9 Cumulative Sum and Exponentially Weighted Moving Average Control Charts

9.1 The Cumulative Sum Control Chart

- The $\overline{x}$-chart is a good method for monitoring a process mean when the magnitude of the shift in the mean to be detected is relatively large.
- If the actual process shift is relatively small (e.g., in the range of $.5\sigma_{\overline{x}}$ to $1\sigma_{\overline{x}}$),the $\overline{x}$-chart will be slow in detecting the shift. This is a major drawback of variables control charts.
- An alternative method to use when the shift in the process mean required to be detected is relatively small is the **cumulative sum (cusum)** procedure.
- The cusum procedure is also effective for detecting large shifts in the process, and its performance is comparable to Shewhart control charts in this situation.
- In general, the cusum procedure can be used to monitor any quality characteristic, say Q, in relation to some standard value Q_0 by cumulating deviations from Q_0. Q could be any statistic of interest (e.g., x, $\overline{x}$, R, s, proportion defectives p, or number of defects c).
- We analyze this situation by computing cusum(n) = for $n = 1, 2, ...$
  - If , the cusum will tend to remain relatively close to 0.
  - If , the cusum will tend to consistently increase from 0 if $E(Q_i) > Q_0$ or derease if $E(Q_i) < Q_0$.
  - Upper and lower limits are imposed to determine if the cusum has drifted too far away from 0.
  - It is also important to be able to determine when a shift away from Q_0 occurred and estimate the magnitude of the shift.
- The cusum is, therefore, a type of *sequential analysis* because it relies upon past data to make a decision as each new Q_i appears. That is, whether to conclude if there has been a positive shift ($E(Q) > Q_0$), a negative shift ($E(Q_i) < Q_0$), or to continue collecting new data.
- Recall: the ARL is the average number of samples taken from a process before an out-of-control signal is detected.
  - The in-control ARL is the average number of samples taken from an in-control process before a false out-of-control signal is detected. The in-control ARL should be chosen to be sufficiently large to reduce unnecessary adjustments to the process due to false out-of-control signals.
  - The out-of-control ARL for a shift in the process mean from μ_0 to $\mu_1 = \mu_0 \pm \delta\sigma$ is the average number of samples taken before a shift in the mean of magnitude $\delta\sigma$ or greater is detected.
- It is desirable to detect a true shift in the process mean in as few samples as possible (small ARL) while the in-control ARL should be large.
- The objective of cusum charts is to quickly indicate true departures from Q_0 but not falsely indicate a departure from Q_0 when no departure has occurred. Therefore, we want the in-control ARL to be long and an out-of-control ARL to be short.

- The principle behind the cusum procedure for individual measurements $Q_i = x_i$ or sample means $Q_i = \overline{x}_i$ is that the difference between a random x or $\overline{x}$ and the aim value μ_0 for the process is expected to be zero *if the process is in the in-control state.*

- The cusum for monitoring the process mean, denoted C_i, is defined as:

$C_i \quad =$ for individual measurements

$C_i \quad =$ for sample means

- For an in-control process ($\mu = \mu_0$), the C_i values should be close to zero.

- If too many positive deviations accumulate, the value of C_i will consistently increase, indicating the process mean is $> \mu_0$.

- If too many negative deviations accumulate, the value of C_i will consistently decrease, indicating the process mean is $< \mu_0$.

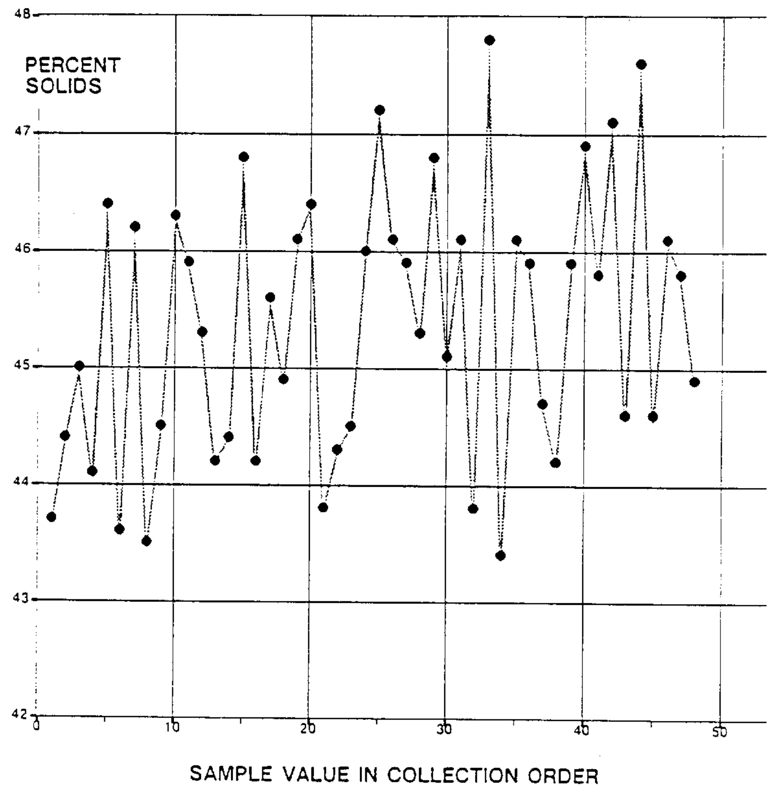
Example of a cusum plot: In an industrial process, the percent solids (x) in a chemical mixture is being monitored. Forty-eight samples were collected and the percent solids x was recorded. When in-control the process aim for x is $\mu_0 = 45\%$ solids. Thus,

