CS203B Assignment 1

Answer Key

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1. In the series mode, the right bit is received if and only if only an even number of channels makes errors during the transmission. Therefore, the probability of transmitting the correct bit is equal to,

$$P_{series}(n,p) = \sum_{\substack{i=0 \\ i \text{ is even}}}^{2\lfloor n/2 \rfloor} \binom{n}{i} p^i (1-p)^{n-i}.$$

In the parallel mode, the right bit is received if and only if at most $\lfloor n/2 \rfloor$ channels makes errors during the transmission. Therefore, the proabaility of transmitting the right bit is,

$$P_{parallel}(n,p) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i} p^i (1-p)^{n-i}.$$

(a) Let $p = \frac{1}{3}$ and $n = 5$, on calculating the probabilities, $P_{series}(5, 1/3) \approx 122/243$ and $P_{parallel}(5, 1/3) \approx 192/243$. Therefore, parallel mode is better.

(b) Let E be the event that the 1 was sent by A. Let F be the event that majority of bits received at B turns out to be 1. Using Bayes Theorem,

$$\begin{aligned} P(E \mid F) &= \frac{P(E)P(F \mid E)}{P(F)} \\ &= \frac{P(E)P(F \mid E)}{P(E)P(F \mid E) + P(E^c)P(F \mid E^c)} \end{aligned}$$

We know that $P(E) = \frac{1}{3}$ and $P(E^c) = \frac{2}{3}$. If 1 was the original bit then the received majority is 1 if and only if at most 1 channel makes an error. Therefore, $P(F \mid E) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} + \binom{3}{1} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{20}{27}$. If 0 was the original bit, then the received majority is 1 if and only if at least two channels makes an error. Thus, $P(F \mid E^c) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} + \binom{3}{2} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{7}{27}$. Substituting these values in the Bayes formula, we get, $P(E \mid F) = \frac{20}{34} = \frac{10}{17}$.

2. (a) Idea: $\int_{-\infty}^{\infty} g(u)\,du = 1 \Rightarrow \int_{-1/2}^{0} (c+u)\,du + \int_{0}^{-1/2} (c-u)\,du = 1 \Rightarrow c = \dfrac{5}{4}$

(b)
   i. $u \leq -\dfrac{1}{2} \Rightarrow G(u) = 0$
   ii. $-1/2 < u \leq 0 \Rightarrow G(u) = u^2/2 + 5u/4 + 1/2$
   iii. $G(0) = 1/2$
   iv. $0 < u \leq 1/2 \Rightarrow$
   $\int_{-\infty}^{0} g(y)\,dy + \int_{0}^{x} g(y)\,dy = 12 + 5/4x - x^2/2$
   v. $G(1/2) = 1$
   vi. $u > 1/2 \Rightarrow G(u) = 1$

$$G(u) = \begin{cases} 0 \quad \text{where } u \leq -1/2, \\ u^2/2 + 5u/4 + 1/2 \quad \text{where } -1/2 < u \leq 0 \\ -u^2/2 + 5u/4 + 1/2 \quad \text{where } 0 < u \leq 1/2 \\ 1 \text{ where } u \geq 1/2 \end{cases}$$

(c) $\Pr(\frac{1}{8} < U < \frac{2}{5} \mid \frac{1}{10} < U < \frac{1}{5}) = \Pr(A|B)$

Here $A = (\frac{1}{8} < U < \frac{2}{5})$ and $B = (\frac{1}{10} < U < \frac{1}{5})$. Here $A \cap B = (\frac{1}{8} < U < \frac{1}{5})$

$\dfrac{\Pr((\frac{1}{8} < U < \frac{2}{5}) \cap (\frac{1}{10} < U < \frac{1}{5}))}{\Pr(\frac{1}{10} < U < \frac{1}{5})}$

$= \dfrac{\Pr(\frac{1}{8} < U < \frac{1}{5})}{\Pr(\frac{1}{10} < U < \frac{1}{5})} = \dfrac{261}{352}$

Computing each part $\Pr(\frac{1}{8} < U < \frac{1}{5}) = \frac{261}{3200}$ and $\Pr(\frac{1}{10} < U < \frac{1}{5}) = \frac{352}{3200}$.

3. (a) Let X denote the number of tosses until 6 consecutive heads are obtained. Assume that the die face (denoted by random variable D) turned up to be 1 and therefor the probability of head turning up during the coin tosses is $p_1 = \frac{1}{6}$. If the first toss turns out to be T, then $X = 1 + X_T$ where X_T denotes the number of tosses until 6 consecutive heads are obtained after the first toss turned out to be tail. Let E_T denote the event that the first toss is T. Since, the number of consecutive heads obtained *resets* to 0 on obtaining a tail and since the tosses are independent, it follows that $\mathbf{E}[X_T \mid D = 1 \wedge E_T] = \mathbf{E}[X \mid D = 1 \wedge E_T] = \mathbf{E}[X \mid D = 1]$. If the first toss is a head and the second one is a tail, $X = 1 + X_{HT}$ where X_{HT} denotes the number of tosses until 6 consecutive heads are obtained after the first two tosses yield HT. As in the previous case, we have $\mathbf{E}[X_{TH} \mid D = 1 \wedge E_{HT}] = \mathbf{E}[X \mid D = 1 \wedge E_{HT}] = \mathbf{E}[X \mid D = 1]$. Continuing this analysis and using conditional expectation formula along with the linearity of expectation,

