# Gromov-Witten Invariants in Algebraic Geometry

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#### Abstract

Gromov-Witten invariants for arbitrary projective varieties and arbitrary genus are constructed using the techniques from [K. Behrend, B. Fantechi. *The Intrinsic Normal Cone.*]

# Introduction

In [2] the problem of constructing the Gromov-Witten invariants of a smooth projective variety V was reduced to defining a 'virtual fundamental class'

$$[\overline{M}_{g,n}(V,\beta)]^{\mathrm{virt}} \in A_{(1-g)(\dim V - 3) - \beta(\omega_V) + n}(\overline{M}_{g,n}(V,\beta))$$

in the Chow group of the algebraic stack

$$\overline{M}_{g,n}(V,\beta)$$

of stable maps of class  $\beta \in H_2(V)$  from an *n*-marked prestable curve of genus q to V.

If g=0 and V is convex (i.e.  $H^1(\mathbb{P}^1, f^*T_V)=0$ , for all  $f:\mathbb{P}^1\to V$ ), then  $\overline{M}_{0,n}(V,\beta)$  is smooth of the expected dimension  $\dim V-3-\beta(\omega_V)+n$  and the usual fundamental class

$$[\overline{M}_{g,n}(V,\beta)]$$

will work. This was proved in [2].

In this paper we treat the general case using the construction from [1]. Recall from [ibid.] that virtual fundamental classes are constructed using an obstruction theory, and the intrinsic normal cone. The obstruction theory serves to give rise to a vector bundle stack  $\mathfrak{E}$ , into which the intrinsic normal cone  $\mathfrak{C}$  can be embedded as a closed subcone stack. The virtual fundamental class is then obtained by intersecting  $\mathfrak{C}$  with the zero section of  $\mathfrak{E}$ .

In our context, this process works as follows. Let  $\mathfrak{M}_{g,n}$  be the algebraic stack of n-marked prestable curves of genus g. This is an algebraic stack, not of Deligne-Mumford (or even finite) type, but smooth of dimension 3(g-1)+n. There is a canonical morphism

$$\overline{M}_{g,n}(V,\beta) \to \mathfrak{M}_{g,n},$$

given by forgetting the map, retaining the curve (but not stabilizing). Then  $\overline{M}_{g,n}(V,\beta) \to \mathfrak{M}_{g,n}$  is an open substack of a stack of morphisms, and as such has a relative obstruction theory, which in this case is  $(R\pi_*f^*T_V)^{\vee}$ , where  $\pi: C \to \overline{M}_{g,n}(V,\beta)$  is the universal curve and  $f: C \to V$  is the universal stable map. Saying that  $(R\pi_*f^*T_V)^{\vee}$  is a relative obstruction theory means that there is a homomorphism

$$\phi: (R\pi_* f^* T_V)^{\vee} \longrightarrow L^{\bullet}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}},$$

(where  $L^{\bullet}$  is the cotangent complex) such that  $h^{0}(\phi)$  is an isomorphism and  $h^{-1}(\phi)$  is surjective.

The homomorphism  $\phi$  induces a closed immersion

$$\phi^{\vee}: \mathfrak{N}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}} \longrightarrow h^1/h^0(R\pi_*f^*T_V)$$

of abelian cone stacks (see [1]) over  $\overline{M}_{g,n}(V,\beta)$ , where  $\mathfrak N$  is the relative intrinsic normal sheaf. The relative intrinsic normal cone  $\mathfrak C_{\overline{M}_{g,n}(V,\beta)/\mathfrak M_{g,n}}$  is a closed subcone stack of  $\mathfrak N_{\overline{M}_{g,n}(V,\beta)/\mathfrak M_{g,n}}$ , and so we get a closed immersion of cone stacks

$$\mathfrak{C}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}} \longrightarrow h^1/h^0(R\pi_*f^*T_V).$$

Now since  $R\pi_*f^*T_V$  has global resolutions (see Proposition 5), we may intersect  $\mathfrak{C}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}}$  with the zero section of the vector bundle stack  $h^1/h^0(R\pi_*f^*T_V)$  to get the virtual fundamental class  $[\overline{M}_{g,n}(V,\beta)]^{\text{virt}}$ .

The fundamental axioms (see [5]) Gromov-Witten invariants need to satisfy to deserve their name are reduced in [2] to five basic compatibilities between the virtual fundamental classes. These follow from the basic properties proved in [1]. The dimension axiom, for example, follows from the basic fact that the intrinsic normal cone always has dimension zero.

We also show that if V = G/P, for a reductive group G and a parabolic subgroup P, there is an alternative construction of the virtual fundamental classes avoiding the intrinsic normal cone. We construct a cone C in the vector bundle  $R^1\pi_*\mathcal{O}\otimes\mathfrak{g}$  on  $\overline{M}_{q,n}(V,\beta)$ , which may then be intersected with

the zero section of  $R^1\pi_*\mathcal{O}\otimes\mathfrak{g}$  to obtain the virtual fundamental class. This cone C is constructed as the normal cone of an embedding of  $\overline{M}_{g,n}(V,\beta)$  into a certain stack of principal P-bundles (which is smooth, but not of Deligne-Mumford type).

A construction of Gromov-Witten invariants using a cone inside a vector bundle has also been announced by J. Li and G. Tian. Their methods differ from ours in that they use analytic methods, including the Kuranishi map.

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#### Preliminaries on Prestable Curves

Let k be a field. We shall work over the category of locally noetherian k-schemes (with the fppf-topology). For a modular graph  $\tau$  (see [2], Definition 1.5) let  $\mathfrak{M}(\tau)$  denote the k-stack of  $\tau$ -marked prestable curves (which are defined in [2], Definition 2.6).

**Lemma 1** The algebraic k-stack  $\overline{M}(\tau)$  of stable  $\tau$ -marked curves is an open substack of  $\mathfrak{M}(\tau)$ .

PROOF. Let  $C_v \to \mathfrak{M}(\tau)$  be the universal curve corresponding to the vertex  $v \in V_{\tau}$ . Let  $\widetilde{C}_v$  be the stabilization. Then  $\overline{M}(\tau)$  is the substack of  $\mathfrak{M}(\tau)$  over which all  $p_v : C_v \to \widetilde{C}_v$  are isomorphisms. This is open because the  $C_v$  are proper over  $\mathfrak{M}(\tau)$ .  $\square$ 

Now consider a modular graph  $\tau'$  obtained from  $\tau$  by adding some tails. We get an induced morphism of k-stacks  $\mathfrak{M}(\tau') \to \mathfrak{M}(\tau)$  which simply forgets the markings corresponding to the tails  $S_{\tau'} - S_{\tau}$ . If  $S_{\tau'} - S_{\tau}$  has cardinality 1, then  $\mathfrak{M}(\tau') \to \mathfrak{M}(\tau)$  is a smooth curve, hence representable and smooth of relative dimension 1. So by induction,  $\mathfrak{M}(\tau') \to \mathfrak{M}(\tau)$  is representable and smooth of relative dimension  $\#(S_{\tau'} - S_{\tau})$ . By Lemma 1 the same is true for  $\overline{M}(\tau') \to \mathfrak{M}(\tau)$ .