$$C_i \;=\; \sum_{j=1}^{i}(x_j - 45) \quad \text{for } i = 1, 2, \ldots, 48.$$

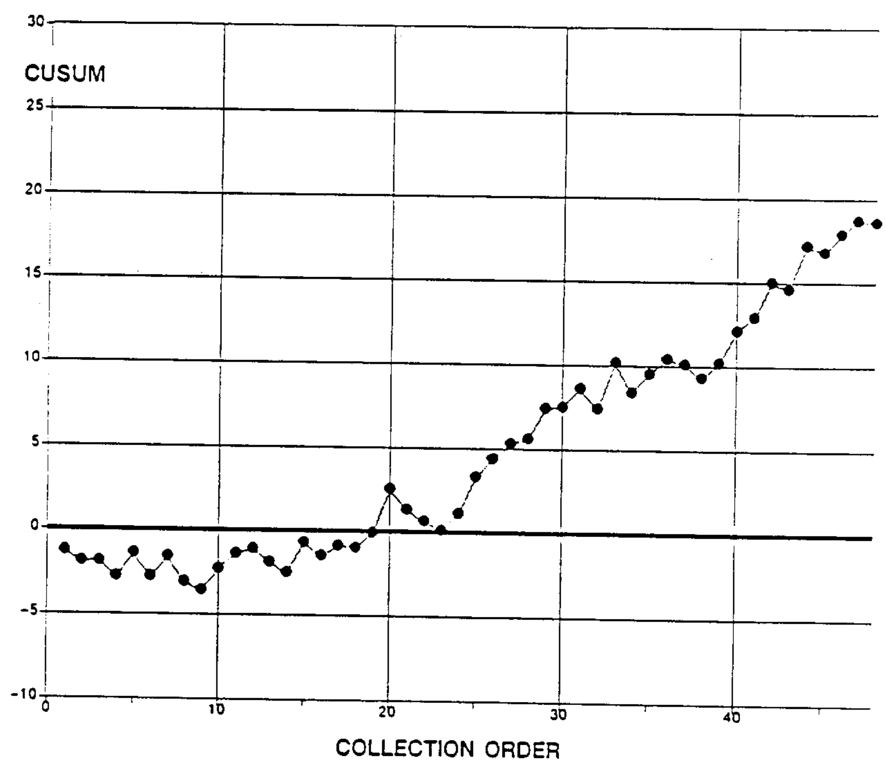
The following table contains the forty-eight y_i values, the deviations from aim ($x_i - 45$), and the cusum values (C_i).

Sample i	x_i	$x_i - 45$	cusum C_i	Sample i	x_i	$x_i - 45$	cusum C_i	Sample i	x_i	$x_i - 45$	cusum C_i
1	43.7	-1.3	-1.3	17	45.6	0.6	-0.9	33	47.8	2.8	9.2
2	44.4	-0.6	-1.9	18	44.9	-0.1	-1.0	34	43.4	-1.6	7.6
3	45.0	0.0	-1.9	19	46.1	1.1	0.1	35	46.1	1.1	8.7
4	44.1	-0.9	-2.8	20	46.4	1.4	1.5	36	45.9	0.9	9.6
5	46.4	1.4	-1.4	21	43.8	-1.2	0.3	37	44.7	-0.3	9.3
6	43.6	-1.4	-2.8	22	44.3	-0.7	-0.4	38	44.2	-0.8	8.5
7	46.2	1.2	-1.6	23	44.5	-0.5	-0.9	39	45.9	0.9	9.4
8	43.5	-1.5	-3.1	24	46.0	1.0	0.1	40	46.9	1.9	11.3
9	44.5	-0.5	-3.6	25	47.2	2.2	2.3	41	45.8	0.8	12.1
10	46.3	1.3	-2.3	26	46.1	1.1	3.4	42	47.1	2.1	14.2
11	45.9	0.9	-1.4	27	45.9	0.9	4.3	43	44.6	-0.4	13.8
12	45.3	0.3	-1.1	28	45.3	0.3	4.6	44	47.6	2.6	16.4
13	44.2	-0.8	-1.9	29	46.8	1.8	6.4	45	44.6	-0.4	16.0
14	44.4	-0.6	-2.5	30	45.1	0.1	6.5	46	46.1	1.1	17.1
15	46.8	1.8	-0.7	31	46.1	1.1	7.6	47	45.8	0.8	17.9
16	44.2	-0.8	-1.5	32	43.8	-1.2	6.4	48	44.9	-0.1	17.8

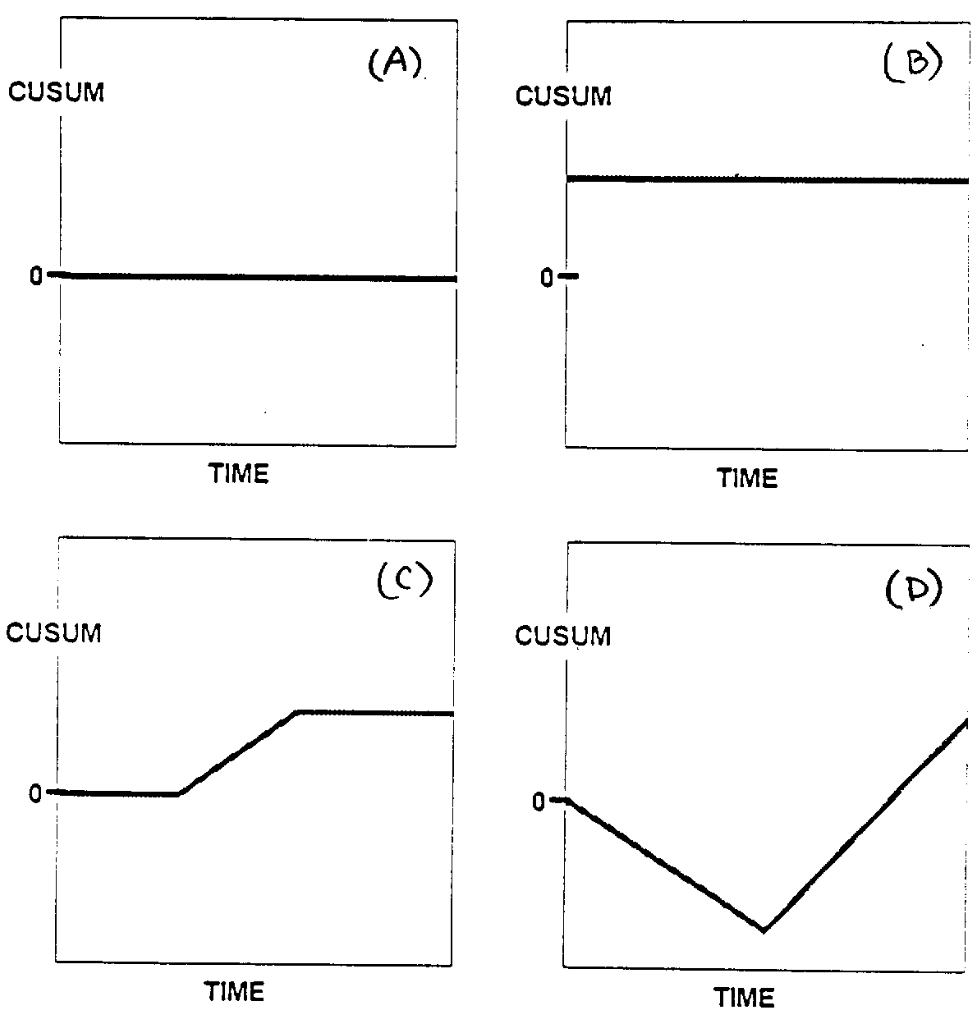
- Suppose the actual process mean μ shifts from being close to 45% up to 46% after sample 23. From the following chart it is unclear when a shift occurs, and if it did, the magnitude of the shift is also unknown.



