Robust Risk Measurement and Model Risk in Worst Case Scenarios

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What is Robust Risk Measurement?

Assess vulnerabilities to models and their potential impact

Quantify the worst case error in measuring model performance

Go beyond parameter sensitivity to consider the effect of changes in probability law

Philosophy of Robust Risk Measurement

$E[V(X)]$ → $E[V_a(X)]$ → $E[m\, V_a(X)]$ → Varying α in α-divergence

Risk Measurement

The general risk measurement when there is no uncertainty

Decision parameter

The parameter ‘a’ represents a decision to be taken which goes inside the function V.

Robust Risk

Hypothetical adversary imposing model error constrained to $m \log m < \eta$

Parametrized Constraint

Adding more complexity in case of heavy tail cases.

How is the Analysis Feasible?

The solutions to the above mentioned optimization problems can be derived.

Closed form solutions of Relative Entropy

$$m^*(\theta, a) = \frac{exp\left(\theta V_a(X)\right)}{E\left[exp\left(\theta V_a(X)\right)\right]}$$

$$a^*(\theta) = \arg\inf_a = \frac{1}{\theta}\log E[exp(\theta V_a(X))]$$

$$\inf_{\theta>0} H(\theta) + \frac{\eta}{\theta} = \frac{1}{\theta}\log E[exp(\theta V_{a^*(\theta)}(X))] + \frac{\eta}{\theta}$$

$$m^*(\theta,\alpha,a) = \left(\theta(\alpha-1)V_a(X) + c(\theta,\alpha,a)\right)^{\frac{1}{\alpha-1}}$$

$$\text{where } c(\theta,\alpha,a) \text{ can be solved using } E\left[m^*(\theta,\alpha,a)\right]=1$$

$$a^*(\theta) = \arg\min_a \frac{\alpha-1}{\alpha}E\left[\left(\theta(\alpha-1)V_a(X) + c(\theta,\alpha,a)\right)^{\frac{1}{\alpha-1}}V_a(X)\right] + \frac{c(\theta,\alpha,a)}{\theta\alpha(1-\alpha)}$$

Implicit solution in the case of α-divergence for m*

- ➔ Generally $\arg\inf_{\theta} H(\theta) \to \infty$
- ➔ Analyse the relationship between **Entropy**(or divergence) and **Risk Measurement.**
- ➔ This is possible because the constant η **does not appear** in the solution and is derived from the obtained m*.

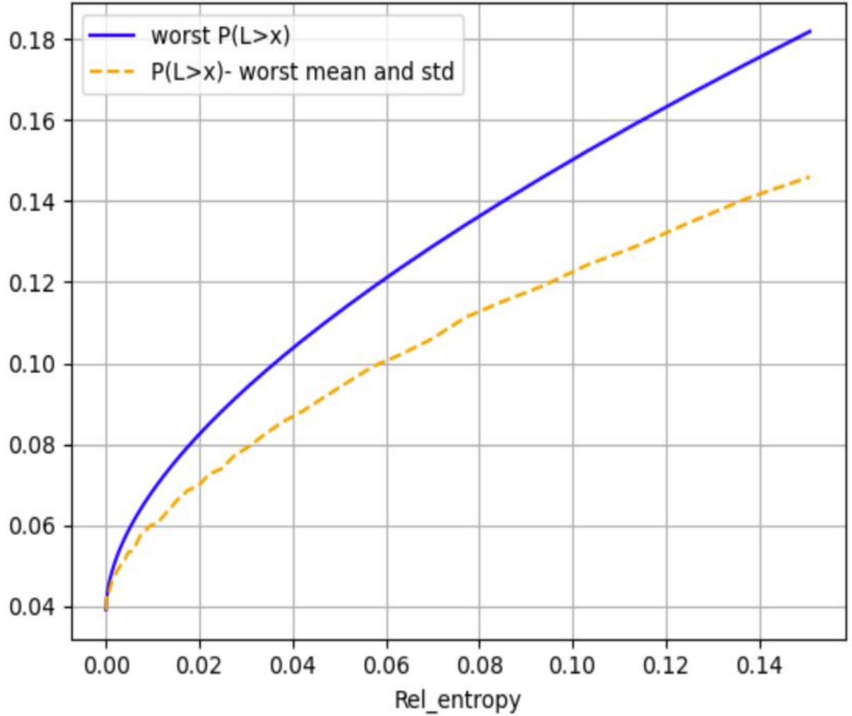


Fig 1- Entropy vs Risk plotted as locus points; the Risk measurement = P(L>x) {Prob. of Loss}