**Proposition 2** The stack  $\mathfrak{M}(\tau)$  is a smooth algebraic k-stack of dimension

$$\dim(\tau) = \#S_{\tau} - \#E_{\tau} - 3\chi(\tau).$$

PROOF. For the definition of  $\dim(\tau)$  and  $\chi(\tau)$  see [2], Definitions 6.1 and 6.2. Note that for every point of  $\mathfrak{M}(\tau)$  there exists a  $\tau'$  as above such that the induced morphism  $\overline{M}(\tau') \to \mathfrak{M}(\tau)$  contains this given point in its image. Thus  $\coprod_{\tau'} \overline{M}(\tau')$  is a presentation of  $\mathfrak{M}(\tau)$  showing that  $\mathfrak{M}(\tau)$  is algebraic.  $\square$ 

Now let  $\tau^s$  be the stabilization of  $\tau$ . Stabilization defines a morphism of algebraic k-stacks

$$s:\mathfrak{M}(\tau)\longrightarrow \overline{M}(\tau^s).$$

If  $\tau'$  is obtained as above by adjoining tails to  $\tau$  such that  $\tau'$  is stable, we have a commutative diagram

$$\overline{M}(\tau') \downarrow \qquad \searrow \overline{M}(\phi) \\ \mathfrak{M}(\tau) \stackrel{s}{\longrightarrow} \overline{M}(\tau^s).$$

Here  $\phi: \tau' \to \tau^s$  is the canonical morphism of stable modular graphs. In fact, one may define s locally by using such diagrams.

**Proposition 3** The morphism  $s: \mathfrak{M}(\tau) \to \overline{M}(\tau^s)$  is flat.

PROOF. This follows by descent since the morphism  $\overline{M}(\phi)$  for various  $\phi: \tau' \to \tau^s$  as above are flat.  $\square$ 

#### The Virtual Fundamental Classes

Over  $\mathfrak{M}(\tau)$  there is a family  $(\mathcal{C}_v)_{v \in V_{\tau}}$  of universal curves, with sections  $x_i$ :  $\mathfrak{M}(\tau) \to \mathcal{C}_{\partial_{\tau}(i)}$ . Let  $\mathcal{C}(\tau) \to \mathfrak{M}(\tau)$  be the curve obtained from  $\coprod_{v \in V_{\tau}} \mathcal{C}_v$  by identifying  $x_i$  and  $x_j$ , for every edge  $\{i,j\} \in E_{\tau}$ . The curve  $\mathcal{C}(\tau)$  has markings  $x_i : \mathfrak{M}(\tau) \to \mathcal{C}(\tau)$ , for each  $i \in S_{\tau}$ . In fact,  $\mathcal{C}(\tau)$  is a  $\tilde{\tau}$ -marked prestable curve, where  $\tilde{\tau}$  is the graph obtained from  $\tau$  by contracting all edges of  $\tau$ . Let us denote the structure morphism by

$$\pi: \mathcal{C}(\tau) \longrightarrow \mathfrak{M}(\tau).$$

We shall also denote any base change of  $\pi$  by  $\pi$ .

Now let V be a smooth projective k-variety,  $(\tau, \beta)$  a stable V-graph and let  $\operatorname{Mor}_{\mathfrak{M}(\tau)}(\tau, V)$  be the  $\mathfrak{M}(\tau)$ -space of morphisms from  $\mathcal{C}(\tau)$  to V. Denote the universal morphism by

$$f: \mathcal{C}(\tau) \times \mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau, V) \longrightarrow V.$$

By [3] the stack  $\operatorname{Mor}_{\mathfrak{M}(\tau)}(\tau, V)$  is an algebraic k-stack and the structure morphism

$$\operatorname{Mor}_{\mathfrak{M}(\tau)}(\tau, V) \longrightarrow \mathfrak{M}(\tau)$$

is representable.

**Proposition 4** The proper Deligne-Mumford stack  $\overline{M}(V,\tau,\beta)$  of stable maps is an open substack of  $\operatorname{Mor}_{\mathfrak{M}(\tau)}(\tau,V)$ .

Proof. The set of points where stabilization is an isomorphism is open.  $\Box$ 

To define the virtual fundamental class on  $\overline{M}(V, \tau, \beta)$  we consider the morphism  $\overline{M}(V, \tau, \beta) \to \mathfrak{M}(\tau)$  and denote the relative intrinsic normal cone (see [1]) by

$$\mathfrak{C}(V,\tau,\beta) = \mathfrak{C}_{\overline{M}(V,\tau,\beta)/\mathfrak{M}(\tau)}$$

The intrinsic normal sheaf [ibid.] of  $\overline{M}(V, \tau, \beta)$  over  $\mathfrak{M}(\tau)$  we shall denote by  $\mathfrak{N}(V, \tau, \beta)$ .

By the relative version of [1] Proposition 6.2 we have a perfect relative obstruction theory [ibid.]

$$\pi_*(e^\vee)^\vee : R\pi_*(f^*T_V)^\vee \longrightarrow L^{\bullet}_{\mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau,V)/\mathfrak{M}(\tau)}.$$

Restricting to the open substack  $\overline{M}(V,\tau,\beta)$  we get a perfect relative obstruction theory

$$\pi_*(e^{\vee})^{\vee}: R\pi_*(f^*T_V)^{\vee} \longrightarrow L^{\bullet}_{\overline{M}(V,\tau,\beta)/\mathfrak{M}(\tau)},$$

which we shall also denote by  $E^{\bullet}(V, \tau, \beta)$ . Thus  $\mathfrak{C}(V, \tau, \beta)$  is embedded as a closed subcone stack in the vector bundle stack

$$\mathfrak{E}(V,\tau,\beta) = h^1/h^0(R\pi_*f^*T_V).$$

Note that the relative virtual dimension of  $\overline{M}(V, \tau, \beta)$  over  $\mathfrak{M}(\tau)$  with respect to the obstruction theory  $R\pi_*(f^*T_V)^{\vee}$  is equal to

$$\operatorname{rk} R\pi_*(f^*T_V)^{\vee} = \chi(f^*T_V)$$

$$= \operatorname{deg} f^*T_V + \operatorname{dim} V \cdot \chi(\mathcal{C}(\tau))$$

$$= \chi(\tau) \operatorname{dim} V - \beta(\tau)(\omega_V).$$

Essential is the following result.

