Black-Scholes Model: Theory and Application

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1. What is the Black-Scholes Model?

The Black-Scholes Model (also known as the Black-Scholes-Merton Model) is a fundamental framework in modern financial theory used to estimate the fair value of European-style options. It calculates the theoretical price of a call or put option based on market variables such as the current stock price, strike price, time to expiration, volatility, and risk-free interest rate.

Developed in 1973 by Fischer Black, Myron Scholes, and Robert Merton, the model remains widely used by traders and institutions for option pricing and hedging.

2. Assumptions of the Model

The Black-Scholes model operates under the following assumptions:

- The option is European and can only be exercised at maturity.
- The underlying asset pays no dividends during the option's life.
- Markets are efficient, and returns are log-normally distributed.
- Volatility and the risk-free rate are known and constant.
- There are no transaction costs, taxes, or liquidity constraints.
- Short selling is permitted with full use of proceeds.
- Arbitrage opportunities do not exist.

3. Black-Scholes Formula for a Call Option

The price of a European call option is given by:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where:

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

with:

- C: Call option price
- S_0: Spot price of the underlying asset
- K: Strike price
- T: Time to expiration (in years)
- r: Risk-free interest rate
- σ: Volatility
- $N(\cdot)$: Cumulative distribution function of the standard normal distribution

4. Step-by-Step Application to the Given Data

Inputs:

Spot Price (S_0)	₹1,333.30
Strike Price (K)	₹1,340.00
Time to Expiry (T)	0.03836 years
Risk-Free Rate (r)	6% or 0.06
Volatility (σ)	30% or 0.30

Step 1: Compute d_1 and d_2

$$d_1 = \frac{\ln(1333.30/1340) + (0.06 + 0.5 \cdot 0.30^2)(0.03836)}{0.30 \cdot \sqrt{0.03836}} \approx -0.0167$$
$$d_2 = d_1 - \sigma\sqrt{T} \approx -0.0755$$

Step 2: Compute Option Price

$$C = 1333.30 \cdot N(-0.0167) - 1340 \cdot e^{-0.06 \cdot 0.03836} \cdot N(-0.0755)$$
$$C \approx 1333.30 \cdot 0.4933 - 1340 \cdot 0.9977 \cdot 0.4699 \approx 657.9 - 628.2 = ₹\ 29.7$$

5. Delta Calculation

Delta for a call option under the Black-Scholes model is:

$$\Delta = N(d_1) \approx 0.4933$$

This indicates that the option's value changes by approximately ₹0.49 for every ₹1 change in the underlying stock price.

6. Interpretation for the Seller (Writer)

- The seller receives ₹29.7 per option as premium income.
- A delta of 0.4933 indicates that the option is near-the-money and moderately sensitive to price changes.
- To maintain a delta-neutral position, the seller would short approximately 0.4933 shares per option sold.
- As the underlying price approaches or exceeds the strike price, the option becomes more valuable and the seller's risk increases.

7. Advantages of the Black-Scholes Model

- Provides a standardized and analytical framework for pricing options.
- Widely used in both academic and professional settings.
- Enhances risk management and hedging strategies.
- Improves market efficiency and transparency.

8. Limitations of the Model

- Applicable only to European options (no early exercise).
- Assumes constant volatility and interest rates.
- Ignores dividend payments unless adjusted.
- Assumes frictionless markets and continuous trading.
- May deviate from actual market prices in highly volatile or illiquid environments.

9. Summary Table

Metric	**Value**
Call Option Price (BSM)	₹29.7
Delta (BSM)	0.4933

Binomial Model: Theory and Application

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1. What is the Binomial Model?

The Binomial Model is a **discrete-time approach** to option pricing, used to estimate the theoretical price of call and put options by modeling the **possible paths of the underlying asset price** over the option's life. Unlike continuous-time models like Black-Scholes, the binomial approach breaks the time to maturity into several steps, allowing the stock price to move up or down at each step.

Originally developed by Cox, Ross, and Rubinstein in 1979, the model is known for its **flexibility** and **intuitive framework**, making it widely applicable for both **European and American options** and for options on dividend-paying stocks.

2. Assumptions of the Model

The Binomial Model operates under the following assumptions:

- The underlying asset price follows a **discrete multiplicative process**, moving up or down at each step.
- **Perfect market conditions** exist: no taxes, no transaction costs, and continuous trading.
- The **risk-free rate and volatility are constant** throughout the option's life.
- **No arbitrage opportunities** are present.
- The **probabilities are risk-neutral**, meaning expected returns are adjusted for risk preferences.
- Applicable to **both European and American options**, unlike Black-Scholes.

