Term Structure of Asset Prices and Returns

David Backus, Nina Boyarchenko and Mikhail Chernov, JFE 2018

Naz Koont, Dhruv Singal and Xiaobo Yu

Columbia Business School

December 6, 2019

Table of Contents

Motivation

- The largest challenge to representative agent models comes from evidence about the term structure of expected returns.
- In an endowment economy, representative agent models have two key components:
  1. An equilibrium-based SDF which prices all assets of the economy
  2. An exogenously specified CF process for a given asset
- The empirical evidence implicitly suggests which features these two components must posses.

Contributions of this paper

1. Introduces evidence on term structure of average log excess returns for US government bonds, foreign-currency bonds, inflation protected bonds, dividend yields.
2. Identifies features that allow affine asset pricing models to be consistent with this evidence
3. Introduces the idea of co-entropy to compute log excess returns in non-normal environments
4. Shows that the empirical evidence is consistent with and implies certain features within an equilibrium asset pricing model

Definitions

- Cash flow d_t with growth rate $g_{t,t+n} = \dfrac{d_{t+n}}{d_t}$ over n periods.
  - Analyze zero coupon claims to $g_{t,t+n}$ with price $p_{t,n}$
  - Yield: $y_{t,t+n} = -\frac{1}{n} \log p_{t,n}$
  - Hold-to-maturity n-period log return:
    $\log R_{t,t+n} = \log(g_{t,t+n}/p_{t,n}) = \log g_{t,t+n} + n y_{t,t+n}$
- Term spread of average per-period returns:
  $\dfrac{1}{n} E[\log R_{t,t+n}] - E[\log R_{t,t+1}] = E[y_{t,t+n} - y_{t,t+1}]$
- Define per-period excess holding return:
  $\log RX_{t,t+n} = \frac{1}{n}(\log R_{t,t+n} - \log R^f_{t,t+n})$
- Average difference in log RX = difference in average term spreads!
  $E(\log RX_{t,t+n} - \log RX_{t,t+1}) = E(y_{t,t+n} - y_{t,t+1}) - E(y^f_{t,t+n} - y^f_{t,t+1})$

We only need to compute the average excess return for $n = 1$ and then propagate it across horizons using the yields.

Log excess returns

- Change in averages with horizon tracks the difference between term spreads of the USD bond yield curve and the asset's yield curve.
  - Returns are difficult to calculate over long horizons, number of observations decrease mechanically
  - Yields are available every period, do not require CF observations
- Natural relation with entropy of SDF. Log excess returns vs sharpe ratios
- Complements evidence on term structure of risk premia, and connects evidence across the different horizons in a more transparent way
  - *Binsbergen et al. 2012, Belo et al. 2015*

Table 1: One period average excess returns

- Quarterly log excess returns: difference of log gross returns between asset and 3 month bond
- Large cross sectional dispersion in returns: 1.36% per quarter
- Short term excess returns are non-normal

Table 2: Yields and departures from US term structure

- US term structure shape
- Difference in term spread relative to US nominal curve

Figure 1: Term structure of excess returns

- $E(\log RX_{t,t+n} - \log RX_{t,t+1}) = E(y_{t,t+n} - y_{t,t+1}) - E(y^f_{t,t+n} - y^f_{t,t+1})$
- Excess returns decline with horizon except for dividend strips
- Large cross-sectional differences in excess returns, increase with horizon (persistence of asset yields vs. interest rates)

What determines level and shape of the term structure?

Tension between shape and level of US nominal term structure:

- Average level of excess returns (large) gives a lower bound on the largest risk premium in the economy $\iff$ entropy of SDF
- How the yield term spread changes with horizon (small) gives an upper bound on horizon dependence. This bounds entropy

Can apply this logic to other assets:

- Redefine pricing kernel as the product of the USD pricing kernel and the cashflow growth of a given asset.
- Term structure driven by differences between USD curve and asset specific yields $\rightarrow$ infer empirically plausible cashflow dynamics for each asset

Will quantitatively characterize this tension using an affine term structure model.

