CSE 400: Fundamentals of Probability and Random Variables

Lecture: Types of Continuous Random Variables

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Types of Continuous Random Variables

1. Uniform Random Variable

Definition

A continuous random variable (Θ) is said to be uniformly distributed over an interval $[a, b]$ if its probability density function is constant over that interval.

Support

$$a \le \Theta < b$$

Probability Density Function (PDF)

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{b-a}, & a \le \theta < b \\ 0, & \text{otherwise} \end{cases}$$

Given

$$\Theta \sim \text{Uniform}[0, 2\pi]$$

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

(a)

$$\Pr(\Theta > \tfrac{3\pi}{4})$$

$$= \int_{3\pi/4}^{2\pi} \frac{1}{2\pi} \, d\theta$$

$$= \frac{1}{2\pi}\left(2\pi - \frac{3\pi}{4}\right)$$

$$= \frac{5}{8}$$

(b)

$$\Pr(\Theta < \pi \mid \Theta > \tfrac{3\pi}{4})$$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$A = \{\Theta < \pi\}, \quad B = \{\Theta > \tfrac{3\pi}{4}\}$$

$$A \cap B = \left\{\tfrac{3\pi}{4} < \Theta < \pi\right\}$$

$$\Pr(A \cap B) = \int_{3\pi/4}^{\pi} \frac{1}{2\pi}\, d\theta = \frac{1}{8}$$

$$\Pr(B) = \frac{5}{8}$$

$$\Pr(\Theta < \pi \mid \Theta > \tfrac{3\pi}{4}) = \frac{1/8}{5/8} = \frac{1}{5}$$

(c)

$$\Pr(\cos(\Theta) < \tfrac{1}{2})$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\Pr\left(\frac{\pi}{3} < \Theta < \frac{5\pi}{3}\right) = \int_{\pi/3}^{5\pi/3} \frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi}\left(\frac{4\pi}{3}\right) = \frac{2}{3}$$

2. Exponential Random Variable

Definition

A random variable (X) is exponential if its PDF is

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$

Support

$$x \geq 0$$

Probability Density Function (PDF)

$$f_X(x) = e^{-x} u(x)$$

(a)

$$\Pr(3X < 5)$$

$$= \Pr\left(X < \frac{5}{3}\right)$$

$$= \int_0^{5/3} e^{-x} \, dx$$

$$= 1 - e^{-5/3}$$

(b)

$$\Pr(3X < y)$$

$$= \Pr\left(X < \frac{y}{3}\right)$$

$$= \int_0^{y/3} e^{-x} \, dx$$

$$= 1 - e^{-y/3}, \quad y \geq 0$$

(c) **Let** $(Y = 3X)$

$$F_Y(y) = \Pr(Y < y) = 1 - e^{-y/3}, \quad y \geq 0$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{3} e^{-y/3} u(y)$$

3. Laplace Random Variable

Definition

A random variable (W) is Laplace distributed if

$$f_W(w) = ce^{-2|w|}$$

Support

$$-\infty < w < \infty$$

(a)

$$\int_{-\infty}^{\infty} ce^{-2|w|}\, dw = 1$$

$$2c \int_0^{\infty} e^{-2w}\, dw = 1$$

$$2c \cdot \frac{1}{2} = 1 \Rightarrow c = 1$$

(b)

$$\Pr(-1 < W < 2)$$

$$= \int_{-1}^{0} e^{2w}\, dw + \int_0^2 e^{-2w}\, dw$$

$$= \left[\frac{1}{2} e^{2w}\right]_{-1}^{0} + \left[-\frac{1}{2} e^{-2w}\right]_0^2$$

$$= \frac{1}{2}(1 - e^{-2}) + \frac{1}{2}(1 - e^{-4})$$

(c)

$$\Pr(W > 0 \mid -1 < W < 2)$$

$$= \frac{\Pr(0 < W < 2)}{\Pr(-1 < W < 2)}$$

$$\Pr(0 < W < 2) = \int_0^2 e^{-2w}\, dw = \frac{1}{2}(1 - e^{-4})$$

$$\Pr(W > 0 \mid -1 < W < 2) = \frac{\frac{1}{2}(1 - e^{-4})}{\frac{1}{2}(1 - e^{-2}) + \frac{1}{2}(1 - e^{-4})}$$

4. Gamma Random Variable

Definition of the Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Mean and Variance

If $(X \sim \text{Gamma}(\alpha, \lambda))$,

$$E[X] = \frac{\alpha}{\lambda}$$

$$\text{Var}(X) = \frac{\alpha}{\lambda^2}$$

5. Properties of the Gamma Function

(a)

$$\Gamma(n) = (n - 1)!$$

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt$$

Using integration by parts repeatedly,

$$\Gamma(n) = (n - 1)\Gamma(n - 1)$$

With ($\Gamma(1) = 1$),

$$\Gamma(n) = (n-1)!$$

(b)

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$$

Integration by parts gives:

$$\Gamma(x+1) = x\Gamma(x)$$

(c)

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\,dt$$

Using the Gaussian integral,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

6. In-Class Activity: Gaussian Simulation and Gaussian Density Estimation

(Google Colab)

Activity Overview

- Gaussian simulation
- Distribution visualization
- Density estimation
- Packet delay modeling

1. Motivation

- Sensor measurements
- Network packet delays
- Image sensor noise

2. Gaussian Simulation (Known Parameters)

```
import numpy as np
import matplotlib.pyplot as plt

N = 10000
mu = 0
sigma = 1

samples = np.random.normal(mu, sigma, N)
```

3. Visualizing the Distribution

```
counts, bins = np.histogram(samples, bins=50, density=True)
centers = (bins[:-1] + bins[1:]) / 2

pdf = (1/(sigma*np.sqrt(2*np.pi))) * np.exp(-(centers-mu)**2/(2*sigma**2))

plt.bar(centers, counts)
plt.plot(centers, pdf)
plt.show()
```

4. Reality Check: Unknown Parameters

- Only samples are available
- Parameters unknown

5. Gaussian Density Estimation from Data

```
mu_hat = np.mean(samples)
sigma_hat = np.std(samples)
```

$$\hat{f}(x) = \frac{1}{\sigma_{\text{hat}}\sqrt{2\pi}} e^{-\frac{(x-\mu_{\text{hat}})^2}{2\sigma_{\text{hat}}^2}}$$

6. Application: Packet Delay in Networks

- Gaussian delay simulation
- Non-negative constraint
- Histogram construction
- Parameter estimation
- Density plotting

7. Key Takeaways

- Gaussian simulation uses randomized algorithms
- Density estimation learns distributions from samples
- Same methodology applies across applications