My first answer on math.stackexchange

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Abstract

How do you construct Hadamard matrices of order 12?

1 First method

- A recursive method to construct Hadamard matrices of order 4.3^n, $n \geq 1$, with the matrices for even n having the additional property of being regular, was provided in the paper A recursive construction for new symmetric designs by Ionin and Kharaghani.

(A regular Hadamard matrix is one where the row sums and column sums are constant)

Theorem 2.1 in the paper states,

¿Let matrices A_n and B_n be defined recursively for $n \geq 1$ by $A_n = B_{n-1} \otimes I$ and $B_n = A_{n-1} \otimes J + B_{n-1} \otimes Q$. Then for each $n \geq 0$, $H_n = A_n + B_n$ and $H_n^{'} = A_n - B_n$ are Hadamard matrices and H_{2n} is a regular Hadamard matrix. Furthermore, each row of every matrix H_n can be represented as a 1×4 block-matrix $[H_{n1}H_{n2}H_{n3}H_{n4}]$, where each block is a 1×3^n matrix, which in turn can be represented as a block-matrix $[X_1, X_2, \cdots, X_{3^{n-1}}]$ with each block being a row of $\pm J$ or a row of $\pm(I + Q)$.

with $Q = \begin{pmatrix} 0 & - & 1 \\ 1 & 0 & - \\ - & 1 & 0 \end{pmatrix}$, $A_0 = \begin{pmatrix} - & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B_0 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & - & 1 \\ 1 & 1 & 0 & - \\ 1 & - & 1 & 0 \end{pmatrix}$

I is the identity matrix of order 4.3^n and J is the matrix of all ones of order 4.3^n

This should give you a family of pairs of Hadamard matrices for each value of n.

2 Second method

Another way to construct a Hadamard matrix of order 12 could be this:

Find a Generalized Hadamard matrix GH of order 12 over an elementary abelian group of order 4 ($\mathbb{Z}_{\not\succeq} \times \mathbb{Z}_{\not\succeq}$). Denote the elements of the group as the set of vectors of dimension 2 over $GF(2)$. As an example, $EA_4 = \{(1,1), (1,-), (-,1), (-,-)\}$, the group operation is the component-wise product of two vectors.

The GH matrix can be "split" to obtain 2 matrices. First, consider the matrix formed by taking the first element of each vector seperately, and another matrix formed by taking the second element of each vector seperately. For eg. for $GH(12, EA(4)) = [GH_{ij}]$, $1 \leq i \leq 12, 1 \leq j \leq 12$, (The notation refers to the Generalized Hadamard matrix of order 12 over an elementary abelian group of order 4); Form two matrices H^1 and H^2 of order 12, such that, if $GH_{ij} = (a, b)$, then $H^1_{ij} = a$ and $H^2_{ij} = b$.

Both H^1 and H^2 are Hadamard matrices. You can find an example of $GH(12, EA(4))$, due to Jennifer Seberry here.