AS5545 - Course Project

Inertia-Free Spacecraft Attitude Control Using Reaction Wheels

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Chapter 1

Introduction and Rotational Dynamics of the Spacecraft

1.1 Introduction

This paper builds upon a previously developed continuous **inertia-free** control law for spacecraft attitude tracking, extending it to accommodate **three axisymmetric reaction wheels**. The term "inertia-free" refers to the scenario where the controller is unaware of the inertia of the spacecraft. The wheels are assumed to be installed in a known, linearly independent configuration that may not be orthogonal, with an arbitrary and unknown orientation relative to the spacecraft's unidentified principal axes. So the controller has information of the orientation of the spacecraft and the inertia of wheels. Nothing is known regarding the inertia of the spacecraft and the disturbance torques. Simulation results are provided for slew maneuvers, accounting for torque and momentum saturation.

Many of the mathematical aspects of the formulation are discussed in more detail in this document compared to the paper, especially the error analogs and lyapunov stability proofs.

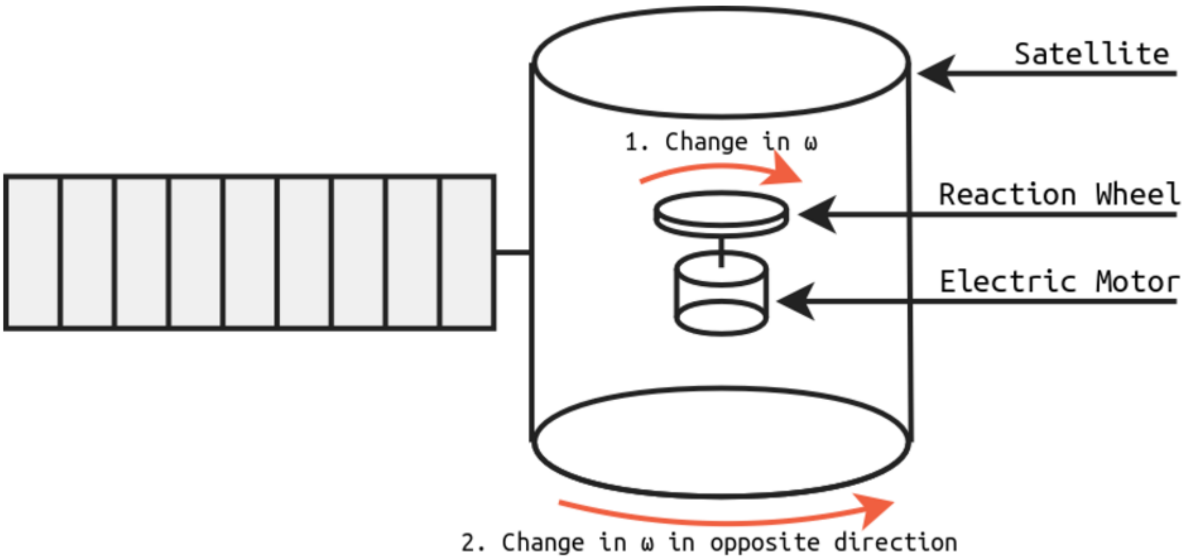


Figure 1.1: General working principle of a reaction wheel

1.2 Rotational Dynamics of the Spacecraft

In this section, we will look to derive the equations governing the rotational dynamics of a spacecraft equipped with 3 reaction wheels. We will start from the very basics and slowly build up. The following assumptions are enforced in this discussion:

- The spacecraft is a rigid body.
- The body frame is selected such that it aligns with the principal axes. (i.e. the inertia matrix is diagonal).
- The inertia matrix is a constant. (i.e. the relative mass distribution remains constant for the spacecraft)
- The 3 reaction wheels are aligned along the body/principal axes.

Let the angular momentum of a Body $\mathcal{B}$ with respect to a point p be denoted by $\mathcal{H}_{\mathcal{B}}^p$.

$$\boxed{\mathcal{H}_{\mathcal{B}}^p = \int_{\mathcal{B}} \vec{r}_{dm_p} \times \vec{v}_{dm_p} dm} \tag{1.1}$$

where $\vec{r}_{dm_p}$ denotes the position vector of dm with respect to point p and $\vec{v}_{dm_p}$ denotes the velocity of dm with respect to point p. If point p is attached to the body $\mathcal{B}$, then the velocity term can be written as $\vec{r}_{dm_p} \times \vec{\omega}$, where ω denotes the angular velocity of the body expressed in the body axis bases. Upon rearranging, we can write:

$$\boxed{\mathcal{H}_{\mathcal{B}}^p = \mathcal{I}_{\mathcal{B}}^p \vec{\omega}} \tag{1.2}$$

where

$$\boxed{\mathcal{I}_{\mathcal{B}}^p = \int_{\mathcal{B}} (\|\vec{r}_{dm_p}\|^2 I_3 - r_{dm_p} r_{dm_p}^T) dm} \tag{1.3}$$

The body under consideration here includes the entire spacecraft (denoted by sc, i.e. the 3 reaction wheels plus the platform. The 'platform' here refers to the spacecraft part, excluding the reaction wheels. We denote the wheels as w_1, w_2, w_3 and the platform as b

$$sc = \sum_{i=1}^{3} w_i + b$$

The point p is taken as the center of mass (c) of the spacecraft.

$$\boxed{\begin{aligned} \mathcal{H}_{sc}^c &= \mathcal{H}_b^c + \sum_{i=1}^{3} \mathcal{H}_{w_i}^c \\ \mathcal{H}_{sc}^c &= \mathcal{I}_b^c \omega + \sum_{i=1}^{3} (\mathcal{I}_{w_i}^c \omega + \alpha_i \omega_{w_i} \vec{a}_i) \end{aligned}} \tag{1.4}$$

where α_i (scalar) is the moment of inertia of the i^{th} wheel for a rotation about its axis of rotation, given by the direction vector $\vec{a}_i$. ω_{w_i} (scalar) is the rotation rate of the i^{th} wheel about it's axis of rotation. If we are representing all the expressions above in the basis of the body axes, $a_1 = [1, 0, 0]^T, a_2 = [0, 1, 0]^T, a_3 = [0, 0, 1]^T$ (due to the assumption that the 3 wheel axes are aligned to the body axes). After rearranging the terms, we get

$$\boxed{\mathcal{H}_{sc}^{c} = \underbrace{(\mathcal{I}_{b}^{c}\omega + \sum_{i=1}^{3} \mathcal{I}_{w_i}^{c}\omega)}_{J_{sc}^{c}\omega} + \underbrace{\sum_{i=1}^{3} \alpha_i \omega_{w_i} \vec{a}_i}_{J_{\alpha}\nu}} \tag{1.5}$$

$$\boxed{\mathcal{H}_{sc}^{c} = J_{sc}^{c}\omega + J_{\alpha}\nu} \tag{1.6}$$

where $\nu = [\omega_{w_1}, \omega_{w_2}, \omega_{w_3}]^T$. Now, let's apply the transport theorem to Eq (1.6).

$$\boxed{\dot{\mathcal{H}}_{sc}^{c} = \tau_{ext} = J_{sc}^{c}\dot{\omega} + J_{\alpha}\dot{\nu} + \omega \times (J_{sc}^{c}\omega + J_{\alpha}\nu)} \tag{1.7}$$

Upon rearranging the terms:

$$\boxed{J_{sc}^{c}\dot{\omega} = (J_{sc}^{c}\omega + J_{\alpha}\nu) \times \omega - J_{\alpha}u + \tau_{dist}} \tag{1.8}$$

where $u = \dot{\nu}$ is the control input to the system. Eq (1.8) is the governing equation of the rotational dynamics of a spacecraft with 3 body-axis-oriented reaction wheels.