**Proposition 5** Let (C, x, f) be a stable map over T to V, where T is a finite type algebraic k-stack. Let E be a vector bundle on C. Then  $R\pi_*E$  has global resolutions, where  $\pi: C \to T$  is the structure map.

PROOF. Let M be an ample invertible sheaf on V and let

$$L = \omega_{C/T}(x_1 + \ldots + x_n) \otimes f^*M^{\otimes 3}.$$

By Proposition 3.9 of [2] the sheaf L is ample on the fibers of  $\pi$ . So for sufficiently large N we have that

- 1.  $\pi^*\pi_*(E \otimes L^{\otimes N}) \to E \otimes L^{\otimes N}$  is surjective,
- $2. R^1\pi_*(E\otimes L^{\otimes N})=0,$
- 3. for all  $t \in T$  we have that  $H^0(C_t, L_t^{\otimes -N}) = 0$ .

Let

$$F = \pi^* \pi_* (E \otimes L^{\otimes N}) \otimes L^{\otimes -N}$$

and let H be the kernel of the map  $F \to E$ . Thus we have a short exact sequence

$$0 \longrightarrow H \longrightarrow F \longrightarrow E \longrightarrow 0$$

of vector bundles on C. Note that for every  $t \in T$  we have

$$H^{0}(C_{t}, F) = H^{0}(C_{t}, \pi_{*}(E \otimes L^{\otimes N})_{t} \otimes L_{t}^{\otimes -N})$$

$$= H^{0}(C_{t}, L_{t}^{\otimes -N}) \otimes \pi_{*}(E \otimes L^{\otimes N})_{t}$$

$$= 0$$

and hence  $H^0(C_t, H) = 0$ , also. Therefore,  $\pi_* H$  and  $\pi_* F$  are zero and  $R^1 \pi_* H$  and  $R^1 \pi_* F$  are locally free. This implies that

$$R\pi_*E \cong [R^1\pi_*H \to R^1\pi_*F].$$

As shown in [1], by Proposition 5 the obstruction theory  $R\pi_*(f^*T_V)^{\vee}$  gives rise to a virtual fundamental class

$$[\overline{M}(V,\tau,\beta), R\pi_*(f^*T_V)^{\vee}] \in A_{\dim(V,\tau,\beta)}(\overline{M}(V,\tau,\beta)),$$

since

$$\dim \mathfrak{M}(\tau) + \operatorname{rk} R \pi_* (f^* T_V)^{\vee}$$

$$= \chi(\tau) (\dim V - 3) - \beta(\tau) (\omega_V) + \# S_{\tau} - \# E_{\tau}$$

$$= \dim(V, \tau, \beta).$$

(See Definition 6.2 in [2] for the definition of  $\dim(V, \tau, \beta)$ .)

**Theorem 6** The system of virtual fundamental classes

$$J(V, \tau, \beta) = [\overline{M}(V, \tau, \beta), R\pi_*(f^*T_V)^{\vee}]$$

is an orientation of  $\overline{M}$  over  $\mathfrak{G}_s(V)$ . If V is convex, on the tree level subcategory  $\mathfrak{T}_s(V)$ , we get back the orientation of [2], Theorem 7.5.

PROOF. If V is convex and  $\tau$  a forest, then  $R^1\pi_*(f^*T_V) = 0$ , so that the virtual fundamental class is the usual fundamental class by [1] Proposition 7.3. Thus the virtual fundamental class agrees with the orientation of [2], Theorem 7.5. To check that J is an orientation, we need to check the five axioms listed in [2], Definition 7.1. This shall be done in the next Section.  $\Box$ 

**Remark** As shown in [2], we get an associated system of Gromov-Witten classes for V.

## Checking the Axioms

#### AXIOM I. Mapping to a point

Let  $\tau$  be a stable V-graph of class zero such that  $|\tau|$  is non-empty and connected. As noted in [2] Section 7 we have

$$\overline{M}(V,\tau,0) = V \times \overline{M}(\tau)$$

which is obviously smooth over  $\mathfrak{M}(\tau)$ . In fact, the morphism  $\overline{M}(V,\tau,0) \to \mathfrak{M}(\tau)$  is just the composition

$$V\times \overline{M}(\tau) \longrightarrow \overline{M}(\tau) \longrightarrow \mathfrak{M}(\tau)$$

of projection followed by inclusion. If  $\widetilde{\pi}: \mathcal{C}(\tau) \to \overline{M}\tau$  is the universal curve over  $\overline{M}(\tau)$ , then  $\mathcal{C}(V,\tau,0) = V \times \mathcal{C}(\tau)$  and  $\pi: \mathcal{C}(V,\tau,0) \to \overline{M}(V,\tau,0)$  is identified with id  $\times \widetilde{\pi}: V \times \mathcal{C}(\tau) \to V \times \overline{M}(\tau)$ . Hence

$$R^{1}\pi_{*}f^{*}T_{V} = T_{V} \boxtimes R^{1}\widetilde{\pi}_{*}\mathcal{O}_{\mathcal{C}(\tau)}$$
$$= \mathcal{T}^{(1)}$$

is locally free. So by [1] Proposition 7.3 we have

$$\begin{split} J(V,\tau,0) &= c_{\operatorname{rk} R^1\pi_*f^*T_V}(R^1\pi_*f^*T_V) \cdot [\overline{M}(V,\tau,0)] \\ &= c_{g(\tau)\dim V}(\mathcal{T}^{(1)}) \cdot [\overline{M}(V,\tau,0)], \end{split}$$

which is Axiom I.

#### AXIOM II. Products

Let  $(\sigma, \alpha)$  and  $(\tau, \beta)$  be stable V-graphs and denote the 'product' by  $(\sigma \times \tau, \alpha \times \beta)$ . Note that

$$E^{\bullet}(V, \sigma \times \tau, \alpha \times \beta) = E^{\bullet}(V, \sigma, \alpha) \boxplus E^{\bullet}(V, \tau, \beta),$$

so by [1] Proposition 7.4 we have

$$J(V, \sigma \times \tau, \alpha \times \beta) = [\overline{M}(V, \sigma \times \tau, \alpha \times \beta), E^{\bullet}(V, \sigma, \alpha) \boxplus E^{\bullet}(V, \tau, \beta)]$$
$$= [\overline{M}(V, \sigma, \alpha), E^{\bullet}(V, \sigma, \alpha)] \times [\overline{M}(V, \tau, \beta), E^{\bullet}(V, \tau, \beta)]$$
$$= J(V, \sigma, \alpha) \times J(V, \tau, \beta),$$

which is the product axiom.

