Proposal

Incorporating Globally Optimal Direction Fields with Index Constraints into Trivial Connections

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1 Project Description and Goals

The goal of this project is to learn and implement a method for constructing smooth vector fields on 3D surfaces with singularities located at "important points". The resulting vector fields will have applications in a variety of areas, such as quadrilateral re-meshing, anisotropic shading, and hatching shading. To achieve this goal, the team will learn and implement the following concepts and skills:

- Optimization techniques for solving eigenvalue problems.
- The index of vector fields and the Hopf-Poincaré Theorem.
- Representation of cross fields, line fields, and n-fields.

The basic goals for this project include:

- Defining energy for the vector field at each point on the surface and finding a vector field that minimizes this energy.
- Creating a non-unit norm vector field to avoid infinite Dirichlet energy at singular points.
- Using holomorphic energies to smooth the vector field, resulting in fewer singularities and a smoother overall field.
- Evaluating the effect of the interpolation parameter on the resulting vector field.

The reach goals for this project include:

- Incorporating index specification constraints, similar to the Trivial Connections paper.
- Comparing the performance of the resulting vector field with other state-of-the-art methods.
- Exploring applications of the resulting vector field in anisotropic shading and hatching shading.

Through this project, the team hopes to gain expertise in the field of geometry processing and to contribute to the development of new techniques for constructing smooth vector fields on 3D surfaces.

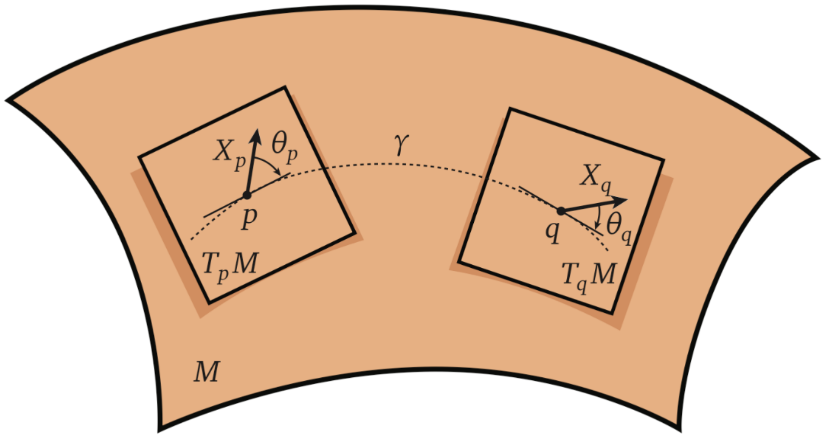

Figure 1: To transport a vector from one tangent space to another it is sufficient to measure the angle it makes against a geodesic connecting the two tangent spaces.

2 Methodology/Process

2.1 Parallel Transport

To calculate the difference between vectors at two different points, we move the vector from the first point to the second point along the geodesic. We basically first find how the vector changes at any given point infinitesimally in any given direction. If we know this infinitesimal change, then we can integrate the change along the geodesic to find where the vector will be in the final position.

This simplifies the difference between these vectors to $|e^{i\theta_{pq}} z_p - z_q|^2$ where z_p and z_q are the complex coefficients which specify the rotation with respect to the basic vectors at points p and q, and $\theta_{pq} = \theta_p - \theta_q$. The transport is also the same for the cross fields, except we use the representation vectors instead of the original cross fields vectors.

2.2 Dirichlet Energy of non-unit vectors

Dirichlet Energy goes to infinity at singular points for unit vectors, so we will try to scale vectors so that the Dirichlet Energy is defined at the singularities. So the Dirichlet Energy definition goes from

$$E_D(\psi) := \frac{1}{2} \int_M |\nabla \psi|^2 dA \qquad \text{to} \qquad \frac{1}{2} \langle\langle (\triangle + |\omega|^2) a, a \rangle\rangle$$

where a is the scale of the vector, ω is the angular velocity of that vector field at that point and $\triangle$ is the Laplacian at that point. Now the Dirichlet energy is defined at every point and the vector fields are non-unit norm and can go to zero at singular points.

Now the goal is to minimize this energy with the constraints of $||a|| = 1$ and $||\psi|| = 1$, which is basically minimizing the energy over all vector fields. This happens to be a simple problem such as finding smallest eigenvalue problem, $\triangle\psi = \lambda\psi$.

2.3 Holomorphic and Anti-Holomorphic splitting Dirichlet Energy

Dirichlet Energy can be split into Holomorphic and AntiHolomorphic parts like $E_D(\psi) = E_H(\psi) + E_A(\psi)$. Generally using the Holomorphic energies gives us more smooth vector fields and have less number of singularities. And using a general Dirichlet Energies gives us more number of singularities and less smooth vectors. So we define a smoothness factor s and create a new energy using the E_H and E_A as $E_s := (1+s)E_H + (1-s)E_A$. As s tends to 1, the vector fields become more smooth. The problem we need to solve to minimize this energy is the same, finding smallest eigenvalue problem. But, for this new smoother energy.

