

Notes on construction of $GH(12, EA(4))$ from 2 Hadamard matrices

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Abstract

Making 2 Hadamard matrices of order 12 such that their Hadamard product is also Hadamard. A $GH(12, EA(4))$ is constructed from the 2 Hadamard matrices.

1 Construction

Let A, B, C, D be the following matrices :

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \\ 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \end{pmatrix}$$

2 circulant matrices of order 3,

$$C1 = \begin{pmatrix} - & 1 & - \\ - & - & 1 \\ 1 & - & - \end{pmatrix},$$

$$C2 = \begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \\ 1 & 1 & - \end{pmatrix}$$

To make the 2 Hadamard matrices from A, B, C, D , the following method was used:

$$H_1 = (J \otimes A) + (C1 \otimes B) + (C1 \otimes C) + (C1 \otimes D)$$

$$H_2 = (J \otimes A) + (C2 \otimes B) + (C2 \otimes C) + (C2 \otimes D)$$

H_1 and H_2 are always Hadamard matrices, because of the structure of A, B, C, D and the construction above.

If the Hadamard product of H_1 and H_2 is Hadamard, a Generalized Hadamard matrix can be constructed from H_1 and H_2 . Otherwise, row and column permutations can be applied on H_1 and H_2 to see if their Hadamard product is Hadamard.

$$H_1 = \begin{pmatrix} (A - B - C - D) & (A + B + C + D) & (A - B - C - D) \\ (A - B - C - D) & (A - B - C - D) & (A + B + C + D) \\ (A + B + C + D) & (A - B - C - D) & (A - B - C - D) \end{pmatrix}$$

$$H_2 = \begin{pmatrix} (A + B + C + D) & (A + B + C + D) & (A - B - C - D) \\ (A - B - C - D) & (A + B + C + D) & (A + B + C + D) \\ (A + B + C + D) & (A - B - C - D) & (A + B + C + D) \end{pmatrix},$$

Matrices H_1 and H_2 can be represented as a 3×3 matrix consisting of block matrices of the order 4×4 .

If $A' = A + B + C + D$, and $B' = A - B - C - D$

$$H_1 = \begin{pmatrix} B' & A' & B' \\ B' & B' & A' \\ A' & B' & B' \end{pmatrix}$$

$$H_2 = \begin{pmatrix} A' & A' & B' \\ B' & A' & A' \\ A' & B' & A' \end{pmatrix}$$

Each of A' and B' is a Hadamard matrix, so, in H_1 and H_2 , permuting rows/columns inside row/column blocks should still yield a Hadamard matrix. Only permutations of this type are considered. Thus the Hadamard matrix after permutation would look like this:

(where α, β, γ represent a permutation matrix of order 4×4)

$$H'_2 = \begin{pmatrix} \alpha A' & \alpha A' & \alpha B' \\ \beta B' & \beta A' & \beta A' \\ \gamma A' & \gamma B' & \gamma A' \end{pmatrix}$$

Similarly for column permutations,

$$H'_2 = \begin{pmatrix} A' \alpha & A' \beta & B' \gamma \\ B' \alpha & A' \beta & A' \gamma \\ A' \alpha & B' \beta & A' \gamma \end{pmatrix}$$

The row(column) permutations can be represented by a block diagonal matrix of the following type.

$$P = \begin{pmatrix} \alpha & & 0 \\ & \beta & \\ 0 & & \gamma \end{pmatrix}$$

Here, α, β, γ represent any permutation matrix of the order 4×4 .

We can assume without loss of generality that any permutation of the rows and columns of H_1 and H_2 can be split as only row permutations on H_1 , and only column permutations on H_2 . By this we mean that instead of performing both row and column permutations on both of the matrices, it is possible to split the permutations as row permutations and column permutations separately, with the row permutations (say) performed on H_1 and column permutations on H_2 . This is done keeping in mind that the Hadamard product of the two matrices does not change if the permutations are split this way.

Thus, two permutation matrices are obtained, a row permutation (say) for H_1 called P_1 and column permutations (say) for H_2 called P_2 .

We take the following Hadamard product and check if the resulting matrix is Hadamard or not.

$$H = (P_1 \times H_1) \odot (H_2 \times P_2)$$

2 Some observations

α, β, γ can take on 24 values each, as there are 24 possible permutations for a 4×4 matrix.

So, there are in total, 24x24x24 permutation matrices of order 12 of the permissible type discussed above.

P_1, P_2, H, GH have the same meaning as in the previous section.

Listed in the appendix are some(not all) permutation matrices P_1 and P_2 for which H is a Hadamard matrix, and GH is a Generalized Hadamard matrix.

But for none of them are α, β, γ all from the set $\{I_2 \otimes I_2, I_2 \otimes R_2, R_2 \otimes I_2, R_2 \otimes R_2\}$

3 Suggestions

4 Appendix

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix},$$

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$$P1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$P1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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