We also define the following quantities that will aid us in the controller formulation and Lyapunov stability proofs later on.

$$\boxed{\gamma = [J_{11}, J_{22}, J_{33}, J_{23}, J_{13}, J_{12}]^T} \tag{1.9}$$

$$\boxed{L(\omega) = \begin{bmatrix} \omega_1 & 0 & 0 & 0 & \omega_3 & \omega_2 \\ 0 & \omega_2 & 0 & \omega_3 & 0 & \omega_1 \\ 0 & 0 & \omega_3 & \omega_2 & \omega_1 & 0 \end{bmatrix}} \tag{1.10}$$

As a consequence:

$$\boxed{J\omega = L(\omega)\gamma} \tag{1.11}$$

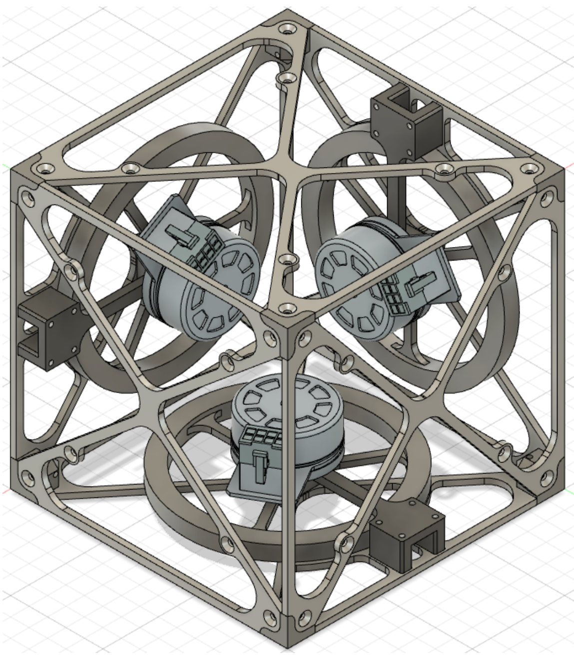


Figure 1.2: A Typical 3 Axis Reaction Wheel System on a Spacecraft

Chapter 2

Control Objectives

Now that we have a working understanding of the governing equations, let us define our control objectives. To do so, let us first define some error analogs. Many of the terms discussed here might feel discontinous at this stage, but will start to make coherent sense in the later parts of the document.

2.1 Errors

Let R be the rotational matrix of the space craft orientation from a reference orientation. R varies with time as $\dot{R} = R\omega^*$ where ω^* is the cross product matrix. The desired states can be computed as follows:

$$\boxed{\begin{aligned} &\dot{R}_d(t) = R_d(t)\omega_d^*(t) \\ &\omega_d(0) = \omega_{d0} \end{aligned}} \tag{2.1}$$

where ω_d is the desired angular velocity (need not be a constant). The attitude error rotation matrix gives the error between R and R_d ($\tilde{R}$).

$$\boxed{\tilde{R} = R_d^T R} \tag{2.2}$$

Let's find out how this attitude error rotation matrix varies with time:

$$\begin{aligned}
\dot{\tilde{R}} &= \dot{R}_d^T R + R_d^T \dot{R} \\
\dot{\tilde{R}} &= (R_d\omega_d^*)^T R + R_d^T R\omega^* \\
\dot{\tilde{R}} &= -\omega_d^* \underbrace{R_d^T R}_{\tilde{R}} + \underbrace{R_d^T R}_{\tilde{R}} \omega^* \\
\dot{\tilde{R}} &= -\omega_d^* \tilde{R} + \tilde{R}\omega^* \text{ as } \tilde{R} \text{ is an orthogonal matrix} \\
\tilde{R}^T \dot{\tilde{R}} &= \omega^* - \underbrace{\tilde{R}^T \omega_d^* \tilde{R}}_{(\tilde{R}^T\omega_d)^*} \\
\tilde{R}^T \dot{\tilde{R}} &= \omega^* - (\tilde{R}^T\omega_d)^* \\
\tilde{R}\tilde{R}^T \dot{\tilde{R}} &= \tilde{R}(\omega - \tilde{R}^T\omega_d)^* \text{ as } \tilde{R} \text{ is an orthogonal matrix}
\end{aligned}$$

$$\boxed{\dot{\tilde{R}} = \tilde{R}\tilde{\omega}^*} \text{ where } \tilde{\omega} = \omega - \tilde{R}^T\omega_d. \tag{2.3}$$

From Eq. (2.3), plugging $\omega = \tilde{\omega} + \tilde{R}^T\omega_d$ into Eq. (1.8)

$$\begin{aligned}
J^c_{sc}(\dot{\tilde{\omega}} + \dot{\tilde{R}}^T\omega_d + \tilde{R}^T\dot{\omega}_d) &= (J^c_{sc}(\tilde{\omega} + \tilde{R}^T\omega_d) + J_\alpha\nu) \times (\tilde{\omega} + \tilde{R}^T\omega_d) - J_\alpha u + \tau_{dist} \\
J^c_{sc}\dot{\tilde{\omega}} &= (J^c_{sc}(\tilde{\omega} + \tilde{R}^T\omega_d) + J_\alpha\nu) \times (\tilde{\omega} + \tilde{R}^T\omega_d) - J^c_{sc}(\underbrace{\dot{\tilde{R}}^T}\omega_d + \tilde{R}^T\dot{\omega}_d) - J_\alpha u + \tau_{dist} \\
J^c_{sc}\dot{\tilde{\omega}} &= (J^c_{sc}(\tilde{\omega} + \tilde{R}^T\omega_d) + J_\alpha\nu) \times (\tilde{\omega} + \tilde{R}^T\omega_d) - J^c_{sc}((\tilde{R}\tilde{\omega}^*)^T\omega_d + \tilde{R}^T\dot{\omega}_d) - J_\alpha u + \tau_{dist} \\
J^c_{sc}\dot{\tilde{\omega}} &= (J^c_{sc}(\tilde{\omega} + \tilde{R}^T\omega_d) + J_\alpha\nu) \times (\tilde{\omega} + \tilde{R}^T\omega_d) - J^c_{sc}(-\tilde{\omega}^*\tilde{R}^T\omega_d + \tilde{R}^T\dot{\omega}_d) - J_\alpha u + \tau_{dist}
\end{aligned}$$

$$\boxed{J^c_{sc}\dot{\tilde{\omega}} = (J^c_{sc}(\tilde{\omega} + \tilde{R}^T\omega_d) + J_\alpha\nu) \times (\tilde{\omega} + \tilde{R}^T\omega_d) + J^c_{sc}(\tilde{\omega} \times \tilde{R}^T\omega_d - \tilde{R}^T\dot{\omega}_d) - J_\alpha u + \tau_{dist}} \tag{2.4}$$

Eq (2.4) represents the dynamics of $\tilde{\omega}(t)$ for some $\omega_d(t)$

We also define another error measurement that goes by the name: Eigneaxis Attitude Error. This is the rotation angle $\theta(t)$ about the eigenaxis needed to rotate the spacecraft from its attitude $R(t)$ to the desired attitude $R_d(t)$.

$$\boxed{\theta(t) = \cos^{-1}\left(\frac{1}{2}[\text{trace}(\tilde{R}(t)) - 1]\right)} \tag{2.5}$$

2.2 Control Objectives:

Now, we are in a position to define our control objectives in terms of the quantities defined above. We wish to formulate a function for the control input u that does the following to the system:

- Drive $\omega \rightarrow \omega_d$.
- Drive $R \rightarrow R_d$.
- Drive $\tilde{R} \rightarrow$ the 3 by 3 Identity matrix.
- Drive $\theta \rightarrow 0$

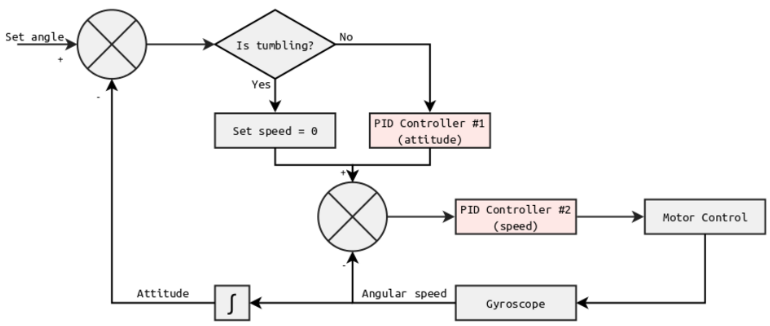


Figure 2.1: Generic PID based control law for Reaction Wheel based Detumbling

The paper **does not** look at the above typical formulation of a controller. It presents 2 sliding mode inspired controller formulations with Lyapunov guarantees of asymptotic performance.

Chapter 3

Controller Design

In this chapter, we shall state the 2 controllers proposed in the paper.

3.1 Controller 1:

This is a relatively simple controller that does not take into consideration any disturbance torques. For some positive-definite matrix $A = \text{diag}(a_1, a_2, a_3)$ and $\tilde{R}$, we define the S vector:

$$\boxed{S = \sum_{i=1}^{3} a_i(\tilde{R}^T e_i) \times e_i} \tag{3.1}$$

where e_i denotes the i^{th} column of the 3 by 3 Identity Matrix.

