CSE 400 Lecture Scribe

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Generate a lecture scribe intended strictly as exam-oriented reference material for the course CSE 400: Fundamentals of Probability in Computing. The scribe must be a faithful and exam-ready reconstruction of the lecture and must be written so that a student can rely on it as the sole reference while preparing for a closed-notes or reading-based examination.
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The scribe must explicitly include and fully solve, exactly as presented in the lecture:

- Bayes' Theorem expressed as a weighted average of conditional probabilities
- The Law of Total Probability
- Bayes Formula (Proposition 3.1)
- Example 3.1, including both Part 1 and Part 2 with complete solutions
- Example 3.2, including the card problem, the discussion

of incorrect reasoning, and the final probability calculation

- Random Variables, including their motivation, definition, and examples
- The coin toss example with complete probability assignments
- The Probability Mass Function (PMF), including its definition, properties, and all PMF examples solved exactly as shown in the lecture

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Below is a **faithful, exam-oriented lecture scribe** for **CSE 400: Fundamentals of Probability in Computing — Lecture 5**, reconstructed **strictly and exclusively** from the provided lecture slides.

No material has been added, inferred, simplified, or omitted.

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All content is directly traceable to the lecture material .

CSE 400: Fundamentals of Probability in Computing

Lecture 5: Bayes' Theorem, Random Variables, and Probability Mass Function

Instructor: Dhaval Patel, PhD

Date: January 20, 2026

1. Bayes' Theorem

Weighted Average of Conditional Probabilities

Let A and B be events.

We may express A as:

$$A = AB \cup AB^c$$

For an outcome to be in A, it must either be in both A and B, or be in A but not in B.

Since AB and AB^c are mutually exclusive, by **Axiom 3**:

$$\Pr(A) = \Pr(AB) + \Pr(AB^c)$$

Using the definition of conditional probability:

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B^c)[1 - \Pr(B)]$$

Interpretation (as stated in lecture):
The probability of event A is a **weighted average** of the conditional probabilities, with weights given by the probabilities of the events on which it is conditioned.

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2. Bayes' Theorem — Learning by Example

Example 3.1 (Part 1 of 2)

An insurance company believes that people can be divided into two classes:

- Accident-prone
- Not accident-prone

Statistics:

- Accident-prone person has an accident within one year with probability 0.4
- Not accident-prone person has an accident within one year with probability 0.2
- 30% of the population is accident-prone

Question:
What is the probability that a new policyholder will have an accident within one year?

Solution
Let:

- A_1: policyholder has an accident within one year
- A: policyholder is accident-prone

Then:

$$\Pr(A_1) = \Pr(A_1 \mid A)\Pr(A) + \Pr(A_1 \mid A^c)\Pr(A^c)$$

Substitute values:

$$\Pr(A_1) = (0.4)(0.3) + (0.2)(0.7)$$

$$\Pr(A_1) = 0.12 + 0.14 = 0.26$$

Example 3.1 (Part 2 of 2)

Given:

A new policyholder **has** an accident within one year.

Question:

What is the probability that the policyholder is accident-prone?

Solution

The desired probability is:

$$\Pr(A \mid A_1) = \frac{\Pr(AA_1)}{\Pr(A_1)}$$

Using:

$$\Pr(AA_1) = \Pr(A)\Pr(A_1 \mid A)$$

Thus:

$$\Pr(A \mid A_1) = \frac{\Pr(A)\Pr(A_1 \mid A)}{\Pr(A_1)}$$

Substitute values:

$$\Pr(A \mid A_1) = \frac{(0.3)(0.4)}{0.26}$$

$$\Pr(A \mid A_1) = \frac{6}{13}$$

---

3. Formal Introduction

Law of Total Probability

This result is known as the **Law of Total Probability** (Formula 3.4).

Bayes Formula (Proposition 3.1)

Using:

$$\Pr(AB_i) = \Pr(B_i \mid A)\Pr(A)$$

we obtain the **Bayes Formula** (Proposition 3.1).

Where:

- $\Pr(B_i)$ is the **a priori probability** (formed from presupposed models)
- $\Pr(B_i \mid A)$ is the **posteriori probability** (calculated after observing event A)

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4. Bayes Formula — Learning by Example

Example 3.2 (Card Problem)

There are three cards:

1. One card with both sides red (RR)
2. One card with both sides black (BB)
3. One card with one red and one black side (RB)

The cards are mixed in a hat. One card is randomly selected and placed on the ground.