#### AXIOM III. Cutting Edges

Use notation as in [2], Section 7, modified as necessary to avoid confusion. Let  $\beta$  denote the  $H_2(V)^+$ -structure on both  $\sigma$  and  $\tau$ . Write  $\mathfrak{M} = \mathfrak{M}(\tau) = \mathfrak{M}(\sigma)$ . Consider the cartesian diagram

$$\overline{M}(V, \sigma, \beta) \xrightarrow{\overline{M}(\Phi)} \overline{M}(V, \tau, \beta) 
g \downarrow \qquad \qquad \downarrow 
\mathfrak{M} \times V \xrightarrow{\Delta} \mathfrak{M} \times V \times V$$

of stacks over  $\mathfrak{M}$ . Let us show that the obstruction theories  $E^{\bullet}(V, \tau, \beta)$  and  $E^{\bullet}(V, \sigma, \beta)$  are compatible over  $\Delta$  (see [1]).

Over  $\overline{M}(V, \sigma, \beta)$  let us consider the following two curves. First the curve  $C = C(V, \sigma, \beta)$  obtained from the universal curves  $(C_v)_{v \in V_{\sigma}}$  by gluing according to the edges of  $\sigma$ . Secondly, we have the curve C', which we obtain from  $(C_v)_{v \in V_{\sigma}}$  by gluing according to the edges of  $\tau$ . In other words,  $C' = \overline{M}(\Phi)^*C(V, \tau, \beta)$ . Moreover, C is obtained from C' by identifying the two sections  $x_1$  and  $x_2$  of C', corresponding to the edge  $\{i_1, i_2\}$  of  $\sigma$  which

is cut by  $\Phi$ . Thus there is a structure morphism  $p:\mathcal{C}'\to\mathcal{C}$  fitting into the commutative diagram

$$\begin{array}{ccc} \mathcal{C}' & \stackrel{p}{\longrightarrow} & \mathcal{C} \\ & & \downarrow \pi \\ & & \overline{M}(V, \sigma, \beta). \end{array}$$

We shall also use the diagram

$$\begin{array}{ccc} \mathcal{C}' & \stackrel{p}{\longrightarrow} & \mathcal{C} \\ f' \searrow & \downarrow f \\ & V. \end{array}$$

where  $f: \mathcal{C} \to V$  is the universal map. Let  $x = p \circ x_1 = p \circ x_2$ .

If E is any locally free sheaf on C, then for i = 1, 2 we have the evaluation homomorphism

$$u_i: p^*E \longrightarrow x_{i*}x_i^*p^*E = x_{i*}x^*E.$$

Applying  $p_*$  we get

$$p_*(u_i): p_*p^*E \longrightarrow x_*x^*E.$$

Letting  $u = p_*(u_2) - p_*(u_1)$  we have a short exact sequence

$$0 \longrightarrow E \longrightarrow p_*p^*E \stackrel{u}{\longrightarrow} x_*x^*E \longrightarrow 0$$

of coherent sheaves on C. Applying  $R\pi_*$  we get a distinguished triangle

$$R\pi_*E \longrightarrow R\pi'_*p^*E \stackrel{R\pi_*(u)}{\longrightarrow} x^*E \longrightarrow R\pi_*E[1]$$

in  $D(\mathcal{O}_{\overline{M}(V,\sigma,\beta)})$ . Taking  $E=f^*T_V$  we get the distinguished triangle

$$R\pi_*f^*T_V \longrightarrow R\pi'_*f'^*T_V \xrightarrow{R\pi_*(u)} x^*f^*T_V \longrightarrow R\pi_*f^*T_V[1],$$

or dually,

$$x^*f^*\Omega_V \xrightarrow{R\pi_*(u)^\vee} (R\pi'_*f'^*T_V)^\vee \longrightarrow (R\pi_*f^*T_V)^\vee \longrightarrow x^*f^*\Omega_V[1]. \tag{1}$$

Note that we have  $E^{\bullet}(V, \sigma, \beta) = (R\pi_* f^* T_V)^{\vee}$  and  $\overline{M}(\Phi)^* (E^{\bullet}(V, \tau, \beta)) = (R\pi'_* f'^* T_V)^{\vee}$ . Moreover,  $L_{\Delta}^{\bullet} = \Omega_V[1] | \mathfrak{M} \times V$ , so that  $g^* L_{\Delta} = x^* f^* \Omega_V[1]$ , since  $f \circ x = p_V \circ g$ . So (1) gives the distinguished triangle

$$g^*L_{\Delta}[-1] \stackrel{R\pi_*(u)^{\vee}}{\longrightarrow} \overline{M}(\Phi)^*E^{\bullet}(V,\tau,\beta) \longrightarrow E^{\bullet}(V,\sigma,\beta) \longrightarrow g^*L_{\Delta},$$

which we may shuffle around to give

$$\overline{M}(\Phi)^* E^{\bullet}(V, \tau, \beta) \longrightarrow E^{\bullet}(V, \sigma, \beta) \longrightarrow g^* L_{\Delta} \xrightarrow{R\pi_*(-u)^{\vee}} \overline{M}(\Phi)^* E^{\bullet}(V, \tau, \beta)[1].$$

Now we have the obstruction morphisms  $E^{\bullet}(V, \tau, \beta) \to L^{\bullet}_{\overline{M}(V, \tau, \beta)/\mathfrak{M}}$  and  $E^{\bullet}(V, \sigma, \beta) \to L^{\bullet}_{\overline{M}(V, \sigma, \beta)/\mathfrak{M}}$ . Moreover, we have the natural homomorphism  $g^*L_{\Delta} \to L^{\bullet}_{\overline{M}(\Phi)}$ . These give rise to a homomorphism of distinguished triangles

showing that  $E^{\bullet}(V, \tau, \beta)$  and  $E^{\bullet}(V, \sigma, \beta)$  are compatible over  $\Delta$ . Hence by [1] Proposition 7.5 we have

$$\Delta^! J(V, \tau, \beta) = J(V, \sigma, \beta)$$

which is Axiom III.