Applications

Investigate the idea of robustness in 3 difference risk measurements

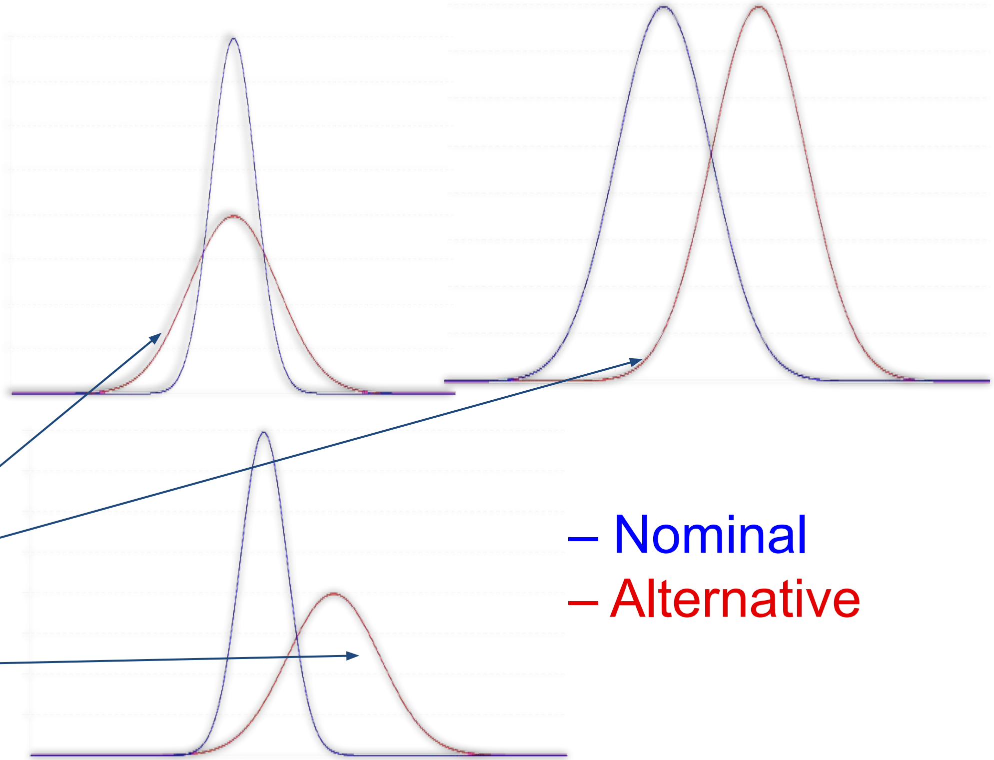
01	**Portfolio Variance**	• Entropy vs Model Risk • Empirical Example
02	**Conditional Value At Risk**	• Relative Entropy • Heavy-tail Case
03	**Portfolio Credit Risk**	• Gaussian Copula Model • Robustness and Model Error

Portfolio Variance

→Assumption - Multivariate normal distribution with mean and standard deviation estimated from historical data.

→Risk measurement does not align with the model.

Alternate distribution deviating with mean or variance constraints



– Nominal

– Alternative

Mean-Variance Objective

$$\inf_{a} \; - \; E\left[-a^T X - \frac{\varsigma}{2} a^T (X - E(X))(X - E(X))^T a \right]$$

1. a = [a_1 , a_2 , …, a_n] with constraints:

   $a_1 + a_2 + \ldots + a_n = 1, \qquad 0 < a_i < 1,\ i=1,2,3,\ldots,n$

2. Find the optimal portfolio using computing algorithms
3. This portfolio weights in called the **Nominal Portfolio**

Robust Portfolio Calculation

1) Constrained mean- optimal decision parameter

$$a^*(\theta) = arg \min_{a \in \mathscr{A}(\theta)} \frac{1}{\theta} E\left[exp\left[\theta\left(V_a(X) - \lambda^T X\right)\right]\right] + \lambda^T \mu$$

$$= arg \min_{a \in \mathscr{A}(\theta)} \frac{1}{\theta} \frac{1}{\sqrt{\| I - \theta\gamma a a^T \Sigma \|}} + a^T \mu$$

2) Along with the sum and bound constraints define the positive definite constraint- ($\Sigma^{-1} - \theta\gamma a a^T > 0$) to solve this optimization. The resulting portfolio is called **Robust Portfolio.**

3) $\Sigma^{-1} \downarrow$ hence covariance $\uparrow$.

Experiment 1- Synthetic Data

$\mu_i = 0.1$, $\sigma_i = 0.3$, $\varrho_{ij} = 0.25$, i,j = 1,2,3,..10, i.e, for 10 stocks.