Control Law (assumed $\omega_d = 0$):

$$\boxed{u = J_\alpha^{-1}(K_p S + K_v \omega)} \tag{3.2}$$

where K_p is a positive scalar and $K_v \in \mathcal{R}^{3\times 3}$ is a positive definite matrix with distinct diagonal entries. For a more general treatment, one can replace ω with $\tilde{\omega}$ in the above expression. Eq. (3.2) represents the equation of the first controller. Note that it does not depend of the inertia matrix of the spacecraft.

3.2 Controller 2:

Recall that one of the main features of the formulations discussed in the paper is that the controller is **inertia-free**, i.e. it is unaware of the inertia matrix of the platform.

- This is a more involved controller compared to the one mentioned above.
- Here on top of trying to control the orientation and rotation rate of the spacecraft, the controller also tries to estimate the inertia matrix (J or γ) of the spacecraft and the disturbance torques acting on the spacecraft.
- It is assumed that each component of τ_{dist} is a linear combination of constant and harmonic signals, for which the frequencies are known but for which the amplitudes and phases are unknown. This implies that $\boxed{\dot{\tau}_{dist} = A_d \tau_{dist}}$. A_d is taken to be skew symmetric.

Since, we are looking at estimating γ and τ_{dist}, we must once again define an error term here.

$$\boxed{\tilde{\gamma} = \gamma - \hat{\gamma}} \tag{3.3}$$

$$\boxed{\tilde{\tau}_{dist} = \tau_{dist} - \hat{\tau}_{dist}} \tag{3.4}$$

The terms with the "hat" symbol on top are the estimated values, and the terms with the "tilda" symbol are corresponding error values (consistent with other sections of the document).

The following estimators are proposed:

$$\boxed{\dot{\hat{\gamma}} = Q^{-1}[L^T(\omega)\omega^* + L^T(K_1\dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T\dot{\omega}_d)](\tilde{\omega} + K_1 S)} \tag{3.5}$$

$$\boxed{\dot{\hat{\tau}}_{dist} = A_d\hat{\tau}_{dist} + D^{-1}(\tilde{\omega} + K_1 S)} \tag{3.6}$$

where $K_1 \in \mathcal{R}^{3\times3}, Q \in \mathcal{R}^{6\times6}, D \in \mathcal{R}^{3\times3}$ are all positive definite matrices and S is as defined in Eq. (3.1).

We can find $\dot{S}$ from Eq. (3.1) as follows:

$$\dot{S} = \sum_{i=1}^{3} a_i(\dot{\tilde{R}}^T e_i) \times e_i$$
$$\dot{S} = \sum_{i=1}^{3} a_i((\tilde{R}\tilde{\omega}^*)^T e_i) \times e_i$$
$$\dot{S} = \sum_{i=1}^{3} a_i(-\tilde{\omega}^*\tilde{R}^T e_i) \times e_i$$

$$\boxed{\dot{S} = \sum_{i=1}^{3} a_i((\tilde{R}^T e_i) \times \tilde{\omega}) \times e_i} \tag{3.7}$$

With these quantities at play, the proposed control law is as follows:

$$\boxed{\begin{aligned} u &= -J_\alpha^{-1}(\nu_1 + \nu_2 + \nu_3) \\ \nu_1 &= -(\hat{J}_{sc}\omega + J_\alpha\nu) \times \omega - \hat{J}_{sc}(K_1\dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T\dot{\omega}_d) \\ \nu_2 &= -\hat{\tau}_{dist} \\ \nu_3 &= -K_v(\tilde{\omega} + K_1 S) - K_p S \end{aligned}} \tag{3.8}$$

where K_p is a positive scalar and $K_v \in \mathcal{R}^{3\times3}$ is a positive definite matrix. Eq. (3.8) represents the equation of the second controller.

With equations (3.2,3.8), we can simulate different scenarios of the spacecraft and observe how the error terms behave. But before looking at the simulation results, lets dive deep into why these expressions for the controllers make sense and whether they guarantee good performance always.

Chapter 4

Lyapunov Stability Guarantees

4.1 Controller 1:

Consider a Lyapunov Candidate Function V as follows:

$$\boxed{V(\omega, \tilde{R}) = \frac{1}{2}\omega^T J_{sc}\omega + K_p \text{trace}(A - A\tilde{R})} \tag{4.1}$$

where A is a diagonal positive definite matrix as defined in Section (3.1).

- The first term is a quadratic term in ω, so its always non negative. It is 0 only when $\omega = 0$.
- The second term is also non negative. This can be proved by noting that since all the terms of R and $R_d \in [-1, 1]$, the same can be said about the elements of $\tilde{R}$ as well. So, the diagonal elements of $A - A\tilde{R}$ will always be greater than or equal to 0. The equality will hold only when all the diagonal elements of $\tilde{R}$ are equal to 1, implying $\tilde{R}$ =Identity Matrix.
- So $V(\omega, \tilde{R})$ is a non negative function that takes the value of 0 iff $\omega = 0, \tilde{R} =$ Identity Matrix.

Now let's look at $\dot{V}(\omega, \tilde{R})$:

$$\begin{aligned}
\dot{V}(\omega, \tilde{R}) &= \omega^T J_{sc}\dot{\omega} + K_p \frac{d}{dt}(\text{trace}(A - A\tilde{R})) \\
\dot{V}(\omega, \tilde{R}) &= \omega^T((J_{sc}^c\omega + J_\alpha\nu) \times \omega - J_\alpha u) + K_p \frac{d}{dt}(\text{trace}(A - A\tilde{R})) \\
&\qquad \text{Expanding } J_{sc}\dot{\omega} \text{ from Eq. (1.8) and assuming } \tau_{dist} = 0 \\
\dot{V}(\omega, \tilde{R}) &= \omega^T(-K_p S - K_v\omega) + K_p \frac{d}{dt}(\text{trace}(A - A\tilde{R})) \\
&\qquad \text{Taking } u \text{ from Eq. (3.2)} \\
\dot{V}(\omega, \tilde{R}) &= -\omega^T K_v\omega - K_p\omega^T S + K_p \underbrace{\frac{d}{dt}(\text{trace}(A - A\tilde{R}))}_{\text{Let's analyze this separately}}
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}(\text{trace}(A - A\tilde{R})) &= \underbrace{\frac{d}{dt}\text{trace}(A)}_{0,\text{constant}} - \frac{d}{dt}\text{trace}(A\tilde{R}) \\
\frac{d}{dt}(\text{trace}(A - A\tilde{R})) &= -\text{trace}(A\dot{\tilde{R}}) \\
\frac{d}{dt}(\text{trace}(A - A\tilde{R})) &= -\sum_{i=1}^{3} a_i e_i^T \dot{\tilde{R}} e_i \text{ ,where } e_i \text{ is the } i^{th} \text{ column of the identity matrix} \\
\frac{d}{dt}(\text{trace}(A - A\tilde{R})) &= -\sum_{i=1}^{3} a_i e_i^T \tilde{R}\omega^* e_i = \sum_{i=1}^{3} a_i e_i^T \tilde{R} e_i^* \omega = -\sum_{i=1}^{3} a_i (e_i^* (\tilde{R}^T e_i))^T \omega \\
\frac{d}{dt}(\text{trace}(A - A\tilde{R})) &= \underbrace{\sum_{i=1}^{3} a_i ((\tilde{R}^T e_i) \times e_i)^T}_{=S} \omega
\end{aligned}
$$

$$
\boxed{\frac{d}{dt}(\text{trace}(A - A\tilde{R})) = S^T\omega = \omega^T S} \tag{4.2}
$$

Now going back to the $\dot{V}(\omega, \tilde{R})$ expression:

$$
\begin{aligned}
\dot{V}(\omega, \tilde{R}) &= -\omega^T K_v \omega - K_p \omega^T S + K_p \frac{d}{dt}(\text{trace}(A - A\tilde{R})) \\
\dot{V}(\omega, \tilde{R}) &= -\omega^T K_v \omega - K_p \omega^T S + K_p \omega^T S
\end{aligned}
$$

$$
\boxed{\dot{V}(\omega, \tilde{R}) = -\omega^T K_v \omega} \tag{4.3}
$$

- $\dot{V}(\omega, \tilde{R})$ is a negative quadratic term in ω. So $\dot{V}(\omega, \tilde{R})$ is always non positive.
- In Eq. (4.1), we showed that $V(\omega, \tilde{R}$ is non negative and equal to 0 only when $\omega = 0, \tilde{R} =$ Identity Matrix.
- The 2 points above imply that as time approaches ∞, we are guaranteed to have $\omega \to 0, \tilde{R} \to$ Identity Matrix.
- So, Controller 1 (Eq. 3.2) is guaranteed to meet our control objectives defined in Section (2.2).