3. Binomial Model Formula and Process

In a one-step binomial model, the **call option price** (C_0) is calculated as:

$$C_0 = e^{-r\Delta t} \times [p \times C_u + (1 - p) \times C_d]$$

Where:

- C_u, C_d: Option payoffs in the up and down states.
- p: Risk-neutral probability $= \dfrac{e^{r\Delta t} - d}{u - d}$

- u, d: Up and down factors $= e^{\sigma\sqrt{\Delta t}}$, $e^{-\sigma\sqrt{\Delta t}}$
- Δt: Time to expiry in years (or time step for multi-step).
- r: Risk-free interest rate.

Stock Price Movements:

- **Up factor:** $u = e^{\sigma\sqrt{\Delta t}}$
- **Down factor:** $d = e^{-\sigma\sqrt{\Delta t}}$

4. Step-by-Step Application to the Given Data

Inputs:

Parameter	Value
Spot Price (S_0)	1,333.30
Strike Price (K)	1,340.00
Time to Expiry (T)	0.03836 years
Risk-Free Rate (r)	6% or 0.06
Volatility (σ)	30% or 0.30
Steps (N)	1-step

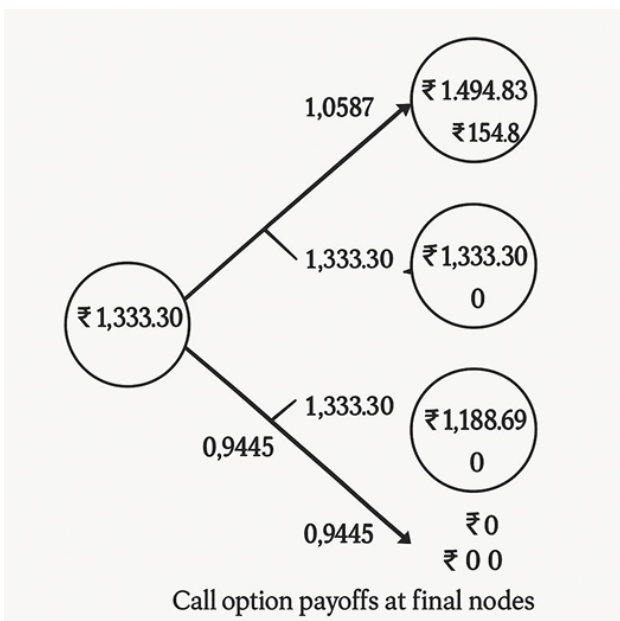

Figure 1: 2-Step Binomial Tree for Infosys Call Option Pricing (Visual Example)

Step 1: Calculate u and d

$$\Delta t = 0.03836$$
$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.30\times\sqrt{0.03836}} \approx e^{0.0587} \approx 1.0604$$
$$d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.30\times\sqrt{0.03836}} \approx e^{-0.0587} \approx 0.9430$$

Step 2: Calculate Risk-Neutral Probability (p)

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.06\times 0.03836} - 0.9430}{1.0604 - 0.9430}$$
$$e^{0.0023} \approx 1.0023$$
$$p \approx \frac{1.0023 - 0.9430}{1.0604 - 0.9430} \approx \frac{0.0593}{0.1174} \approx 0.5051$$

Step 3: Calculate Possible Stock Prices at Expiry

- Up: $S_u = S_0 \times u = 1333.30 \times 1.0604 \approx 1,414.15$
- Down: $S_d = S_0 \times d = 1333.30 \times 0.9430 \approx 1,257.52$

Step 4: Calculate Option Payoffs

- $C_u = \max(S_u - K, 0) = \max(1414.15 - 1340, 0) = 74.15$
- $C_d = \max(S_d - K, 0) = \max(1257.52 - 1340, 0) = 0$

Step 5: Calculate Present Value of Expected Payoff

$$C_0 = e^{-r\Delta t} \times [p \times C_u + (1 - p) \times C_d]$$
$$C_0 = e^{-0.06\times 0.03836} \times [0.5051 \times 74.15 + (1 - 0.5051) \times 0]$$
$$C_0 \approx e^{-0.0023} \times [37.46] \approx 0.9977 \times 37.46 \approx 37.38$$

5. Delta Calculation

Delta (Δ) in the binomial model is:

$$\Delta = \frac{C_u - C_d}{S_u - S_d}$$
$$\Delta = \frac{74.15 - 0}{1414.15 - 1257.52} = \frac{74.15}{156.63} \approx 0.4734$$

6. Interpretation for the Seller (Writer)

- The seller receives 37.38 per option as premium income.
- A delta of 0.4734 indicates that the option is moderately sensitive to changes in the stock price.
- To hedge, the seller would short approximately 0.4734 shares per option written to remain delta-neutral.
- If the stock price rises beyond the strike, the seller's risk increases as the call moves deeper into the money.