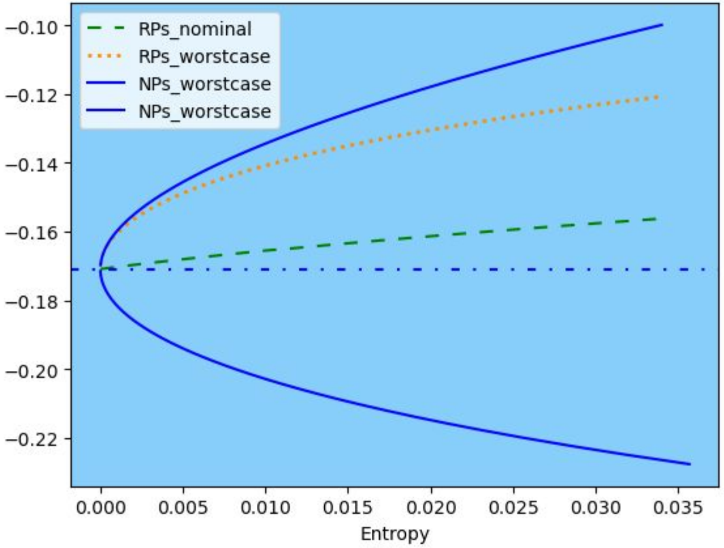


Fig 1: Performance of NP and RP in both the scenarios

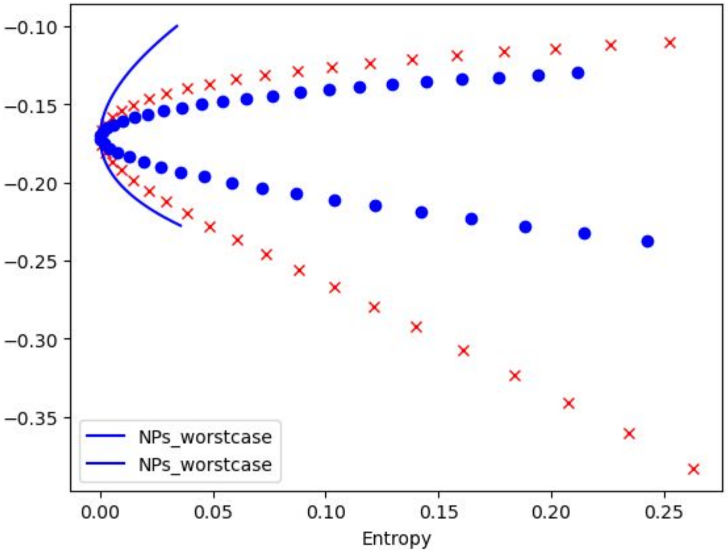


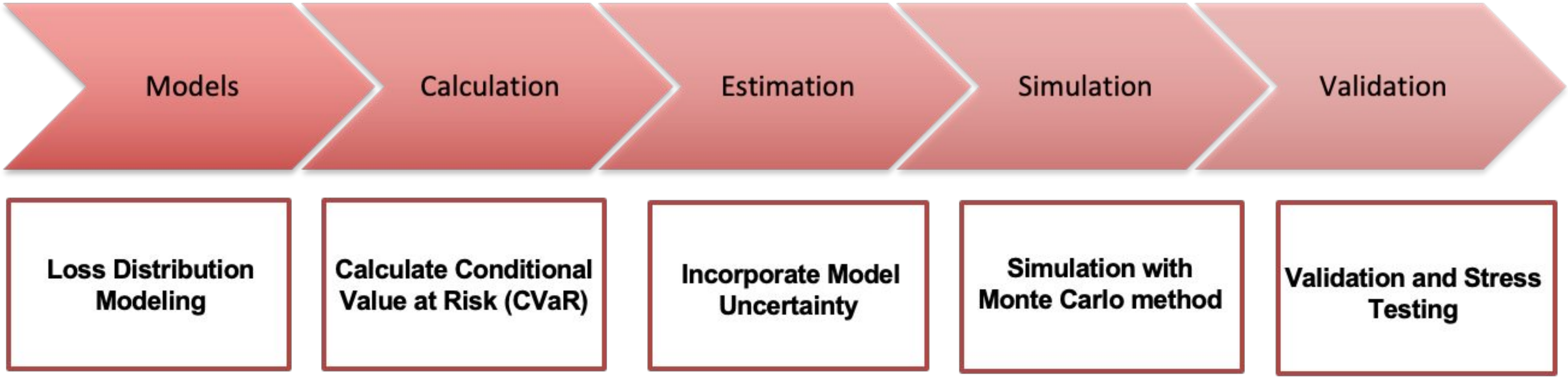
Fig2: Including Model error caused by parameter perturbations

Experiment 2 - Empirical Data

Realized_variance **2x Std error**	**9.21e-04** **(7.58e-04, 10.84e-04)**
forcasted_variance 2x Std error	**1.44e-04** 3.19e-05
theta_ = 200 (with Model error)	**1.45e-04** (1.12e-04, 1.78e-04)
theta_ = 400 (with Model error)	1.45e-04 (1.13e-04, 1.76e-04)
theta_ = 700 (with Model error)	1.44e-04 (1.12e-04, 1.77e-04)

Realized_variance **2x Std error**	**9.65e-04** **(7.94e-04, 11.4e-04)**
forcasted_variance 2x Std error	**2.34e-04** 3.18e-05
theta_ = 200 (with Model error)	**2.58e-04** (1.81e-04 3.34e-04)
theta_ = 400 (with Model error)	**2.65e-04** (1.76e-04 3.54e-04)
theta_ = 700 (with Model error)	**2.63e-04** (1.82e-04 3.46e-04)

Outline : Conditional Value at Risk



Conditional Value at Risk

Definition: $CVaR_\beta = \mathrm{E}[X|X > VaR_\beta]$

$$\Updownarrow$$

Objective Function: $\min_a\{\frac{1}{1-\beta}E[(x-a)^+] + a\}$

Double Exponential Distribution

Double Exponential Distribution DE(μ, b):

- μ: Location parameter
- b: Scale parameter

Preconditions and Parameters:

- Confidence Level β
- Uncertainty Parameter θ
- Likelihood ratio m

Double Exponential Distribution

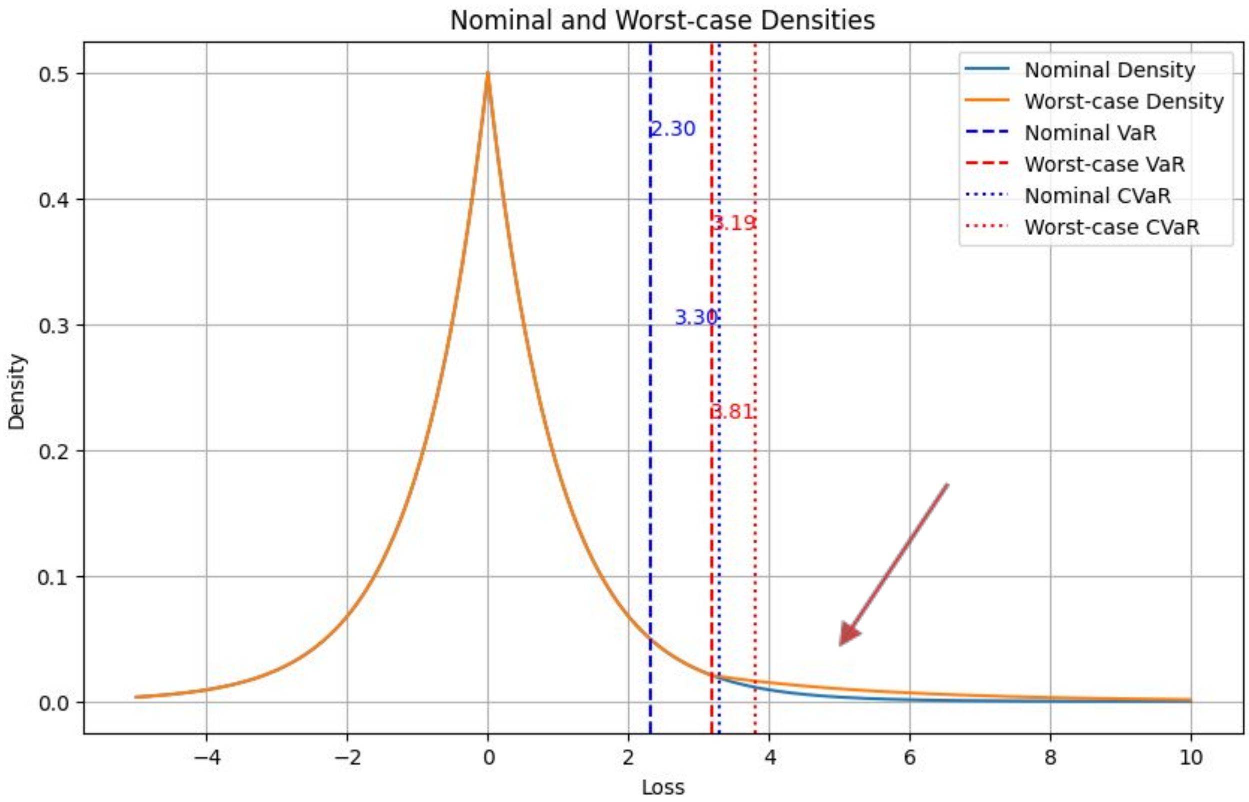
Modified CVaR

$$CVaR_{\beta,\theta} = a^*(\theta) + \frac{1}{\frac{1}{b} + \frac{\theta}{1-\beta}}$$

Relative Entropy

$$\begin{aligned}\eta(\theta) &= E\left[m^*_{a^*(\theta),\theta} \log m^*_{a^*(\theta),\theta}\right] \\ &= \theta \frac{E\left[V_{a^*(\theta)} exp\left(\theta V_{a^*(\theta)}(X)\right)\right]}{E\left[exp\left(\theta V_{a^*(\theta)}(X)\right)\right]} \\ &\quad - \log E\left[exp\left(\theta V_{a^*(\theta)}(X)\right)\right] \\ &= \theta\kappa'_{a^*(\theta)}(\theta) - \kappa_{a^*(\theta)}(\theta)\end{aligned}$$

Double Exponential Distribution



Heavy-Tail Case

Optimal likelihood ratio: $m^*_{\theta,a}(X) = (\theta(\alpha-1)V_a(X) + c(\theta,\alpha,a))^{\frac{1}{\alpha-1}}$