4.2 Controller 2:

Consider a Lyapunov Candidate Function V as follows:

$$
\boxed{V(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau}) = \frac{1}{2}(\tilde{\omega} + K_1 S)^T J_{sc}(\tilde{\omega} + K_1 S) + K_p \text{trace}(A - A\tilde{R}) + \frac{1}{2}\tilde{\gamma}^T Q \tilde{\gamma} + \frac{1}{2}\tilde{\tau}^T D \tilde{\tau}} \tag{4.4}
$$

where Q, D, K_1, K_v and A are as defined in Section (3.2).

- The first term is a quadratic term in $(\tilde{\omega} + K_1 S)$, so its always non negative. It is 0 only when $(\tilde{\omega} + K_1 S) = 0$.
- Second term is always non negative as well and 0 only when $\tilde{R}$ = the Identity Matrix. (proved in the previous section)
- $\tilde{R}$ = the Identity Matrix implies that S is 0. (Eq. 3.1. So, for the first term and second term being 0 implies $\tilde{\omega} = 0$.
- Third and Fourth terms are also quadratic terms in $\tilde{\gamma}$ and $\tilde{\tau}$ respectively. It takes the value of 0 only when $\tilde{\gamma} = 0$ and $\tilde{\tau} = 0$ respectively.
- So, $V(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau})$ is proved to be a non negative function that takes the value of 0 iff $\tilde{\omega} = 0, \tilde{R} =$ the Identity Matrix, $\tilde{\gamma} = 0$ and $\tilde{\tau} = 0$.

Now let's look at $\dot{V}(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau})$:

$$\dot{V}(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau}) = \underbrace{(\tilde{\omega} + K_1 S)^T J_{sc}(\dot{\tilde{\omega}} + K_1 \dot{S})}_{\text{Term I}} + K_p \tilde{\omega}^T S + \underbrace{\tilde{\gamma}^T Q \dot{\tilde{\gamma}}}_{\text{Term II}} + \underbrace{\tilde{\tau}^T D \dot{\tilde{\tau}}}_{\text{Term III}} \text{ from Eq. (4.2)}$$

Let's analyze each term one by one and then sum them up together.

Term I:

$$(\tilde{\omega} + K_1 S)^T J_{sc}(\dot{\tilde{\omega}} + K_1 \dot{S})$$
$$= (\tilde{\omega} + K_1 S)^T [(J_{sc}(\tilde{\omega} + \tilde{R}^T \omega_d) + J_\alpha \nu) \times (\tilde{\omega} + \tilde{R}^T \omega_d) + J_{sc}(\tilde{\omega} \times \tilde{R}^T \omega_d - \tilde{R}^T \dot{\omega}_d) - J_\alpha u + \tau + K_1 J_{sc} \dot{S}]$$

From Eq. (2.4)

$$= (\tilde{\omega} + K_1 S)^T [(J_{sc}(\omega) + J_\alpha \nu) \times (\omega) + J_{sc}(\tilde{\omega} \times \tilde{R}^T \omega_d - \tilde{R}^T \dot{\omega}_d) + (\nu_1 + \nu_2 + \nu_3) + \tau + K_1 J_{sc} \dot{S}]$$

From Eq. (3.8)

$$= (\tilde{\omega} + K_1 S)^T [(J_{sc}(\omega) + J_\alpha \nu) \times (\omega) + J_{sc}(\tilde{\omega} \times \tilde{R}^T \omega_d - \tilde{R}^T \dot{\omega}_d) +$$
$$(-(\hat{J}_{sc}\omega + J_\alpha \nu) \times \omega - \hat{J}_{sc}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d) - \hat{\tau} - K_v(\tilde{\omega} + K_1 S) - K_p S) + \tau + K_1 J_{sc} \dot{S}]$$
$$= (\tilde{\omega} + K_1 S)^T [\tilde{\tau} + K_1 \tilde{J}_{sc} \dot{S} - K_v(\tilde{\omega} + K_1 S) - K_p S + \tilde{J}_{sc}(\tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)]$$

Since $\tilde{\omega} \times \omega = \tilde{\omega} \times \tilde{R}^T \omega_d$

$$= -(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) + (\tilde{\omega} + K_1 S)^T [\tilde{\tau} + K_1 \tilde{J}_{sc} \dot{S} + \tilde{J}_{sc}(\tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d) - K_p S]$$

Term II:

$$\tilde{\gamma}^T Q \dot{\tilde{\gamma}}$$
$$= -\tilde{\gamma}^T Q \dot{\hat{\gamma}}, \text{ From Eq. (3.3)}$$
$$= \tilde{\gamma}^T [-L^T(\omega)\omega^* - L^T(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)](\tilde{\omega} + K_1 S), \text{ From Eq. (3.5)}$$
$$= ((-\tilde{J}_{sc}\omega)^T \omega^* + (-\tilde{J}_{sc}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d))^T)(\tilde{\omega} + K_1 S)$$
$$= ((\omega^* \tilde{J}_{sc}\omega) + (-\tilde{J}_{sc}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)))^T (\tilde{\omega} + K_1 S)$$
$$= \underbrace{\overset{T}{(\tilde{\omega} + K_1 S)}}_{\omega}(\omega \times \tilde{J}_{sc}\omega) - (\tilde{\omega} + K_1 S)^T \tilde{J}_{sc}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)$$
$$= -(\tilde{\omega} + K_1 S)^T \tilde{J}_{sc}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)$$

Term III:

$$
\begin{aligned}
&\tilde{\tau}^T D\dot{\tilde{\tau}} \\
&= \tilde{\tau}^T D(\dot{\tau} - \dot{\hat{\tau}}), \text{ From Eq.(3.4)} \\
&= \tilde{\tau}^T D(A_d\tau - A_d\hat{\tau} - D^{-1}(\tilde{\omega} + K_1 S)), \text{ From Eq. (3.6)} \\
&= \tilde{\tau}^T \underbrace{DA_d}_{\frac{1}{2}(DA_d + A_d^T D)} \tilde{\tau} - (\tilde{\omega} + K_1 S)^T\tilde{\tau} \\
&= \frac{1}{2}\tilde{\tau}^T(DA_d + A_d^T D)\tilde{\tau} - (\tilde{\omega} + K_1 S)^T\tilde{\tau}
\end{aligned}
$$

Summing Up Together:

$$
\begin{aligned}
\dot{V}(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau}) = &-(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) + (\tilde{\omega} + K_1 S)^T[\tilde{\tau} + K_1\tilde{J}_{sc}\dot{S} + \tilde{J}_{sc}(\tilde{\omega} \times \omega - \tilde{R}^T\dot{\omega}_d) - K_p S] \\
&+ K_p\tilde{\omega}^T S - (\tilde{\omega} + K_1 S)^T\tilde{J}_{sc}(K_1\dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T\dot{\omega}_d) + \frac{1}{2}\tilde{\tau}^T(DA_d + A_d^T D)\tilde{\tau} - (\tilde{\omega} + K_1 S)^T\tilde{\tau} \\
\dot{V}(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau}) = &-(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - (K_1 S)^T K_p S + \frac{1}{2}\tilde{\tau}^T(DA_d + A_d^T D)\tilde{\tau}
\end{aligned}
$$

$$
\boxed{\dot{V}(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau}) = -(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - K_p S^T K_v^T S + \frac{1}{2}\tilde{\tau}^T(DA_d + A_d^T D)\tilde{\tau}} \quad (4.5)
$$

- The first two terms in Eq. (4.5) are negative quadratic terms in $(\tilde{\omega} + K_1 S)$ and S respectively.
- In the paper, it is assumed that $(DA_d + A_d^T D)$ is a negative definite matrix. This makes the third term also a negative quadratic term in $\tilde{\tau}$.
- $V(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{\tau})$ expression in Eq. (4.4) is proven to be non negative and zero iff $\tilde{\omega} = 0, \tilde{R} =$ the Identity Matrix, $\tilde{\gamma} = 0$ and $\tilde{\tau} = 0$.
- The 2 points above imply that as time approaches ∞, we are guaranteed to have $\tilde{\omega} \to 0, \tilde{R} \to$ the Identity Matrix, $\tilde{\gamma} \to 0$ and $\tilde{\tau} \to 0$.
- So, Controller 2 (Eq. 3.8) is guarantee to meet our control objectives defined in Section (2.2).