- With the cusum procedure, we will be able to detect the shift relatively quickly after sample 23 and estimate the magnitude of the shift.



- With a cusum chart, it is much easier to see a shift from the process aim that it is with a sequence plot of the response (like a Shewhart-type chart).
- Cusum charts also dampen out random variation compared to a sequence plot with interpretable patterns:
  - Any sequence of points on the cusum chart that are close to horizontal indicates the process mean is running near the aim during that sequence.
  - Any sequence of points on the cusum chart that are increasing (decreasing) <u>linearly</u> indicates the process mean is constant but is above (below) the aim during that sequence.
  - A change in the slope in the cusum chart indicates a change in the process mean.
- Consider the following plots.
  - The pattern in Plot A indicates a process that is on-aim (in control).
  - The pattern in Plot C indicates the process is initially on-aim, then the mean increases by a positive amount, but shifts back again to being on aim.
  - The pattern in Plot D indicates a process with a mean less than the aim but then a shift in the mean to above the aim occurs.
  - What does Plot B indicate?



9.2 The Tabular Cusum Procedure

- To determine if C_i is too large or too small to have reasonably occurred from an in-control process, we use a **tabular form** of the cusum which is a simple computational procedure.
- The tabular form for the cusum can be either two-sided (detecting a shift in either direction from the aim value) or a one-sided upper cusum or lower cusum (detecting a shift in one specified direction from the aim value).
- For visual interpretation of the results, the tabular form is complemented with a cusum plot.
- The tabular form requires specification of 3 values: k, h, and σ (or, $\widehat{\sigma}$).
- Once acceptable values of k, h, and σ have been found, $K = k\sigma$ and $H = h\sigma$ can be computed and the cusum table constructed.
- The tabular form of the cusum procedure uses two one-sided cusums.
  - The **upper one-sided cusum** accumulates deviations from the aim value if the tabular deviations are greater than zero.
  - The **lower one-sided cusum** accumulates deviations from the aim value if the tabular deviations are less than zero.
- We will now introduce k, the first cusum parameter. Denote the upper one-sided cusum by C_i^+ and the lower one-sided cusum by C_i^-. These two tabular cusums are defined as:

$$C_i^+ =$$

$$C_i^- =$$

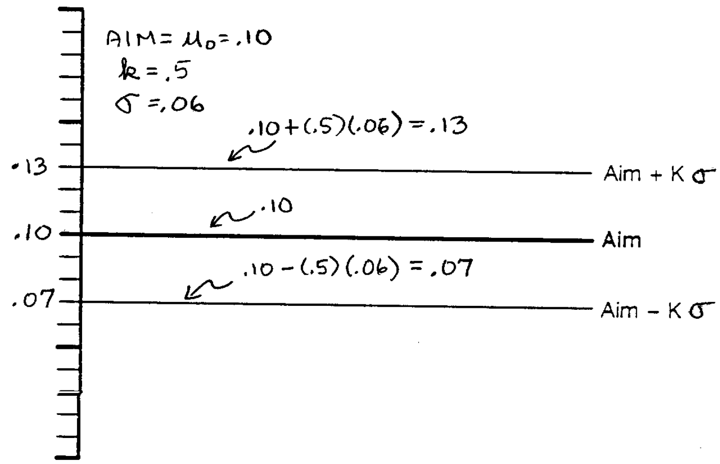
  where μ_0 is the aim value and $K = k\sigma$ for a specified value k.

- Note that $C_i^+ \geq 0$ and $C_i^- \geq 0$.
- The basic principle behind these formulas is that, if the difference between the observed value of x and μ_0 is changing at a rate greater than the allowable rate of change K, then the differences between x and μ_0 will accumulate.
- That is, if $E(X) > \mu_0 + k\sigma$ then C_i^+ will show an increasing trend, or if $E(X) < \mu_0 - k\sigma$ then C_i^- will show an increasing trend. Otherwise, C_i^+ and C_i^- will tend toward 0.
- The interval $(\mu_0 - K, \mu_0 + K) = (\mu_0 - k\sigma, \mu_0 + k\sigma)$ is often referred to as the **slack band**. If $x_i < \mu_0 - K$, then C_i^- will increase and if $x_i > \mu_0 + K$, then C_i^+ will increase.

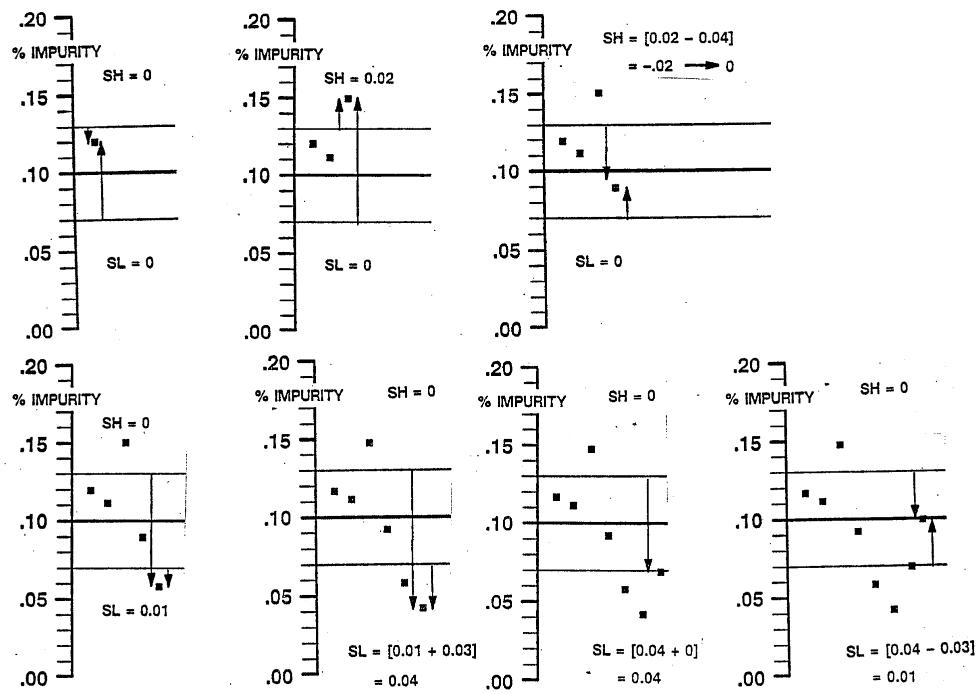
  If x_i is outside the slack band then one of the one-sided cusums increases while the other decreases (or stays at zero).

