

# Unbiasing Review Ratings with Tendency based Collaborative Filtering

## Appendix

**Pranshi Yadav\***

IIIT, Hyderabad

pranshiyadav@gmail.com

**Pegah Nokhiz**

University of Utah

pnokhiz@cs.utah.edu

**Priya Yadav\***

JSSATE, Noida

priya.yadav252@gmail.com

**Vivek Gupta**

University of Utah

vgupta@cs.utah.edu

### A Notation and Definition

#### A.1 Known Parameters

1. User Mean:  $\bar{r}_u = \frac{1}{n_i} \sum_{i=1}^{n_i} r_{ui}$
2. Product Mean:  $\bar{r}_i = \frac{1}{n_u} \sum_{u=1}^{n_u} r_{ui}$
3. User Tendency:  $\tau_u = \frac{1}{n_i} \sum_{i=1}^{n_i} (r_{ui} - \bar{r}_i)$
4. Item Tendency:  $\tau_i = \frac{1}{n_u} \sum_{u=1}^{n_u} (r_{ui} - \bar{r}_u)$

Where,

$n_i$ - number of items rated by the user ( $u$ ).

$n_u$ - number of user who have rated item ( $i$ ).

$r_{ui}$ - rating by user ( $u$ ) to the item ( $i$ ).

**Enough Rating Assumption:** To calculate adequate invariant estimates of user/item means and tendencies from the rated data, we need to make the following assumption: The prediction example is based in an online setting. Thus, the distribution means and tendencies (both users/items) will not change significantly, when we predict ratings for new user-item pairs. This helps us in deciding the case on the estimated mean and tendency from the given labeled ratings. Similarly, the case for reverse estimation function will be decided on the prior data's means and tendencies.

#### A.2 Derived Parameters

1.  $\sum_{i=1}^{n_i} r_{ui} = \bar{r}_u * n_i$  : Sum of all the ratings rated by the user ( $u$ ) for all items.
2.  $\sum_{u=1}^{n_u} r_{ui} = \bar{r}_i * n_u$  : Sum of all the ratings rated by user for an item ( $i$ ) for all users.
3.  $\sum_{i=1}^{n_i} \bar{r}_i = n_i(\bar{r}_u - \tau_u)$  : Sum of all item means (rated by user ( $u$ )).
4.  $\sum_{u=1}^{n_u} \bar{r}_u = n_u(\bar{r}_i - \tau_i)$  : Sum of all user means (rated for item ( $i$ )).

#### A.3 Derivation of Derived Parameters

$$1. \bar{r}_u = \frac{1}{n_i} \sum_{i=1}^{n_i} r_{ui}$$

$$\sum_{i=1}^{n_i} r_{ui} = \bar{r}_u * n_i$$

$$2. \bar{r}_i = \frac{1}{n_u} \sum_{u=1}^{n_u} r_{ui}$$

$$\sum_{u=1}^{n_u} r_{ui} = \bar{r}_i * n_u$$

$$3. \tau_u = \frac{\sum_{i=1}^{n_i} r_{ui} - \sum_{i=1}^{n_i} \bar{r}_i}{n_i}$$

$$\tau_u * n_i = \sum_{i=1}^{n_i} r_{ui} - \sum_{i=1}^{n_i} \bar{r}_i$$

$$\sum_{i=1}^{n_i} \bar{r}_i = \sum_{i=1}^{n_i} r_{ui} - (\tau_u * n_i)$$

$$\sum_{i=1}^{n_i} \bar{r}_i = (\bar{r}_u * n_i) - (\tau_u * n_i) \text{ -- From eq. 1}$$

$$\sum_{i=1}^{n_i} \bar{r}_i = n_i(\bar{r}_u - \tau_u)$$

$$4. \tau_i = \frac{\sum_{u=1}^{n_u} r_{ui} - \sum_{u=1}^{n_u} \bar{r}_u}{n_u}$$

$$\tau_i * n_u = \sum_{u=1}^{n_u} r_{ui} - \sum_{u=1}^{n_u} \bar{r}_u$$

$$\sum_{u=1}^{n_u} \bar{r}_u = \sum_{u=1}^{n_u} r_{ui} - (\tau_i * n_u)$$

$$\sum_{u=1}^{n_u} \bar{r}_u = (\bar{r}_i * n_u) - (\tau_i * n_u) \text{ -- From eq. 2}$$

$$\sum_{u=1}^{n_u} \bar{r}_u = n_u(\bar{r}_i - \tau_i)$$

### B Unbiased Review Scores

Case	UnBias Function ( $\widehat{r}_{ui}$ )
1. $\tau_u > 0, \tau_i > 0$	$\max(\bar{r}_u + \tau_i, \bar{r}_i + \tau_u)$
2. $\tau_u < 0, \tau_i < 0$	$\min(\bar{r}_u + \tau_i, \bar{r}_i + \tau_u)$
3. $\tau_u < 0, \tau_i > 0, \bar{r}_u < \bar{r}_i$	$\min(\max(\bar{r}_u, (\bar{r}_i + \tau_u)\beta + (\bar{r}_u + \tau_i)(1 - \beta)), \bar{r}_i)$
4. $\tau_u < 0, \tau_i > 0, \bar{r}_u > \bar{r}_i$	$\bar{r}_i\beta + \bar{r}_u(1 - \beta)$
5. $\tau_u > 0, \tau_i < 0, \bar{r}_u > \bar{r}_i$	$\min(\max(\bar{r}_i, (\bar{r}_u + \tau_i)\beta + (\bar{r}_i + \tau_u)(1 - \beta)), \bar{r}_u)$
6. $\tau_u > 0, \tau_i < 0, \bar{r}_u < \bar{r}_i$	$\bar{r}_u\beta + \bar{r}_i(1 - \beta)$

Table 1: Unbias functions

### C Derivation of Reverse functions

This sections contains the details about the derivations of the reverse functions. The reverse tendency estimated functions are listed in Table 2. Here, (train) in subscript mean using examples from training set, (new) in subscript mean a new examples from test/valid test, and (final) in subscript representing the final prediction.