Matching empirical evidence with an affine model

1. How can we match shape?
   $\longrightarrow$ Let expected cashflow growth depend on two state variables. One which also effects the expected consumption growth, i.e. the "L.R. risk" component, and one which is asset specific.
   $\longrightarrow$ generates cross sectional differences in cashflow persistence, allows cashflow to have different persistence than pricing kernel
2. How can we match level?
   $\longrightarrow$ The pricing kernel and cashflow growth process should have non-normal coincident iid jumps. Resolves tension between level and shape of yield curve, generates realistic risk premia.

Implications for rep agent endowment economy models

Can reverse-engineer rep agent model with recursive preferences to have a similar functional form of the transformed pricing kernel. Highlights how certain features of the data translate into an equilibrium model.

1. Volatility, rather than variance, of consumption growth AR(1)
   - generates upward sloping yield curves
2. Consumption growth features iid jump
   - resolves tension between level and shape in generating realistic risk premia
3. Expected cashflow growth depends on two state variables, one similar to traditional LR risk component, and the other asset specific
   - allows differences in persistence to match shape of term structure

Related Literature

- Interaction between cashflow and kernel at infinite horizon
  - *Hansen 2012, Hansen et al. 2008*
  - In contrast, this paper relies on finite horizon evidence to characterize transition in log excess returns
- Value premium in the cross-section of equities
  - *Lettau and Wachter 2007*
  - These papers confront a different set of facts that do not have explicit horizon dependence, make different modeling choices.
- Impact of jumps
  - *Merton 1976, Backus et al. 2011*
  - The ability of jumps to explain asset risk premia

Table of Contents

Entropy

Definition (Khinchin axioms and Shannon Entropy)

Let X be a random variable with with density p, the Shannon Entropy

$$H(p) = -\mathbb{E}_t\left[\log p\right]$$

is the unique (up to normalization) function from $\Delta(X)$ to $\mathbb{R}$ that satisfies the Khinchin Axoims: continuity, expandability, additivity and maximality for the uniform distribution.

Definition (Relative Entropy (a.k.a. Kullback-Leibler divergence))

Let $p_{t,t+1} = p(x_{t+1}|x_t)$ be the transition state transition density, and $p_{t,t+n} = \Pi_{j=1}^{n} p(x_{t+j}|x_{t+j-1})$. The *relative entropy* of the risk-adjusted distribution is defined as

$$L_t\left(\frac{\tilde{p}_{t,t+n}}{p_{t,t+n}}\right) = \log \mathbb{E}_t\left[\frac{\tilde{p}_{t,t+n}}{p_{t,t+n}}\right] - \mathbb{E}_t\left[\log\left(\frac{\tilde{p}_{t,t+n}}{p_{t,t+n}}\right)\right]$$

while $\tilde{p}_{t,t+n}$ is the same transition density under the risk neutral measure.

Entropy

Definition (Conditional Entropy)

$$L_t(M_{t,t+n}) = \log \mathbb{E}_t\left[M_{t,t+n}\right] - \mathbb{E}_t\left[\log M_{t,t+n}\right]$$

Entropy

Definition (Conditional Entropy)

$L_t(M_{t,t+n}) = \log \mathbb{E}_t\left[M_{t,t+n}\right] - \mathbb{E}_t\left[\log M_{t,t+n}\right]$

Linkage: $M_{t,t+n} = q_{t,t+n}\frac{\tilde{p}_{t,t+n}}{p_{t,t+n}}$ where $q_{t,t+n} := \mathbb{E}_t\left[M_{t,t+n}\right] = R^f_{t,t+1}$ is the price of an $n-$period bond.

Entropy

Definition (Conditional Entropy)

$L_t(M_{t,t+n}) = \log \mathbb{E}_t \left[M_{t,t+n}\right] - \mathbb{E}_t \left[\log M_{t,t+n}\right]$

Linkage: $M_{t,t+n} = q_{t,t+n} \frac{\tilde{p}_{t,t+n}}{p_{t,t+n}}$ where $q_{t,t+n} := \mathbb{E}_t \left[M_{t,t+n}\right] = R^f_{t,t+1}$ is the price of an $n-$period bond.

