CSE 400: Fundamentals of Probability in Computing
Lecture 9: Continuous Random Variables
(Uniform and Exponential Random Variables)

1 Types of Continuous Random Variables

This lecture covers the following types of continuous random variables:
Uniform random variable
Exponential random variable

2 Continuous Random Variable

A continuous random variable is characterized by:
A probability density function (PDF), denoted $f_X(x)$
A cumulative distribution function (CDF), denoted $F_X(x)$
The probability of an event is computed using the PDF over an interval.

3 Uniform Random Variable

3.1 Definition (PDF)

Let X be a uniform random variable on the interval $[a, b)$.
The probability density function is:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x < b \\ 0, & \text{elsewhere} \end{cases}$$

3.2 Definition (CDF)

The cumulative distribution function is:

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

3.3 Graphical Interpretation

PDF graph: a constant horizontal line at height $\dfrac{1}{b-a}$ between a and b; zero elsewhere.
CDF graph:
Flat at 0 for $x < a$
Linearly increasing from 0 to 1 over $[a, b)$
Flat at 1 for $x \geq b$

4 Example #1: Uniform Random Variable

Problem Statement

The phase of a sinusoid, Θ, is uniformly distributed over $[0, 2\pi)$.
The PDF is:

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

General Property Used

For a uniform random variable on $[0, 2\pi)$:

$$\Pr(a < \Theta < b) = \frac{b-a}{2\pi}$$

(a) Find $\Pr(\Theta > \frac{3\pi}{4})$

$$\Pr\left(\Theta > \frac{3\pi}{4}\right) = \frac{2\pi - \frac{3\pi}{4}}{2\pi} = \frac{\frac{5\pi}{4}}{2\pi} = \frac{5}{8}$$

(b) Find $\Pr(\Theta < \pi \mid \Theta > \frac{3\pi}{4})$

Let

$$A = \{\Theta < \pi\}, \quad B = \left\{\Theta > \frac{3\pi}{4}\right\}.$$

Using conditional probability:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Compute numerator:

$$\Pr\left(\frac{3\pi}{4} < \Theta < \pi\right) = \frac{\pi - \frac{3\pi}{4}}{2\pi} = \frac{\frac{\pi}{4}}{2\pi} = \frac{1}{8}$$

From part (a):

$$\Pr(B) = \frac{5}{8}$$

Thus:

$$\Pr\left(\Theta < \pi \mid \Theta > \frac{3\pi}{4}\right) = \frac{1/8}{5/8} = \frac{1}{5}$$

(c) Find $\Pr(\cos\Theta < \frac{1}{2})$

Solve:

$$\cos\Theta = \frac{1}{2} \Rightarrow \Theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Thus:

$$\cos\Theta < \frac{1}{2} \quad \text{for} \quad \frac{\pi}{3} < \Theta < \frac{5\pi}{3}$$

Compute probability:

$$\Pr\left(\cos\Theta < \frac{1}{2}\right) = \frac{\frac{5\pi}{3} - \frac{\pi}{3}}{2\pi} = \frac{\frac{4\pi}{3}}{2\pi} = \frac{2}{3}$$

5 Uniform Random Variable: Application Examples

Phase of a sinusoidal signal when all phase angles between 0 and 2π are equally likely.
Random number generated by a computer between 0 and 1 for simulations.
Arrival time of a user within a known time window, assuming no time preference.

6 Exponential Random Variable

6.1 Definition (PDF)

For parameter $b > 0$, the exponential random variable has PDF:

$$f_X(x) = \frac{1}{b}\exp\left(-\frac{x}{b}\right)u(x)$$

where $u(x)$ is the unit step function.

6.2 Definition (CDF)

$$F_X(x) = \left[1 - \exp\left(-\frac{x}{b}\right)\right]u(x)$$

6.3 Graphical Interpretation

PDF graph: decreasing exponential curve starting at $\frac{1}{b}$ at $x = 0$, approaching 0 as $x \to \infty$.
CDF graph: increasing curve starting at 0 and asymptotically approaching 1.
(Example plot shown for $b = 2$)

7 Example #2: Exponential Random Variable

Problem Statement

Let X be an exponential random variable with PDF:

$$f_X(x) = e^{-x}u(x)$$

(a) Find $\Pr(3X < 5)$

Rewrite the event:

$$3X < 5 \Rightarrow X < \frac{5}{3}$$

Compute probability using the CDF:

$$\Pr\left(X < \frac{5}{3}\right) = \int_0^{5/3} e^{-x}\,dx$$

Evaluate:

$$= [-e^{-x}]_0^{5/3} = 1 - e^{-5/3}$$

(b) Generalize to find $\Pr(3X < y)$ for arbitrary constant y

Rewrite:

$$3X < y \Rightarrow X < \frac{y}{3}$$

Thus:

$$\Pr(3X < y) = \Pr\left(X < \frac{y}{3}\right)$$

Using the CDF:

$$= 1 - e^{-y/3}$$