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Notebook

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Q1.

Normal distribution

$$PMF(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\mu = \theta_1 \quad , \quad \sigma^2 = \theta_2$$

$$\therefore f(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$f(x_i \mid \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} - \frac{(x_i-\theta_1)^2}{2\theta_2}$$

Likelihood function,

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i \mid \theta_1, \theta_2)$$

$$\Rightarrow L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} (\theta)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{\frac{-(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta i)^2}$$

$$\ln(L(\theta_1, \theta_2)) = \ln\left[(\theta_2)^{-n/2} (2\pi)^{-n/2} e^{\frac{-1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}\right]$$

$$= -\frac{n}{2}\ln\theta_2 - \frac{n}{2}\ln 2\pi - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \quad \text{—(1)}$$

differentiate w.r.t. θ_1

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)$$

Now, $\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = 0$

$$\therefore \quad \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1) = 0$$

$$\Rightarrow \boxed{\frac{1}{\theta_2}\left(\sum_{i=1}^{n}x_i - n\theta_1\right) = 0}$$

$$\sum_{i=1}^{n}x_i - n\theta_1 = 0$$

$$\theta_1 = \frac{\sum_{i=1}^{n}x_i}{n}$$

$$\Rightarrow \quad \theta_1 = \overline{x}_n$$

$$\therefore \quad \boxed{\theta_{1MLE} = \overline{x}_n} \quad \text{—— (2)}$$

differentiating (1) w.r.t (2),

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_2} = \frac{-n}{2\theta_2} - \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

now $\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_2} = 0$

$$\therefore -\frac{n}{2\theta_2} - \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = 0$$

$$\frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = \frac{-n}{2\theta_2}$$

$$\boxed{\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

From equation ②,

$$\boxed{\theta_{2MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}$$

Q2. For binomial distribution,

$$PMF(x_i) = {}^{n}C_{x_i}\, p^{x_i}(1-p)^{n-x_i}$$

$$n = m,\ p = \theta$$

$$L(P) = \prod_{i=1}^{n} P(x_i | m, \theta)$$

$$= \prod_{i=1}^{n}\left({}^{m}C_{x_i}\,\theta^{x_i}(1-\theta)^{m-x_i}\right)$$

$$= \prod_{i=1}^{n}{}^{m}C_{x_i}\prod_{i=1}^{n}\theta^{x_i}\prod_{i=1}^{n}(1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n}{}^{m}C_{x_i}\,\theta^{\sum_{i=1}^{n}x_i}(1-\theta)^{nm-\sum_{i=1}^{n}x_i}$$

$$\ln L(P) = \ln\left(\prod_{i=1}^{n}{}^{m}C_{x_i}\prod_{i=1}^{n}\theta^{x_i}(1-\theta)^{nm-\sum_{i=1}^{n}x_i}\right)$$

$$= \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \ln\left(\theta^{\sum_{i=1}^{n} x_i}\right) + \ln\left((1-\theta)^{nm - \sum_{i=1}^{n} x_i}\right)$$

$$= \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \ln(\theta) \cdot \sum_{i=1}^{n} x_i + \ln(1-\theta)\left(nm - \sum_{i=1}^{n} x_i\right)$$

diff. w.r.t θ

$$\frac{\partial L(P)}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{-1}{1-\theta}\right) nm - \sum_{i=1}^{n} x_i$$

$$= \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right)$$

now $\frac{\partial L(P)}{\partial \theta} = 0$

$$\therefore \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right) = 0$$

$$\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right)$$

$$\frac{1-\theta}{\theta} = \frac{nm - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \frac{nm}{\sum_{i=1}^{n} x_i} - 1 = \frac{1}{\theta} - 1$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} x_i}{nm}$$

$$\Rightarrow \theta = \frac{\overline{x_n}}{m}$$

$$\therefore \boxed{\theta_{MLE} \in (0,1) = \frac{\overline{x_n}}{m}}$$