- Generalized Pareto distribution

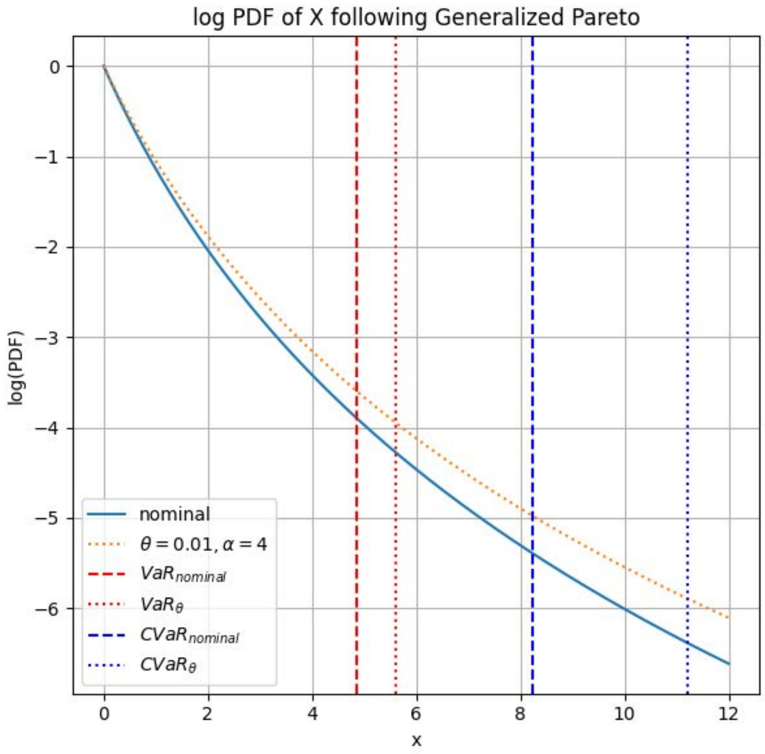
$$f(x) = \frac{1}{b}\left(1 + \frac{\xi_{gp}}{b_{gp}}x\right)^{-\frac{1}{\xi_{gp}}-1}, \quad \text{for } x \geq 0, \text{ some } b_{gp} > 0 \text{ and } \xi_{gp} > 0$$

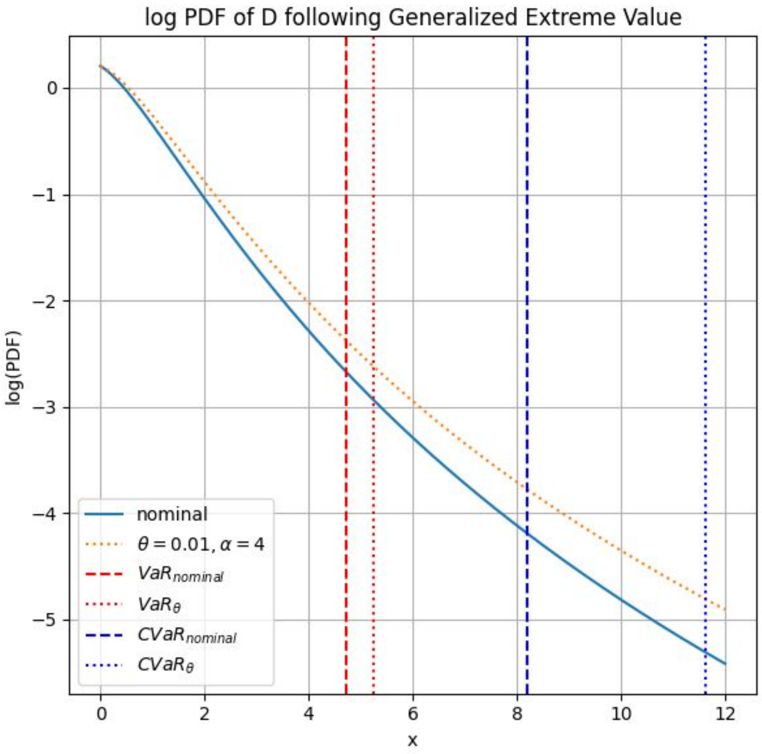
- Generalized Extreme Value distribution

$$f(x) = \frac{1}{\xi_{\text{gev}}}(1+\xi_{\text{gev}}x)^{-\frac{1}{\xi_{\text{gev}}}-1}\exp\left(-(1+\xi_{\text{gev}}x)^{-\frac{1}{\xi_{\text{gev}}}}\right),$$

$$\text{for } \; \mathrm{x} \geq 0 \text{ and } \xi_{\text{gev}} > 0$$

Heavy-Tail Case





Outline: Portfolio Credit Risk

Data	Target	Model	Simulation	Results Comparison
100 obligors 1% default rate	Calculate the probability of P(L>X) under worst senario	Gaussian Copula Model	Monte Carlo method	Compare models' results under two different methods

Gaussian Copula Model(Vaciek)

- Z as a broad risk factor that affects all obligors
- ε_i is an idiosyncratic risk associated with the i-th obligor only
- Z and ε_i are independent normal variables

single-factor homogeneous model

$$X_i = \rho Z + \sqrt{1-\rho^2}\epsilon_i$$

VaR: x_i

$$Y_i = I\left(X_i > x_i\right)$$

Loss: c_i

$$L = \sum_{i=1}^{n} c_i Y_i$$

Other obligators/ bonds/securities, total n

Worst Case Scenario: Numerical Experiments

The worst-case change of measure at parameter θ is:

$$m_\theta^* \sim exp(\theta I_{L>x})$$

$$\hat{m}_i^\theta = \frac{exp(\theta I_i)}{\sum_{i=1}^n exp(\theta I_i)/N}$$

$$\tilde{p} = m^* p$$

- $\sum_i^n exp(\theta I)/N$ is a normalization constant.
- Takes x = 5, which yields P(L > x) = 3.8%.
- The results are based on simulation with N = 10^6 samples.