  If x_i is inside the slack band then both of the one-sided cusums decrease (or stay at zero).

Example: For a chemical process, assume the impurity aim is $\mu_0 = .10$ with $\sigma_0 = .06$. If cusum parameter $k = .5$, then $K = k\sigma = .03$. Thus, the slack band is (.07,.13).



- Suppose the first 8 samples yield $x = .12, .11, .15, .09, .06, .04, .07, .10$. The following plot shows geometrically what occurs for the tabular cusum. (Note: the plot for sample 2 is skipped.)



		Upper Cusum		Lower Cusum	
Sample i	Impurity x_i	$x_i - .13$	C_i^+	$.07 - x_i$	C_i^-
1	.12	$.12 - .13 = -.01$	0	$.07 - .12 = -.05$	0
2	.11	$.11 - .13 = -.02$	0	$.07 - .11 = -.04$	0
3	.15	$.15 - .13 = +.02$	.02	$.07 - .15 = -.09$	0
4	.09	$.09 - .13 = -.04$	0	$.07 - .09 = -.02$	0
5	.06	$.06 - .13 = -.07$	0	$.07 - .06 = +.01$	.01
6	.04	$.04 - .13 = -.09$	0	$.07 - .04 = +.03$	.04
7	.07	$.07 - .13 = -.06$	0	$.07 - .07 = 0$	.04
8	.10	$.10 - .13 = -.03$	0	$.07 - .10 = -.03$	.01

- The next step is to set a bound $H = h\sigma$ for C_i^+ and C_i^- for signalling an out-of-control process. Thus, we are now considering a choice of h, the second tabular cusum parameter.
- The rule for detection of a shift in the process mean is based on the second cusum parameter h. If a value in either the C_i^+ or C_i^- columns exceeds H, where $H = h\sigma$, then an out-of-control signal is indicated. An investigation for an assignable cause should be carried out and the process should be adjusted accordingly.

Example: Reconsider the percent solids example where a shift in the mean to $\mu = 46$ occurred after sample 23. The following table contains a summary of the first 29 samples.

If $h = 4$ and $\sigma = 1$, then $H = (4)(1) = 4$. Thus, if $C_i^+ \geq 4$ or $C_i^- \geq 4$, we get an out-of-control signal. This occurs for the first time on sample 29 when $C_{29}^+ = 4.3$.

Sample i	x_i	$x_i - 45.5$	C_i^+	N_i^+	$44.5 - x_i$	C_i^-	N_i^-
1	43.7	-1.8	0.0	0	0.8	0.8	1
2	44.4	-1.1	0.0	0	0.1	0.9	2
3	45.0	-0.5	0.0	0	-0.5	0.4	3
4	44.1	-1.4	0.0	0	0.4	0.8	4
5	46.4	0.9	0.9	1	-1.9	0.0	0
6	43.6	-1.9	0.0	0	0.9	0.9	1
7	46.2	0.7	0.7	1	-1.7	0.0	0
8	43.5	-2.0	0.0	0	1.0	1.0	1
9	44.5	-1.0	0.0	0	0.0	1.0	2
10	46.3	0.8	0.8	1	-1.8	0.0	0
11	45.9	0.4	1.2	2	-1.4	0.0	0
12	45.3	-0.2	1.0	3	-0.8	0.0	0
13	44.2	-1.3	0.0	0	0.3	0.3	1
14	44.4	-1.1	0.0	0	0.1	0.4	2
15	46.8	1.3	1.3	1	-2.3	0.0	0
16	44.2	-1.3	0.0	0	0.3	0.3	1
17	45.6	0.1	0.1	1	-1.1	0.0	0
18	44.9	-0.6	0.0	0	-0.4	0.0	0
19	46.1	0.6	0.6	1	-1.6	0.0	0
20	46.4	0.9	1.5	2	-1.9	0.0	0
21	43.8	-1.7	0.0	0	0.7	0.7	1
22	44.3	-1.2	0.0	0	0.2	0.9	2
23	44.5	-1.0	0.0	0	0.0	0.9	3
24	46.0	0.5	0.5	1	-1.5	0.0	0
25	47.2	1.7	2.2	2	-2.7	0.0	0
26	46.1	0.6	2.8	3	-1.6	0.0	0
27	45.9	0.4	3.2	4	-1.4	0.0	0
28	45.3	-0.2	3.0	5	-0.8	0.0	0
29	46.8	1.3	**4.3**	6	-2.3	0.0	0

- To estimate the new mean value of the process characteristic, use:

$$\widehat{\mu} = \begin{cases} & \text{if } C_i^+ > H \\ & \text{if } C_i^- > H. \end{cases} \tag{23}$$

N^+ is a count of the number of consecutive samples for which $C_i^+ > 0$.

N^- is a count of the number of consecutive samples for which $C_i > 0$. where N^+ or N^- is the sample at which the out of control signal was detected.

The quantity $\frac{C_i^+}{N^+}$ is an estimate of the amount the current mean is above $\mu_0 + K$ when a signal occurs with C_i^+.

The quantity $\frac{C_i^-}{N^-}$ is an estimate of the amount the current mean is below $\mu_0 - K$ when a signal occurs with C_i^-.

- On sample 29, we have $N_{29}^+ = 6$ consecutive samples with $C_i^+ > 0$ beginning at sample 24. Thus, the cusum indicates the shift began at sample 24.
- The estimated process mean beginning at sample 24 is

$$\widehat{\mu} \;=\; \mu_0 + K + \frac{C_i^+}{N^+} \;=$$

- Typically, both cusums are reset to zero after an out-of-control signal and the cusum procedure is restarted once the adjustments have been made.
- The in-control ARL is denoted ARL_0, and when the process is out-of-control, the ARL is denoted ARL_1.
- These correspond to a null hypothesis H_0 and an alternative hypothesis H_1. For 3σ Shewhart charts:
  - The in-control $ARL_0 \approx$
  - If a 1σ shift occurs in the process, the out-of-control $ARL_1 \approx$
  - If a 2σ shift occurs in the process, the out-of-control $ARL_1 \approx$
- If we can set up a cusum chart such that $ARL_0 = 370$ and $ARL_1 < 44$ for a 1σ shift, the cusum chart have the same α as the Shewhart chart but would be more powerful (smaller β) in detecting a 1σ shift.
- The same is true of any $\delta\sigma$ shift. For example, if we can set up a cusum chart such that $ARL_0 = 370$ and $ARL_1 < 6.3$ for a 2σ shift, the cusum chart have the same α as the Shewhart chart but would be more powerful (smaller β) in detecting a 2σ shift.
- In general, cusum charts are better for detecting small shifts in the process.
- Initially, we will concentrate on cusum procedures for x and $\overline{x}$.
- The h and k parameters of the cusum are specified by the user. Choosing these values will be discussed later.
- The following table (from the *SAS-QC* documentation) gives cusum chart ARL's for given values of h and k across various values of δ.