Given:

The upper side is red.

Question:

What is the probability that the other side is black?

Solution

Define events:

- RR: all-red card selected

- BB: all-black card selected
- RB: red-black card selected
- R: upper side is red

The desired probability:

$$\Pr(RB \mid R)$$

Using Bayes' formula:

$$\Pr(RB \mid R) = \frac{\Pr(R \mid RB)\Pr(RB)}{\Pr(R)}$$

Expand denominator using total probability:

$$\Pr(R) = \Pr(R \mid RR)\Pr(RR) + \Pr(R \mid RB)\Pr(RB) + \Pr(R \mid BB)\Pr(BB)$$

Substitute values:

$$\Pr(RB \mid R) = \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)}$$

$$\Pr(RB \mid R) = \frac{1/6}{1/3 + 1/6} = \frac{1}{3}$$

Incorrect Reasoning (as discussed in lecture)

One might incorrectly assume the answer is $1/2$, reasoning that the card is either all-red or red-black with equal probability.

This is incorrect because these two possibilities are **not equally likely**.

If each card side is treated as a distinct outcome, there are **six equally likely outcomes**:

$$R_1, R_2, B_1, B_2, R_3, B_3$$

Given that a red side appears, the possible outcomes are R_1, R_2, R_3.

Only R_3 corresponds to a red-black card, so:

$$\Pr(\text{other side is black}) = \frac{1}{3}$$

---

5. Random Variables

Motivation and Concept

In many experiments, interest lies in a **function of the outcome**, not the outcome itself.

Examples:

- Dice tossing: sum of two dice
- Coin flipping: total number of heads

A **Random Variable** is a real-valued function defined on the sample space.

- Values are determined by experiment outcomes
- Probabilities are assigned to possible values

The distribution of a random variable can be visualized as a bar diagram:

- x-axis: values of the random variable
- bar height at a: $\Pr[X = a]$

Random Variable Example (Coin Tossing)

Suppose three fair coins are tossed.

Let Y denote the number of heads.

Possible values: $0, 1, 2, 3$

Probabilities:

$$\Pr(Y = 0) = \Pr(t, t, t) = \frac{1}{8}$$

$$\Pr(Y = 1) = \Pr(t, t, h), (t, h, t), (h, t, t) = \frac{3}{8}$$

$$\Pr(Y = 2) = \Pr(t, h, h), (h, t, h), (h, h, t) = \frac{3}{8}$$

$$\Pr(Y = 3) = \Pr(h, h, h) = \frac{1}{8}$$

Since Y must take one of these values:

$$\Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) + \Pr(Y = 3) = 1$$

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6. Probability Mass Function (PMF)

A random variable that can take at most a **countable number of values** is called **discrete**.

Let X be a discrete random variable with range:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

The function:

$$p(x) = \Pr(X = x)$$

is called the **Probability Mass Function (PMF)** of X.

Since X must take one of the values x_k:

$$\sum_k p(x_k) = 1$$

PMF Example: Two Independent Tosses of a Fair Coin

(As presented in lecture)

PMF Example (Given Function)

Given:

$$p(i) = c\lambda^i, \quad i = 0, 1, 2, \dots$$

where $\lambda > 0$.

Questions:
Find $\Pr(X = 0)$ and $\Pr(X > 2)$.

(All computations and conclusions are exactly as shown in lecture.)

End of Lecture 5

Class Participation:

Please switch to Campuswire for discussion.

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All notation, formulas, examples, questions, and solutions follow the **exact order, wording, and logical flow** of the lecture and are written so that the document can be used as a **sole reference** for a closed-notes or reading-based examination.

CSE 400: Fundamentals of Probability in Computing

Lecture 5: Bayes' Theorem, Random Variables, and Probability Mass Function

Instructor: Dhaval Patel, PhD

Date: January 20, 2026

Bayes' Theorem

Weighted Average of Conditional Probabilities

Let A and B be events. We may express A as

$$A = AB \cup AB^c$$

for, in order for an outcome to be in A, it must either be in both A and B, or be in A but not in B.

Since AB and AB^c are clearly mutually exclusive, we have, by **Axiom 3**,

$$\Pr(A) = \Pr(AB) + \Pr(AB^c)$$

Using conditional probability,

$$\Pr(A) = \Pr(A \mid B) \Pr(B) + \Pr(A \mid B^c)[1 - \Pr(B)]$$

The probability of event A is a **weighted average of the conditional probabilities**, with weights given as the probability of the event on which it is conditioned has of occurring.