Case	Sub-case	Reverse tendency estimation function
1. $\tau_u > 0$ , $\tau_i > 0$	I. $\widehat{r_{ui}} = \overline{r_u} + \tau_i(new)$ II. $\widehat{r_{ui}} = \overline{r_i} + \tau_u(new)$	$r_{ui(new)} = (\frac{1}{n_i + n_u + 1})[\widehat{r_{ui}}(n_u + 1)(n_i + 1) - (n_i + 1)(n_u \times \tau_i) - (n_i \times n_u \times \overline{r_u})]$ $r_{ui(new)} = (\frac{1}{n_i + n_u + 1})[\widehat{r_{ui}}(n_u + 1)(n_i + 1) - (n_u + 1)(n_i \times \tau_u) - (n_i \times n_u \times \overline{r_i})]$
2. $\tau_u < 0$ , $\tau_i < 0$	I. $\widehat{r_{ui}} = \overline{r_u} + \tau_i(new)$ II. $\widehat{r_{ui}} = \overline{r_i} + \tau_u(new)$	$r_{ui(new)} = (\frac{1}{n_i + n_u + 1})[\widehat{r_{ui}}(n_u + 1)(n_i + 1) - (n_i + 1)(n_u \times \tau_i) - (n_i \times n_u \times \overline{r_u})]$ $r_{ui(new)} = (\frac{1}{n_i + n_u + 1})[\widehat{r_{ui}}(n_u + 1)(n_i + 1) - (n_u + 1)(n_i \times \tau_u) - (n_i \times n_u \times \overline{r_i})]$
3. $\tau_u < 0$ , $\tau_i > 0$ , $\overline{r_u} < \overline{r_i}$	I. $\widehat{r_{ui}} = \overline{r_u}(final)$ II. $\widehat{r_{ui}} = (\overline{r_i} + \tau_u)\beta + (\overline{r_u} + \tau_i)(1 - \beta)$ III. $\widehat{r_{ui}} = \overline{r_i}(final)$	$r_{ui(new)} = (n_i + 1)\widehat{r_{ui}} - (\overline{r_u} \times n_i)$ $r_{ui(new)} = (\frac{1}{n_i + n_u + 1})[(n_u + 1)(n_i + 1)\widehat{r_{ui}} - (n_u)(\overline{r_u} \times n_i)(1 - \beta) - (n_i + 1)(1 - \beta)(n_u \times \tau_i) - \beta(n_i)(\overline{r_i} \times n_u) - \beta(n_u + 1)(n_i \times \tau_u)]$ $r_{ui(new)} = (n_u + 1)\widehat{r_{ui}} - (\overline{r_i} \times n_u)$
4. $\tau_u < 0$ , $\tau_i > 0$ , $\overline{r_u} > \overline{r_i}$	$\widehat{r_{ui}} = \overline{r_i}\beta + \overline{r_u}(1 - \beta)$	$r_{ui(new)} = \frac{1}{[\overline{r_u}(1 - \beta) + 1 + \beta n_i]} \times [(n_u + 1)(n_i + 1)\widehat{r_{ui}} - \beta(n_i + 1)(\overline{r_i} \times n_u) - (1 - \beta)(n_u + 1)(\overline{r_u} \times n_i)]$
5. $\tau_u > 0$ , $\tau_i < 0$ , $\overline{r_u} > \overline{r_i}$	I. $\widehat{r_{ui}} = \overline{r_i}(final)$ II. $\widehat{r_{ui}} = (\overline{r_u} + \tau_i)\beta + (\overline{r_i} + \tau_u)(1 - \beta)$ III. $\widehat{r_{ui}} = \overline{r_u}(final)$	$r_{ui(new)} = (n_u + 1)\widehat{r_{ui}} - (\overline{r_i} \times n_u)$ $r_{ui(new)} = (\frac{1}{n_i + n_u + 1})[(n_u + 1)(n_i + 1)\widehat{r_{ui}} - (n_i)(\overline{r_i} \times n_u)(1 - \beta) - (n_u + 1)(1 - \beta)(n_i \times \tau_u) - \beta(n_u)(\overline{r_u} \times n_i) - \beta(n_i + 1)(n_u \times \tau_i)]$ $r_{ui(new)} = (n_i + 1)\widehat{r_{ui}} - (\overline{r_u} \times n_i)$
6. $\tau_u > 0$ , $\tau_i < 0$ , $\overline{r_u} < \overline{r_i}$	$\widehat{r_{ui}} = \overline{r_u}\beta + \overline{r_i}(1 - \beta)$	$r_{ui(new)} = \frac{1}{[\overline{r_i}(1 - \beta) + 1 + \beta n_u]} \times [(n_u + 1)(n_i + 1)\widehat{r_{ui}} - \beta(n_u + 1)(\overline{r_u} \times n_i) - (1 - \beta)(n_i + 1)(\overline{r_i} \times n_u)]$

Table 2: Reverse Estimation Functions to predict review rating (with bias) for a new user-item pair

### C.1 Case 1: User Tendency is positive and Product Tendency is positive ( $\tau_u > 0$ , $\tau_i > 0$ )

Unbias Rating :  $\widehat{r_{ui}} = \max(\overline{r_u} + \tau_i, \overline{r_i} + \tau_u)$

Reverse Function

1. If  $\widehat{r_{ui}} = \overline{r_u} + \tau_i$  (new)

$$\tau_i = \widehat{r_{ui}} - \overline{r_u} \text{ ----- } -1$$

And

$$\tau_i = \frac{\sum_{u \in U_i} (r_{ui} - \overline{r_u})}{n_u + 1} \text{ ----- } -2$$

Now, equating 1 and 2

$$\widehat{r_{ui}} - \overline{r_u} = \frac{\sum_{u \in U_i} (r_{ui} - \overline{r_u})}{n_u + 1}$$

$$\widehat{r_{ui}} - \frac{\sum_{n_i+1}^{items} r_{ui}(final)}{n_i+1} = \frac{\sum_{n_u+1}^{users} r_{ui}(final) - \sum_{n_u+1}^{users} \overline{r_u}(final)}{n_u+1}$$

Where,

$n_u + 1$  - number of users who have rated the item (including the final user)

$n_i + 1$  - numbers of items rated by the user (including the final item)

$\sum_{n_i+1}^{items} r_{ui}(final)$  - sum of all the scores rated by the user  $u$  (final)

$\sum_{n_u+1}^{users} r_{ui}(final)$  - sum of the scores rated by the users for the item  $i$  (final)

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_i+1}^{items} r_{ui}(train) + r_{ui}(new)}{n_i+1} = \frac{\sum_{n_u+1}^{users} r_{ui}(train) + r_{ui}(new) - \sum_{n_u+1}^{users} \overline{r_u}(final)}{n_u+1}$$

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_i+1}^{items} r_{ui}(train)}{n_i+1} - \frac{r_{ui}(new)}{n_i+1} = \frac{\sum_{n_u+1}^{users} r_{ui}(train) + r_{ui}(new) - [\sum_{n_u+1}^{users} \overline{r_u}(train) + \overline{r_u}(new)]}{n_u+1}$$

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_i+1}^{items} r_{ui}(train)}{n_i+1} - \frac{r_{ui}(new)}{n_i+1} = \frac{\sum_{n_u+1}^{users} r_{ui}(train) + r_{ui}(new) - \sum_{n_u+1}^{users} \overline{r_u}(train) - \sum_{n_i+1}^{items} r_{ui}(final)}{n_u+1}$$

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_i+1}^{items} r_{ui}(train)}{n_i+1} - \frac{r_{ui}(new)}{n_i+1} = \frac{\sum_{n_u+1}^{users} r_{ui}(train) + r_{ui}(new) - \sum_{n_u+1}^{users} \overline{r_u}(train) - \sum_{n_i+1}^{items} r_{ui}(train) - \frac{r_{ui}(new)}{n_i+1}}{n_u+1}$$

$$\Rightarrow \widehat{r_{ui}}(n_u + 1) - \frac{(n_u + 1) \sum_{n_i+1}^{items} r_{ui}(train)}{n_i+1} - \frac{(n_u + 1) r_{ui}(new)}{n_i+1} = \sum_{n_u+1}^{users} r_{ui}(train) + r_{ui}(new) - \sum_{n_u+1}^{users} \overline{r_u}(train) - \frac{\sum_{n_i+1}^{items} r_{ui}(train)}{n_i+1} - \frac{r_{ui}(new)}{n_i+1}$$

$$\begin{aligned}
&\Rightarrow \widehat{r_{ui}}(n_u + 1) - \frac{(n_u+1) \sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} - \frac{(n_u+1)r_{ui(new)}}{n_i+1} = \sum^{users} r_{ui(train)} - \sum^{users} \overline{r_{u(train)}} - \\
&\frac{\sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} + \left[ \frac{(n_i+1)r_{ui(new)} - r_{ui(new)}}{n_i+1} \right] \\
&\Rightarrow \widehat{r_{ui}}(n_u + 1) - \frac{(n_u+1) \sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} - \frac{(n_u+1)r_{ui(new)}}{n_i+1} = \sum^{users} (r_{ui(train)} - \overline{r_{u(train)}}) - \frac{\sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} + \\
&\frac{(n_i)r_{ui(new)}}{n_i+1}
\end{aligned}$$