Average run lengths for cusum charts.

Parameters		δ (shift in mean)										
h	k	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00
2.50	0.25	13.64	11.22	7.67	5.38	4.06	2.71	2.06	1.68	1.42	1.11	1.01
4.00	0.25	38.54	24.71	13.20	8.38	6.06	3.91	2.93	2.38	2.05	1.61	1.23
6.00	0.25	125.40	50.33	20.89	12.37	8.73	5.51	4.07	3.26	2.74	2.13	1.90
8.00	0.25	368.39	83.63	28.76	16.37	11.39	7.11	5.21	4.15	3.48	2.67	2.14
10.00	0.25	1035.75	124.55	36.71	20.37	14.06	8.71	6.36	5.04	4.20	3.20	2.65
2.00	0.50	19.27	15.25	9.63	6.27	4.44	3.74	1.99	1.58	1.32	1.07	1.01
3.00	0.50	58.80	36.24	17.20	9.67	6.40	3.75	2.68	2.12	1.77	1.31	1.07
4.00	0.50	167.68	74.22	26.63	13.29	8.38	4.75	3.34	2.62	2.19	1.71	1.31
5.00	0.50	465.44	139.49	38.00	17.05	10.38	5.75	4.01	3.11	2.57	2.01	1.69
6.00	0.50	1276.55	249.26	51.34	20.90	12.37	6.75	4.68	3.62	2.98	2.24	1.95
1.50	0.75	21.28	17.22	11.01	7.00	4.77	2.73	1.90	1.48	1.24	1.04	1.00
2.25	0.75	69.85	45.97	22.04	11.63	7.13	3.73	2.51	1.91	1.56	1.16	1.02
3.00	0.75	221.40	110.95	39.31	17.34	9.68	4.73	3.12	2.36	1.93	1.41	1.11
3.75	0.75	687.85	251.56	65.58	24.16	12.37	5.73	3.71	2.79	2.27	1.72	1.31
4.50	0.75	2125.85	552.11	105.09	32.09	15.15	6.73	4.31	3.21	2.59	1.97	1.60
1.00	1.00	17.65	15.03	10.39	6.88	4.72	2.63	1.78	1.38	1.17	1.02	1.00
1.50	1.00	46.92	35.70	20.31	11.49	7.07	3.50	2.24	1.66	1.34	1.07	1.01
2.00	1.00	129.34	84.00	37.93	18.14	10.00	4.45	2.74	1.99	1.58	1.16	1.02
2.50	1.00	358.00	191.48	67.76	27.25	13.43	5.42	3.25	2.34	1.85	1.31	1.07
3.00	1.00	981.39	423.29	117.32	39.47	17.35	6.40	3.75	2.68	2.12	1.52	1.16
3.50	1.00	2670.70	917.89	199.40	55.69	21.76	7.39	4.25	3.01	2.37	1.73	1.31
0.70	1.50	33.86	28.41	18.90	11.84	7.59	3.66	2.18	1.55	1.25	1.04	1.00
1.10	1.50	92.14	71.41	40.91	22.29	12.71	5.17	2.80	1.86	1.43	1.08	1.01
1.50	1.50	274.84	191.58	91.58	42.39	21.07	7.09	3.50	2.24	1.66	1.16	1.02
1.90	1.50	881.05	536.07	208.31	80.41	34.25	9.38	4.26	2.64	1.92	1.29	1.05
2.30	1.50	2948.65	1523.15	474.09	150.96	54.47	12.00	5.03	3.04	2.20	1.45	1.12

Using SAS:

- **Piston Ring Diameter Cusum Example (σ known)**: Previously, we made $\overline{X}/R$ and $\overline{X}/S$ charts for the piston ring diameter data. The data set contained 25 samples with $n = 5$.
- If $h = 4$, $\sigma = .005$, and $n = 5$, then $\sigma_{\overline{x}} =$ $\approx$.

  and $H = h\sigma_{\overline{x}} = (4)(.002236) \approx .008944$.
- Thus, if $C_i^+ \geq .008944$ or $C_i^- \geq .008944$, we get an out-of-control signal.
- The following table contains the tabular cusum values with resets after each signal.

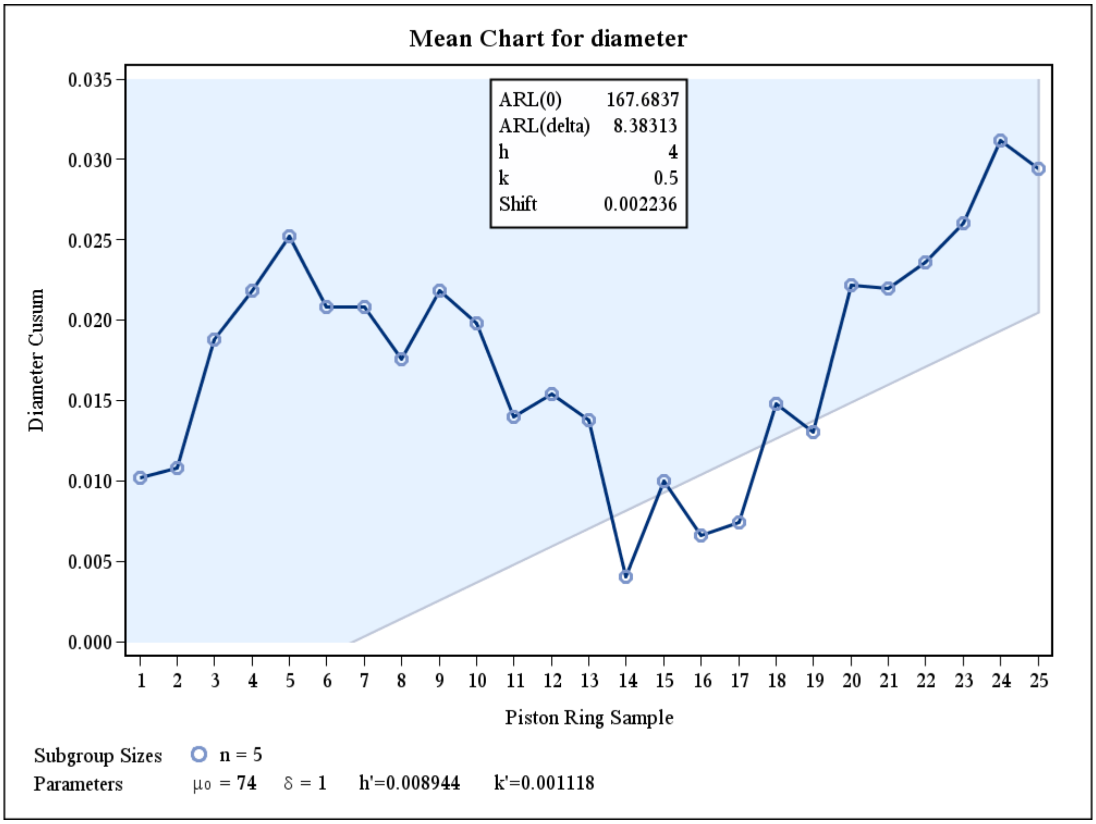
CUSUM with Reset after Signal (sigma known)