Estimate Mean and Std of Variables Under m*

Distribution of Parameters (ε_i , Z) under worst case measurement:

$$\mu_{\tilde{z}} = \frac{1}{N}\sum_{i=1}^{N} z_i \times m^{*}{}_{i}^{\theta} \qquad \mu_{\tilde{\varepsilon}_i} = \frac{1}{N}\sum_{i=1}^{N} \varepsilon_i \times m^{*}{}_{i}^{\theta}$$

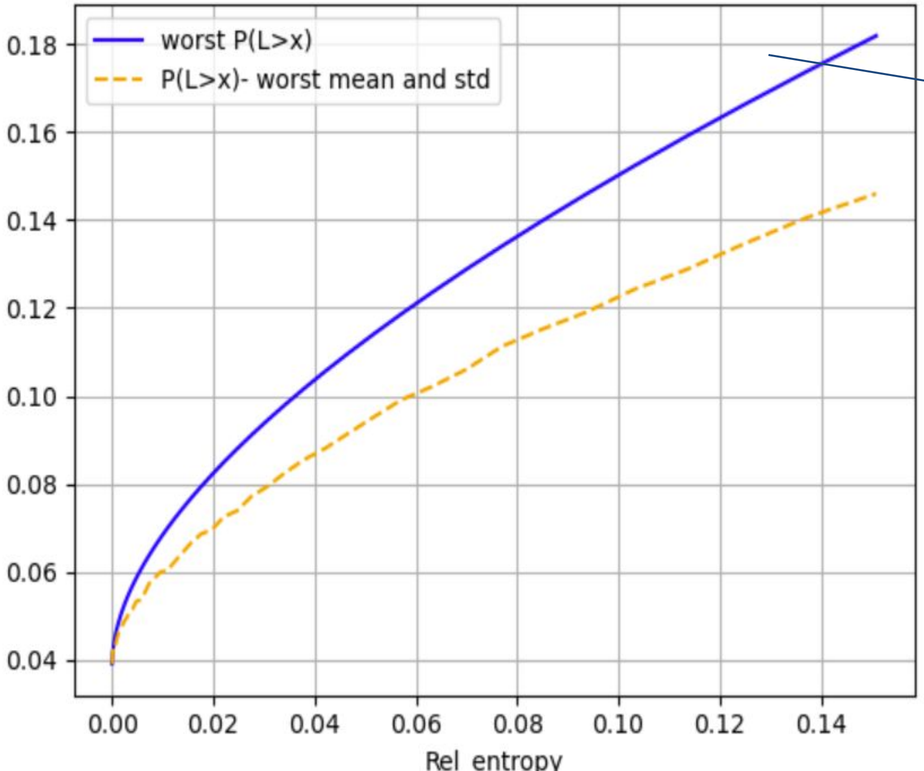
$$cov(\tilde{\varepsilon}, z) = cov\left(m^{*}\varepsilon, m^{*} z\right)$$

m* is not a constant

Generate new variables then estimate mean and std:

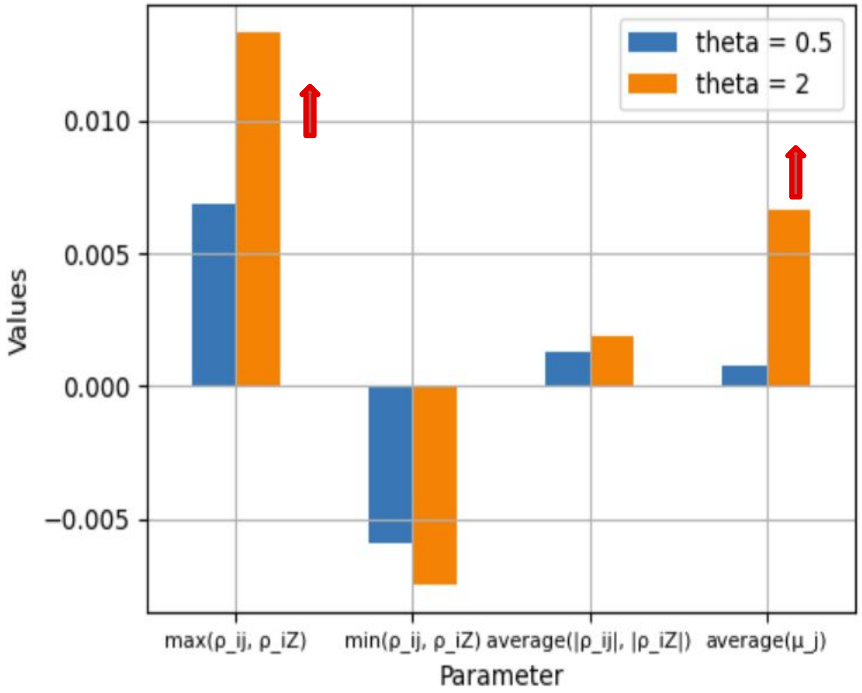
$$\tilde{X}_i = \rho\tilde{Z} + \sqrt{1-\rho^2}\,\tilde{\varepsilon}_i$$