Reshuffling,

$$\begin{aligned}
&\Rightarrow \frac{(n_i)r_{ui(new)}}{n_i+1} + \frac{(n_u+1)r_{ui(new)}}{n_i+1} = \frac{\sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} - \sum^{users} (r_{ui(train)} - \overline{r_{u(train)}}) + \widehat{r_{ui}}(n_u + 1) - \\
&\frac{(n_u+1) \sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} \\
&\Rightarrow \frac{(n_i+n_u+1)r_{ui(new)}}{n_i+1} = \widehat{r_{ui}}(n_u + 1) - \sum^{users} (r_{ui(train)} - \overline{r_{u(train)}}) - \frac{(n_u) \sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} \\
&\Rightarrow r_{ui(new)} = \left( \frac{n_i+1}{n_i+n_u+1} \right) \left[ \widehat{r_{ui}}(n_u + 1) - \sum^{users} (r_{ui(train)} - \overline{r_{u(train)}}) - \frac{(n_u) \sum_{n_i+1}^{items} r_{ui(train)}}{n_i+1} \right] \\
&\Rightarrow r_{ui(new)} = \left( \frac{n_i+1}{n_i+n_u+1} \right) \left[ \widehat{r_{ui}}(n_u + 1) - (\overline{r_i} * n_u) + (\overline{r_i} * n_u) - (n_u * \tau_i) - \frac{(n_u)(\overline{r_u} * n_i)}{n_i+1} \right] \\
&\Rightarrow r_{ui(new)} = \left( \frac{n_i+1}{n_i+n_u+1} \right) \left[ \widehat{r_{ui}}(n_u + 1) - (n_u * \tau_i) - \frac{(n_u)(\overline{r_u} * n_i)}{n_i+1} \right] \\
&\Rightarrow r_{ui(new)} = \left( \frac{1}{n_i+n_u+1} \right) \left[ \widehat{r_{ui}}(n_u + 1)(n_i + 1) - (n_i + 1)(n_u * \tau_i) - (n_i * n_u * \overline{r_u}) \right]
\end{aligned}$$

2. If  $\widehat{r_{ui}} = \overline{r_i} + \tau_u$  (new)

$$\tau_u = \widehat{r_{ui}} - \overline{r_i} \text{ ----- -1}$$

And

$$\tau_u = \frac{\sum_{u \in I} (r_{ui} - \overline{r_i})}{n_i+1} \text{ ----- -2}$$

Now, equating 1 and 2

$$\widehat{r_{ui}} - \overline{r_i} = \frac{\sum_{u \in I} (r_{ui} - \overline{r_i})}{n_i+1}$$

$$\widehat{r_{ui}} - \frac{\sum_{n_u+1}^{users} r_{ui(final)}}{n_u+1} = \frac{\sum_{n_i+1}^{items} r_{ui(final)} - \sum_{n_i+1}^{items} \overline{r_i(final)}}{n_i+1}$$

Where,

$n_{u+1}$  - number of users who have rated the item (including the final user)

$n_{i+1}$  - numbers of items rated by the user (including the final item)

$\sum_{n_i+1}^{items} r_{ui(final)}$  - sum of all the scores rated by the user  $u$  (final)

$\sum_{n_u+1}^{users} r_{ui(final)}$  - sum of the scores rated by the users for the item  $i$  (final)

$$\begin{aligned}
&\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_u+1}^{users} r_{ui(train)} + r_{ui(new)}}{n_u+1} = \frac{\sum_{n_i+1}^{items} r_{ui(train)} + r_{ui(new)} - \sum_{n_i+1}^{items} \overline{r_i(final)}}{n_i+1} \\
&\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{r_{ui(new)}}{n_u+1} = \frac{\sum_{n_i+1}^{items} r_{ui(train)} + r_{ui(new)} - \left[ \sum_{n_i+1}^{items} \overline{r_i(train)} + \overline{r_i(new)} \right]}{n_i+1} \\
&\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{r_{ui(new)}}{n_u+1} = \frac{\sum_{n_i+1}^{items} r_{ui(train)} + r_{ui(new)} - \sum_{n_i+1}^{items} \overline{r_i(train)} - \frac{\sum_{n_u+1}^{users} r_{ui(final)}}{n_u+1}}{n_i+1} \\
&\Rightarrow \widehat{r_{ui}} - \frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{r_{ui(new)}}{n_u+1} = \frac{\sum_{n_i+1}^{items} r_{ui(train)} + r_{ui(new)} - \sum_{n_i+1}^{items} \overline{r_i(train)} - \frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{r_{ui(new)}}{n_u+1}}{n_i+1} \\
&\Rightarrow \widehat{r_{ui}}(n_i + 1) - \frac{(n_i+1) \sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{(n_i+1)r_{ui(new)}}{n_u+1} = \sum_{n_i+1}^{items} r_{ui(train)} + r_{ui(new)} - \sum_{n_i+1}^{items} \overline{r_i(train)} - \\
&\frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{r_{ui(new)}}{n_u+1} \\
&\Rightarrow \widehat{r_{ui}}(n_i + 1) - \frac{(n_i+1) \sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{(n_i+1)r_{ui(new)}}{n_u+1} = \sum_{n_i+1}^{items} r_{ui(train)} - \sum_{n_i+1}^{items} \overline{r_i(train)} - \\
&\frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} + \left[ \frac{(n_u+1)r_{ui(new)} - r_{ui(new)}}{n_u+1} \right] \\
&\Rightarrow \widehat{r_{ui}}(n_i + 1) - \frac{(n_i+1) \sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \frac{(n_i+1)r_{ui(new)}}{n_u+1} = \sum_{n_i+1}^{items} (r_{ui(train)} - \overline{r_i(train)}) - \frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} + \\
&\frac{(n_u)r_{ui(new)}}{n_u+1}
\end{aligned}$$

Reshuffling,

$$\begin{aligned}
&\Rightarrow \frac{(n_u)r_{ui(new)}}{n_u+1} + \frac{(n_i+1)r_{ui(new)}}{n_u+1} = \frac{\sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} - \sum_{n_u+1}^{items} (r_{ui(train)} - \overline{r_i(train)}) + \widehat{r_{ui}}(n_i + 1) - \\
&\frac{(n_i+1) \sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1} \\
&\Rightarrow \frac{(n_i+n_u+1)r_{ui(new)}}{n_u+1} = \widehat{r_{ui}}(n_i + 1) - \sum_{n_u+1}^{items} (r_{ui(train)} - \overline{r_i(train)}) - \frac{(n_i) \sum_{n_u+1}^{users} r_{ui(train)}}{n_u+1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow r_{ui(new)} &= \left( \frac{n_u+1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_i+1) - \sum^{items} (r_{ui(train)} - \overline{r_{i(train)}}) - \frac{(n_i) \sum^{users} r_{ui(train)}}{n_u+1}] \\
\Rightarrow r_{ui(new)} &= \left( \frac{n_u+1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_i+1) - (\overline{r_u} * n_i) + (\overline{r_u} * n_i) - (n_i * \tau_u) - \frac{(n_i)(\overline{r_i} * n_u)}{n_u+1}] \\
\Rightarrow r_{ui(new)} &= \left( \frac{n_u+1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_i+1) - (n_i * \tau_u) - \frac{(n_i)(\overline{r_i} * n_u)}{n_u+1}] \\
\Rightarrow r_{ui(new)} &= \left( \frac{1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_u+1)(n_i+1) - (n_u+1)(n_i * \tau_u) - (n_i * n_u * \overline{r_i})]
\end{aligned}$$

