Topics: Normal distribution, Functions of Random Variables

1. The time required for servicing transmissions is normally distributed with $\mu = 45$ minutes and $\sigma = 8$ minutes. The service manager plans to have work begin on the transmission of a customer's car 10 minutes after the car is dropped off and the customer is told that the car will be ready within 1 hour from drop-off. What is the probability that the service manager cannot meet his commitment?

    A. 0.3875
    B. 0.2676
    C. 0.5
    D. 0.6987

Ans: We have a normal distribution with $\mu = 45$ and $\sigma = 8.0$. Let X be the amount of time it takes to complete the repair on a customer's car. To finish in one hour you must have $X \leq 50$ so the question is to find $\Pr(X > 50)$.
$\Pr(X > 50) = 1 - \Pr(X \leq 50)$.
$Z = (X - \mu)/\sigma = (X - 45)/8.0$
Thus, the question can be answered by using the normal table to find
$\Pr(X \leq 50) = \Pr(Z \leq (50 - 45)/8.0) = \Pr(Z \leq 0.625)=73.4\%$
Probability that the service manager will not meet his demand will be = 100-73.4 = 26.6% or 0.2676

2. The current age (in years) of 400 clerical employees at an insurance claims processing center is normally distributed with mean $\mu = 38$ and Standard deviation $\sigma = 6$. For each statement below, please specify True/False. If false, briefly explain why.
    A. More employees at the processing center are older than 44 than between 38 and 44.
    B. A training program for employees under the age of 30 at the center would be expected to attract about 36 employees.

Ans:

We have a normal distribution with $\mu = 38$ and $\sigma = 6$. Let X be the number of employees. So according to question

[A] Probability of employees greater than age of 44= Pr(X>44)

$\Pr(X > 44) = 1 - \Pr(X \leq 44)$.

$Z = (X - \mu)/\sigma = (X - 38)/6$

Thus, the question can be answered by using the normal table to find

$\Pr(X \leq 44) = \Pr(Z \leq (44 - 38)/6) = \Pr(Z \leq 1)$=84.1345%

Probability that the employee will be greater than age of 44 = 100-84.1345=15.86%

So, the probability of number of employees between 38-44 years of age = Pr(X<44)-0.5=84.1345-0.5= 34.1345%

Therefore, the statement that "More employees at the processing center are older than 44 than between 38 and 44" is TRUE.

[B] Probability of employees less than age of 30 = Pr(X<30).

$Z = (X - \mu)/\sigma = (30 - 38)/6$

Thus, the question can be answered by using the normal table to find

$\Pr(X \leq 30) = \Pr(Z \leq (30 - 38)/6) = \Pr(Z \leq -1.333)$=9.12%

So the number of employees with probability 0.912 of them being under age 30 = 0.0912*400=36.48(or 36 employees).

Therefore, the statement B of the question is also TRUE.

3. If $X_1 \sim N(\mu, \sigma^2)$ and $X_2 \sim N(\mu, \sigma^2)$ are *iid* normal random variables, then what is the difference between 2 X_1 and $X_1 + X_2$? Discuss both their distributions and parameters.

Ans: The difference between 2X1 and X1 + X2 is X1 - X2. This is the difference between two independent and identically distributed normal random variables, and its distribution is also normal.

The mean of X1 - X2 is the difference between the means of X1 and X2, which is 0 since they have the same mean. The variance of X1 - X2 is the sum of the variances of X1 and X2, which is 2σ2 since they have the same variance. Therefore, the distribution of X1 - X2 is normal with mean 0 and variance 2σ2.

In contrast, the distribution of 2X1 is normal with mean 2μ and variance 4σ2, and the distribution of X1 + X2 is normal with mean 2μ and variance 2σ2. Both 2X1 and X1 + X2 have the same mean, but their variances are different. Specifically, the variance of 2X1 is twice the variance of X1 + X2.

4. Let X ~ N(100, 20^2). Find two values, *a* and *b*, symmetric about the mean, such that the probability of the random variable taking a value between them is 0.99.

   A. 90.5, 105.9
   B. 80.2, 119.8
   C. 22, 78
   D. 48.5, 151.5
   E. 90.1, 109.9

Ans:
Since we need to find out the values of a and b, which are symmetric about the mean, such that the probability of random variable taking a value between them is 0.99, we have to work out in reverse order.
The Probability of getting value between a and b should be 0.99.
So, the Probability of going wrong, or the Probability outside the a and b area is 0.01 (ie. 1-0.99).
The Probability towards left from a = -0.005 (ie. 0.01/2).
The Probability towards right from b = +0.005 (ie. 0.01/2).
So, since we have the probabilities of a and b, we need to calculate X, the random variable at a and b which has got these probabilities.
By finding the Standard Normal Variable Z (Z Value), we can calculate the X values.
Z= (X- μ) / σ
For Probability 0.005 the Z Value is -2.57 (from Z Table).
Z * σ + μ = X
Z(-0.005)*20+100 = -(-2.57)*20+100 = 151.4
Z(+0.005)*20+100 = (-2.57)*20+100 = 48.6
So, option D is correct

5. Consider a company that has two different divisions. The annual profits from the two divisions are independent and have distributions $Profit_1 \sim N(5, 3^2)$ and $Profit_2 \sim N(7, 4^2)$ respectively. Both the profits are in \$ Million. Answer the following questions about the total profit of the company in Rupees. Assume that \$1 = Rs. 45
    A. Specify a Rupee range (centered on the mean) such that it contains 95% probability for the annual profit of the company.
    B. Specify the 5^{th} percentile of profit (in Rupees) for the company
    C. Which of the two divisions has a larger probability of making a loss in a given year?

Ans: Attached Jupyter Notebook Python File.