Chapter 5

Numerical Simulations

5.1 Controller 1 Simulations:

Scenario Description

- The spacecraft body inertia (without wheels), $J_b = diag(10, 25/3, 5)$.
- The entire spacecraft inertia, $J_{sc} = diag(10.75, 9.08, 5.75)$.
- $J_\alpha = diag(0.5, 0.5, 0.5)$.
- $A = diag(1, 2, 3)$.
- $K_p = \frac{5}{\text{trace}(A)}$.
- $K_v = 5diag(\frac{1}{1+|\omega_1|}, \frac{1}{1+|\omega_2|}, \frac{1}{1+|\omega_3|})$
- $\omega(0) = [1, -1, 0.5]^T$ rad/s.
- $\omega_d = 0$.
- $R(0) =$ Identity Matrix.
- $R_d = diag(1, -1, -1)$.
- $\nu(0) = [0, 0, 0]^T$.
- Controller 1 (Eq. 3.2) is employed with the given simulation conditions.
- Baseline Simulation results are shown in images (5.1).
- Bounds are imposed on the control inputs (Figs. 5.2,5.3) and Wheel rotation rates (Figs. 5.4, 5.5).

5.2 Controller 2 Simulations:

Scenario Description:
Almost everything is same as the previous case, here we define the extra constants/minor changes required to define the controller 2:

- $K_p = \frac{1}{\text{trace}(A)}$.
- $K_v = 5diag(\frac{1}{1+|\omega_1|}, \frac{1}{1+|\omega_2|}, \frac{1}{1+|\omega_3|})$
- $K_1 =$ Identity Matrix.
- $D =$ Identity Matrix.
- $Q =$ Identity Matrix (order 6).
- Controller 2 (Eq. 3.8) is employed with the given simulation conditions.
- Baseline Simulation results are shown in images (5.6).
- Bounds are imposed Wheel rotation rates (Fig. 5.7).

5.3 Settling Time Metric:

Setting Time Metric is defined in the paper as follows-

$$\boxed{k_0 = \min_{k>100}\{k : \text{for all} i \in \{1, 2, ...100\}, \theta((k-i)T_s) < 0.05\}} \tag{5.1}$$

where k is the simulation step, T_s is the integration step size, and $\theta(kT_s)$ is the eigen-axis attitude error at the k_{th} step.

5.4 Standard Inertia Matrices of Different Shapes

Few standard shapes and their Inertia Matrices are given below:

- $J_1 = diag([10, 10, 10])$: Sphere
- $J_2 = diag([10, 10, 5])$: Cylinder
- $J_3 = diag([10, 25/3, 5])$: Centroid. (This is the J_b taken previously)
- $J_4 = diag([10, 5, 5])$: Thin Disk
- $J_5 = diag([10, 10, 0.1])$: Thin Cylinder

$$\boxed{J_b = (1-\lambda)J_3 + \lambda J_i, \text{ i = 1,4,5}} \tag{5.2}$$

Settling time variation with λ is shown for the 2 controllers in Fig. (5.8)

5.5 Effect of Misalignment:

Let $\mathcal{O}$ be a rotation matrix about one of the principal axes, the altered J_b due to misalignment of ϕ degrees about this axis is given by:

$$\boxed{J_b(\phi) = \mathcal{O}^T J_3 \mathcal{O}} \tag{5.3}$$

Settling time variation with ϕ about the 3 axes of rotation is shown for the 2 controllers in Fig. (5.9).