## C.2 Case 2: User Tendency is negative and Product Tendency is negative ( $\tau_u < 0, \tau_i < 0$ )

Unbias Rating :  $\widehat{r_{ui}} = \min(\overline{r_u} + \tau_i, \overline{r_i} + \tau_u)$

Reverse Function

1. If  $\widehat{r_{ui}} = \overline{r_u} + \tau_i$  (new)

$$\tau_i = r_{ui} - \overline{r_u} \text{ ----- -1}$$

And

$$\tau_i = \frac{\sum_{u \in U_i} (r_{ui} - \overline{r_u})}{n_u+1} \text{ ----- -2}$$

Now, equating 1 and 2

$$\widehat{r_{ui}} - \overline{r_u} = \frac{\sum_{u \in U_i} (r_{ui} - \overline{r_u})}{n_u+1}$$

$$\widehat{r_{ui}} - \frac{\sum^{items} r_{ui(final)}}{n_i+1} = \frac{\sum^{users} r_{ui(final)} - \sum^{users} \overline{r_u(final)}}{n_u+1}$$

Where,

$n_u+1$  - number of users who have rated the item (including the final user)

$n_i+1$  - numbers of items rated by the user (including the final item)

$\sum^{items} r_{ui(final)}$  - sum of all the scores rated by the user  $u$  (final)

$\sum^{users} r_{ui(final)}$  - sum of the scores rated by the users for the item  $i$  (final)

$$\begin{aligned}
\Rightarrow \widehat{r_{ui}} - \frac{\sum^{items} r_{ui(train)} + r_{ui(new)}}{n_i+1} &= \frac{\sum^{users} r_{ui(train)} + r_{ui(new)} - \sum^{users} \overline{r_u(final)}}{n_u+1} \\
\Rightarrow \widehat{r_{ui}} - \frac{\sum^{items} r_{ui(train)}}{n_i+1} - \frac{r_{ui(new)}}{n_i+1} &= \frac{\sum^{users} r_{ui(train)} + r_{ui(new)} - [\sum^{users} \overline{r_u(train)} + \overline{r_u(new)}]}{n_u+1} \\
\Rightarrow \widehat{r_{ui}} - \frac{\sum^{items} r_{ui(train)}}{n_i+1} - \frac{r_{ui(new)}}{n_i+1} &= \frac{\sum^{users} r_{ui(train)} + r_{ui(new)} - \sum^{users} \overline{r_u(train)} - \sum^{items} \frac{r_{ui(final)}}{n_i+1}}{n_u+1} \\
\Rightarrow \widehat{r_{ui}} - \frac{\sum^{items} r_{ui(train)}}{n_i+1} - \frac{r_{ui(new)}}{n_i+1} &= \frac{\sum^{users} r_{ui(train)} + r_{ui(new)} - \sum^{users} \overline{r_u(train)} - \sum^{items} \frac{r_{ui(train)}}{n_i+1} - \frac{r_{ui(new)}}{n_i+1}}{n_u+1} \\
\Rightarrow \widehat{r_{ui}}(n_u+1) - \frac{(n_u+1) \sum^{items} r_{ui(train)}}{n_i+1} - \frac{(n_u+1)r_{ui(new)}}{n_i+1} &= \sum^{users} r_{ui(train)} + r_{ui(new)} - \sum^{users} \overline{r_u(train)} - \frac{\sum^{items} r_{ui(train)}}{n_i+1} - \frac{r_{ui(new)}}{n_i+1} \\
\Rightarrow \widehat{r_{ui}}(n_u+1) - \frac{(n_u+1) \sum^{items} r_{ui(train)}}{n_i+1} - \frac{(n_u+1)r_{ui(new)}}{n_i+1} &= \sum^{users} r_{ui(train)} - \sum^{users} \overline{r_u(train)} - \frac{\sum^{items} r_{ui(train)}}{n_i+1} + \left[ \frac{(n_i+1)r_{ui(new)} - r_{ui(new)}}{n_i+1} \right] \\
\Rightarrow \widehat{r_{ui}}(n_u+1) - \frac{(n_u+1) \sum^{items} r_{ui(train)}}{n_i+1} - \frac{(n_u+1)r_{ui(new)}}{n_i+1} &= \sum^{users} (r_{ui(train)} - \overline{r_u(train)}) - \frac{\sum^{items} r_{ui(train)}}{n_i+1} + \frac{(n_i)r_{ui(new)}}{n_i+1}
\end{aligned}$$

Reshuffling,

$$\begin{aligned}
\Rightarrow \frac{(n_i)r_{ui(new)}}{n_i+1} + \frac{(n_u+1)r_{ui(new)}}{n_i+1} &= \frac{\sum^{items} r_{ui(train)}}{n_i+1} - \sum^{users} (r_{ui(train)} - \overline{r_u(train)}) + \widehat{r_{ui}}(n_u+1) - \frac{(n_u+1) \sum^{items} r_{ui(train)}}{n_i+1} \\
\Rightarrow \frac{(n_i+n_u+1)r_{ui(new)}}{n_i+1} &= \widehat{r_{ui}}(n_u+1) - \sum^{users} (r_{ui(train)} - \overline{r_u(train)}) - \frac{(n_u) \sum^{items} r_{ui(train)}}{n_i+1} \\
\Rightarrow r_{ui(new)} &= \left( \frac{n_i+1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_u+1) - \sum^{users} (r_{ui(train)} - \overline{r_u(train)}) - \frac{(n_u) \sum^{items} r_{ui(train)}}{n_i+1}] \\
\Rightarrow r_{ui(new)} &= \left( \frac{n_i+1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_u+1) - (\overline{r_i} * n_u) + (\overline{r_i} * n_u) - (n_u * \tau_i) - \frac{(n_u)(\overline{r_u} * n_i)}{n_i+1}] \\
\Rightarrow r_{ui(new)} &= \left( \frac{n_i+1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_u+1) - (n_u * \tau_i) - \frac{(n_u)(\overline{r_u} * n_i)}{n_i+1}] \\
\Rightarrow r_{ui(new)} &= \left( \frac{1}{n_i+n_u+1} \right) [\widehat{r_{ui}}(n_u+1)(n_i+1) - (n_i+1)(n_u * \tau_i) - (n_i * n_u * \overline{r_u})]
\end{aligned}$$

2. If  $\widehat{r_{ui}} = \bar{r_i} + \tau_u$  (new)

$$\tau_u = \widehat{r_{ui}} - \bar{r_i} \text{ ----- } -1$$

And

$$\tau_u = \frac{\sum_{u \in I} (r_{ui} - \bar{r_i})}{n_{i+1}} \text{ ----- } -2$$