# AXIOM IV. Forgetting Tails

Let us deal with the incomplete case, leaving the tripod losing cases to the reader. Letting  $\mathcal{C} \to \mathfrak{M}(\tau)$  be the universal curve corresponding to the vertex  $w \in V_{\tau}$  (notation from [2], Section 7). We have a cartesian diagram of algebraic k-stacks

$$\begin{array}{ccc} \overline{M}(V,\sigma,\beta) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(V,\tau,\beta) \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathfrak{M}(\tau). \end{array}$$

By [1] Proposition 7.2 we have

$$\overline{M}(\Phi)^*J(V,\tau,\beta) = [\overline{M}(V,\sigma,\beta),\overline{M}(\Phi)^*E^{\bullet}(V,\tau,\beta)].$$

Here the class on the right hand side is the virtual fundamental class defined by the relative intrinsic normal cone of the morphism d and the relative obstruction theory  $\overline{M}(\Phi)^*E^{\bullet}(V,\tau,\beta)$ . Note that the structure morphism  $\overline{M}(V,\sigma,\beta) \to \mathfrak{M}(\sigma)$  factors through  $d:\overline{M}(V,\sigma,\beta) \to \mathcal{C}$ .

$$\overline{M}(V,\sigma,\beta) \xrightarrow{d} \mathcal{C} \\
\searrow \qquad \downarrow \\
\mathfrak{M}(\sigma)$$

The morphism  $d: \overline{M}(V, \sigma, \beta) \to \mathcal{C}$  associates to the stable map (C, x, h) the pair ((C', x'), y), where (C', x', h') is the image of (C, x, h) under  $\overline{M}(\Phi)$  and (C', x') the underlying  $\tau$ -marked prestable curve. Letting  $x_f$  be the section of  $C_v$  corresponding to the flag f, we obtain (C', x', h') by forgetting  $x_f$  and stabilizing. Moreover, y is the image of the forgotten section  $x_f$  in  $C'_w$ .

The morphism  $\mathcal{C} \to \mathfrak{M}(\sigma)$  associates to the pair ((C,x),y), where (C,x) is a  $\tau$ -marked prestable curve and y a section of  $C_w$ , the  $\sigma$ -marked prestable curve  $(\widetilde{C},\widetilde{x})$  obtained as follows. For  $v'\neq v$  we have  $\widetilde{C}_{v'}=C_{w'}$ , where w' is the vertex of  $\tau$  corresponding to v'. The curve  $(\widetilde{C}_v,(\widetilde{x}_j)_{j\in F_{\sigma}(v)})$  is obtained from  $((C_w,(x_j)_{j\in F_{\tau}(w)}),y)$  by 'prestabilizing' (i.e. separating the special points) as in [4], Definition 2.3.

#### **Lemma 7** The morphism $\mathcal{C} \to \mathfrak{M}(\sigma)$ is étale.

PROOF. We will use the formal criterion for étaleness. Without loss of generality assume that w is the only vertex of  $\tau$ . So let ((C,x),y) be a  $\tau$ -marked prestable curve with section over the scheme  $T,T\to T'$  a square zero extension and (C',x') a  $\sigma$ -marked prestable curve over T' such that (C',x')|T is the prestabilization of ((C,x),y). We may assume that we may choose additional sections s of C over T, making (C,x,s) a stable marked curve. Then we extend the sections s to sections s' of C' over T'. Taking the stabilization of (C',x',s') after forgetting the section  $x'_f$  gives an extension of ((C,x),y) to T' whose prestabilization is (C',x').  $\square$ 

Consider the natural morphism  $p: \mathcal{C}(V, \sigma, \beta) \to \overline{M}(\Phi)^*\mathcal{C}(V, \tau)$ , which fits into the two commutative diagrams

$$\begin{array}{ccc} \mathcal{C}(V,\sigma,\beta) & \stackrel{p}{\longrightarrow} & \overline{M}(\Phi)^*\mathcal{C}(V,\tau,\beta) \\ & & \downarrow \pi' \\ & & \overline{M}(V,\sigma,\beta) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C}(V,\sigma,\beta) & \xrightarrow{p} & \overline{M}(\Phi)^*\mathcal{C}(V,\tau,\beta) \\ & & \downarrow f' \\ & & V & . \end{array}$$

Whenever E is a locally free sheaf on  $\overline{M}(\Phi)^*\mathcal{C}(V,\tau,\beta)$  the canonical homomorphism  $E \to p_*p^*E$  is an isomorphism. Applying this principle to  $E = f'^*T_V$  we get an isomorphism

$$f'^*T_V \longrightarrow p_*f^*T_V.$$

Applying  $R\pi'_*$  to this, gives an isomorphism

$$R\pi'_*f'^*T_V \longrightarrow R\pi_*f^*T_V.$$

Noting that  $R\pi'_*f'^*T_V = \overline{M}(\Phi)^*E^{\bullet}(V,\tau,\beta)$  we get an isomorphism

$$\overline{M}(\Phi)^* E^{\bullet}(V, \tau, \beta) \longrightarrow E^{\bullet}(V, \sigma, \beta)$$

and whence an isomorphism

$$\mathfrak{E}(V,\sigma,\beta) \longrightarrow \overline{M}(\Phi)^* \mathfrak{E}(V,\tau,\beta).$$

By [1] Proposition 7.1 there is a natural isomorphism

$$\mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathcal{C}} \longrightarrow \overline{M}(\Phi)^* \mathfrak{C}_{\overline{M}(V,\tau,\beta)/\mathfrak{M}(\tau)}.$$

By Lemma 7 we have a canonical isomorphism

$$\mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathcal{C}} \longrightarrow \mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathfrak{M}(\sigma)},$$

such that the diagram

$$\begin{array}{ccc} \mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathfrak{M}(\sigma)} & \stackrel{\sim}{\longleftarrow} & \mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathcal{C}} \\ & \cap & & \cap \\ \mathfrak{E}(V,\sigma,\beta) & \stackrel{\sim}{\longrightarrow} & \overline{M}(\Phi)^*\mathfrak{E}(V,\tau,\beta) \end{array}$$

commutes. So finally, we have

$$\overline{M}(\Phi)^* J(V, \tau, \beta) = [\overline{M}(V, \sigma, \beta), \overline{M}(\Phi)^* E^{\bullet}(V, \tau, \beta)]$$
$$= [\overline{M}(V, \sigma, \beta), E^{\bullet}(V, \sigma, \beta)]$$
$$= J(V, \sigma, \beta),$$

which is Axiom IV.

#### AXIOM V. Isogenies

Before we start with the proof, some general remarks. Let  $\Phi: \tau \to \sigma$  be an elementary contraction of stable modular graphs, contracting the edge  $\{f, \overline{f}\}$  of  $\tau$ . Let  $a: \tau \to \tau'$  and  $b: \sigma \to \sigma'$  be combinatorial morphisms of modular graphs identifying  $\tau$  and  $\sigma$  as the stabilizations of  $\tau'$  and  $\sigma'$ , respectively. Finally, let  $\Phi': \tau' \to \sigma'$  be as follows. We require  $\{a(f), a(\overline{f})\}$  to be an edge of  $\tau'$  and  $\Phi': \tau' \to \sigma'$  to be the elementary contraction contracting the edge

 $\{a(f), a(\overline{f})\}$ . Moreover, we require  $\Phi$  to be the stabilization of  $\Phi'$ . To fix notation, denote the vertex onto which  $\Phi'$  contracts the edge  $\{a(f), a(\overline{f})\}$  by  $v_0 \in V_{\sigma'}$  and let  $v_1 = \partial_{\tau'}(a(f))$  and  $v_2 = \partial_{\tau'}(a(\overline{f}))$ .