Proposition (Entropy Bound)

The pricing equation $\mathbb{E}_t \left[M_{t,t+1} R_{t,t+1}\right] = 1$ *implies*

$$\mathbb{E}\left[\log R_{t,t+1} - \log R^f_{t,t+1}\right] \leq \mathbb{E}\left[L_t(M_{t,t+1})\right]$$

Proof.

Apply Jensen's inequality to the pricing equation

$$\mathbb{E}_t \left[\log M_{t,t+1}\right] + \mathbb{E}_t \left[\log R_{t,t+1}\right] \leq 0$$

add $L_t(M_{t,t+1})$ to both sides and take unconditional expectation. $\square$

Cumulant Generating Function

Definition (Cumulant Generating Function)

$$k_t(s; \log(X_{t+1})) = \log \mathbb{E}_t \left[e^{s \log X_{t+1}} \right]$$

Linkage:

$$\begin{aligned} L_t(M_{t,t+1}) &= k_t(1; \log M_{t,t+1}) - \mathbb{E}_t \left[\log M_{t,t+1} \right] \\ &= \frac{\kappa_{2,t}(\log M_{t,t+1})}{2!} + \underbrace{\sum_{j=3}^{\infty} \frac{\kappa_{j,t}(\log M_{t,t+1})}{j!}}_{Nonnormal part} \end{aligned} \tag{1}$$

where $\kappa_{j,t}(X) := \frac{\partial^j}{\partial s^j} k_t(s; X)|_{s=0}$ and $k_t(s; X) = \sum_{j=1}^{\infty} \kappa_{j,t}(X) \frac{s^j}{j!}$.

Coentropy

Definition (Coentropy)

$C(X_1, X_2) = L(X_1 X_2) - L(X_1) - L(X_2)$

Relation to CGFs:

$k(s_1, s_2) = \log \mathbb{E}\left[e^{s_1 \log X_1 + s_2 \log X_2}\right] = \sum_{j=1}^{\infty} \sum_{p=0}^{j} \frac{\kappa^{j-p,p}}{j!} \frac{j!}{p!(j-p)!} s_1^{j-p} s_2^{p}$

where $\kappa^{i,j} = \frac{\partial^{i+j}}{\partial s_1^i \partial s_2^j} k(s_1, s_2)\Big|_{s_1 = s_2 = 0}$ is the joint cumulant.

$$C(X_1, X_2) = \underbrace{\kappa^{1,1}}_{\text{log-normal}} + \underbrace{\sum_{j=3}^{\infty} \sum_{p=1}^{j-1} \frac{\kappa^{j-p,p}}{p!(j-p)!}}_{\text{higher-order joint cumulants}}$$

Intuition: Coentropy focuses on the joint distribution of two r.v. by removing all the terms pertaining to the respective marginal distributions.

Horizon dependence

Proposition (Horizon dependence and bond yield)

Assuming stationarity, $\mathbb{E}\left[\log M_{t,t+1}\right] = \mathbb{E}\left[\log M_{s,s+1}\right], \forall s,$ *we have* $\mathcal{H}_m(n) := \mathcal{L}_m(n) - \mathcal{L}_m(1) = -\mathbb{E}\left[y^f_{t,t+n} - y^f_{t,t+1}\right]$ *where* $\mathcal{L}_m(n) = \frac{1}{n}\mathbb{E}\left[L_t(M_{t,t+n})\right]$ *and* $y^f_{t,t+n} = -\frac{1}{n}\log q_{t,t+n}$ *is the yield*

Intuition: If $M_{t,t+1}$ are iid, then $\mathbb{E}\left[L_t(M_{t,t+n})\right] = n\mathbb{E}\left[L_t(M_{t,t+1})\right]$. So $\mathcal{H}_m(n)$ measures the departure from the iid case which is observed in data.

Proof.

$L_t(M_{t,t+n}) = \log \mathbb{E}_t\left[M_{t,t+n}\right] - \mathbb{E}_t\left[\log M_{t,t+n}\right]$
$L_t(M_{t,t+n}) = \log q_{t,t+n} - \mathbb{E}_t\left[\sum_{j=1}^{n} \log M_{t+j-1,t+j}\right]$
$\mathcal{L}_m(n) = \frac{1}{n}\mathbb{E}\left[\log q_{t,t+n}\right] - \mathbb{E}\left[\log M_{t,t+1}\right]$ by stationarity
$\mathcal{H}_m(n) = -\mathbb{E}\left[y_{t,t+n} - y_{t,t+1}\right]$ by $y^f_{t,t+n} = -\frac{1}{n}\log q_{t,t+n}$ □

Implication: Increasing yield curve ⇒ negative horizon dependence.

