PShaji_Assignment3

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Problem set 1

Question 1)

What is the rank of the matrix A?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

Answer 1)

Let us create an upper matrix of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 6 & 17 & 23 \end{bmatrix}, R2^* < -R1 + R2$$
$$R4^* < -(5)R1 - R4$$
$$R3^* < -R2^* - 2R3$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & 8 & 5 \\ 0 & 6 & 17 & 23 \end{bmatrix}, R4^{**} < -3R2 - R4^*$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & 8 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}, R4^{***} < -5R3^* + 8R4^{**}$$

We can calculate the upper matrix by counting the non-zero rows, In upper matrix matrix A, the number of non-zero rows are 4, therefore `rank of matrix A is 4`.

Question 2)

Given an mxn matrix where m > n, what can be the maximum rank? The minimum rank, assuming that the matrix is non-zero?

Answer 2)

As per definition, number of non-zero rows of the matrix is the rank of the matrix and also maximum rank is min(m,n). Therefore, `maximum rank is n, minimum rank is 1`, since it is a non-zero matrix and there will be at least one row which will have non-zero value.

Question 3) What is the rank of matrix B?

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$

Answer 3)

Creating an upper traingular matrix.

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R2^* < -R2 - 3R1$$
$$R3^* < -R3 - 2R1$$

As we can see, there is only 1 row with non-zero values, therefore, `rank of matrix B is 1`.

Problem set 2

Question 1)

Compute the eigenvalues and eigenvectors of the matrix A. You'll need to show your work. You'll need to write out the characteristic polynomial and show your solution.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Answer 1)

As per definition, $(\lambda I - A)V = 0$, where V is a non-zero vector if matrix A has a non-trivial null space, and only non-trival matrices have non-trivial null space or only matrices having determinant `0` has non-trivial null space.

Let's find the characteristic polynomial of matrix A:

$$A = det(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix})$$

$$A = det(\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

Now by applying the `sarrus rule`, add the product of diagonal elements from LHS, then subtract the product of diagonal elements from RHS.

$(1-\lambda)((4-\lambda)(6-\lambda) + (-2)(-5)(0) + (-3)(0)(0) - (-2)(0)(\lambda-6) - (\lambda-1)(-5)(0) - (-3)(\lambda-4)(0)) = 0$
$(1-\lambda)(\lambda^2 - 10\lambda + 24) = 0$ $\lambda^3 - 11\lambda^2 + 34\lambda - 24 = 0$

Therfore , characteristic polynomial is:

$\lambda^3 - 11\lambda^2 + 34\lambda - 24 = 0$

Now, let us solve the characteristic polynimial to get the eigen values:

$\lambda^3 - 11\lambda^2 + 34\lambda - 24 = 0$

$= (1-\lambda)(\lambda^2 - 10\lambda + 24) = 0$

$= (1-\lambda)(\lambda-4)(\lambda-6) = 0$

Therefore, eigen values are:

$ λ = 1,4,6$

Now, let's solve for eigen vectors of A:

$(\lambda I - A)V = 0$

$\lambda I =$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$\lambda I - A =$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

Now , λ = 1

$$B = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} V = 0$$

Eigen vector for eigen value 1, is null space for the above given matrix. Values for V, which will satisfy the matrix make up eigen vectors of eigen space whose $\lambda = 1$

$$= \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} V1 \\ V2 \\ V3 \end{bmatrix} = 0$$

Solving the matrix:

$5V3 = 0$

$V3 = 0$

$V2 = (-5/3)V3$

$V2 = 0$

Therefore, for eigenvalue λ=1, eigenvector is

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now , λ = 4

$$B = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} V = 0$$

Eigen vector for eigen value 4, is null space for the above given matrix. Values for V, which will satisfy the matrix make up eigen vectors of eigen space whose $\lambda = 4$

$$= \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} V1 \\ V2 \\ V3 \end{bmatrix} = 0$$

Solving the matrix:

$5V3 = 0$

$V3 = 0$

$-3V1 + 2V2 = 0$

$V2 = (3/2)V1$

$V1 = 1$ $V2 = 3/2$ $V3 = 0$

Therefore, for eigenvalue λ=4, eigenvector is

$$= \begin{bmatrix} 1 \\ 3/2 \\ 0 \end{bmatrix}$$

Now , λ = 6

$$B = \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} V = 0$$

Eigen vector for eigen value 6, is null space for the above given matrix. Values for V, which will satisfy the matrix make up eigen vectors of eigen space whose $\lambda = 6$

$$= \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V1 \\ V2 \\ V3 \end{bmatrix} = 0$$

Solving the matrix:

$-5V1 + 2V2 + 3V3 = 0$

$-2V2 + 5V3 = 0$

$V2 = (5/2)V3$

$V1 = (8/5)V3$

$V1 = 8/5$ $V2 = 5/2$ $V3 = 1$

Therefore, for eigenvalue λ=6, eigenvector is

$$= \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}$$