Now, equating 1 and 2

$$\widehat{r_{ui}} - \bar{r_i} = \frac{\sum_{u \in I} (r_{ui} - \bar{r_i})}{n_{i+1}}$$

$$\widehat{r_{ui}} - \frac{\sum_{u+1}^{users} r_{ui}(final)}{n_{u+1}} = \frac{\sum_{i+1}^{items} r_{ui}(final) - \sum_{i+1}^{items} \bar{r_i}(final)}{n_{i+1}}$$

Where,

$n_{u+1}$  - number of users who have rated item (including the final user)

$n_{i+1}$  - numbers of items rated by the user (including the final item)

$\sum_{i+1}^{items} r_{ui}(final)$  - sum of all the scores rated by the user  $u$  (final)

$\sum_{u+1}^{users} r_{ui}(final)$  - sum of the scores rated by the users for the item  $i$  (final)

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{u+1}^{users} r_{ui}(train) + r_{ui}(new)}{n_{u+1}} = \frac{\sum_{i+1}^{items} r_{ui}(train) + r_{ui}(new) - \sum_{i+1}^{items} \bar{r_i}(final)}{n_{i+1}}$$

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \frac{r_{ui}(new)}{n_{u+1}} = \frac{\sum_{i+1}^{items} r_{ui}(train) + r_{ui}(new) - [\sum_{i+1}^{items} \bar{r_i}(train) + r_{ui}(new)]}{n_{i+1}}$$

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \frac{r_{ui}(new)}{n_{u+1}} = \frac{\sum_{i+1}^{items} r_{ui}(train) + r_{ui}(new) - \sum_{i+1}^{items} \bar{r_i}(train) - \sum_{u+1}^{users} r_{ui}(final)}{n_{i+1}}$$

$$\Rightarrow \widehat{r_{ui}} - \frac{\sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \frac{r_{ui}(new)}{n_{u+1}} = \frac{\sum_{i+1}^{items} r_{ui}(train) + r_{ui}(new) - \sum_{i+1}^{items} \bar{r_i}(train) - \sum_{u+1}^{users} r_{ui}(train) - \frac{r_{ui}(new)}{n_{u+1}}}{n_{i+1}}$$

$$\Rightarrow \widehat{r_{ui}}(n_i + 1) - \frac{(n_i + 1) \sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \frac{(n_i + 1) r_{ui}(new)}{n_{u+1}} = \sum_{i+1}^{items} r_{ui}(train) + r_{ui}(new) - \sum_{i+1}^{items} \bar{r_i}(train) - \frac{\sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \frac{r_{ui}(new)}{n_{u+1}}$$

$$\Rightarrow \widehat{r_{ui}}(n_i + 1) - \frac{(n_i + 1) \sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \frac{(n_i + 1) r_{ui}(new)}{n_{u+1}} = \sum_{i+1}^{items} r_{ui}(train) - \sum_{i+1}^{items} \bar{r_i}(train) - \frac{\sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} + \left[ \frac{(n_u + 1) r_{ui}(new) - r_{ui}(new)}{n_{u+1}} \right]$$

$$\Rightarrow \widehat{r_{ui}}(n_i + 1) - \frac{(n_i + 1) \sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \frac{(n_i + 1) r_{ui}(new)}{n_{u+1}} = \sum_{i+1}^{items} (r_{ui}(train) - \bar{r_i}(train)) - \frac{\sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} + \frac{(n_u) r_{ui}(new)}{n_{u+1}}$$

Reshuffling,

$$\Rightarrow \frac{(n_u) r_{ui}(new)}{n_{u+1}} + \frac{(n_i + 1) r_{ui}(new)}{n_{u+1}} = \frac{\sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}} - \sum_{i+1}^{items} (r_{ui}(train) - \bar{r_i}(train)) + \widehat{r_{ui}}(n_i + 1) - \frac{(n_i + 1) \sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}}$$

$$\Rightarrow \frac{(n_i + n_u + 1) r_{ui}(new)}{n_{u+1}} = \widehat{r_{ui}}(n_i + 1) - \sum_{i+1}^{items} (r_{ui}(train) - \bar{r_i}(train)) - \frac{(n_i) \sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}}$$

$$\Rightarrow r_{ui}(new) = \left( \frac{n_u + 1}{n_i + n_u + 1} \right) [\widehat{r_{ui}}(n_i + 1) - \sum_{i+1}^{items} (r_{ui}(train) - \bar{r_i}(train)) - \frac{(n_i) \sum_{u+1}^{users} r_{ui}(train)}{n_{u+1}}]$$

$$\Rightarrow r_{ui}(new) = \left( \frac{n_u + 1}{n_i + n_u + 1} \right) [\widehat{r_{ui}}(n_i + 1) - (\bar{r_u} * n_i) + (\bar{r_u} * n_i) - (n_i * \tau_u) - \frac{(n_i)(\bar{r_i} * n_u)}{n_{u+1}}]$$

$$\Rightarrow r_{ui}(new) = \left( \frac{n_u + 1}{n_i + n_u + 1} \right) [\widehat{r_{ui}}(n_i + 1) - (n_i * \tau_u) - \frac{(n_i)(\bar{r_i} * n_u)}{n_{u+1}}]$$

$$\Rightarrow r_{ui}(new) = \left( \frac{1}{n_i + n_u + 1} \right) [\widehat{r_{ui}}(n_u + 1)(n_i + 1) - (n_u + 1)(n_i * \tau_u) - (n_i * n_u * \bar{r_i})]$$

### C.3 Case 3: User Tendency is negative and Product Tendency is Positive; User mean < Item mean ( $\tau_u < 0$ , $\tau_i > 0$ , $\bar{r_u} < \bar{r_i}$ , $\widehat{r_{ui}}$ )

Unbias Rating :  $\widehat{r_{ui}} = \min(\max(\bar{r_u}, (\bar{r_i} + \tau_u)\beta + (\bar{r_u} + \tau_i)(1 - \beta)), \bar{r_i})$

Reverse Function

1. If  $\widehat{r_{ui}} = \bar{r_u}(final)$

$$\Rightarrow \widehat{r_{ui}} = \frac{\sum_{i+1}^{items} r_{ui}(final)}{n_{i+1}}$$

$$\Rightarrow \widehat{r_{ui}} = \frac{\sum_{i+1}^{items} r_{ui}(train) + r_{ui}(new)}{n_{i+1}}$$

$$\Rightarrow (n_i + 1) \widehat{r_{ui}} = \sum_{i+1}^{items} r_{ui}(train) + r_{ui}(new)$$

$$\begin{aligned}\Rightarrow r_{ui(new)} &= (n_i + 1)\widehat{r_{ui}} - \sum^{items} r_{ui(train)} \\ \Rightarrow r_{ui(new)} &= (n_i + 1)\widehat{r_{ui}} - (\overline{r_u} * n_i)\end{aligned}$$

2. If  $\widehat{r_{ui}} = (\overline{r_i} + \tau_u)\beta + (\overline{r_u} + \tau_i)(1 - \beta)$

$$\widehat{r_{ui}} = \left( \frac{\sum^{users} r_{ui(final)}}{n_u + 1} + \frac{\sum^{items} r_{ui(final)} - \sum^{items} \overline{r_i(final)}}{n_i + 1} \right) \beta + \left( \frac{\sum^{items} r_{ui(final)}}{n_i + 1} + \frac{\sum^{users} r_{ui(final)} - \sum^{users} \overline{r_u(final)}}{n_u + 1} \right) (1 - \beta)$$