Excess Return and Coentropy

Proposition ((log) risk premium and expected (log) excess return)

$$\log \mathbb{E}_t\left[R_{t,t+1}\right] - \log R^f_{t,t+1} = -C_t(M_{t,t+1}, R_{t,t+1})$$
$$\mathbb{E}_t\left[\log R - \log R^f\right] = L_t(M) - L_t(MR) = -L_t(R) - C_t(M, R)$$

Proof.

$$\begin{aligned} C_t(M,R) =& \log \mathbb{E}_t\left[MR\right] - \mathbb{E}_t\left[\log(MR)\right] - \left(\log \mathbb{E}_t\left[M\right] - \mathbb{E}_t\left[\log M\right]\right) \\ & - \left(\log \mathbb{E}_t\left[R\right] - \mathbb{E}_t\left[\log R\right]\right) \\ =& -\log \mathbb{E}_t\left[R\right] - \log \mathbb{E}_t\left[M\right] \\ & + \left(\mathbb{E}_t\left[\log R\right] + \mathbb{E}_t\left[\log M\right] - \mathbb{E}_t\left[\log(MR)\right]\right) \\ =& -\log \mathbb{E}_t\left[R\right] + \log R^f \\ L_t(M) - L_t(MR) =& \log \mathbb{E}_t\left[M\right] + \mathbb{E}_t\left[\log MR\right] - \mathbb{E}_t\left[\log M\right] - \log \mathbb{E}_t\left[MR\right] \end{aligned} \quad (2)$$

□

Example

Example 1: $\log M_{t,t+1} = \log\beta + a_0 w_{t+1} + a_1 w_t$ with w_t i.i.d. standard normal

- $L_t(M_{t,t+1}) = \frac{a_0^2}{2}$
- $\log M_{t,t+n} = ns \log\beta + a_0 w_{t+n} + (a_0 + a_1)\sum_{j=1}^{n-1} w_{t+j} + a_1 w_t$
- $k_t(s; \log M_{t,t+n}) = ns\log\beta + (n-1)s^2\frac{(a_0+a_1)^2}{2} + s^2\frac{a_0^2}{2} + sa_1 w_t$
- $\log q_{t,t+n} = k_t(1; \log M_{t,t+n})$
- $y^f_{t,t+1} = -\log q_{t,t+1} = -\log\beta - a_0^2/2 - a_1 w_t$
- $\mathcal{H}_m(n) = \frac{n-1}{n}\frac{(a_0+a_1)^2 - a_0^2}{2}$

Example

Example 1: $\log M_{t,t+1} = \log\beta + a_0 w_{t+1} + a_1 w_t$ with w_t i.i.d. standard normal

- $L_t(M_{t,t+1}) = \frac{a_0^2}{2}$
- $\log M_{t,t+n} = ns\log\beta + a_0 w_{t+n} + (a_0 + a_1)\sum_{j=1}^{n-1} w_{t+j} + a_1 w_t$
- $k_t(s; \log M_{t,t+n}) = ns\log\beta + (n-1)s^2\frac{(a_0+a_1)^2}{2} + s^2\frac{a_0^2}{2} + sa_1 w_t$
- $\log q_{t,t+n} = k_t(1; \log M_{t,t+n})$
- $y^f_{t,t+1} = -\log q_{t,t+1} = -\log\beta - a_0^2/2 - a_1 w_t$
- $\mathcal{H}_m(n) = \frac{n-1}{n}\frac{(a_0+a_1)^2 - a_0^2}{2}$

Tension: volatility of $y^f_{t,t+1}$ pins down $a_1 \Rightarrow$ level of $y^f_{t,t+1}$ determines $a_0 \Rightarrow$ need to match $\mathcal{H}_m(n)$ simultaneously.