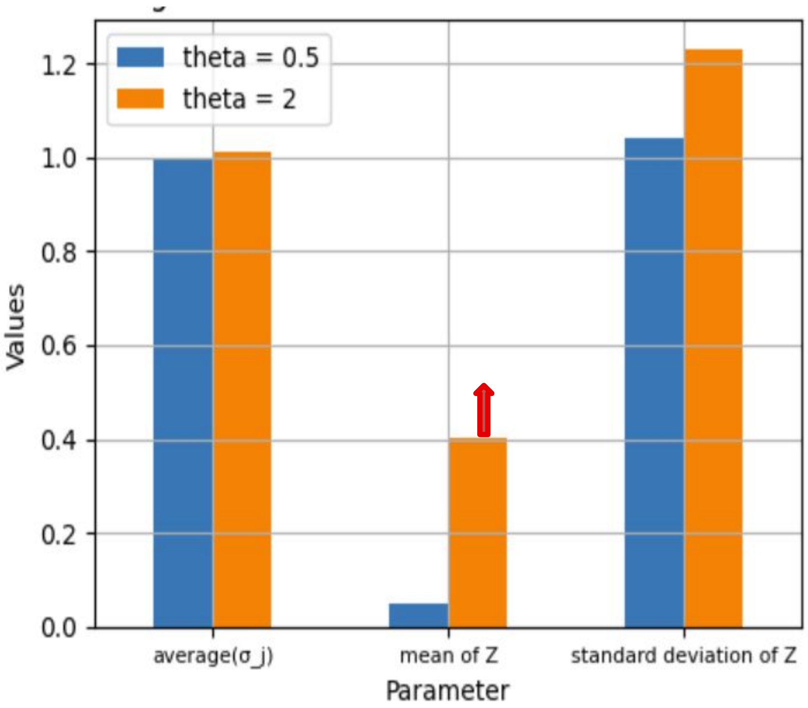
Observations



higher risk from worst case change of measure denotes full entropy budget has been used