2.4 Alignment

In some cases, when the generated vector fields don't align to what we feel is more natural or intuitive, we can specify alignment vectors ϕ at every point and define a new alignment energy as $E_l(\psi) = \int_M Re(\langle\phi, \psi\rangle)dA = Re(\langle\phi, \psi\rangle)$. The new final energy we need to minimize will be a mix of the smoothness energy and the alignment energy defined as $E_{s,t}(\psi) := (1-t)E_s(\psi) - tE_l(\psi)$ where $t \in [0, 1]$ and controls the strength of alignment.

Now we just need to minimize $E_{s,t}$ which becomes solving the linear system $(A - \lambda_t I)\tilde{\psi} = \phi$.

2.5 Finite Element Discretization

To discretize the problem, we can use finite element methods. We first partition the surface into small triangles and then approximate the vector field on each triangle by a linear combination of basis functions. We can use piece-wise linear functions or higher-order basis functions for a better approximation. We can also use different basis functions for different components of the vector field, such as cross fields, line fields, or n-fields.

Once we have the approximation of the vector field on each triangle, we can compute the energy of the vector field by summing up the energies on each triangle. We can then minimize the energy by finding the coefficients of the basis functions that minimize the energy.

To solve the eigenvalue problem $\triangle\psi = \lambda\psi$ with constraints $||a|| = 1$ and $||\psi|| = 1$, we can use the Rayleigh-Ritz method. We can approximate the eigenfunction and eigenvalue by a linear combination of basis functions and then find the coefficients that minimize the Rayleigh quotient. We can also use different basis functions for the different components of the vector field.

Once we have the approximation of the vector field and its singularities, we can edit the singularities by moving them to nearby points. To ensure that the vector field remains smooth after editing the singularities, we can use the trivial connection to transport the vector field from the original singularities to the new singularities while preserving the direction of the vector field. We can also put index constraints on the singularities to control the behavior of the vector field around the singularities.

2.6 Solution in the Discrete Setting

In the discrete setting, we seek to solve the system of equations that represents the discretized version of the original differential equation problem. This system is typically a large linear or nonlinear system of algebraic equations that we can solve using numerical methods.

Using the finite element method as an example, we first define a finite-dimensional function space, such as the space of piecewise linear functions defined on a mesh of triangles or quadrilaterals. We then approximate the solution of the original differential equation by a function in this space that satisfies the weak form of the equation. This weak form is obtained by multiplying the original equation by a test function in the same function space and integrating over the domain.

The resulting system of equations is typically a large sparse linear system that can be solved using iterative methods, such as the conjugate gradient method or the GMRES method. In some cases, the system may be nonlinear, in which case iterative methods for nonlinear equations, such as Newton's method, can be used.

Once we have obtained the numerical solution, we can then post-process the solution to obtain quantities of interest, such as the values of the solution at particular points in the domain, the maximum and minimum values of the solution, or the flux of a quantity across a particular boundary. We can also use the numerical solution to visualize the behavior of the solution, for example by plotting the solution as a contour plot or a surface plot.

It is important to note that the numerical solution obtained in the discrete setting is only an approximation to the true solution of the original differential equation problem. The accuracy of the solution depends on the choice of the discretization method, the mesh size, and the numerical method used to solve the resulting system of equations. It is therefore important to carefully validate the numerical solution and assess its accuracy, for example by comparing it with analytical solutions or with experimental data.

2.7 Achieving the reach goal

We defined the reach goal as being able to edit the singularity points. We define editing as moving an already present point to a newer position or to add a new point given index of the vector field at the new point.

This can be done by taking all the singularity points given by solving the linear sparse system, calculating the index of the vector field at the singular points and then taking all these points, along with the newly specified point and solving for the smoothest vector fields using the Trivial Connections.

In addition, we can also use the vector fields, at the edited point, as alignment vectors to the geometry and then try to re-calculate the optimal solution again with the newly specified alignments. This mostly should not change the other singular points if they're far away. But we can let the energies decide if it really wants to move other singular points a little bit.

3 Applications

3.1 Shape Analysis

Shape analysis is a broad area of study in mathematics, computer science, and engineering that deals with the quantitative analysis of shapes. Vector fields are a powerful tool in shape analysis, as they can capture both the global and local geometric features of a shape. One important application of vector fields in shape analysis is shape matching. Given two shapes, the goal is to find a transformation that maps one shape to the other in a way that preserves their geometric properties. Vector fields can be used to compute a deformation field that maps one shape to the other by aligning their vector fields. The deformation field can then be used to transfer properties from one shape to the other, such as texture, color, or shape features.