Example: Coentropy

Example 2: Let $\log(X)|J = j \sim \mathcal{N}(j\mu, j\Delta)$ where $J \sim Poi(\omega)$ and $\Delta = (\delta_{ij})_{ij} \in \mathbb{R}^{2\times 2}$ is a 2 dimensional symmetric positive definite matrix.

- $k(s) = \omega(e^{s^\top \mu + \frac{1}{2}s^\top \Delta s} - 1)$
- $L(X_i) = \omega(e^{\mu_i + \frac{1}{2}\delta_{ii}} - 1) - \omega\mu_i$
- $L(X_1 X_2) = \omega(e^{\mu_1+\mu_2+\frac{1}{2}(\delta_{11}+\delta_{11}+2\delta_{12})} - 1) - \omega(\mu_1 + \mu_2)$
- $C(X_1, X_2) = \omega(e^{\mu_1+\mu_2+\frac{1}{2}(\delta_{11}+\delta_{11}+2\delta_{12})} - e^{\mu_1+\frac{1}{2}\delta_{11}} - e^{\mu_2+\frac{1}{2}\delta_{22}} + 1)$
- $Cov(\log X_1, \log X_2) = \sum_{j=0}^{\infty}(\mu_1\mu_2 + \delta_{12})j\frac{\omega^j e^{-\omega}}{j!} = (\mu_1\mu_2 + \delta_{12})\mathbb{E}[J]$

Example: Coentropy

Figure: Coentropy and covariance: $\mu_1 = \mu_2 = -0.5, \Delta = \frac{1}{\omega}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The $\frac{1}{\omega}$ term is to ensure the variance doesn't vary with ω.

Term Structure of assets

Let $\hat{M}_{t,t+1} = M_{t,t+1} g_{t,t+1}$ be a transformed pricing kernel, we have

$$q_{t,t+n} = \mathbb{E}_t \left[M_{t,t+1} g_{t,t+1} q_{t+1,t+n} \right] = \mathbb{E}_t \left[\hat{M}_{t,t+1} q_{t+1,t+n} \right]$$

Definition (Excess Return)

$$\log RX_{t,t+n} = \frac{1}{n} (\log R_{t,t+n} - \log R^f_{t,t+n})$$

Proposition

$$\mathbb{E}_t \left[\log RX_{t,t+n} - \log RX_{t,t+1} \right] = \mathcal{H}_m(n) - \mathcal{H}_{\hat{m}}(n)$$

Proof.

$$\mathbb{E}_t \left[\log RX_{t,t+n} - \log RX_{t,t+1} \right] =$$
$$\mathbb{E}_t \left[y_{t,t+n} - y^f_{t,t+n} \right] - \mathbb{E}_t \left[y_{t,t+1} - y^f_{t,t+1} \right] =$$
$$\mathbb{E}_t \left[y_{t,t+n} - y_{t,t+1} \right] - \mathbb{E}_t \left[y^f_{t,t+n} - y^f_{t,t+1} \right] = -\mathcal{H}_{\hat{m}}(n) + \mathcal{H}_m(n) \qquad \square$$

Simplified KLV Model

Setup

$$\begin{cases} \log M_{t,t+1} = \log\beta + \theta_m^\top x_t - \lambda_t^2/2 + \lambda_t w_{t+1} \\ \log g_{t,t+1} = \log\gamma + \theta_g^\top x_t + \eta_0 w_{t+1} \\ x_{t+1} = \Phi x_t + w_{t+1} e \end{cases} \tag{3}$$

where $x_t = (x_{1,t}, x_{2,t})^\top$, $\lambda_t = \lambda_0 + \lambda_1 x_{1,t}$, $\theta_m = (\theta_{m1}, 0)^\top$, $\Phi = diag(\phi_1, \phi_2)$ and $\{w_t\}$ is i.i.d. normal.
Conditional Entropy $L_t(M_{t,t+1}) = (\lambda_0 + \lambda_1 x_{1,t})^2/2$
Entropy $\mathcal{L}_m(1) = \frac{1}{2}\left(\lambda_0^2 + \frac{\lambda_1^2}{1-\phi^2}\right)$