Break it into 2 parts

$$\begin{aligned}\text{I. } & \left( \frac{\sum^{users} r_{ui(final)}}{n_u + 1} + \frac{\sum^{items} r_{ui(final)} - \sum^{items} \overline{r_i(final)}}{n_i + 1} \right) \beta \\ &= \left( \frac{\sum^{users} r_{ui(train)} + r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(final)} - \sum^{items} \overline{r_i(final)}}{n_i + 1} \right) \beta \\ &= \left( \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{\sum^{items} \overline{r_i(train)}}{n_i + 1} - \frac{\overline{r_i(new)}}{n_i + 1} \right) \beta \\ &= \left( \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{\sum^{items} \overline{r_i(train)}}{n_i + 1} - \frac{\sum^{users} r_{ui(train)} + r_{ui(new)}}{n_u + 1} \right) \beta \\ &= \left( \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{\sum^{items} \overline{r_i(train)}}{n_i + 1} - \frac{\sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\ &= \left( \left[ \frac{\sum^{users} r_{ui(train)}}{n_u + 1} - \frac{\sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} \right] + \frac{\sum^{items} (r_{ui(train)} - \overline{r_i(train)})}{n_i + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\ &= \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \overline{r_i(train)})}{n_i + 1} + \frac{((n_i + 1) + (n_u + 1) - 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\ &= \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \overline{r_i(train)})}{n_i + 1} + \frac{(n_i + n_u + 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\ \text{II. } & \left( \frac{\sum^{items} r_{ui(final)}}{n_i + 1} + \frac{\sum^{users} r_{ui(final)} - \sum^{users} \overline{r_u(final)}}{n_u + 1} \right) (1 - \beta) \\ &= \left( \frac{\sum^{items} r_{ui(train)} + r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(final)} - \sum^{users} \overline{r_u(final)}}{n_u + 1} \right) (1 - \beta) \\ &= \left( \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{\sum^{users} \overline{r_u(train)}}{n_u + 1} - \frac{\overline{r_u(new)}}{n_u + 1} \right) (1 - \beta) \\ &= \left( \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{\sum^{users} \overline{r_u(train)}}{n_u + 1} - \frac{\sum^{items} r_{ui(train)} + r_{ui(new)}}{n_i + 1} \right) (1 - \beta) \\ &= \left( \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{\sum^{users} \overline{r_u(train)}}{n_u + 1} - \frac{\sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\ &= \left( \left[ \frac{\sum^{items} r_{ui(train)}}{n_i + 1} - \frac{\sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} \right] + \frac{\sum^{users} (r_{ui(train)} - \overline{r_u(train)})}{n_u + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\ &= \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \overline{r_u(train)})}{n_u + 1} + \frac{((n_i + 1) + (n_u + 1) - 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\ &= \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \overline{r_u(train)})}{n_u + 1} + \frac{(n_i + n_u + 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta)\end{aligned}$$

Now,

$$\begin{aligned}\Rightarrow \widehat{r_{ui}} &= \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \overline{r_i(train)})}{n_i + 1} + \frac{(n_i + n_u + 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta + \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \overline{r_u(train)})}{n_u + 1} + \frac{(n_i + n_u + 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\ \Rightarrow \widehat{r_{ui}} &= \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \overline{r_i(train)})}{n_i + 1} + \frac{(n_i + n_u + 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta + \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \overline{r_u(train)})}{n_u + 1} + \frac{(n_i + n_u + 1) r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta)\end{aligned}$$

Reshuffling

$$\begin{aligned}
\Rightarrow \frac{(n_i+n_u+1)r_{ui(new)}}{(n_u+1)(n_i+1)} &= \widehat{r_{ui}} - \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u+1)(n_i+1)} (1-\beta) - \frac{\sum^{users} (r_{ui(train)} - \overline{r_u(train)})}{n_u+1} (1-\beta) - \\
&\frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u+1)(n_i+1)} \beta - \\
&\frac{\sum^{items} (r_{ui(train)} - \overline{r_i(train)})}{n_i+1} \beta \\
\Rightarrow r_{ui(new)} &= \left( \frac{(n_i+1)(n_u+1)}{n_i+n_u+1} \right) [\widehat{r_{ui}} - \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u+1)(n_i+1)} (1-\beta) - \frac{\sum^{users} (r_{ui(train)} - \overline{r_u(train)})}{n_u+1} (1-\beta) - \\
&\frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u+1)(n_i+1)} \beta - \\
&\frac{\sum^{items} (r_{ui(train)} - \overline{r_i(train)})}{n_i+1} \beta] \\
\Rightarrow r_{ui(new)} &= \left( \frac{1}{n_i+n_u+1} \right) [(n_u+1)(n_i+1)\widehat{r_{ui}} - ((n_u)(1-\beta) \sum^{items} r_{ui(train)}) - ((n_i+1)(1-\beta) \sum^{users} (r_{ui(train)} - \overline{r_u(train)})) - \beta(n_i) \sum^{users} r_{ui(train)} - (\beta(n_u+1) \sum^{items} (r_{ui(train)} - \overline{r_i(train)}))] \\
\Rightarrow r_{ui(new)} &= \left( \frac{1}{n_i+n_u+1} \right) [(n_u+1)(n_i+1)\widehat{r_{ui}} - ((n_u)(\overline{r_u} * n_i)(1-\beta)) - ((n_i+1)(1-\beta)(n_u * \tau_i)) - \beta(n_i)(\overline{r_i} * n_u) - (\beta(n_u+1)(n_i * \tau_u))]
\end{aligned}$$

3. If  $\widehat{r_{ui}} = \overline{r_i(final)}$

$$\begin{aligned}
\Rightarrow \widehat{r_{ui}} &= \frac{\sum^{users} r_{ui(final)}}{n_u+1} \\
\Rightarrow \widehat{r_{ui}} &= \frac{\sum^{users} r_{ui(train)} + r_{ui(new)}}{n_u+1} \\
\Rightarrow (n_u+1)\widehat{r_{ui}} &= \sum^{users} r_{ui(train)} + r_{ui(new)} \\
\Rightarrow r_{ui(new)} &= (n_u+1)\widehat{r_{ui}} - \sum^{users} r_{ui(train)} \\
\Rightarrow r_{ui(new)} &= (n_u+1)\widehat{r_{ui}} - (\overline{r_i} * n_u)
\end{aligned}$$

#### C.4 Case 4: User Tendency is negative and Product Tendency is Positive; User mean > Item mean ( $\tau_u < 0, \tau_i > 0, \overline{r_u} > \overline{r_i} \widehat{r_{ui}}$ )

Unbias Rating :  $\widehat{r_{ui}} = \overline{r_i}\beta + \overline{r_u}(1-\beta)$

Reverse functions

$$\begin{aligned}
\widehat{r_{ui}} &= \overline{r_i}\beta + \overline{r_u}(1-\beta) \\
\Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{users} r_{ui(final)}}{n_u+1} + \frac{(1-\beta) \sum^{items} r_{ui(final)}}{n_i+1} \\
\Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{users} r_{ui(train)}}{n_u+1} + \frac{(\beta)r_{ui(new)}}{n_u+1} + \frac{(1-\beta) \sum^{items} r_{ui(train)}}{n_i+1} + \frac{(1-\beta)r_{ui(new)}}{n_i+1} \\
\Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{users} r_{ui(train)}}{n_u+1} + \frac{(1-\beta) \sum^{items} r_{ui(train)}}{n_i+1} + r_{ui(new)} \left[ \frac{\beta}{n_u+1} + \frac{1-\beta}{n_i+1} \right] \\
\Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{users} r_{ui(train)}}{n_u+1} + \frac{(1-\beta) \sum^{items} r_{ui(train)}}{n_i+1} + r_{ui(new)} \left[ \frac{\beta n_i + \beta + n_u + 1 - \beta n_u - \beta}{(n_u+1)(n_i+1)} \right] \\
\Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{users} r_{ui(train)}}{n_u+1} + \frac{(1-\beta) \sum^{items} r_{ui(train)}}{n_i+1} + r_{ui(new)} \left[ \frac{n_u(1-\beta) + (1+\beta n_i)}{(n_u+1)(n_i+1)} \right]
\end{aligned}$$