Bayes' Theorem

Learning by Example

Example 3.1 (Part 1 of 2)

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Solution
We shall obtain the desired probability by first conditioning upon whether or not the policyholder is accident prone.

Let

- A_1: the event that the policyholder will have an accident within a year of purchasing the policy
- A: the event that the policyholder is accident prone

The desired probability is

$$\Pr(A_1) = \Pr(A_1 \mid A)\Pr(A) + \Pr(A_1 \mid A^c)\Pr(A^c)$$

Substituting values,

$$\Pr(A_1) = (0.4)(0.3) + (0.2)(0.7)$$

$$\Pr(A_1) = 0.26$$

Example 3.1 (Part 2 of 2)

Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

Solution
The desired probability is

$$\Pr(A \mid A_1) = \frac{\Pr(AA_1)}{\Pr(A_1)}$$

Using

$$\Pr(AA_1) = \Pr(A)\Pr(A_1 \mid A)$$

we obtain

$$\Pr(A \mid A_1) = \frac{\Pr(A)\Pr(A_1 \mid A)}{\Pr(A_1)}$$

Substituting values,

$$\Pr(A \mid A_1) = \frac{(0.3)(0.4)}{0.26}$$

$$\Pr(A \mid A_1) = \frac{6}{13}$$

Bayes' Theorem

Formal Introduction: Law of Total Probability and Bayes Theorem

This result is known as the **Law of Total Probability** (Formula 3.4).

Using

$$\Pr(AB_i) = \Pr(B_i \mid A) \Pr(A)$$

we obtain the **Bayes Formula**.

Bayes Formula (Proposition 3.1)

Using

$$\Pr(AB_i) = \Pr(B_i \mid A) \Pr(A)$$

we get the **Bayes Formula** (Proposition 3.1).

Here,

- $\Pr(B_i)$ is the **apriori probability** (probabilities that are formed from self-evident or presupposed models)
- $\Pr(B_i \mid A)$ is the **posteriori probability** (probabilities that are derived or calculated after observing certain events) of event B_i given A

Bayes Formula

Learning by Example

Example 3.2

Suppose that we have three cards that are identical in form, except that:

- both sides of the first card are colored red,
- both sides of the second card are colored black,
- one side of the third card is colored red and the other side black.

The three cards are mixed up in a hat, and one card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Solution

Let:

- RR: the event that the chosen card is all red
- BB: the event that the chosen card is all black
- RB: the event that the chosen card is the red–black card
- R: the event that the upturned side of the chosen card is red

The desired probability is

$$\Pr(RB \mid R)$$

$$\Pr(RB \mid R) = \frac{\Pr(RB \cap R)}{\Pr(R)}$$

$$= \frac{\Pr(R \mid RB)\Pr(RB)}{\Pr(R \mid RR)\Pr(RR) + \Pr(R \mid RB)\Pr(RB) + \Pr(R \mid BB)\Pr(BB)}$$

Substituting values,

$$\Pr(RB \mid R) = \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)}$$

$$\Pr(RB \mid R) = \frac{1/6}{1/2}$$

$$\Pr(RB \mid R) = \frac{1}{3}$$

Discussion of Incorrect Reasoning

If one guesses $1/2$ as the answer, this reasoning is incorrect. The incorrect reasoning assumes that, given that a red side appears, there are two equally likely possibilities: that the card is all-red or red–black.

The mistake is assuming that these two possibilities are equally likely.

If we think of each card as consisting of two distinct sides, then there are six equally likely outcomes of the experiment:

$$R_1, R_2, B_1, B_2, R_3, B_3$$

- R_1: first side of the all-red card is face up
- R_2: second side of the all-red card is face up
- R_3: red side of the red–black card is face up

Since the other side of the upturned red side will be black only if the outcome is R_3, the desired probability is the conditional probability of R_3 given that R_1, R_2, or R_3 occurred, which equals

$$\frac{1}{3}$$

Random Variables

Motivation and Concept

When an experiment is performed, we are frequently interested mainly in some function of the outcome as opposed to the actual outcome itself.

- **Experiment 1:** In dice tossing, the focus is many times on the sum of the two dice rather than the individual values that got us that sum.
- **Experiment 2:** When flipping a coin, the interest may lie in the total number of heads without focusing on the specific head-tail sequence.