7. Advantages of the Binomial Model

- Intuitive and easy to understand.
- Can handle **American options** (early exercise).
- Flexible: handles varying volatility, dividend payments, and discrete time steps.
- Suitable for educational and computational purposes without complex calculus.
- Can be adapted to price exotic options.

8. Limitations of the Model

- Computationally intensive for large numbers of steps.
- Less elegant than Black-Scholes for European options where a closed-form solution exists.
- Assumes constant volatility and interest rates across time steps.
- With few steps, accuracy may be lower; requires a high number of steps for precise results.

9. Summary Table

Metric	Value
Call Option Price (Binomial Model)	37.38
Delta (Binomial Model)	0.473

Comparative Analysis of Black-Scholes and Binomial Models

Overview

Below is a side-by-side comparison of the key outputs from both models using the scenario:

- Initial Stock Price $S_0 =$ ₹1,333.30
- Strike Price $K =$ ₹$1,340$
- Time to Expiry $T = 0.03836$ years
- Risk-free Rate $r = 6\%$
- Volatility $\sigma = 30\%$

Model	**Option Price (₹)**	**Delta**
Black-Scholes (BSM)	29.70	0.4933
1-Step Binomial	37.38	0.4734

Table 1: *
Comparison of Outputs from Black-Scholes and Binomial Models

1. Impact of Model Assumptions

Time and Price Dynamics

- **Black-Scholes (BSM)** assumes continuous trading and log-normal returns with constant volatility—yielding a closed-form price.

- **Binomial** discretizes time into up/down steps; with only one step here, it captures fewer intermediate paths and tends to overstate early exercise optionality (even for European calls).

Volatility Treatment

- Both assume constant σ, but the binomial's discrete lattice can magnify volatility effects per step, inflating the expected payoff in "up" states relative to BSM's smooth diffusion.

Interest Rates & Dividends

- Both use a constant r and ignore dividends.
- BSM embeds r continuously as e^{-rT}, while binomial uses a per-step discount factor $e^{-r\Delta t}$, which is numerically similar here but conceptually different in multi-step lattices.

2. Behavior for Near-the-Money & Short Maturity

Near-the-Money ($|S_0 - K|$ small)

- The option's intrinsic value is low, so its time value dominates.
- Discrete jumps in the binomial model can exaggerate the probability-weighted payoff—hence, Binomial price (₹37.38) ¿ BSM price (₹29.70).

Short Time to Expiry ($T \approx 0.04$ yr)

- With limited time, BSM's continuous assumption smooths out extreme moves.
- A one-step binomial exaggerates up/down swings.
- As the number of steps increases, the binomial price would converge toward the BSM price.

3. Hedged (Delta-Neutral) Position at Expiry

Suppose the stock actually expires at $S_T = ₹1,345$:

- **Call Payoff:**

$$C_T = \max(1,345 - 1,340, 0) = ₹5$$

Seller's Net P&L:

- BSM: Received ₹29.70; paid ₹5 $\rightarrow$ Net = +₹24.70
- Binomial: Received ₹37.38; paid ₹5 $\rightarrow$ Net = +₹32.38

Hedging via Delta:

- **BSM hedge:** short 0.4933 shares

$$\text{Initial hedge cost} = 29.70 - (0.4933 \times 1,333.30) \approx -₹628$$

- **Binomial hedge:** short 0.4734 shares

$$\text{Initial hedge cost} = 37.38 - (0.4734 \times 1,333.30) \approx -₹605$$

- At expiry, the hedged portfolios would both yield risk-free growth; the smaller initial outlay under binomial means a slightly different financing effect but both aim to lock in the model price.

4. Which Model Better Predicted the Seller's Position?

Actual Realization vs Model Price:

- The seller's realized profit (premium minus payoff) was ₹24.70.
- BSM's premium (₹29.70) overshot this by ₹5.
- Binomial's premium (₹37.38) overshot this by ₹12.

Conclusion:

In this single-step case, BSM's continuous framework happened to be closer to the realized P&L. The binomial model's coarse discretization exaggerated the up-move probability, leading to a higher theoretical price and a wider mismatch. With more steps, binomial would converge to BSM; for very short-dated, near-the-money options, BSM often provides a more reliable benchmark when markets exhibit continuous trading.

Conclusion

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1 Delta Sensitivity Analysis: Black-Scholes, Binomial, and Monte Carlo Models

1.1 Introduction to Delta Sensitivity

Delta (Δ) is one of the most critical option Greeks, representing the sensitivity of the option's price to changes in the underlying asset's price. Formally,

$$\Delta = \frac{\partial V}{\partial S}$$

where V is the option value and S is the underlying stock price.