sample	xbar	n	cusum_l	hsigma	cusum_h	flag
1	74.0102	5	0.000000	.008944272	0.009082	upper
2	74.0006	5	0.000000	.008944272	0.000000	
3	74.0080	5	0.000000	.008944272	0.006882	
4	74.0030	5	0.000000	.008944272	0.008764	
5	74.0034	5	0.000000	.008944272	0.011046	upper
6	73.9956	5	0.003282	.008944272	0.000000	
7	74.0000	5	0.002164	.008944272	0.000000	
8	73.9968	5	0.004246	.008944272	0.000000	
9	74.0042	5	0.000000	.008944272	0.003082	
10	73.9980	5	0.000882	.008944272	0.000000	
11	73.9942	5	0.005564	.008944272	0.000000	
12	74.0014	5	0.003046	.008944272	0.000282	
13	73.9984	5	0.003528	.008944272	0.000000	
14	73.9902	5	0.012210	.008944272	0.000000	lower
15	74.0060	5	0.000000	.008944272	0.004882	
16	73.9966	5	0.002282	.008944272	0.000364	
17	74.0008	5	0.000364	.008944272	0.000046	
18	74.0074	5	0.000000	.008944272	0.006328	
19	73.9982	5	0.000682	.008944272	0.003410	
20	74.0092	5	0.000000	.008944272	0.011492	upper
21	73.9998	5	0.000000	.008944272	0.000000	
22	74.0016	5	0.000000	.008944272	0.000482	
23	74.0024	5	0.000000	.008944272	0.001764	
24	74.0052	5	0.000000	.008944272	0.005846	
25	73.9982	5	0.000682	.008944272	0.002928	

- We will now make a cusum plot for the 25 sample means ($n = 5$) assuming a process in-control mean $\mu_0 = 74$ with process standard deviation $\sigma = .005$ (known or specified prior to data collection).

CUSUM for Piston-Ring Diameters (sigma known)

The CUSUM Procedure



CUSUM for Piston-Ring Diameters (sigma known)

The CUSUM Procedure

Cumulative Sum Chart Summary for diameter

sample	Subgroup Sample Size	Subgroup Mean	Subgroup Std Dev	V-Mask Lower Limit	Cusum	V-Mask Upper Limit
1	5	74.010200	0.01477159	-0.0064	0.01020000	0.0652
2	5	74.000600	0.00750333	-0.0053	0.01080000	0.0641
3	5	74.008000	0.01474788	-0.0041	0.01880000	0.0629
4	5	74.003000	0.00908295	-0.0030	0.02180000	0.0618
5	5	74.003400	0.01221884	-0.0019	0.02520000	0.0607
6	5	73.995600	0.00870632	-0.0008	0.02080000	0.0596
7	5	74.000000	0.00552268	0.0003	0.02080000	0.0585
8	5	73.996800	0.01225561	0.0014	0.01760000	0.0574
9	5	74.004200	0.00554076	0.0026	0.02180000	0.0562
10	5	73.998000	0.00628490	0.0037	0.01980000	0.0551
11	5	73.994200	0.00286356	0.0048	0.01400000	0.0540
12	5	74.001400	0.00421900	0.0059	0.01540000	0.0529
13	5	73.998400	0.01045466	0.0070	0.01380000	0.0518
14	5	73.990200	0.01530359	0.0082	0.00400000	0.0506
15	5	74.006000	0.00731437	0.0093	0.01000000	0.0495
16	5	73.996600	0.00779744	0.0104	0.00660000	0.0484
17	5	74.000800	0.01056882	0.0115	0.00740000	0.0473
18	5	74.007400	0.00698570	0.0126	0.01480000	0.0462
19	5	73.998200	0.00846759	0.0137	0.01300000	0.0451
20	5	74.009200	0.00798123	0.0149	0.02220000	0.0439
21	5	73.999800	0.00816701	0.0160	0.02200000	0.0428
22	5	74.001600	0.00743640	0.0171	0.02360000	0.0417
23	5	74.002400	0.01192896	0.0182	0.02600000	0.0406
24	5	74.005200	0.00870057	0.0193	0.03120000	0.0395
25	5	73.998200	0.01617714	0.0205	0.02940000	0.0383

- We will now generate the **upper and lower tabular cusums** for the 25 sample means followed by one-sided cusum plots.