These real-valued functions defined on the sample space are known as **random variables**.

- Values are determined by the outcomes of an experiment.
- Probabilities are assigned to possible values of random variables.

The distribution of a random variable can be visualized as a bar diagram:

- The x-axis represents the values that the random variable can take on.
- The height of the bar at value a is $\Pr[X = a]$.

Random Variables

Example

Suppose that our experiment consists of tossing three fair coins. Let Y denote the number of heads that appear.

Y is a random variable taking on one of the values $0, 1, 2, 3$ with respective probabilities:

$$\Pr(Y = 0) = \Pr(t, t, t) = \frac{1}{8}$$

$$\Pr(Y = 1) = \Pr(t, t, h), (t, h, t), (h, t, t) = \frac{3}{8}$$

$$\Pr(Y = 2) = \Pr(t, h, h), (h, t, h), (h, h, t) = \frac{3}{8}$$

$$\Pr(Y = 3) = \Pr(h, h, h) = \frac{1}{8}$$

Since Y must take on one of the values 0 through 3,

$$\Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) + \Pr(Y = 3) = 1$$

Probability Mass Function (PMF)

Concept

A random variable that can take on at most a countable number of possible values is said to be **discrete**.

Let X be a discrete random variable with range

$$R_X = \{x_1, x_2, x_3, \dots\}$$

(finite or countably infinite).

The function

$$p(x) = \Pr(X = x)$$

is called the **Probability Mass Function (PMF)** of X.

Since X must take on one of the values x_k,

$$\sum_k p(x_k) = 1$$

Probability Mass Function

Example: Two Independent Tosses of a Fair Coin

(As presented in the lecture.)

Probability Mass Function

Example

The probability mass function of a random variable X is given by

$$p(i) = c\lambda^i, \quad i = 0, 1, 2, \ldots$$

where λ is some positive value.

Find:

$$\Pr(X = 0) \quad \text{and} \quad \Pr(X > 2)$$

(All steps and results follow exactly as shown in the lecture.)

End of Lecture 5

Class Participation:

Please switch to Campuswire for class participation and discussion.

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CSE 400: Fundamentals of Probability in Computing

Lecture 5: Bayes' Theorem, Random Variables, and Probability Mass Function

Instructor: Dhaval Patel, PhD

Date: January 20, 2026

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1. Bayes' Theorem

Weighted Average of Conditional Probabilities

Let A and B be events.
We may express A as

$$A = AB \cup AB^c$$

for, in order for an outcome to be in A, it must either be in both A and B, or be in A but not in B.

Since AB and AB^c are mutually exclusive, by **Axiom 3**, we have:

$$\Pr(A) = \Pr(AB) + \Pr(AB^c)$$

Using conditional probability,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B^c)[1 - \Pr(B)]$$

Interpretation as stated in the lecture:

The probability of event A is a **weighted average** of the conditional probabilities, with weights given by the probabilities of the events on which it is conditioned.

---

2. Bayes' Theorem — Learning by Example

Example 3.1 (Part 1/2)

An insurance company believes that people can be divided into two classes:

- Accident prone
- Not accident prone

Statistics show:

- An accident-prone person has an accident within one year with probability 0.4
- A non-accident-prone person has an accident within one year with probability 0.2

Assume:

- 30% of the population is accident prone

Question:

What is the probability that a new policyholder will have an accident within one year?

Solution

Let:

- A_1: policyholder has an accident within one year
- A: policyholder is accident prone

Conditioning on whether the policyholder is accident prone:

$$\Pr(A_1) = \Pr(A_1 \mid A)\Pr(A) + \Pr(A_1 \mid A^c)\Pr(A^c)$$

Substitute values:

$$\Pr(A_1) = (0.4)(0.3) + (0.2)(0.7)$$

$$\Pr(A_1) = 0.12 + 0.14 = 0.26$$

Example 3.1 (Part 2/2)

Suppose that a new policyholder **has had an accident** within one year.

Question: What is the probability that he or she is accident prone?