Simplified KLV Model: Bond Price

Bond Price

1. Suppose $q_{t,t+n} = A_n + B_n x_t$
2. $q_{t,t+n} = \mathbb{E}_t \left[e^{\log\beta + \theta_m^\top x_t - \lambda_t^2/2 + \lambda_t w_{t+1}} e^{A_{n-1} + B_{n-1}(\Phi x_t + w_{t+1} e)} \right]$
3. Match the coefficient of x_t

$$\begin{cases} B_n = \theta_m^\top + B_{n-1}\Phi + B_{n-1}[\lambda_1 e, 0] \\ A_n = \log\beta + \lambda_0 B_{n-1}^\top e + \frac{1}{2}(B_{n-1}^\top e)^2 \end{cases}$$

4. $q_{t,t+1} = \mathbb{E}_t \left[e^{\log\beta + \theta_m^\top x_t - \lambda_t^2/2 + \lambda_t w_{t+1}} \right] = e^{\log\beta + \theta_m^\top x_t}$
5.

$$\begin{cases} B_n = \theta_m^\top (I - \Phi^*)^{-1}(I - \Phi^{*n}); \Phi^* = \Phi + [\lambda_1 e, 0] \\ A_n = n\log\beta + \lambda_0 \sum_{j=0}^{n-1} B_j^\top e + \frac{1}{2}\sum_{j=0}^{n-1}(B_j^\top e)^2 \end{cases}$$

Simplified KLV Model: Bond Price

Bond Price

1. Suppose $q_{t,t+n} = A_n + B_n x_t$
2. $q_{t,t+n} = \mathbb{E}_t \left[e^{\log\beta + \theta_m^\top x_t - \lambda_t^2/2 + \lambda_t w_{t+1}} e^{A_{n-1} + B_{n-1}(\Phi x_t + w_{t+1} e)} \right]$
3. Match the coefficient of x_t

$$\begin{cases} B_n = \theta_m^\top + B_{n-1}\Phi + B_{n-1}[\lambda_1 e, 0] \\ A_n = \log\beta + \lambda_0 B_{n-1}^\top e + \frac{1}{2}(B_{n-1}^\top e)^2 \end{cases}$$

4. $q_{t,t+1} = \mathbb{E}_t \left[e^{\log\beta + \theta_m^\top x_t - \lambda_t^2/2 + \lambda_t w_{t+1}} \right] = e^{\log\beta + \theta_m^\top x_t}$
5. $$\begin{cases} B_n = \theta_m^\top (I - \Phi^*)^{-1}(I - \Phi^{*n}); \Phi^* = \Phi + [\lambda_1 e, 0] \\ A_n = n\log\beta + \lambda_0 \sum_{j=0}^{n-1} B_j^\top e + \frac{1}{2}\sum_{j=0}^{n-1}(B_j^\top e)^2 \end{cases}$$

Horizon Dependence $\mathcal{H}_m(n) = \frac{1}{n}\left[\lambda_0 \sum_{j=0}^{n-1} B_j^\top e + \frac{1}{2}\sum_{j=0}^{n-1}(B_j^\top e)^2\right]$

Simplified KLV Model: Transformed Pricing Kernel

Transformed Pricing Kernel and its Horizon Dependence

- $\log \hat{M}_{t,t+1} = \log \hat{\beta} + \hat{\theta}_m^\top x_t - \hat{\lambda}_t^2/2 + \hat{\lambda}_t w_{t+1}$ where
  - $\log \hat{\beta} = \log \beta + \log \gamma + \lambda_0 \eta_0 + \frac{\eta_0^2}{2}$
  - $\hat{\theta}_m^\top = (\theta_{m1} + \theta_{g1} + \eta_0 \lambda_1, \theta_{g2})$
  - $\hat{\lambda}_0 = \lambda_0 + \eta_0, \hat{\lambda}_t = \hat{\lambda}_0 + \lambda_1 x_{1,t}$
- $\mathcal{H}_{\hat{m}}(n) = \frac{1}{n}\left[\hat{\lambda}_0 \sum_{j=0}^{n-1} \hat{B}_j^\top e + \frac{1}{2}\sum_{j=0}^{n-1}(\hat{B}_j^\top e)^2\right]$
  - $\hat{B}_n = \hat{\theta}_m^\top (I - \Phi^*)^{-1}(I - \Phi^{*n})$

