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Question

Take a $GH(2^k, 1)$ matrix and write each entry as a k-tuple consisting of +1 and -1. Then make k matrices $H_i^{2^k}$ $1 \leq i \leq k$, each formed by taking the i^{th} component of the tuple for each element. Are the $H_i^{2^k}$'s Hadamard matrices? Also, is the Hadamard product of any two $H_i^{2^k}$ a Hadam ard matrix?

1 Sample for k=3

1.1 Making the Generalized Hadamard matrices

The $GH(2^k, 1)$ matrix, for each order was made using the multiplication table of $GF(2^k)$. Two rows u and v of GH are said to be difference balanced if the set $\{u_i - v_i\colon 1 \leq i \leq 2^k\}$, (where $-v_i$ represents the additive inverse of v_i) contains each element of the group once.

An index, $(1 - 2^k)$ was assigned to each element of $GF(2^k)$.
An example of a generalized Hadamard matrix of order 2^3, formed over the elementary abelian 2-group of order 2^3, $GH(2^3, 1)$ is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 7 & 4 & 2 & 8 & 6 \\ 1 & 4 & 7 & 6 & 8 & 5 & 2 & 3 \\ 1 & 5 & 4 & 8 & 7 & 3 & 6 & 2 \\ 1 & 6 & 2 & 5 & 3 & 8 & 4 & 7 \\ 1 & 7 & 8 & 2 & 6 & 4 & 3 & 5 \\ 1 & 8 & 6 & 3 & 2 & 7 & 5 & 4 \end{pmatrix} \tag{1}$$

1.2 Making the Hadamard matrices from the products

Writing each element as a 3-tuple, we get :

$$\begin{pmatrix} (1,1,1) & (1,1,1) & (1,1,1) & (1,1,1) & (1,1,1) & (1,1,1) & (1,1,1) & (1,1,1) \\ (1,1,1) & (1,1,-) & (1,-,1) & (1,-,-) & (-,1,1) & (-,1,-) & (-,-,1) & (-,-,-) \\ (1,1,1) & (1,-,1) & (-,1,1) & (-,-,1) & (1,-,-) & (1,1,-) & (-,-,-) & (-,1,-) \\ (1,1,1) & (1,-,-) & (-,-,1) & (-,1,-) & (-,-,-) & (-,1,1) & (1,1,-) & (1,-,1) \\ (1,1,1) & (-,1,1) & (1,-,-) & (-,-,-) & (-,-,1) & (1,-,1) & (-,1,-) & (1,1,-) \\ (1,1,1) & (-,1,-) & (1,1,-) & (-,1,1) & (1,-,1) & (-,-,-) & (1,-,-) & (1,1,-) \\ (1,1,1) & (-,-,1) & (-,-,-) & (1,1,-) & (-,1,-) & (1,-,-) & (1,-,1) & (-,1,1) \\ (1,1,1) & (-,-,-) & (-,1,-) & (1,-,1) & (1,1,-) & (-,-,1) & (-,1,1) & (1,-,-) \end{pmatrix} \tag{2}$$

The three matrices obtained from the above matrix are:

$H_1 =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

$H_2 =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

$H_3 =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

The pairwise Hadamard products are:

$H_1 * H_2 =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

$H_2 * H_3 =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix}$$

$H_3 * H_1 =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

which are all Hadamard matrices.

The above property was examined for $GH(2^k, 1)$, with $2 \leq k \leq 10$,
For orders $n = 2^2, 2^3, 2^4, 2^6, 2^7, 2^9$, $H_i^{2^k}$ were Hadamard matrices, and the pairwise Hadamard product of $H_i^{2^k}$ was found to be Hadamard.
For orders $n = 2^5, 2^8, 2^{10}$, $H_i^{2^k}$ were Hadamard matrices, however their pairwise Hadamard product did not always turn out to be a Hadamard matrix.

1.3 Doubts

However, I'm not sure if the latter orders will always fail to satisfy the property. Probably, be re-assigning the tuples to the indices in (1), or by considering some other inequivalent GH matrix, the property might be satisfied.