3.2 Computer Graphics

Vector fields are widely used in computer graphics for a variety of tasks, such as animation, simulation, and modeling. In animation, vector fields can be used to generate realistic motion, such as wind blowing through trees or water flowing in a river. In simulation, vector fields can be used to simulate fluid flow, electromagnetic fields, or particle dynamics. In modeling, vector fields can be used to generate geometric structures, such as hair, fur, or grass. Vector fields can also be used in texture synthesis, where they can be used to generate seamless textures by aligning the orientations of the vector fields.

3.3 Medical Imaging

Vector fields are also used in medical imaging for a variety of tasks, such as image registration, segmentation, and tracking. In image registration, vector fields can be used to align images from different modalities, such as MRI and CT scans. In segmentation, vector fields can be used to extract features, such as boundaries, surfaces, or regions. In tracking, vector fields can be used to track the motion of organs, such as the heart or the lungs, over time. Vector fields can also be used in image processing, where they can be used to filter images, such as removing noise or enhancing edges.

3.4 Robotics

Vector fields are also used in robotics for a variety of tasks, such as motion planning, path optimization, and control. In motion planning, vector fields can be used to generate trajectories that avoid obstacles and reach a goal. In path optimization, vector fields can be used to optimize the path of a robot by minimizing a cost function, such as time or energy. In control, vector fields can be used to design feedback controllers that stabilize a robot at a desired position or track a desired trajectory.

3.5 Physics

Vector fields are also used in physics for a variety of tasks, such as modeling electromagnetic fields, fluid flow, and particle dynamics. In electromagnetics, vector fields can be used to model the behavior of electric and magnetic fields, such as in antennas or motors. In fluid flow, vector fields can be used to model the motion of fluids, such as in aerodynamics or hydrodynamics. In particle dynamics, vector fields can be used to model the motion of particles, such as in molecular dynamics or plasma physics. Vector fields can also be used in quantum mechanics, where they can be used to represent the wave function of a particle.

4 Timeline

- **Week 1 (Mar 12 - Mar 18)**

  Read and discuss the paper "Globally Optimal Direction Fields" by Crane et al. Learn about optimization for eigenvalue problems and its relevance to the paper. Familiarize yourselves with the code in Geometry Central related to direction fields.

- **Week 2 (Mar 19 - Mar 25)**

  Read and discuss the paper "Trivial Connections" by Crane et al. Learn about the index of vector fields and the Hopf-Poincaré Theorem. Explore the code in Geometry Central related to trivial connections.

- **Week 3 (Mar 26 - Apr 1)**

  Discuss how to incorporate different energies (between holomorphic and anti-holomorphic) into the algorithm based on the papers read. Begin implementing the changes to the direction field algorithm in the codebase.

- **Week 4 (Apr 2 - Apr 8)**

  Continue implementing changes to the direction field algorithm and run some initial tests. Begin drafting the final report.

- **Week 5 (Apr 9 - Apr 15)**

  Analyze the results of the tests and make necessary modifications to the algorithm. Continue drafting the final report.

- **Week 6 (Apr 16 - Apr 22)**

  Implement index specification constraints in the algorithm, if possible. Continue drafting the final report.

- **Week 7 (Apr 23 - Apr 29)**

  Prepare the final presentation based on the work done so far. Finalize the report and submit it on May 3rd.

5 Concerns and Pitfalls

The project entails various challenges and potential pitfalls. While the implementation of the energies and finding the optimal vector fields may seem straightforward due to the availability of prior work in the field, the achievement of the reach goal may be impeded by unforeseen artifacts, as it has not yet been accomplished in a discrete setting.

One of the crucial challenges to overcome is to determine the index of vector fields or degrees of singular points. The team acknowledges the need to acquire more knowledge and understanding of the topic to determine the best approach for computing the indices in the discrete setting. The indices are essential inputs to the trivial connections algorithm, which will enable the team to achieve the reach goal of obtaining smoother vector fields from already given singular points and their degrees.

Another challenge is to determine how much the other singular points might move if the team adds extra alignments from the trivial connections after moving a point. The team needs to decide how much of the surrounding area it should consider to use as alignment vectors of the edited point from the trivial connections.

In addition, it is also unclear how much the other singular points will change if the team uses the vector fields obtained from the trivial connections as alignment vectors and recalculates the singular points. These uncertainties may impede the team's ability to achieve the reach goal, and further research may be required to mitigate these potential limitations.

In summary, while the project is ambitious and feasible, the team anticipates several challenges and potential limitations in achieving its goals. Addressing these challenges requires further research and experimentation, and the team is committed to learning and gaining expertise in the necessary concepts to achieve the desired outcomes.

6 Bibliography

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