Table of Contents

Calibration of KLV

US nominal bonds

- Short rate, $y^f_{t,t+1} = -\log q_{t,t+1} = -\log \beta - \theta_{m1} x_{1t} \rightarrow$ match empirical **short rate std dev and autocorrelation** $\rightarrow \theta_{m1}, \phi_1$
- Match empirical **average of term spread** $y^f_{t,t+40} - y^f_{t,t+1}$ using duality of horizon dependence and average term spread $\rightarrow \lambda_0$
- Match empirical **level** of yield curve $\rightarrow \beta$
- Match maturity dependence of term spread with **curvature of yield curve** $\rightarrow \lambda_1$

Calibration of KLV

FX bonds and equity dividend strips

- Assume $\theta_{g1} = -\theta_{m1} - \lambda_1 \eta_0$
- One-period yield $y_{t,t+1} = -\log \widehat{\beta} - \theta_{g2} x_{2t} \rightarrow$ match empirical **std dev and autocorrelation** $\rightarrow \theta_{g2}, \phi_2$
- Match empirical **average of term spread** $\rightarrow \eta_0$
- Match empirical **level** of one-period yield $\rightarrow \gamma$

KLV results

- Contrary to literature, FX rates are not random walks
- The domestic and foreign pricing kernels are not symmetric

In KLV lognormal setting,

$$\begin{aligned}\mathbb{E}_t\left[\log RX_{t,t+1}\right] &= -\frac{1}{2}\mathrm{var}_t\left(\log RX_{t,t+1}\right) - \mathrm{cov}_t\left(\log M_{t,t+1}, \log RX_{t,t+1}\right)\\ &= -\frac{\eta_0^2}{2} - \left(\lambda_0 + \lambda_1 x_{1t}\right)\eta_0\end{aligned}$$

The unconditional average excess returns $-\frac{\eta_0^2}{2} - \lambda_0\eta_0$ **do not match** the data after calibration for the other moments!

KLV results

KLV Extension I: iid normal shocks

Idea: iid terms affecting both SDF and cashflow growth do not affect term spreads.

$$\begin{cases} \log M_{t,t+1} = \log\beta + \theta_m^\top x_t - \lambda_t^2/2 + \lambda_t w_{t+1} + \lambda_2 \varepsilon_{t+1} \\ \log g_{t,t+1} = \log\gamma + \theta_g^\top x_t + \eta_0 w_{t+1} + \eta_2 \varepsilon_{t+1} \\ x_{t+1} = \Phi x_t + w_{t+1} e \end{cases} \tag{4}$$

This yields,

$$\operatorname{var}\log RX_{t,t+1} = \eta_0^2\left[1 + \lambda_1^2\left(1 - \phi_1^2\right)^{-1}\right] + \eta_2^2 \;\rightarrow \eta_2$$

$$\mathbb{E}\left[\log RX_{t,t+1}\right] = -\frac{\eta_0^2}{2} - \frac{\eta_2^2}{2} - \lambda_0\eta_0 - \lambda_2\eta_2 \;\rightarrow \lambda_2$$

λ_2 (unique loading on the SDF) varies in the cross-section – failure

KLV Extension II: iid Poisson shocks

$$
\begin{cases}
\log M_{t,t+1} = \log \beta + \theta_m^\top x_t - \lambda_t^2/2 + \lambda_t w_{t+1} + \lambda_2 z_{t+1}^m \\
\log g_{t,t+1} = \log \gamma + \theta_g^\top x_t + \eta_0 w_{t+1} + \eta_2 z_{t+1}^g \\
x_{t+1} = \Phi x_t + w_{t+1} e
\end{cases}
\tag{5}
$$

Here, z^g and z^m are both compound Poisson processes (with coincident jumps) with arrival rate ω and jump size distribution $\mathcal{N}(\mu_m, \delta_m^2)$ and $\mathcal{N}(\mu_d, \delta_g^2)$, respectively.