In this situation we get a commutative diagram of algebraic stacks

$$\mathfrak{M}(\tau') \stackrel{\mathfrak{M}(\Phi')}{\longrightarrow} \mathfrak{M}(\sigma') 
\downarrow s \qquad \qquad \downarrow s 
\overline{M}(\tau) \stackrel{\overline{M}(\Phi)}{\longrightarrow} \overline{M}(\sigma).$$

Define  $\mathfrak{P}$  to be the fibered product

$$\mathfrak{P} \longrightarrow \mathfrak{M}(\sigma')$$

$$\downarrow s$$

$$\overline{M}(\tau) \stackrel{\overline{M}(\Phi)}{\longrightarrow} \overline{M}(\sigma).$$

Consider the induced morphism  $l: \mathfrak{M}(\tau') \to \mathfrak{P}$ .

**Proposition 8** We have  $l_*[\mathfrak{M}(\Phi')] = s^*[\overline{M}(\Phi)].$ 

PROOF. First note that  $\mathfrak{M}(\tau')$  is irreducible, since  $\mathfrak{M}(\tau')$  is a product of stacks of the form  $\mathfrak{M}_{g,n}$ , which are irreducible since the stacks  $\overline{M}_{g,n}$  are. Moreover,  $\mathfrak{M}(\tau') \to \mathfrak{P}$  is surjective, so that  $\mathfrak{P}$  is irreducible, too.

Secondly, let us remark that there exist non-empty (hence dense) open substacks  $\mathfrak{M}(\tau')^0 \subset \mathfrak{M}(\tau')$  and  $\mathfrak{P}^0 \subset \mathfrak{P}$  such that l induces an isomorphism  $l^0: \mathfrak{M}(\tau')^0 \overset{\sim}{\to} \mathfrak{P}^0$ . In fact, let  $\mathfrak{M}(\tau')^0$  be the open substack of  $\mathfrak{M}(\tau')$  characterized by the requirement that the marked curves  $C_{v_1}$  and  $C_{v_2}$  be stable. To construct  $\mathfrak{P}^0$ , let  $\mathfrak{M}(\sigma')^0$  be the open substack of  $\mathfrak{M}(\sigma')$  where the marked curve  $C_{v_0}$  is stable. Then set

$$\mathfrak{P}^0 = \overline{M}(\tau) \times_{\overline{M}(\sigma)} \mathfrak{M}(\sigma')^0.$$

These facts imply the claim.  $\Box$ 

Now let  $(\Phi, m): \tau \to \sigma$  be an elementary isogeny of type forgetting a tail. Let  $f \in F_{\tau}$  be the forgotten tail. Let  $a: \tau \to \tau'$  and  $b: \sigma \to \sigma'$  be as above. Finally, let  $\Phi': \tau' \to \sigma'$  be the 'adjoint' of a combinatorial morphism of graphs, such that there exists a tail map m', a semigroup A and A-structures on  $\tau'$  and  $\sigma'$  making  $(\Phi', m')$  the elementary isogeny of stable A-graphs forgetting the tail a(f). Moreover, we require  $\Phi$  to be the stabilization of  $\Phi'$ .

Let  $\mathfrak{P}$  be the fibered product

$$\begin{array}{ccc} \mathfrak{P} & \longrightarrow & \mathfrak{M}(\sigma') \\ \downarrow & & \downarrow s \\ \overline{M}(\tau) & \longrightarrow & \overline{M}(\sigma) \end{array}$$

and  $\mathcal{C}$  the universal curve over  $\mathfrak{M}(\sigma')$  corresponding to  $w \in V_{\sigma'}$ , where w is the vertex of the forgotten tail. (If w does not exist, i.e. if  $\Phi'$  is complete, then  $\mathcal{C} = \mathfrak{M}(\sigma')$ .) As in the proof of Axiom IV we have a morphism  $\mathcal{C} \to \mathfrak{M}(\tau')$  giving rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\pi'} & \mathfrak{M}(\sigma') \\ \downarrow & & \downarrow s \\ \mathfrak{M}(\tau') & & \downarrow s \\ \downarrow s & & \\ \overline{M}(\tau) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(\sigma) \end{array}$$

and hence to a morphism  $l: \mathcal{C} \to \mathfrak{P}$ .

**Proposition 9** We have  $l_*[\pi'] = s^*[\overline{M}(\Phi)]$ .

PROOF. Again,  $\mathcal{C}$  and  $\mathfrak{P}$  are irreducible and l induces an isomorphism  $l^0: \mathcal{C}^0 \to \mathfrak{P}^0$ , where  $\mathcal{C}^0$  is the restriction of  $\mathcal{C}$  to  $\mathfrak{M}(\sigma')^0$  and  $\mathfrak{P}^0 = \overline{M} \times_{\overline{M}(\sigma)} \mathfrak{M}(\sigma')^0$ . Here  $\mathfrak{M}(\sigma')^0 \subset \mathfrak{M}(\sigma')$  is the open substack where  $C_w$  is stable.  $\square$ 

Now let us prove Axiom V. According to [2], Remark 7.2, it suffices to do this for the case that  $\Phi: \tau \to \sigma$  is an elementary isogeny, #J = 1 and  $(a_i, \tau_i, \Phi_i)_{i \in I}$  a pullback. So we shall use notation as in the Definition of pullback ([2], Definition 6.10). We shall include the  $H_2(V)^+$ -structures on  $\sigma'$  and  $\tau_i$  ( $i \in I$ ) in the notation. They shall be denoted by  $\beta'$  and  $\beta_i$  ( $i \in I$ ), respectively. The underlying graph of  $(\tau_i, \beta_i)$  is the same for all  $i \in I$ . Let us call it simply  $\tau'$ .

Let us first consider the case where  $\Phi$  is a contraction.

Lemma 10 We have a cartesian diagram

$$\prod_{i \in I} \overline{M}(V, \tau', \beta_i) \longrightarrow \overline{M}(V, \sigma', \beta')$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathfrak{M}(\tau') \stackrel{\mathfrak{M}(\Phi')}{\longrightarrow} \mathfrak{M}(\sigma')$$

of algebraic k-stacks. Moreover,

$$\mathfrak{M}(\Phi')^!J(V,\sigma',\beta') = \sum_{i \in I} J(V,\tau',\beta').$$

PROOF. The first fact follows immediately from the definitions. The second fact is [1] Proposition 7.2.  $\Box$ 

Axiom V will follow by putting Lemma 10 and Proposition 8 together as follows. By Lemma 10 all squares in the following diagram are cartesian.

