College Admissions
&
the Stability of Marriage

The Ultimate Question:

N students apply to L colleges, where i^{th} college can at max have q_i students

We need to find the best matching where every student gets what he 'deserves'

Here, I will be talking about the paper by Gale and Shapley titled:

"Colleges Admissions and the Stability of Marriages"

Where they have proposed an algorithm to prove that we can always find a "stable" and "optimal" Solution through the Deferred Acceptance Method

Stable Vs Unstable

Unstable Pair: Student S and College C are Unstable if:

- S prefers C over his assigned college, and
- C prefers S over at least one of it’s students.

An Assignment with no unstable pairs is called as a Stable Assignment.

What is OPTIMAL?

Stable assignment is called as Optimal if every applicant is at least as well off under it under any other stable assignment.

Stable Marriages

This is a simplified version of the College admissions problem.
Here, there are N men and N women, and
each person has his/her preference order of the other gender.
In the graph theory sense, we are trying to find a stable matching for a bipartite graph.

Theorem 1: There always exists a stable set of marriages.
The proof to theorem 1 is given by the Gale Shapley Proposal Algorithm.

Below is a 4 Men, 4 Women preference list, listed as a matrix H where
h_{ij} represents i^{th} preference of j , j^{th} preference of I
The Underlined h_{ij} represents a pair (i,j).
Note that this matching is a stable matching.

	W	X	Y	Z
A	1,3	2,3	$\underline{3,2}$	4,3
B	1,4	4,1	3,3	$\underline{2,2}$
C	$\underline{2,2}$	1,4	3,4	4,1
D	4,1	$\underline{2,2}$	3,1	1,4

Gale Shapley Proposal Algorithm:

- Every man Proposes to his 1st preference.
- Every girl who was proposed says <u>maybe</u> to one man. This man is the one who is placed highest among all those who proposed to her in her list of preferences.
- Else, all men , who have been rejected propose to the next girl in their priority.
- Each girl who has been proposed compares the current proposal to the one whom she has said maybe to, and decides based on her preferences.
- If every girl has said maybe to a man (at present), then we have achieved a stable assignment of marriages and, the algorithm terminates.

Proof that the Algorithm gives a stable assignment:

Firstly, the algorithm terminates because, at every stage of proposals, the number of women who haven’t said maybe is non- increasing and infact, the maximum number of stages required to terminate the proposal is $n^2 – 2n + 3$.

The Assignment is stable since,

If John prefers Mary over his wife, then he must have proposed but gotten rejected.

But this is possible only when, Mary was proposed by a man whom Mary preferred more than John. So, if John is Unhappy, then Mary is

Getting Closer to College Admissions:
An extension of Stable Marriage Problem

The number of Men M is not equal to the number of Women W

Modifying the Gale Shapley Proposal algorithm still provides us a Stable solution to both cases

M<W

The procedure ends when all the M men have been accepted by their respective women i.e., the algorithm terminates when the first M women have been proposed to.

M>W

The procedure ends when for each man, he has been either accepted by a woman or he has been rejected by all women.

If men propose, then the stable assignment is optimal for men

And

If women propose, then the stable assignment is optimal for women

Final Objective
College Admissions
Deferred Acceptance Algorithm

The Deferred Acceptance Algorithm is an extension to the Gale Shapley proposal Algorithm

The algorithm goes as follows:

- All students apply to their respective first choice college. The i^{th} college selects students less than or equal to their maximum capacity q_i from among all it’s applicants and puts them in it’s ‘waiting list’. This is done by all colleges which received an application.
- The students not selected earlier apply to their 2^{nd} choice college. Again, each i^{th} college (i less than or equal to the number of colleges) selects students (according to it’s preferences) less than or equal to their maximum capacity q_i from among all it’s applicants (including the ones already in the waiting list) and puts them in it’s ‘waiting list’.
- The procedure continues until every applicant is either on a waiting list or has been rejected by all the colleges he/she has applied to.

Theorem 2: This Deferred acceptance procedure not only yield a stable assignment of applicants,
but it also yields an optimal assignment of applicants, i.e., every applicant is at least as well off under this assignment as he would be under any other stable assignment.

Proof: The proof follows by induction. Lets call a college "possible" for an applicant, if there is a stable assignment that sends him there.

- Assume that, up to a given point in the procedure, no applicant has yet been turned away from a college that is possible for him.
- At this point, suppose, college A having received a full quota of better qualified applicants b_1, b_2,..., b_q rejects student 'a'.
- For the theorem to hold true, we must show that A is impossible for 'a'.
- We know that each b_i prefers A than any other college that has not rejected him (the colleges that are not impossible for him/her).
- Now, consider a hypothetical assignment where 'a' is sent to A and every one else to the colleges that are possible for them.
- This implies, at least one b_i must go to a college less desirable to it. Which further implies that this hypothetical assignment is unstable as both b_i and A prefer each other more than their current assignment.

Conclusion:

Our Procedure only rejects applicants from colleges which they couldn’t be admitted to in any stable assignment.

Hence,

The resulting Stable assignment (from the deferred acceptance procedure) is Optimal.

Thank You!