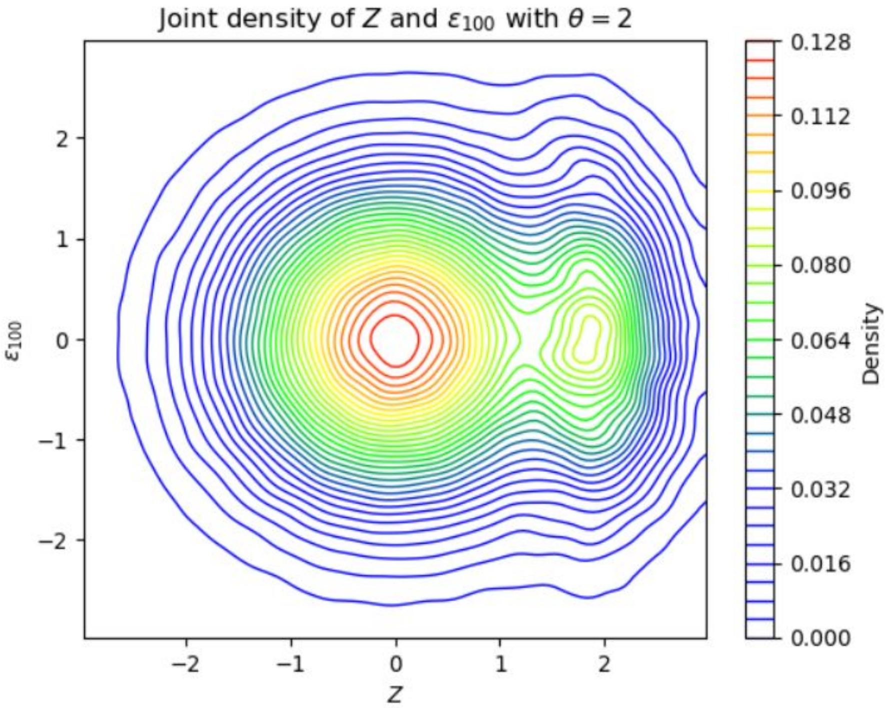
Parameter Estimation- Worst Case Model

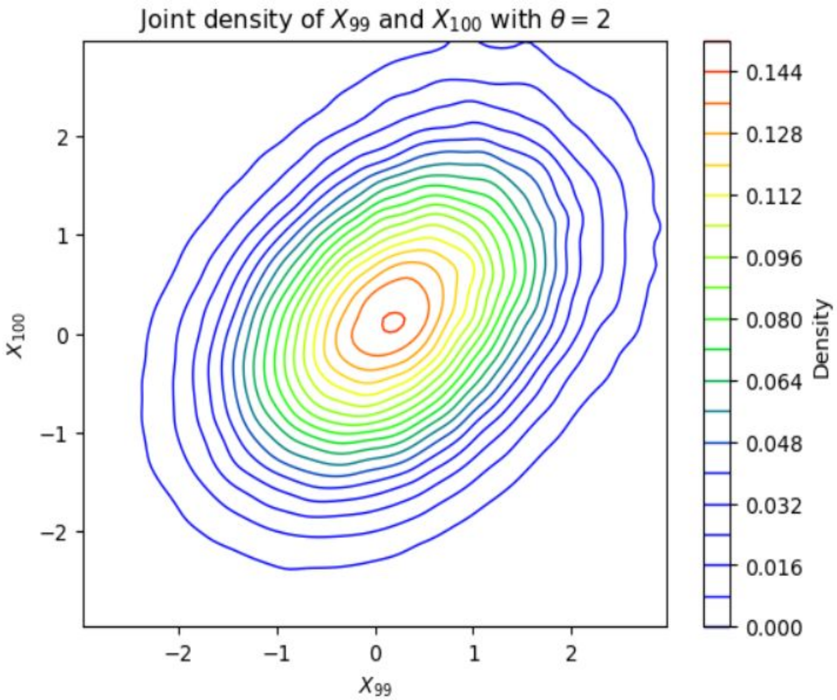




Worst Case Scenarios with θ = 0.5 and 2

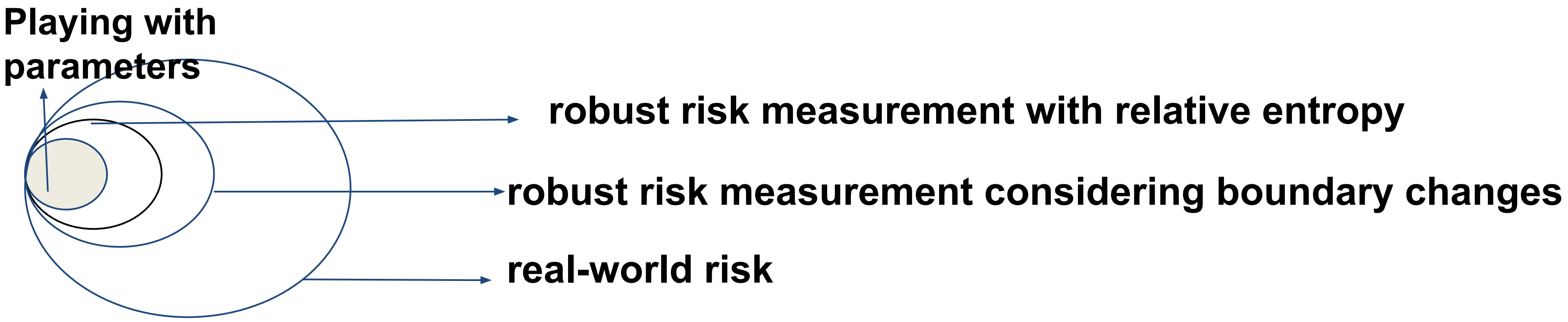
Distribution Behaviour- Worst Case Model



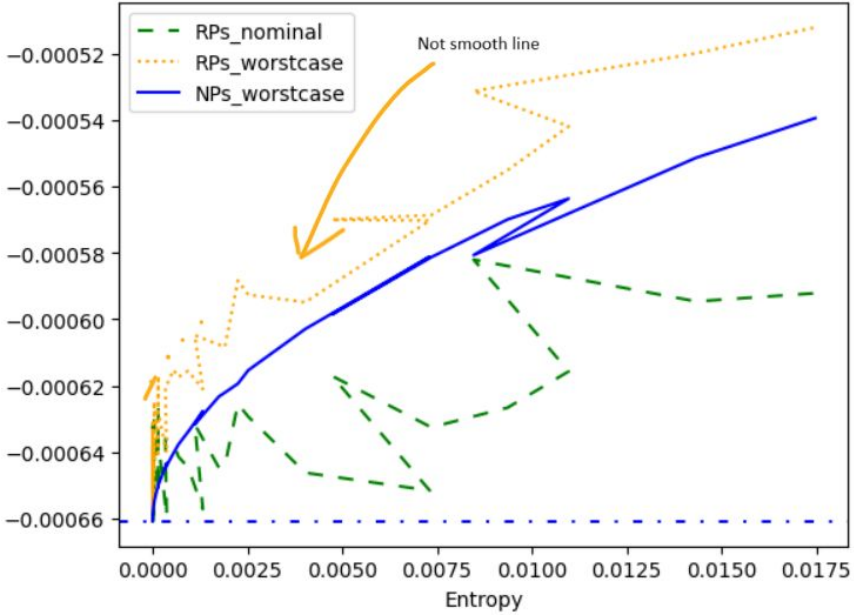


Summary

- The vulnerability of a model has far reaching implications
- Parameter perturbations may miss out some scope of model error which can be covered by Robust risk measurement.



Future Study



1) Find better solvers and optimization tools. This will also solve the problem of the empirical example
2) Design a portfolio of bonds along with the structure of their default probabilities. Analysis the change in distribution in the worst-case scenario.

NP RP analysis done on historical data of stocks.

Thank You!

Q&A