UPPER ONE-SIDED CUSUM

The CUSUM Procedure

Cumulative Sum Chart Summary for diameter						
sample	**Subgroup Sample Size**	**Subgroup Mean**	**Subgroup Std Dev**	**Cusum**	**Decision Interval**	**Decision Interval Exceeded**
1	5	74.010200	0.01477159	0.00908197	0.0089	Upper
2	5	74.000600	0.00750333	0.00856393	0.0089	
3	5	74.008000	0.01474788	0.01544590	0.0089	Upper
4	5	74.003000	0.00908295	0.01732786	0.0089	Upper
5	5	74.003400	0.01221884	0.01960983	0.0089	Upper
6	5	73.995600	0.00870632	0.01409180	0.0089	Upper
7	5	74.000000	0.00552268	0.01297376	0.0089	Upper
8	5	73.996800	0.01225561	0.00865573	0.0089	
9	5	74.004200	0.00554076	0.01173769	0.0089	Upper
10	5	73.998000	0.00628490	0.00861966	0.0089	
11	5	73.994200	0.00286356	0.00170163	0.0089	
12	5	74.001400	0.00421900	0.00198359	0.0089	
13	5	73.998400	0.01045466	0.00000000	0.0089	
14	5	73.990200	0.01530359	0.00000000	0.0089	
15	5	74.006000	0.00731437	0.00488197	0.0089	
16	5	73.996600	0.00779744	0.00036393	0.0089	
17	5	74.000800	0.01056882	0.00004590	0.0089	
18	5	74.007400	0.00698570	0.00632786	0.0089	
19	5	73.998200	0.00846759	0.00340983	0.0089	
20	5	74.009200	0.00798123	0.01149180	0.0089	Upper
21	5	73.999800	0.00816701	0.01017376	0.0089	Upper
22	5	74.001600	0.00743640	0.01065573	0.0089	Upper
23	5	74.002400	0.01192896	0.01193769	0.0089	Upper
24	5	74.005200	0.00870057	0.01601966	0.0089	Upper
25	5	73.998200	0.01617714	0.01310163	0.0089	Upper

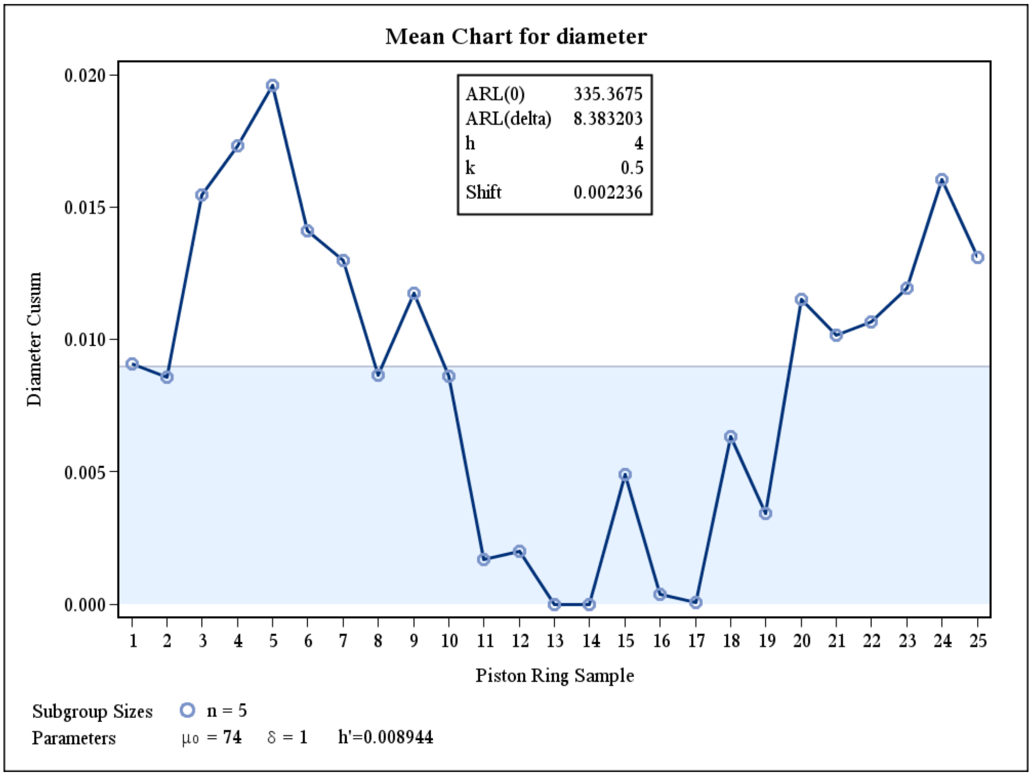
LOWER ONE-SIDED CUSUM

The CUSUM Procedure

Cumulative Sum Chart Summary for diameter						
sample	**Subgroup Sample Size**	**Subgroup Mean**	**Subgroup Std Dev**	**Cusum**	**Decision Interval**	**Decision Interval Exceeded**
1	5	74.010200	0.01477159	0.00000000	0.0089	
2	5	74.000600	0.00750333	0.00000000	0.0089	
3	5	74.008000	0.01474788	0.00000000	0.0089	
4	5	74.003000	0.00908295	0.00000000	0.0089	
5	5	74.003400	0.01221884	0.00000000	0.0089	
6	5	73.995600	0.00870632	0.00328197	0.0089	
7	5	74.000000	0.00552268	0.00216393	0.0089	
8	5	73.996800	0.01225561	0.00424590	0.0089	
9	5	74.004200	0.00554076	0.00000000	0.0089	
10	5	73.998000	0.00628490	0.00088197	0.0089	
11	5	73.994200	0.00286356	0.00556393	0.0089	
12	5	74.001400	0.00421900	0.00304590	0.0089	
13	5	73.998400	0.01045466	0.00352786	0.0089	
14	5	73.990200	0.01530359	0.01220983	0.0089	Upper
15	5	74.006000	0.00731437	0.00509180	0.0089	
16	5	73.996600	0.00779744	0.00737376	0.0089	
17	5	74.000800	0.01056882	0.00545573	0.0089	
18	5	74.007400	0.00698570	0.00000000	0.0089	
19	5	73.998200	0.00846759	0.00068197	0.0089	
20	5	74.009200	0.00798123	0.00000000	0.0089	
21	5	73.999800	0.00816701	0.00000000	0.0089	
22	5	74.001600	0.00743640	0.00000000	0.0089	
23	5	74.002400	0.01192896	0.00000000	0.0089	
24	5	74.005200	0.00870057	0.00000000	0.0089	
25	5	73.998200	0.01617714	0.00068197	0.0089	