Reshuffling.

$$\begin{aligned}
\Rightarrow r_{ui(new)} \left[ \frac{n_u(1-\beta) + 1 + \beta n_i}{(n_u+1)(n_i+1)} \right] &= \widehat{r_{ui}} - \frac{\beta \sum^{users} r_{ui(train)}}{n_u+1} - \frac{(1-\beta) \sum^{items} r_{ui(train)}}{n_i+1} \\
\Rightarrow r_{ui(new)} &= \left[ \frac{(n_u+1)(n_i+1)}{n_u(1-\beta) + (1+\beta n_i)} \right] \left( \widehat{r_{ui}} - \frac{\beta \sum^{users} r_{ui(train)}}{n_u+1} - \frac{(1-\beta) \sum^{items} r_{ui(train)}}{n_i+1} \right) \\
\Rightarrow r_{ui(new)} &= \left[ \frac{1}{n_u(1-\beta) + (1+\beta n_i)} \right] ((n_u+1)(n_i+1)\widehat{r_{ui}} - \beta(n_i+1) \sum^{users} r_{ui(train)} - (1-\beta)(n_u+1) \sum^{items} r_{ui(train)}) \\
\Rightarrow r_{ui(new)} &= \left[ \frac{1}{n_u(1-\beta) + (1+\beta n_i)} \right] ((n_u+1)(n_i+1)\widehat{r_{ui}} - \beta(n_i+1)(\overline{r_i} * n_u) - (1-\beta)(n_u+1)(\overline{r_u} * n_i))
\end{aligned}$$

#### C.5 Case 5: User Tendency is Positive and Product Tendency is Negative; User mean > Item mean ( $\tau_u > 0, \tau_i < 0, \overline{r_u} > \overline{r_i} \widehat{r_{ui}}$ )

Unbias Rating :  $\widehat{r_{ui}} = \min(\max(\overline{r_i}, (\overline{r_u} + \tau_i)\beta + (\overline{r_i} + \tau_u)(1-\beta)), \overline{r_u})$

Reverse Functions

1. If  $\widehat{r_{ui}} = \overline{r_i(final)}$

$$\Rightarrow \widehat{r_{ui}} = \frac{\sum^{users} r_{ui(final)}}{n_u+1}$$

$$\begin{aligned}
&\Rightarrow \widehat{r_{ui}} = \frac{\sum^{users} r_{ui(train)} + r_{ui(new)}}{n_u + 1} \\
&\Rightarrow (n_u + 1)\widehat{r_{ui}} = \sum^{users} r_{ui(train)} + r_{ui(new)} \\
&\Rightarrow r_{ui(new)} = (n_u + 1)\widehat{r_{ui}} - \sum^{users} r_{ui(train)} \\
&\Rightarrow r_{ui(new)} = (n_i + 1)\widehat{r_{ui}} - (\bar{r}_i * n_u)
\end{aligned}$$

2. If  $\widehat{r_{ui}} = (\bar{r}_u + \tau_i)\beta + (\bar{r}_i + \tau_u)(1 - \beta)$

$$\begin{aligned}
\widehat{r_{ui}} &= \left( \frac{\sum^{items} r_{ui(final)}}{n_i + 1} + \frac{\sum^{users} r_{ui(final)} - \sum^{users} \bar{r}_u(final)}{n_u + 1} \right) \beta + \left( \frac{\sum^{users} r_{ui(final)}}{n_u + 1} + \frac{\sum^{items} r_{ui(final)} - \sum^{items} \bar{r}_i(final)}{n_i + 1} \right) (1 - \beta)
\end{aligned}$$

Break it into 2 parts

$$\begin{aligned}
\text{I. } &\left( \frac{\sum^{items} r_{ui(final)}}{n_i + 1} + \frac{\sum^{users} r_{ui(final)} - \sum^{users} \bar{r}_u(final)}{n_u + 1} \right) \beta \\
&= \left( \frac{\sum^{items} r_{ui(train)} + r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(final)} - \sum^{users} \bar{r}_u(final)}{n_u + 1} \right) \beta \\
&= \left( \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{\sum^{users} \bar{r}_u(train)}{n_u + 1} - \frac{\bar{r}_u(new)}{n_u + 1} \right) \beta \\
&= \left( \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{\sum^{items} \bar{r}_u(train)}{n_u + 1} - \frac{\sum^{items} r_{ui(train)} + r_{ui(new)}}{n_i + 1} \right) \beta \\
&= \left( \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{\sum^{users} \bar{r}_u(train)}{n_u + 1} - \frac{\sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\
&= \left( \left[ \frac{\sum^{items} r_{ui(train)}}{n_i + 1} - \frac{\sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} \right] + \frac{\sum^{users} (r_{ui(train)} - \bar{r}_u(train))}{n_u + 1} + \frac{r_{ui(new)}}{n_i + 1} + \frac{r_{ui(new)}}{n_u + 1} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\
&= \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \bar{r}_u(train))}{n_u + 1} + \frac{((n_u + 1) + (n_i + 1) - 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\
&= \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \bar{r}_u(train))}{n_u + 1} + \frac{(n_i + n_u + 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta \\
\text{II. } &\left( \frac{\sum^{users} r_{ui(final)}}{n_u + 1} + \frac{\sum^{items} r_{ui(final)} - \sum^{items} \bar{r}_i(final)}{n_i + 1} \right) \beta \\
&= \left( \frac{\sum^{users} r_{ui(train)} + r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(final)} - \sum^{items} \bar{r}_i(final)}{n_i + 1} \right) (1 - \beta) \\
&= \left( \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{\sum^{items} \bar{r}_i(train)}{n_i + 1} - \frac{\bar{r}_i(new)}{n_i + 1} \right) (1 - \beta) \\
&= \left( \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{\sum^{items} \bar{r}_i(train)}{n_i + 1} - \frac{\sum^{users} r_{ui(train)} + r_{ui(new)}}{n_u + 1} \right) (1 - \beta) \\
&= \left( \frac{\sum^{users} r_{ui(train)}}{n_u + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{\sum^{items} r_{ui(train)}}{n_i + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{\sum^{items} \bar{r}_i(train)}{n_i + 1} - \frac{\sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\
&= \left( \left[ \frac{\sum^{users} r_{ui(train)}}{n_u + 1} - \frac{\sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} \right] + \frac{\sum^{items} (r_{ui(train)} - \bar{r}_i(train))}{n_i + 1} + \frac{r_{ui(new)}}{n_u + 1} + \frac{r_{ui(new)}}{n_i + 1} - \frac{r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\
&= \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \bar{r}_i(train))}{n_i + 1} + \frac{((n_i + 1) + (n_u + 1) - 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\
&= \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \bar{r}_i(train))}{n_i + 1} + \frac{(n_i + n_u + 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta)
\end{aligned}$$