So we may calculate as follows.

$$\overline{M}(\Phi)^! J(V, \sigma', \beta') = a^* s^* [\overline{M}(\Phi)] \cdot J(V, \sigma', \beta')$$

$$= a^* l_* [\mathfrak{M}(\Phi')] \cdot J(V, \sigma', \beta')$$
(by Proposition 8)
$$= h_* \mathfrak{M}(\Phi')^! J(V, \sigma', \beta')$$

$$= h_* \sum_{i \in I} J(V, \tau', \beta_i)$$

by Lemma 10. This is the context of Axiom V.

The case that  $\Phi$  is of type forgetting a tail is similar. Instead of Lemma 10 one uses Axiom IV, and Proposition 8 is replaced by Proposition 9.

This finishes the proof of Axiom V and hence the proof of Theorem 6.

#### Homogeneous Spaces

In the case where V is a generalized flag variety, we can give a more explicit construction of Gromov-Witten invariants as follows.

#### Curves and Principal Bundles

For a smooth algebraic k-group G with Lie algebra  $\mathfrak{g}$ , we denote by

$$\mathfrak{H}^1(\tau,G)$$

the k-stack of G-torsors on  $\tau$ -marked prestable curves. More precisely, for a k-scheme T, the category  $\mathfrak{H}^1(\tau, G)(T)$  is the category of pairs (C, E), where  $C = (C_v)_{v \in V_{\tau}}$  is a  $\tau$ -marked prestable curve over T, giving rise to a morphism  $f: T \to \mathfrak{M}(\tau)$ , and E is a G-torsor on  $f^*\mathcal{C}(\tau)$ .

Let (C, E) be such a pair. Denote by  $E_v$ , for  $v \in V_\tau$ , the G-bundle induced by E on  $C_v$ . We call

$$\deg_v(E) = \deg(E_v) = \deg(E_v \times_{G,Ad} \mathfrak{g})$$

the degree of E at the vertex  $v \in V_{\tau}$ . The degree thus defines a  $\mathbb{Z}_{\geq 0}$ -structure on  $\tau$ , which is locally constant on T. (See [2], Definition 1.6, for  $\mathbb{Z}_{\geq 0}$ -structures.)

In this way, we get for every  $\mathbb{Z}_{\geq 0}$ -structure  $\alpha$  on  $\tau$  an open and closed substack  $\mathfrak{H}^1_{\alpha}(\tau,G)\subset\mathfrak{H}^1(\tau,G)$ , the substack of G-torsors of degree  $\alpha$ .

**Proposition 11** For every  $\mathbb{Z}_{\geq 0}$ -structure  $\alpha$  on  $\tau$  the stack  $\mathfrak{H}^1_{\alpha}(\tau, G)$  is an algebraic k-stack. The canonical morphism

$$\mathfrak{H}^1_{\alpha}(\tau,G) \longrightarrow \mathfrak{M}(\tau)$$

is smooth of relative dimension

$$-\chi(\tau)\dim G - \alpha(\tau),$$

where 
$$\alpha(\tau) = \sum_{v \in V_{\tau}} \alpha(v)$$
.

PROOF. To prove that  $\mathfrak{H}^1(\tau,G)$  is algebraic, choose a suitable embedding  $G \hookrightarrow GL_n$  to reduce the case of G-bundles to the case of vector bundles, for which it is well-known. The smoothness of  $\mathfrak{H}^1(\tau,G)$  follows from the fact that  $H^2(C, E \times_{G,Ad} \mathfrak{g}) = 0$  for any G-torsor E on a  $\tau$ -marked prestable curve G. The dimension of  $\mathfrak{H}^1(\tau,G)$  is equal to

$$-\chi(E \times_{G,Ad} \mathfrak{g}) = -\deg(E \times_{G,Ad} \mathfrak{g}) - \chi(\mathcal{O}_C) \operatorname{rk}(E \times_{G,Ad} \mathfrak{g})$$
$$= -\alpha(\tau) - \chi(\tau) \operatorname{dim} G$$

by Riemann-Roch. □

## Maps to G/P

Now let G be a reductive algebraic group over k and P a parabolic subgroup of G. Then G/P is a smooth projective variety over k. Let us assume for

simplicity that G is split over k. The morphism  $G \to G/P$  is a principal P-bundle, which we shall denote by F.

Let  $U_1, \ldots, U_r$  be the elementary representations of P over  $k, V_1, \ldots, V_r$  the corresponding vector bundles on G/P and  $L_1, \ldots, L_r$  their determinants. For every  $i = 1, \ldots, r$  we have

$$V_i = F \times_P U_i.$$

Note that  $\operatorname{Pic}(G/P) \otimes \mathbb{Q}$  is spanned by  $L_1, \ldots, L_r$  and that  $L_1^{-1} \otimes \ldots \times L_r^{-1}$  is ample.

Let  $H_2(G/P)^+$  be the set of homomorphisms of abelian groups  $\psi$ :  $\operatorname{Pic}(G/P) \to \mathbb{Z}$ , which are non-negative on ample line bundles. Then we get a canonical injection

$$H_2(G/P)^+ \longrightarrow (\mathbb{Z}_{\geq 0})^r$$
  
 $\psi \longmapsto (\psi(L_1^{-1}), \dots, \psi(L_r^{-1})).$ 

Using this injection we shall think of classes in  $H_2(G/P)^+$  as r-tuples of non-negative integers.

Let  $\mathfrak g$  and  $\mathfrak p$  be the Lie algebras of G and P, respectively. We will consider these only as adjoint representations, ignoring the Lie algebra structure. Denote by  $\mathfrak p$  also the induced vector bundle

$$F \times_{P,Ad} \mathfrak{p}$$

on G/P. Evaluating on the inverse of its determinant defines a morphism

$$\deg: H_2(G/P)^+ \longrightarrow \mathbb{Z}_{\geq 0}$$

$$\psi \longmapsto \psi(\det(\mathfrak{p})^{-1}).$$

This morphism has the property that  $deg(\psi) = 0$  implies  $\psi = 0$ .

**Remark** We have det  $\mathfrak{p} \cong \omega_{G/P}$ . In particular, deg  $\psi = -\psi(\omega_{G/P})$ .

Now fix an  $H_2(G/P)^+$ -graph  $(\tau, \beta)$ , with underlying modular graph  $\tau$ . Let  $(\tilde{\tau}, \tilde{\beta})$  be the  $H_2(G/P)^+$ -graph obtained by contracting all edges of  $\tau$ .