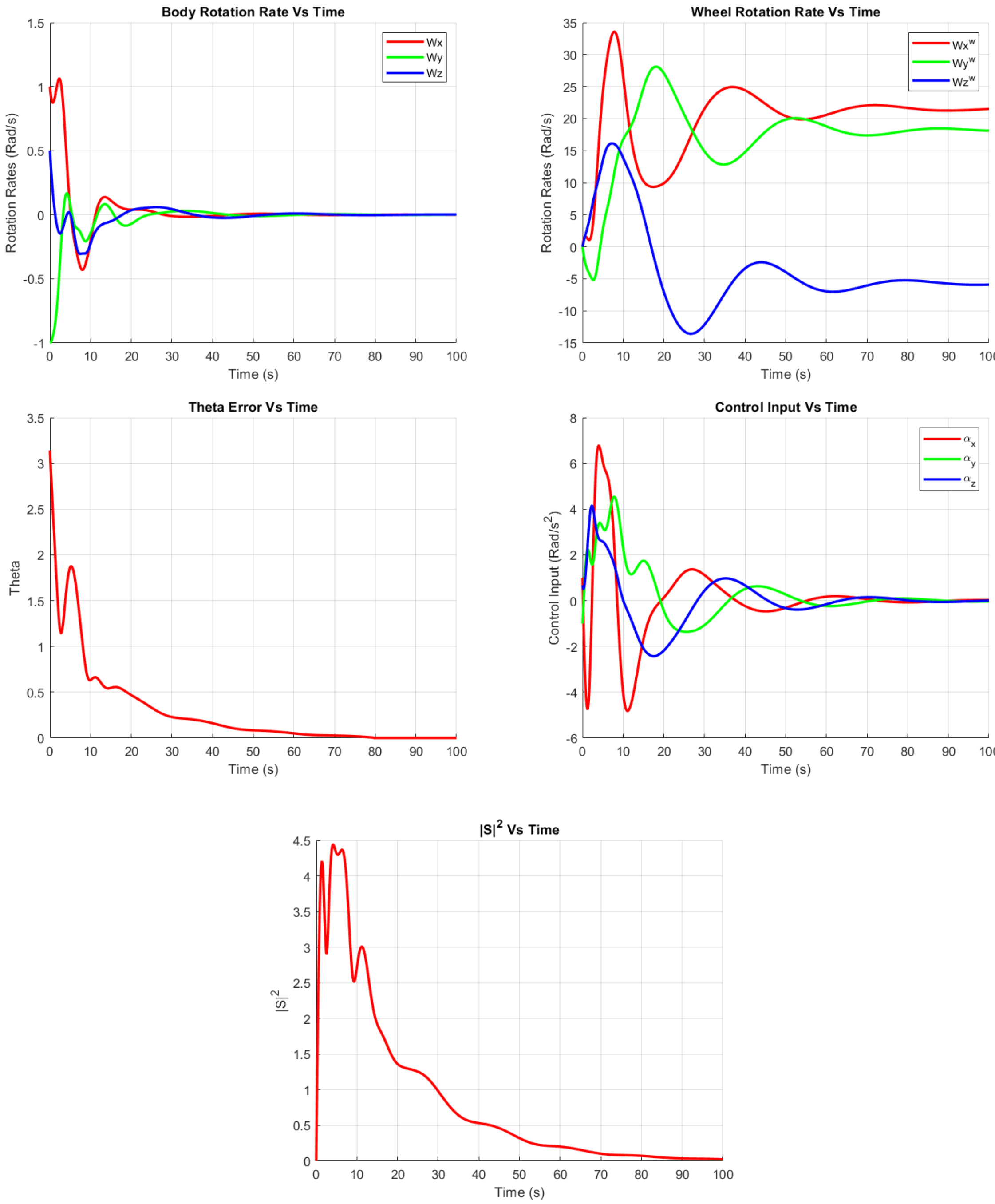


Figure 5.1: Baseline Controller 1

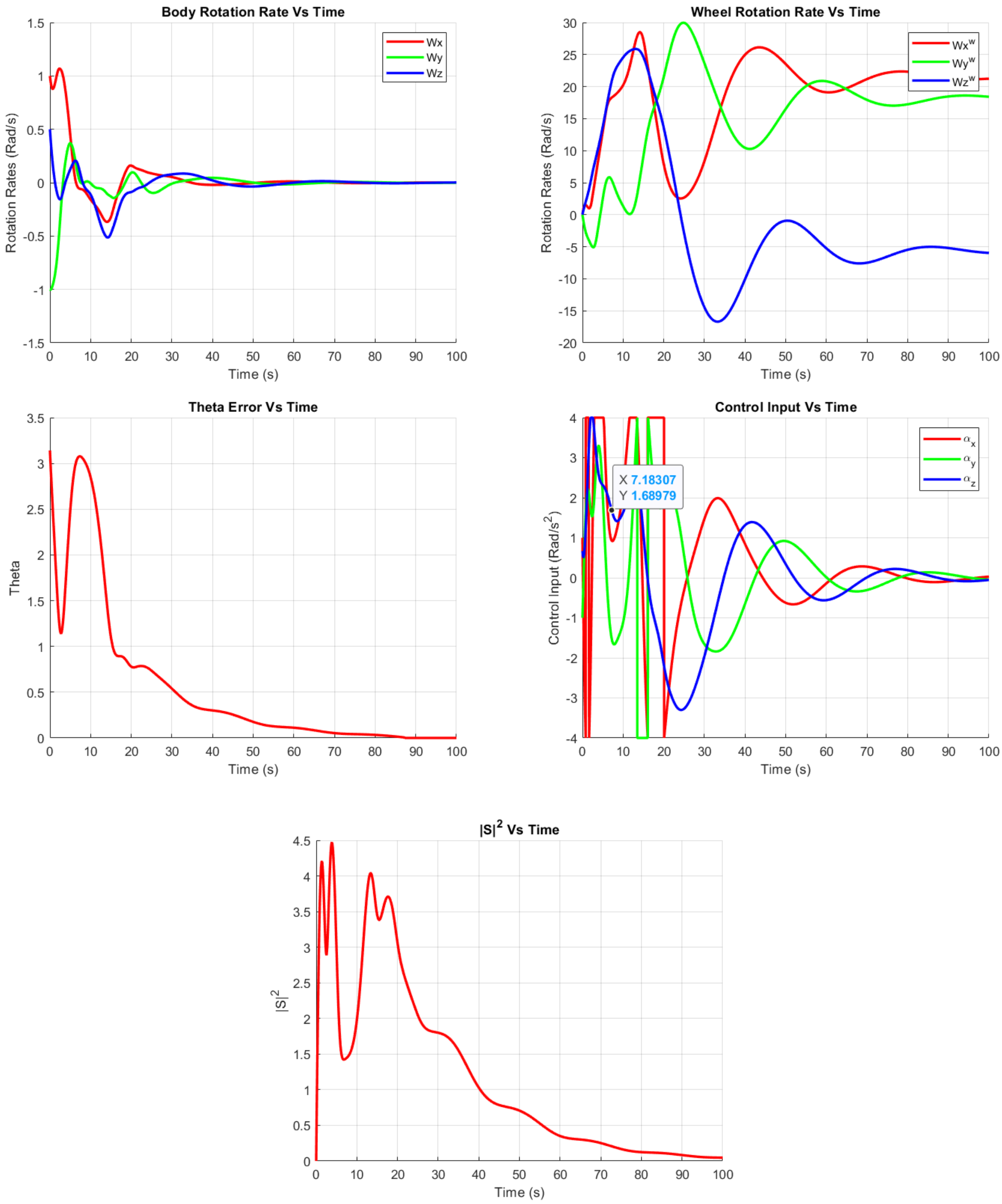


Figure 5.2: $|u| \leq 4$

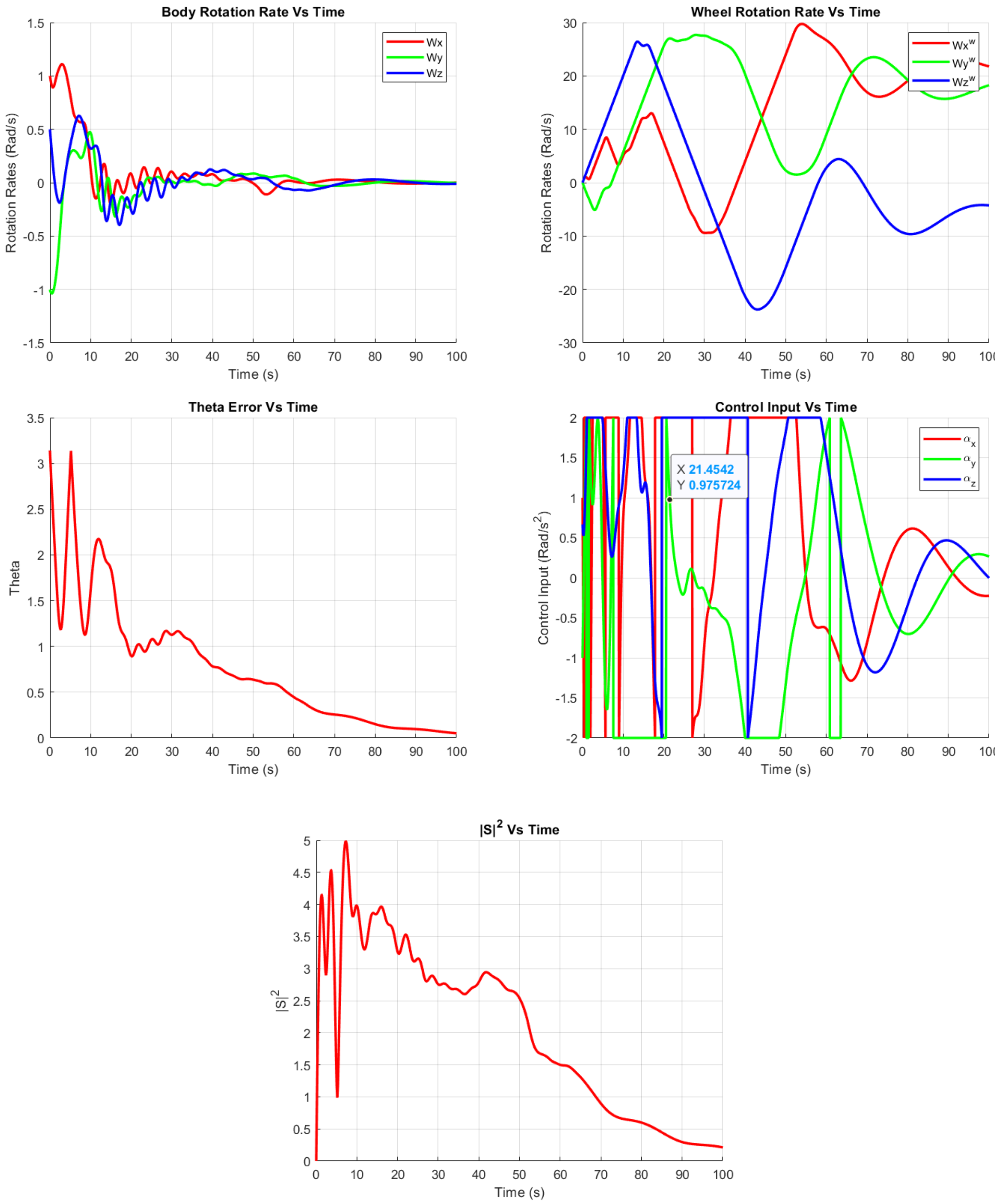


Figure 5.3: $|u| \leq 2$

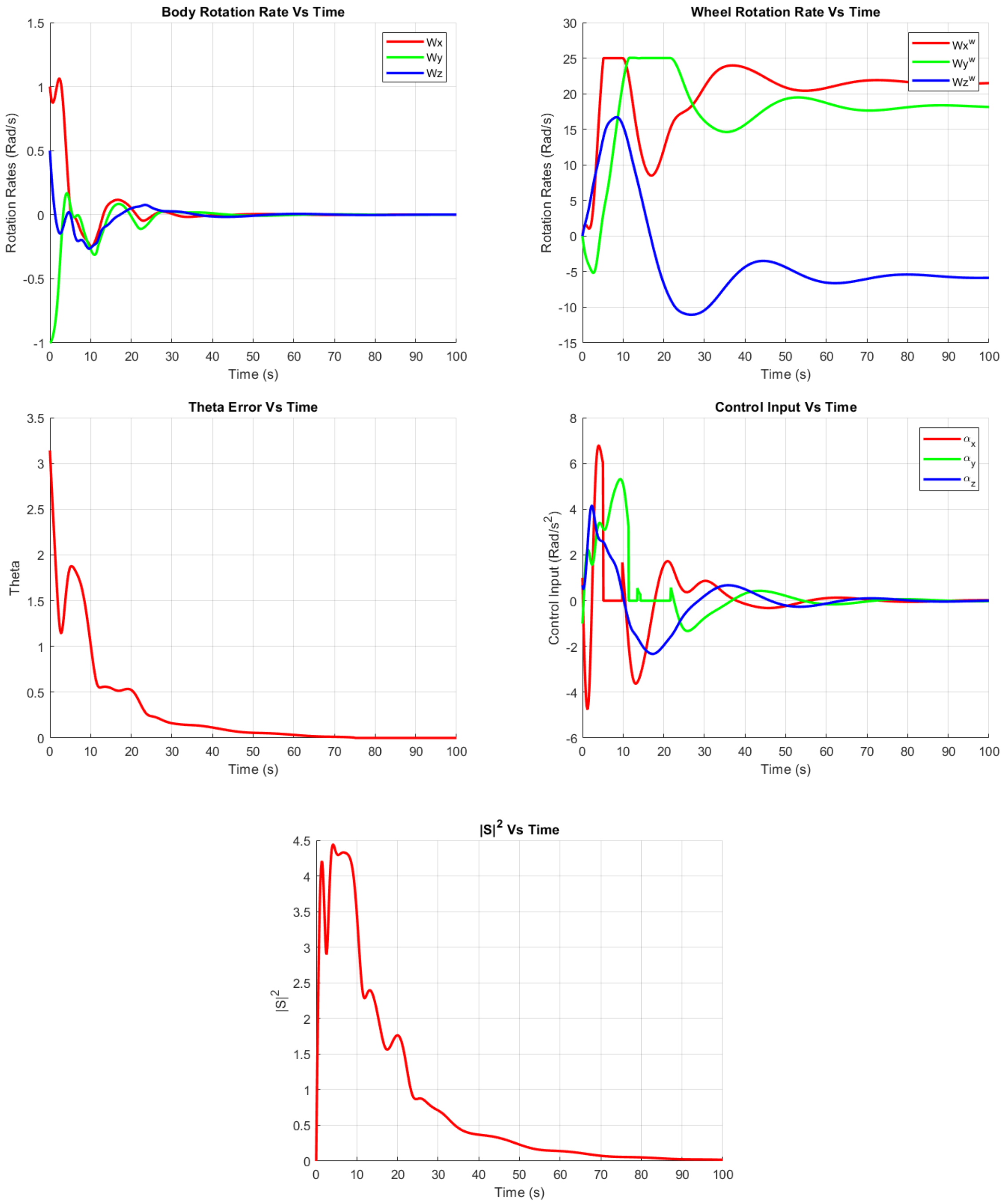


Figure 5.4: $|\omega_i| \leq 25$

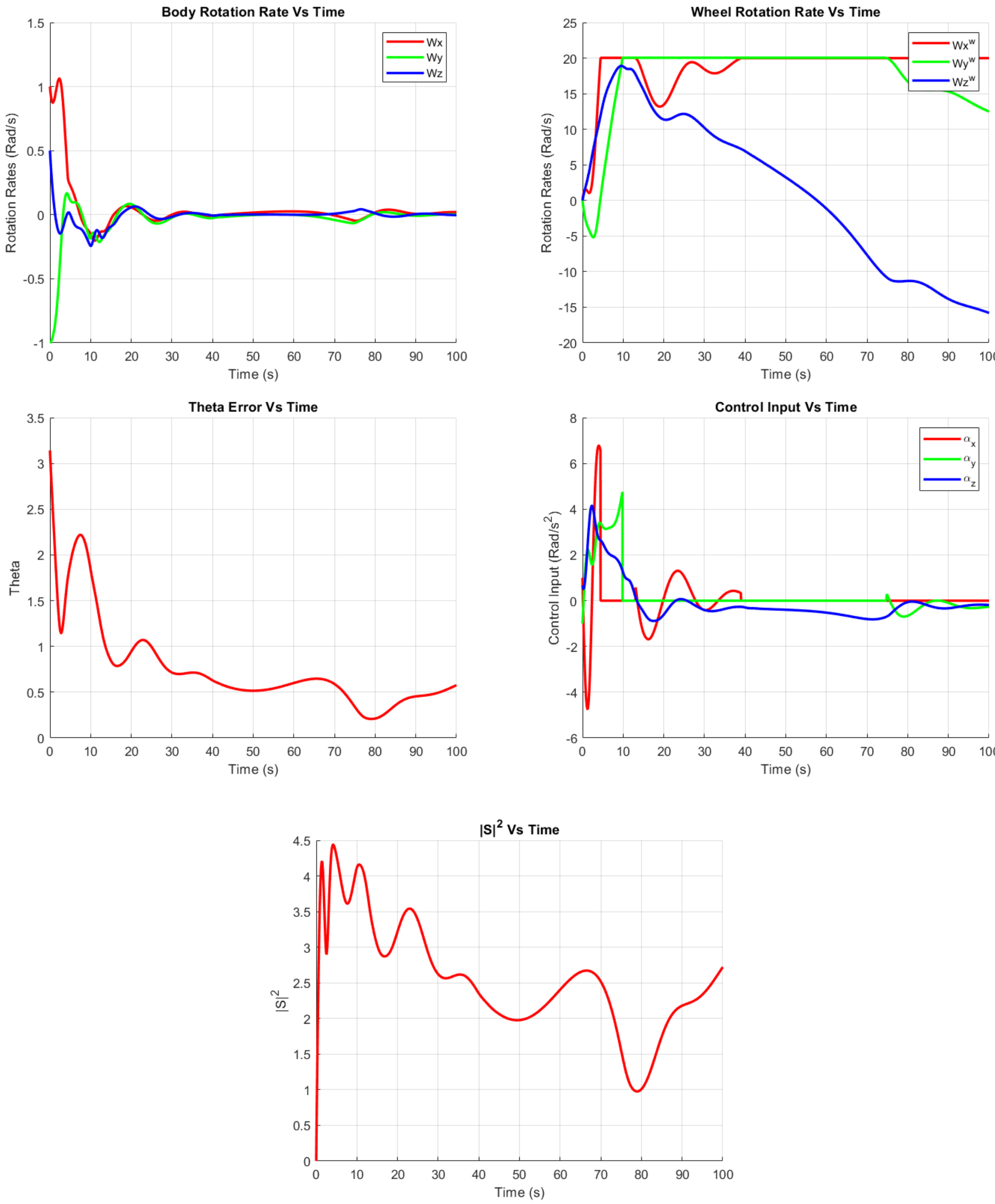


Figure 5.5: $|\omega_i| \leq 20$

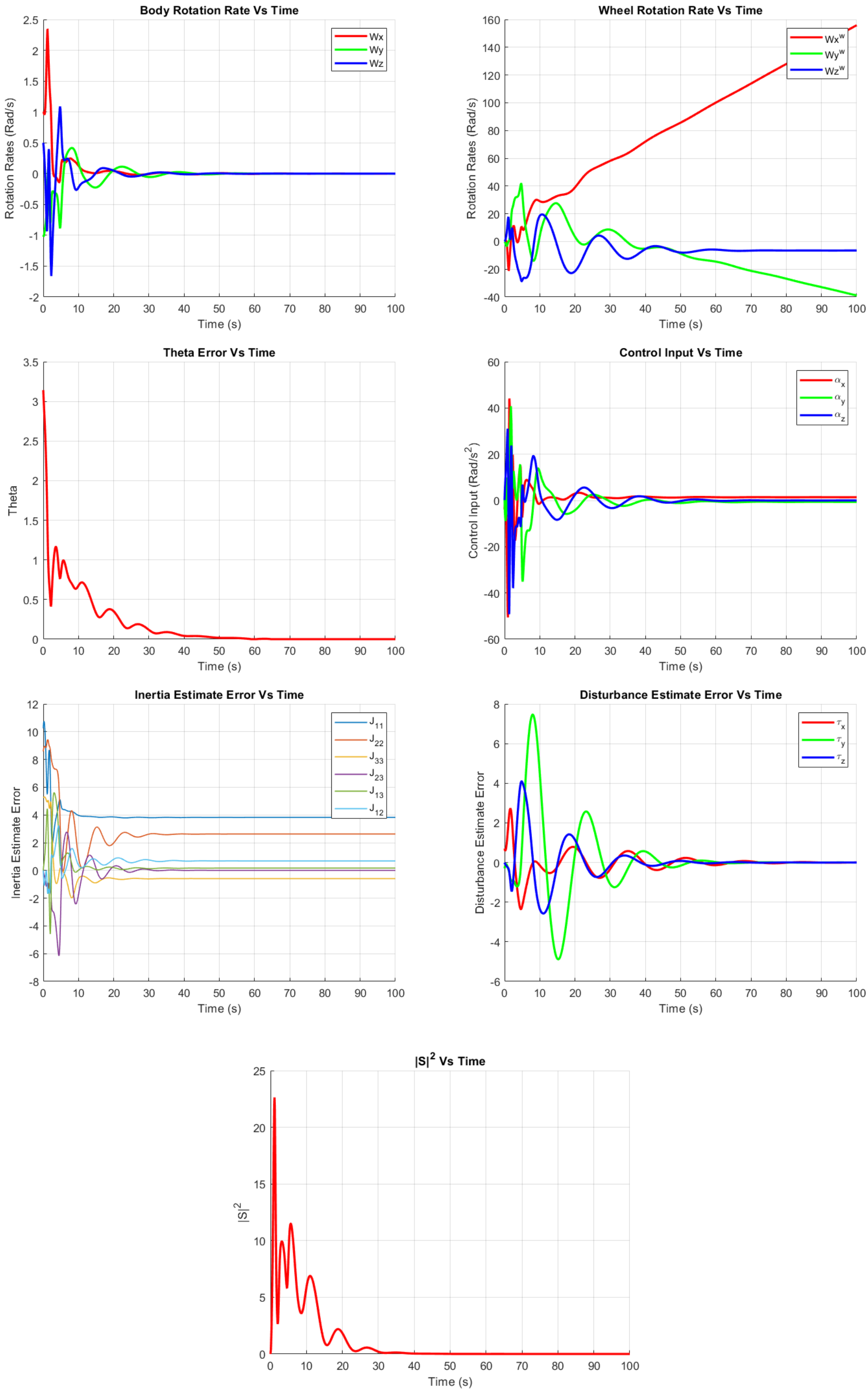