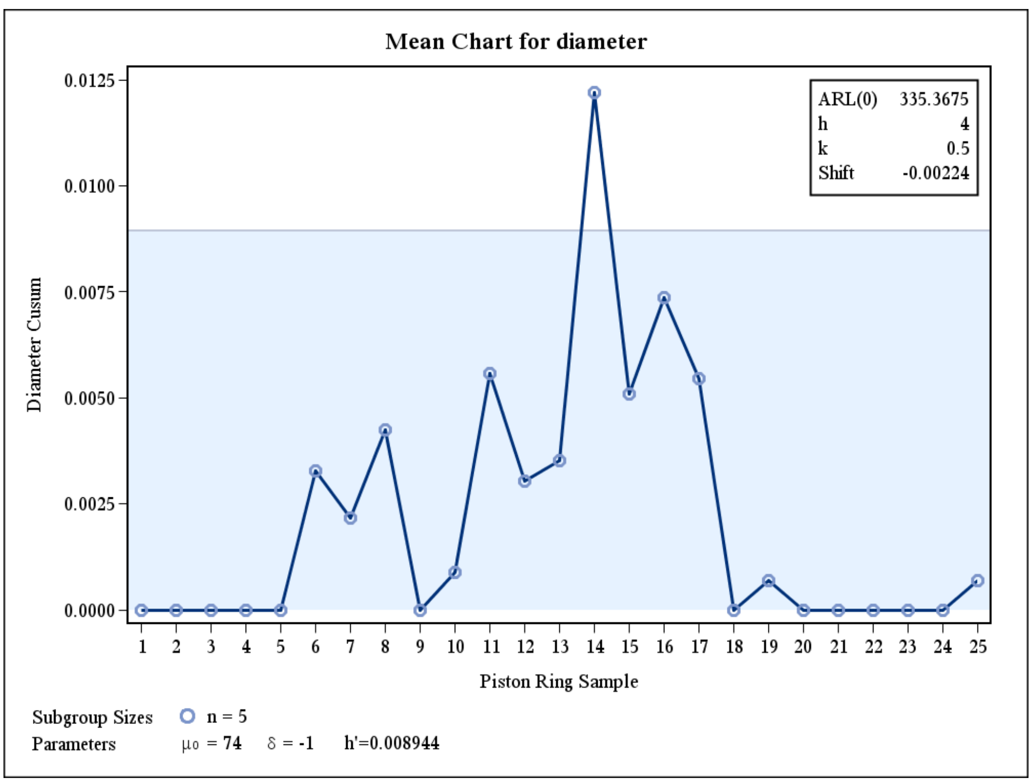
UPPER ONE-SIDED CUSUM

The CUSUM Procedure



LOWER ONE-SIDED CUSUM

The CUSUM Procedure



SAS Cusum Code for Piston-Ring Diameter Data

```
DM 'LOG; CLEAR; OUT; CLEAR;';
OPTIONS NODATE NONUMBER;

DATA piston;
     DO sample=1 TO 25;
     DO item=1 TO 5;
        INPUT diameter @@;
        diameter = diameter+70; OUTPUT;
     END; END;
 LINES;
 4.030 4.002 4.019 3.992 4.008   3.995 3.992 4.001 4.011 4.004
 3.988 4.024 4.021 4.005 4.002   4.002 3.996 3.993 4.015 4.009
 3.992 4.007 4.015 3.989 4.014   4.009 3.994 3.997 3.985 3.993
 3.995 4.006 3.994 4.000 4.005   3.985 4.003 3.993 4.015 3.988
 4.008 3.995 4.009 4.005 4.004   3.998 4.000 3.990 4.007 3.995
 3.994 3.998 3.994 3.995 3.990   4.004 4.000 4.007 4.000 3.996
 3.983 4.002 3.998 3.997 4.012   4.006 3.967 3.994 4.000 3.984
 4.012 4.014 3.998 3.999 4.007   4.000 3.984 4.005 3.998 3.996
 3.994 4.012 3.986 4.005 4.007   4.006 4.010 4.018 4.003 4.000
 3.984 4.002 4.003 4.005 3.997   4.000 4.010 4.013 4.020 4.003
 3.988 4.001 4.009 4.005 3.996   4.004 3.999 3.990 4.006 4.009
 4.010 3.989 3.990 4.009 4.014   4.015 4.008 3.993 4.000 4.010
 3.982 3.984 3.995 4.017 4.013
 ;
SYMBOL1 v=dot width=3;

PROC CUSUM DATA=piston;
     XCHART diameter*sample='1'
           / MU0=74 SIGMA0=.005  H=4.0 K=0.5  DELTA=1.0
             DATAUNITS HAXIS = 1 TO 25
             TABLESUMMARY OUTTABLE = qsum  ;
     INSET ARL0 ARLDELTA H K SHIFT / POS = n;
     LABEL diameter='Diameter Cusum'
           sample = 'Piston Ring Sample';
TITLE 'CUSUM for Piston-Ring Diameters (sigma known)';

PROC CUSUM DATA=piston;
     XCHART diameter*sample='1'
           / MU0=74 SIGMA0=.005  H=4.0 K=0.5  DELTA=1.0
             DATAUNITS HAXIS=1 TO 25
             SCHEME=onesided TABLESUMMARY TABLEOUT;
                     INSET ARL0 ARLDELTA H K SHIFT / POS = n;
     LABEL diameter='Diameter Cusum'
           sample = 'Piston Ring Sample';
TITLE 'UPPER ONE-SIDED CUSUM';
```

```
PROC CUSUM DATA=piston;
     XCHART diameter*sample='1'
           / MU0=74 SIGMA0=.005  H=4.0 K=0.5  DELTA=-1.0
             DATAUNITS HAXIS=1 TO 25
             SCHEME=onesided TABLESUMMARY TABLEOUT;
     INSET ARL0 H K SHIFT / POS = ne;
     LABEL diameter='Diameter Cusum'
           sample = 'Piston Ring Sample';
TITLE 'LOWER ONE-SIDED CUSUM';



*** The following code will make a table with resetting ***;
***     after an out-of-control signal is detected      ***;

DATA qsum; SET qsum;
     h=4;
     k=.5;
     sigma=.005;
     aim=74;          ** enter values **;

     xbar=_subx_;  n=_subn_;
     hsigma=h*sigma/SQRT(_subn_);
     ksigma=k*sigma/SQRT(_subn_);
     RETAIN cusum_l 0 cusum_h 0;
     IF (-hsigma < cusum_l < hsigma)            THEN DO;
             cusum_l = cusum_l + (aim - ksigma) - xbar;
             IF cusum_l < 0 then cusum_l=0;     END;
     IF (-hsigma < cusum_h < hsigma)            THEN DO;
             cusum_h = cusum_h + xbar - (aim + ksigma);
             IF cusum_h < 0 then cusum_h=0;     END;

     IF MAX(cusum_l,cusum_h) ge hsigma   THEN DO;
        IF (cusum_l ge hsigma) THEN DO;
                             flag='lower';  OUTPUT; END;
        IF (cusum_h ge hsigma) THEN DO;
                             flag='upper';  OUTPUT; END;
        cusum_l=0; cusum_h=0;            END;
     ELSE OUTPUT;

PROC PRINT DATA=qsum;
     ID sample;
     VAR xbar n cusum_l hsigma cusum_h flag;
TITLE 'CUSUM with Reset after Signal (sigma known)';

RUN;
```