(s,t)v(s,t) + \hat{V}(s)] + [\mathbb{E}_S V^*(s,t) - \mathbb{E}_{S,T}\hat{V}(s,t)] = \mathbb{E}_S[x^*(s,t)]$
- Contradiction!

Seller Incentive Compatibility

- SPO is S-IC under own-type certifiability.
- SOP is S-IC for all certifiability structure under private values (buyer utility doesn't depend on seller type.)
- If $V(s'|s) = V(s), \forall s'$ then SPO is S-IC iff $R(s) \subset \{s' \in S : V(s') \leq V(s)\}$ (rich certifiability)
- Let's $\{S_k\}_k^K = S/_{u(\cdot,t)}, \forall t$, and $\forall s \in S_k : s' \notin R(s), \forall s \notin S_k$ (Two-way disprovability)

SPO vs Full Information

If full information allocation is SPO, and the seller can certify his own type, it is the unique equilibrium vector.

Definition

A profit vector V is π-buyer-feasible ex-ante optimal if it maximize $\sum_{s \in S} \pi(s) V(s)$ under the B-IC, B-IR for the interim beliefπ.

Proposition

The full-information profit vector is SPO if and only if it is π-buyer-feasible ex-ante optimal for all π.

- $u(s,t) = u(s,t') = u(s), \forall t, t' \in T$
- $u(s,t) = u(s',t) = u(t), \forall s, s' \in S$

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Are all equilibria SPO?Yes, under own type certifiability.

Thank You!