Figure 5.6: Baseline Controller 2

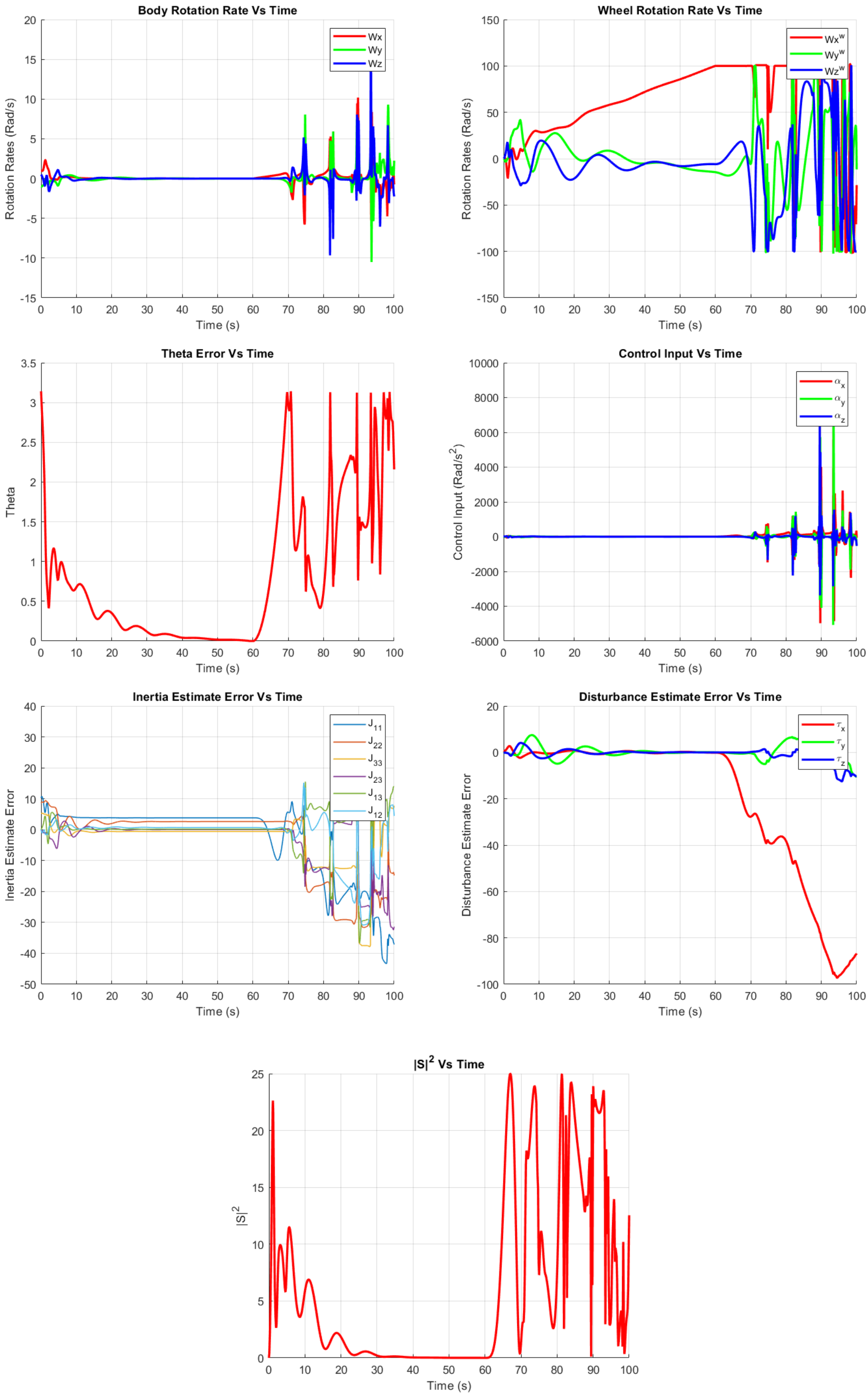
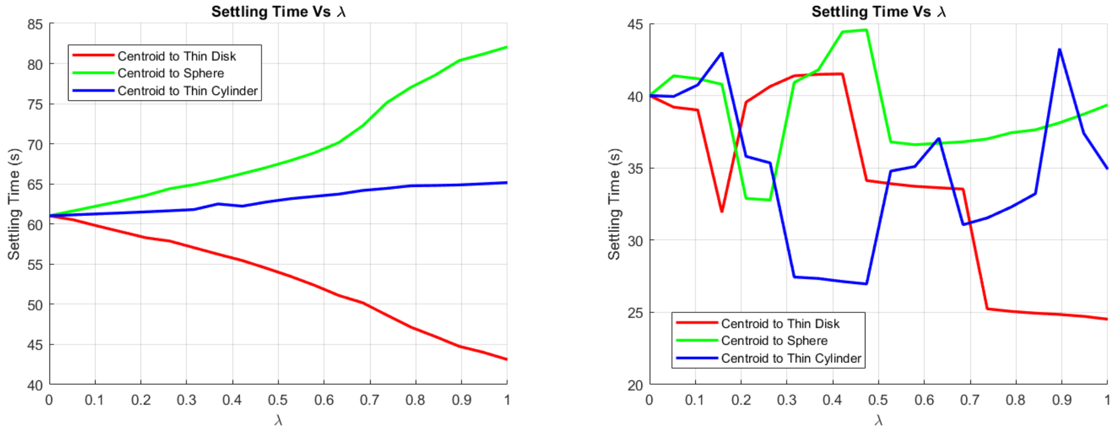


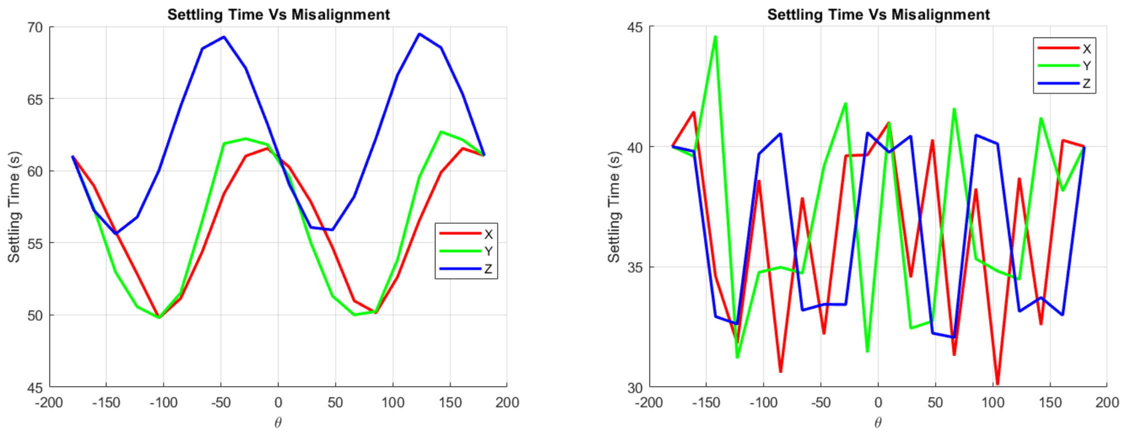
Figure 5.7: $|\omega_i| \leq 100$



(a) Controller 1

(b) Controller 2

Figure 5.8: Settling Time Variation with λ



(a) Controller 1

(b) Controller 2

Figure 5.9: Settling Time Variation with Misalignment ϕ

Chapter 6

Ending Remarks

- The paper used for this course project is: Weiss, A., Kolmanovsky, I., Bernstein, D.S. and Sanyal, A., 2013. Inertia-free spacecraft attitude control using reaction wheels. Journal of Guidance, Control, and Dynamics, 36(5), pp.1425-1439.
- All simulations are self reproduced on MATLAB 2023b. To compare results, refer to the paper above.
- All codes can be accessed from Github here.

THANK YOU!