$$\begin{aligned}
\mathbf{E}[X \mid D = 1] = (1 - p_1)&(1 + \mathbf{E}[X_T \mid D = 1 \wedge E_T]) \\
&+ p_1(1 - p_1)(2 + \mathbf{E}[X_{HT} \mid D = 1 \wedge E_{HT}]) \\
&+ p_1^2(1 - p_1)(3 + \mathbf{E}[X_{HHT} \mid D = 1 \wedge E_{HHT}]) \\
&+ p_1^3(1 - p_1)(4 + \mathbf{E}[X_{HHHT} \mid D = 1 \wedge E_{HHT}]) \\
&+ p_1^4(1 - p_1)(5 + \mathbf{E}[X_{HHHHT} \mid D = 1 \wedge E_{HHHT}]) \\
&+ p_1^5(1 - p_1)(6 + \mathbf{E}[X_{HHHHHT} \mid D = 1 \wedge E_{HHHHT}]) \\
&+ p_1^6 6
\end{aligned}$$

where the random variables $X_{HHT}, X_{HHHT}, X_{HHHHT}$ and X_{HHHHHT} (and the events E_{HHHT} etc.) are defined similarly. All the new random variables have the same expectation as that of X (conditioned on the first few tosses) since the possibility of obtaining consecutive heads is *reset* after obtaining a tail at any point. Therefore, representing $\mathbf{E}[X \mid D = 1]$ by e,

$$\begin{aligned}
e = &(1 - p_1)(1 + e) + p_1(1 - p_1)(2 + e) + p_1^2(1 - p_1)(3 + e) \\
&+ p_1^3(1 - p_1)(4 + e) + p_1^4(1 - p_1)(5 + e) \\
&+ p_1^5(1 - p_1)(6 + e) + p_1^6 6
\end{aligned}$$

Substituting the values of p_1 and solving this linear equation for e, we get,

$$\mathbf{E}[X \mid D = 1] = 55986.$$

(b) Repeating the analysis above for other i's and solving the linear equation above for every i, we get,

$$\begin{aligned}
\mathbf{E}[X \mid D=1] &= 55986\\
\mathbf{E}[X \mid D=2] &= 1092\\
\mathbf{E}[X \mid D=3] &= 126\\
\mathbf{E}[X \mid D=4] &\approx 31.17\\
\mathbf{E}[X \mid D=5] &\approx 11.91.
\end{aligned}$$

Now, using conditional expectation formula,

$$\begin{aligned}
\mathbf{E}[X] &= \frac{1}{5}\left(\mathbf{E}[X \mid D=1] + \mathbf{E}[X \mid D=2] + \mathbf{E}[X \mid D=3] + \mathbf{E}[X \mid D=4] + \mathbf{E}[X \mid D=5]\right)\\
&\approx 11449.
\end{aligned}$$

(c) The required probability is,

$$\frac{1}{5}\left(\sum_{i=1}^{6}\left(\frac{i}{6}\right)^{n}\right).$$

(d) Let E_n denote the event that n consecutive heads are obtained in the first n tosses. Let random variable D denote the die outcome. Then for any i from 1 to 5, using Bayes Theorem,

$$P(D=i \mid E_n) = \frac{P(D=i)P(E_n \mid D=i)}{\sum_{k=1}^{5} P(D=k)P(E_n \mid D=k)}.$$

Hence, we get,

$$P(D=i \mid E_n) = \frac{i^n}{1^n + 2^n + 3^n + 4^n + 5^n}.$$

Dividing by the numerator in each case, it easily follows that $P(D=i \mid E_n) \to 0$ as $n \to \infty$ for every i except $i=5$. For $i=5$, $P(D=5 \mid E_n) \to 1$ as $n \to \infty$. This supports the intuitive fact that if n is large and if n consecutive heads are obtained, then it is quite probable that the die had turned up 5 during the roll and the $\frac{5}{6}$ probability coin is being used for the tosses, compared to the possibility that the other coins which are more biased towards turning up tails are being used for the tosses.

4. (a) The codes should be verified to see the random sampling, averaging and plotting operations are done properly.

   (b) i. The plots should approach the curve of a Normal distribution (Gaussian curve) as $n \to \infty$ in both the cases.

   ii. The peak of the Gaussian curve is centered around the mean of the random variables in both the cases. The spread of the Gaussian curve around the mean is proportional to $\frac{\sigma}{\sqrt{n}}$ where σ is the variance of the random variable considered.