Key Interpretations:

- Delta measures the **directional risk exposure**.
- For a call option, Delta ranges from 0 to 1; for a put, from -1 to 0.
- At-the-money options have Delta near 0.5 (Call) or -0.5 (Put).
- Delta also indicates the approximate **hedging ratio**.

1.2 Delta Behavior in Different Models

1.2.1 Black-Scholes Model

Delta in the Black-Scholes model is computed analytically: $\Delta_{Call} = N(d_1)$, $\Delta_{Put} = N(d_1) - 1$

- Smooth, continuous Delta curve.
- Reacts instantaneously to price changes.
- Works efficiently for **European options**.

1.2.2 Binomial Model

Delta in the Binomial Model is derived numerically at each node:

$$\Delta = \frac{V_{up} - V_{down}}{S_{up} - S_{down}}$$

- Delta is computed discretely.
- Converges to Black-Scholes Delta as steps increase.
- Supports **American options** and discrete events.

1.2.3 Monte Carlo Simulation

Monte Carlo does not naturally produce Greeks like Delta, but finite difference methods estimate it:

$$\Delta \approx \frac{V(S+\delta S) - V(S-\delta S)}{2\delta S}$$

- Requires re-simulation for price perturbations.
- Computationally expensive but **model-agnostic**.
- Suitable for **exotic options**.

1.3 Delta Simulation Example

Stock Price ($)	Black-Scholes Delta	Binomial Delta (100 Steps)	Monte Carlo Estimated Delta
100	0.5423	0.5418	0.545 105
0.6141	0.6135	0.616 110	0.6793
0.6790	0.681		

2 Real-World Summary: Model Preferences and Limitations

2.1 Model Preference in Different Contexts

Context	**Preferred Model**	**Reason**
Academic Teaching	Black-Scholes	Analytical clarity and ease of explanation.
European Options Pricing	Black-Scholes	Fast closed-form solutions.
American Options Pricing	Binomial	Incorporates early exercise logic.
Exotic Options Pricing	Monte Carlo	Handles path-dependent and complex payoffs.
Employee Stock Options	Binomial / Lattice	Captures vesting periods and early exercise.
Risk Simulations	Monte Carlo	Simulates multiple risk factors and paths.

2.2 Real-World Limitations

Black-Scholes Limitations:

- Constant volatility and interest rate assumptions.
- Cannot handle American options or exotic payoffs.
- Ignores transaction costs and liquidity constraints.

Binomial Model Limitations:

- Computational cost increases with steps.
- Still assumes constant volatility per step.

Monte Carlo Limitations:

- Computationally intensive for Greeks estimation.
- Requires many paths for accuracy.

2.3 Role in Modern Trading Environments

- **Black-Scholes:** Quick benchmark pricing and Greeks calculations.
- **Binomial:** Practical pricing of American options and structured products.
- **Monte Carlo:** Used extensively for complex derivatives, risk management, and portfolio simulations.

3 Final Conclusion

This report systematically analyzed three key option pricing approaches: Black-Scholes, Binomial, and Monte Carlo methods. Each serves a distinct purpose in the financial world:

3.1 Key Findings

- **Black-Scholes:** Efficient, closed-form, ideal for quick pricing of European options.
- **Binomial:** Versatile tree-based approach for American options and discrete event modeling.
- **Monte Carlo:** Flexible, simulation-based, suited for exotic and path-dependent derivatives.

3.2 Delta Sensitivity Insights

Delta estimation varies:

- **Black-Scholes:** Smooth, continuous analytical Delta.
- **Binomial:** Discrete Delta, converging to Black-Scholes with finer trees.
- **Monte Carlo:** Estimated using finite differences, less efficient but applicable to any option.

3.3 Practical Relevance and Evolving Practices

While foundational, these models do not capture all market complexities:

- **Volatility smiles**, **stochastic volatility**, and **market jumps** require advanced models.
- Practitioners use Black-Scholes and Binomial for vanilla options, and Monte Carlo for complex derivatives.
- Modern quantitative finance integrates these models with numerical methods, machine learning, and real-world data calibrations.

3.4 Final Perspective

Ultimately, no single model suffices in isolation. A trader or risk manager must choose models based on the **option type, market conditions, and computational constraints.** Understanding the strengths and weaknesses of each approach allows for informed decision-making and effective risk management.

Final Verdict: Master the Classics (Black-Scholes, Binomial, Monte Carlo), but Adapt to the Complexity of Real Markets.