Now,

$$\begin{aligned}
&\Rightarrow \widehat{r_{ui}} = \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \bar{r}_u(train))}{n_u + 1} + \frac{(n_i + n_u + 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta + \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \bar{r}_i(train))}{n_i + 1} + \frac{(n_i + n_u + 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta) \\
&\Rightarrow \widehat{r_{ui}} = \left( \frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{users} (r_{ui(train)} - \bar{r}_u(train))}{n_u + 1} + \frac{(n_i + n_u + 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) \beta + \left( \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u + 1)(n_i + 1)} + \frac{\sum^{items} (r_{ui(train)} - \bar{r}_i(train))}{n_i + 1} + \frac{(n_i + n_u + 1)r_{ui(new)}}{(n_u + 1)(n_i + 1)} \right) (1 - \beta)
\end{aligned}$$



$$\frac{(n_i+n_u+1)r_{ui(new)}}{(n_u+1)(n_i+1)})\beta$$

Reshuffling

$$\begin{aligned} \Rightarrow \frac{(n_i+n_u+1)r_{ui(new)}}{(n_u+1)(n_i+1)} &= \widehat{r_{ui}} - \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u+1)(n_i+1)}(1-\beta) - \frac{\sum^{items} (r_{ui(train)} - \overline{r_{i(train)}})}{n_i+1}(1-\beta) - \\ &\frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u+1)(n_i+1)}\beta - \\ &\frac{\sum^{users} (r_{ui(train)} - \overline{r_{u(train)}})}{n_u+1}\beta \\ \Rightarrow r_{ui(new)} &= \left( \frac{(n_i+1)(n_u+1)}{n_i+n_u+1} \right) [\widehat{r_{ui}} - \frac{(n_i) \sum^{users} r_{ui(train)}}{(n_u+1)(n_i+1)}(1-\beta) - \frac{\sum^{items} (r_{ui(train)} - \overline{r_{i(train)}})}{n_i+1}(1-\beta) - \\ &\frac{(n_u) \sum^{items} r_{ui(train)}}{(n_u+1)(n_i+1)}\beta - \\ &\frac{\sum^{users} (r_{ui(train)} - \overline{r_{u(train)}})}{n_u+1}\beta] \\ \Rightarrow r_{ui(new)} &= \left( \frac{1}{n_i+n_u+1} \right) [(n_u+1)(n_i+1)\widehat{r_{ui}} - ((n_i)(1-\beta) \sum^{users} r_{ui(train)}) - ((n_u+1)(1-\beta) \sum^{items} (r_{ui(train)} - \overline{r_{i(train)}})) - \beta(n_u) \sum^{items} r_{ui(train)} - (\beta(n_i+1) \sum^{users} (r_{ui(train)} - \overline{r_{u(train)}}))] \\ \Rightarrow r_{ui(new)} &= \left( \frac{1}{n_i+n_u+1} \right) [(n_u+1)(n_i+1)\widehat{r_{ui}} - ((n_i)(\overline{r_i} * n_u)(1-\beta)) - ((n_u+1)(1-\beta)(n_i * \tau_u)) - \beta(n_u)(\overline{r_u} * n_i) - (\beta(n_i+1)(n_u * \tau_i))] \end{aligned}$$

3. If  $\widehat{r_{ui}} = \overline{r_{u(final)}}$

$$\begin{aligned} \Rightarrow \widehat{r_{ui}} &= \frac{\sum^{items} r_{ui(final)}}{n_i+1} \\ \Rightarrow \widehat{r_{ui}} &= \frac{\sum^{items} r_{ui(train)} + r_{ui(new)}}{n_i+1} \\ \Rightarrow (n_i+1)\widehat{r_{ui}} &= \sum^{items} r_{ui(train)} + r_{ui(new)} \\ \Rightarrow r_{ui(new)} &= (n_i+1)\widehat{r_{ui}} - \sum^{items} r_{ui(train)} \\ \Rightarrow r_{ui(new)} &= (n_i+1)\widehat{r_{ui}} - (\overline{r_u} * n_i) \end{aligned}$$

### C.6 Case 6: User Tendency is Positive and Product Tendency is Negative; User mean < Item mean ( $\tau_u > 0, \tau_i < 0, \overline{r_u} < \overline{r_i} \widehat{r_{ui}}$ )

Unbias Rating :  $\widehat{r_{ui}} = \overline{r_u}\beta + \overline{r_i}(1-\beta)$

Reverse functions

$$\begin{aligned} \widehat{r_{ui}} &= \overline{r_u}\beta + \overline{r_i}(1-\beta) \\ \Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{items} r_{ui(final)}}{n_i+1} + \frac{(1-\beta) \sum^{users} r_{ui(final)}}{n_u+1} \\ \Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{items} r_{ui(train)}}{n_i+1} + \frac{(\beta)r_{ui(new)}}{n_i+1} + \frac{(1-\beta) \sum^{users} r_{ui(train)}}{n_u+1} + \frac{(1-\beta)r_{ui(new)}}{n_u+1} \\ \Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{items} r_{ui(train)}}{n_i+1} + \frac{(1-\beta) \sum^{users} r_{ui(train)}}{n_u+1} + r_{ui(new)} \left[ \frac{\beta}{n_i+1} + \frac{1-\beta}{n_u+1} \right] \\ \Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{items} r_{ui(train)}}{n_i+1} + \frac{(1-\beta) \sum^{users} r_{ui(train)}}{n_u+1} + r_{ui(new)} \left[ \frac{\beta n_u + \beta + n_i + 1 - \beta n_i - \beta}{(n_u+1)(n_i+1)} \right] \\ \Rightarrow \widehat{r_{ui}} &= \frac{\beta \sum^{items} r_{ui(train)}}{n_i+1} + \frac{(1-\beta) \sum^{users} r_{ui(train)}}{n_u+1} + r_{ui(new)} \left[ \frac{n_i(1-\beta) + (1+\beta n_u)}{(n_u+1)(n_i+1)} \right] \end{aligned}$$

Reshuffling,

$$\begin{aligned} \Rightarrow r_{ui(new)} \left[ \frac{n_i(1-\beta) + 1 + \beta n_u}{(n_u+1)(n_i+1)} \right] &= \widehat{r_{ui}} - \frac{\beta \sum^{items} r_{ui(train)}}{n_i+1} - \frac{(1-\beta) \sum^{users} r_{ui(train)}}{n_u+1} \\ \Rightarrow r_{ui(new)} &= \left[ \frac{(n_u+1)(n_i+1)}{n_i(1-\beta) + (1+\beta n_u)} \right] \left( \widehat{r_{ui}} - \frac{\beta \sum^{items} r_{ui(train)}}{n_i+1} - \frac{(1-\beta) \sum^{users} r_{ui(train)}}{n_u+1} \right) \\ \Rightarrow r_{ui(new)} &= \left[ \frac{1}{n_i(1-\beta) + (1+\beta n_u)} \right] ((n_u+1)(n_i+1)\widehat{r_{ui}} - \beta(n_u+1) \sum^{items} r_{ui(train)} - (1-\beta)(n_i+1) \sum^{users} r_{ui(train)}) \\ \Rightarrow r_{ui(new)} &= \left[ \frac{1}{n_i(1-\beta) + (1+\beta n_u)} \right] ((n_u+1)(n_i+1)\widehat{r_{ui}} - \beta(n_u+1)(\overline{r_u} * n_i) - (1-\beta)(n_i+1)(\overline{r_i} * n_u)) \end{aligned}$$