KLV Extension II: iid Poisson shocks

Define $\mathcal{C}_{mg}(n) = n^{-1}\mathbb{E}\left[C_t\left(M_{t,t+n}, g_{t,t+n}\right)\right]$.
Hence,

$$\mathcal{C}_{mg}(n) - \mathcal{C}_{mg}(1) = \mathcal{H}_{\widehat{m}}(n) - \mathcal{H}_m(n) - \mathcal{H}_g(n)$$

Horizon dependence $\mathcal{H}$ for any of $\{\widehat{m}, m, g\}$ is not affected by any iid shocks! This gives,

KLV Extension II: iid Poisson shocks

Define $\mathcal{C}_{mg}(n) = n^{-1}\mathbb{E}\left[C_t\left(M_{t,t+n}, g_{t,t+n}\right)\right]$.
Hence,

$$\mathcal{C}_{mg}(n) - \mathcal{C}_{mg}(1) = \mathcal{H}_{\widehat{m}}(n) - \mathcal{H}_m(n) - \mathcal{H}_g(n)$$

Horizon dependence $\mathcal{H}$ for any of $\{\widehat{m}, m, g\}$ is not affected by any iid shocks! This gives,

$$\mathcal{C}_{mg}(1) = \lambda_0\eta_0 + k_z(\lambda_2, \eta_2) - k_z(\lambda_2, 0) - k_z(0, \eta_2)$$

$$\text{where, } k_z(s_1, s_2) = \omega\left(e^{s_1\mu_m + s_2\mu_g + \frac{1}{2}\left(s_1^2\delta_m^2 + s_2^2\delta_g^2\right)} - 1\right)$$

KLV Extension II: iid Poisson shocks

Define $\mathcal{C}_{mg}(n) = n^{-1}\mathbb{E}\left[C_t\left(M_{t,t+n}, g_{t,t+n}\right)\right]$.
Hence,

$$\mathcal{C}_{mg}(n) - \mathcal{C}_{mg}(1) = \mathcal{H}_{\widehat{m}}(n) - \mathcal{H}_m(n) - \mathcal{H}_g(n)$$

Horizon dependence $\mathcal{H}$ for any of $\{\widehat{m}, m, g\}$ is not affected by any iid shocks! This gives,

$$\mathcal{C}_{mg}(1) = \lambda_0\eta_0 + k_z(\lambda_2, \eta_2) - k_z(\lambda_2, 0) - k_z(0, \eta_2)$$
$$\text{where, } k_z(s_1, s_2) = \omega\left(e^{s_1\mu_m + s_2\mu_g + \frac{1}{2}\left(s_1^2\delta_m^2 + s_2^2\delta_g^2\right)} - 1\right)$$

But the entropy differs now:

$$\mathcal{L}_m(1) = \frac{\lambda_0^2}{2} + \frac{\lambda_1^2}{2}\left(1 - \phi_1^2\right)^{-1} - \omega\lambda_2\mu_m + \omega\left(e^{\lambda_2\mu_m + \frac{1}{2}\lambda_2^2\delta_m^2}\right)$$

We also get $\mathcal{C}_{mg}(n)$ and $\mathcal{L}_m(n)$.

Calibration of KLV Extn II

- Normalize $\lambda_2 = \eta_2 = 1$.
- Use CI2 model from **?** to get *reasonable* ω, μ_m, δ_m
- $$\mathbb{E}\left[\log RX_{t,t+1}\right] = -\frac{\eta_0^2}{2} - \lambda_0\eta_0 - k_z(\lambda_2, \eta_2) + k_z(\lambda_2, 0) + \omega\eta_2\mu_g \ \rightarrow \mu_g$$
$$\operatorname{var}\log RX_{t,t+1} = \eta_0^2\left(1 + \lambda_1^2\left(1 - \phi_1^2\right)^{-1}\right) + \eta_2^2\omega\left(\mu_g + \delta_g^2\right) \ \rightarrow \delta_g$$

KLV Extn II: iid Poisson shocks

Results:

- Covariance and coentropy vary significantly
- Variance and autocorrelation of simulated cash-flows (*not matched in calibration*) line up with the data
- Equity premium (infinite portfolio of strips) is matched as well
- Further analysis: non-normality in SDF largely drives the risk-premium BUT needs the non-normality in cash flow (even if it's small) to get any traction

33333

A Representative Agent Model

References