Consider the algebraic k-stacks  $\mathfrak{H}^1(\tau, G)$  and  $\mathfrak{H}^1(\tau, P)$ . Since G is reductive, any G-torsor on a curve has degree zero, and thus

$$\mathfrak{H}^1(\tau,G) \longrightarrow \mathfrak{M}(\tau)$$

is smooth of relative dimension

$$-\chi(\tau)\dim G$$
.

If E is a P-torsor, then associated to  $U_1, \ldots, U_r$  we have associated vector bundles  $E_i = E \times_P U_i$ , for  $i = 1, \ldots, r$ , and thus we may associate to E the multi-degree

$$\operatorname{mult-deg}(E) = (-\operatorname{deg}(E_1), \dots, -\operatorname{deg}(E_r)).$$

Let  $\mathfrak{H}^1_{\beta}(\tau, P)$  be the open and closed substack of  $\mathfrak{H}^1(\tau, P)$  of P-torsors whose multi-degree is equal to  $\beta$ .

Let  $\alpha = \deg \beta$  be the  $\mathbb{Z}_{>0}$ -structure on  $\tau$  associated to  $\beta$ . Then we have

$$\mathfrak{H}^1_{\beta}(\tau, P) \subset \mathfrak{H}^1_{-\alpha}(\tau, P),$$

so that by Proposition 11 the stack  $\mathfrak{H}^1_{\beta}(\tau,P)$  is smooth of relative dimension

$$-\chi(\tau) \dim P - \beta(\tau)(\omega_{G/P})$$

over  $\mathfrak{M}(\tau)$ .

Now let  $\mathfrak{M}(G/P, \tau, \beta)$  be the stack of maps from  $\tau$ -marked prestable curves to G/P of class  $\beta$ . More precisely, for a k-scheme T, the objects of  $\mathfrak{M}(G/P, \tau, \beta)(T)$  are triples (C, x, f), where (C, x) is a  $\tau$ -marked prestable curve over T and  $f = (f_v)_{v \in V_{\tau}}$  is a family of k-morphisms  $f_v : C_v \to G/P$  such that

- 1. for all  $i \in F_{\tau}$  we have  $f_{\partial(i)}(x_i) = f_{\partial(j_{\tau}(i))}(x_{j_{\tau}(i)})$ ,
- 2. for all  $v \in V_{\tau}$  we have  $f_{v*}[C_v] = \beta(v)$ .

**Remark** If  $(\tau, \beta)$  is stable, then  $\overline{M}(G/P, \tau, \beta)$  is an open substack of  $\mathfrak{M}(G/P, \tau, \beta)$ .

Note that  $G^{V_{\widetilde{\tau}}}$  acts on  $\mathfrak{M}(G/P,\tau,\beta)$  as follows. An element  $(g_w)_{w\in V_{\widetilde{\tau}}}$  of  $G^{V_{\widetilde{\tau}}}$  takes  $(C,x,(f_v)_{v\in V_{\tau}})$  to  $(C,x,(g_{\phi(v)}\circ f_v)_{v\in V_{\tau}})$ , where  $\phi:\tau\to\widetilde{\tau}$  is the structure contraction. Let

$$\mathfrak{M}(G/P, \tau, \beta)/G^{V_{\widetilde{\tau}}}$$

be the stack-theoretic quotient of this action. This is an abuse of notation, since this is a left and not a right action.

We shall let  $G^{V_{\widetilde{\tau}}}$  act trivially on  $\mathfrak{M}(\tau)$  and denote by

$$\mathfrak{M}(\tau)/G^{V_{\widetilde{\tau}}}$$

the quotient.

**Proposition 12** There is a natural cartesian diagram of algebraic k-stacks

$$\begin{array}{cccc} \mathfrak{M}(G/P,\tau,\beta)/G^{V_{\widetilde{\tau}}} & \stackrel{\kappa}{\longrightarrow} & \mathfrak{H}^1_{\beta}(\tau,P) \\ & & \downarrow & & \downarrow \\ & \mathfrak{M}(\tau)/G^{V_{\widetilde{\tau}}} & \stackrel{\iota}{\longrightarrow} & \mathfrak{H}^1(\tau,G). \end{array}$$

The vertical maps are representable, the horizontal maps are local immersions.

PROOF. This is essentially the fact that a map to G/P is the same as a principal P-bundle with a trivialization of the associated G-bundle.  $\square$ 

The morphism  $\iota$  is a local regular immersion with normal bundle  $R^1\pi_*\mathcal{O}\otimes\mathfrak{g}$ . Thus the normal cone  $C(\tau,\beta)$  of  $\mathfrak{M}(G/P,\tau,\beta)/G^{V_{\widetilde{\tau}}}$  in  $\mathfrak{H}^1_{\beta}(\tau,P)$  is a cone in

$$\mathfrak{n}(\tau,\beta) = \eta^* R^1 \pi_* \mathcal{O} \otimes \mathfrak{g}.$$

Pulling back to  $\mathfrak{M}(G/P,\tau,\beta)$  and, if  $(\tau,\beta)$  is stable, to  $\overline{M}(G/P,\tau,\beta)$  defines  $G^{V_{\widetilde{\tau}}}$ -equivariant cones, which we shall still denote  $C(\tau,\beta)$ , inside equivariant vector bundles, which we shall still denote by  $\mathfrak{n}(\tau,\beta)$ .

Let us now assume that  $(\tau, \beta)$  is stable. Then we may intersect the cone  $C(\tau, \beta)$  over  $\overline{M}(G/P, \tau, \beta)$  with the zero section of the vector bundle  $\mathfrak{n}(\tau, \beta)$ , to define a cycle class

$$J(\tau,\beta) \in A_{\dim(G/P,\tau,\beta)}(\overline{M}(G/P,\tau,\beta))$$

with rational coefficients. Note that  $C(\tau, \beta)$  is pure of the correct dimension, since it is constructed as a normal cone inside a smooth stack of the correct dimension.

**Proposition 13** The collection of cycle classes  $J(\tau, \beta)$  is the orientation of  $\overline{M}$  over  $\mathfrak{G}_s(G/P)$  defined using the intrinsic normal cone.

PROOF. This follows from [1] Example 7.6, since

$$(R\pi_* f^* T_{G/P})^{\vee} = \kappa^* L^{\bullet}_{\mathfrak{H}^{1}_{\beta}(\tau, P)/\mathfrak{H}^{1}(\tau, G)}.$$

**Remark** As a corollary we get that the orientation classes  $J(\tau, \beta)$  are  $G^{V_{\tau}}$ -invariant. The same is then true for the Gromov-Witten invariants.

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