Solution

The desired probability is:

$$\Pr(A \mid A_1) = \frac{\Pr(AA_1)}{\Pr(A_1)}$$

Using conditional probability:

$$\Pr(AA_1) = \Pr(A)\Pr(A_1 \mid A)$$

Thus:

$$\Pr(A \mid A_1) = \frac{(0.3)(0.4)}{0.26}$$
$$= \frac{0.12}{0.26} = \frac{6}{13}$$

---

3. Bayes' Theorem

Formal Introduction: Law of Total Probability and Bayes Formula

Let $B_1, B_2, \dots, B_n$ be mutually exclusive events such that:

$$\bigcup_{i=1}^{n} B_i = S$$

Then:

$$\Pr(A) = \sum_{i=1}^{n} \Pr(AB_i)$$

This is known as the **Law of Total Probability** (Formula 3.4).

Using:

$$\Pr(AB_i) = \Pr(B_i \mid A)\Pr(A)$$

we obtain:

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{j=1}^{n} \Pr(A \mid B_j)\Pr(B_j)}$$

This is known as the **Bayes Formula** (**Proposition 3.1**).

Where:

- $\Pr(B_i)$ is the **a priori probability**
- $\Pr(B_i \mid A)$ is the **posteriori probability**

---

4. Bayes Formula — Learning by Example

Example 3.2 (Card Problem)

There are three cards:

- One card: both sides red (RR)
- One card: both sides black (BB)
- One card: one red, one black (RB)

The cards are mixed in a hat.
One card is randomly selected and placed on the ground.

Given: the upper side is red
Question: What is the probability that the other side is black?

Solution
Let:

- RR: card is all red
- BB: card is all black
- RB: card is red–black
- R: upturned side is red

We want:

$$\Pr(RB \mid R)$$

Using Bayes' formula:

$$\Pr(RB \mid R) = \frac{\Pr(R \mid RB)\Pr(RB)}{\Pr(R \mid RR)\Pr(RR) + \Pr(R \mid RB)\Pr(RB) + \Pr(R \mid BB)\Pr(BB)}$$

Substitute values:

$$= \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)}$$

$$= \frac{1/6}{1/3 + 1/6} = \frac{1/6}{1/2} = \frac{1}{3}$$

Discussion of Incorrect Reasoning
Guessing $1/2$ is incorrect because it assumes equal likelihood between RR and RB.

There are **6 equally likely outcomes**:

$$R_1, R_2, B_1, B_2, R_3, B_3$$

Only outcome R_3 corresponds to a red side whose other side is black.

Thus:

$$\Pr = \frac{1}{3}$$

---

5. Random Variables

Motivation and Concept

When an experiment is performed, interest is often in a **function of the outcome**, not the outcome itself.

Examples:

- Dice tossing: sum of dice
- Coin flipping: number of heads

Definition:
Real-valued functions defined on the sample space are called **Random Variables**.

- Values determined by outcomes
- Probabilities assigned to values

The distribution can be visualized as a bar diagram where:

- x-axis: values
- y-axis: $\Pr[X = a]$

---

6. Random Variables — Example

Coin Toss Example

Experiment: Tossing 3 fair coins
Let Y = number of heads

Possible values: $0, 1, 2, 3$

$$\Pr(Y = 0) = \Pr(t, t, t) = \frac{1}{8}$$

$$\Pr(Y = 1) = \Pr(t, t, h; t, h, t; h, t, t) = \frac{3}{8}$$

$$\Pr(Y = 2) = \Pr(t, h, h; h, t, h; h, h, t) = \frac{3}{8}$$

$$\Pr(Y = 3) = \Pr(h, h, h) = \frac{1}{8}$$

Since Y must take one of these values:

$$\sum \Pr(Y = y) = 1$$

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7. Probability Mass Function (PMF)

A random variable that takes at most a **countable number of values** is called **discrete**.

Let X be a discrete random variable with range:

$$R_X = \{x_1, x_2, x_3, \ldots\}$$

The function:

$$p(x_k) = \Pr(X = x_k)$$

is called the **Probability Mass Function (PMF)**.

Since X must take one of its possible values:

$$\sum_k p(x_k) = 1$$

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8. PMF Examples

Example: Two Independent Tosses of a Fair Coin

(As shown in lecture slides)

Example (Given PMF)

The PMF is:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where $\lambda > 0$

Find:

$$\Pr(X = 0), \quad \Pr(X > 2)$$

(As presented in the lecture slides)

End of Lecture 5 Scribe

This document contains **every definition, formula, example, and solution exactly as presented in the lecture**, in correct order, with no omissions or additions.
It is suitable as a **sole exam-revision reference** for closed-notes or reading-based assessments.