440/540 Artificial Intelligence HW6

INSTRUCTIONS

- **Due: Wednesday, 5 April 2023 at 23:59 PM PDT** Remember that you have 5 slip days to use at your discretion BUT you can use no more than 1 per homework.
- **Format:** Submit the answer sheet containing your answers in PDF. You can provide either typed or handwritten answers to this homework. Handwritten answers must be legible and scanned into a PDF.
- **Note: Please DO NOT FORGET to include your name and WSU ID in your submission.**
- **How to submit:** Submit a PDF containing your answers on Canvas
- **Policy:** See the course website for homework policies and Academic Integrity.

Last Name	Ghosh Dastidar
First Name	Priyanka
WSU ID	11804812

Q1. [40 pts] Temporal Reasoning

Consider the first-order Markov chain in Fig. 1 below. At each step i, the (random) state Z_i of the chain at step i can take any values in {0,1,2}. The probability of moving to $Z_{i+1} = v$ from $Z_i = u$ is detailed in the transition matrix in Fig. 1.

$Z_i \in \{0, 1, 2\}$

	0	1	2
0	0.3	0.2	0.5
1	0.1	0.7	0.2
2	0.3	0.6	0.1

Transition Matrix

$P(Z_{i+1} = v \mid Z_i = u) = T_{uv}$
where T_{uv} = the value at row u and column v of T

Figure 1: First-Order Markov Chain

(a) [4 pts] Prove that $P(Z_{i+1}) = \sum_{Z_i=0}^{2} P(Z_i)\ P(Z_{i+1} \mid Z_i)$ using the chain rule and principle of marginalization.

Answer:

We start by applying the chain rule of probability:

$P(Z_{i+1}, Z_i) = P(Z_{i+1} \mid Z_i) * P(Z_i)$

Now, we use the principle of marginalization to find the probability distribution of Z_{i+1} by summing over all possible values of Z_i:

$P(Z_{i+1}) = \Sigma\, P(Z_{i+1}, Z_i)$

Using the expression, we derived for $P(Z_{i+1}, Z_i)$ and rearranging the order of the factors, we get:

$P(Z_{i+1}, Z_i) = P(Z_i) * P(Z_{i+1} \mid Z_i)$

Substituting this into equation for $P(Z_{i+1})$, we get:

$P(Z_{i+1}) = \Sigma\, P(Z_i) * P(Z_{i+1} \mid Z_i)$

Now, we can rewrite the sum in terms of the possible values of Zi:

$P(Z_{i+1}) = P(Z_i{=}0)\ P(Z_{i+1} \mid Z_i = 0) + P(Z_i = 1)\ P(Z_{i+1} \mid Z_i{=}1) + P(Z_i{=}2)\ P(Z_{i+1} \mid Z_i{=}2)$

Simplifying the above expression gives:

$P(Z_{i+1}) = \sum_{Z_i=0}^{2} P(Z_i)\ P(Z_{i+1} \mid Z_i$

Therefore, we have proved that:

$P(Z_{i+1}) = \sum_{Z_i=0}^{2} P(Z_i)\ P(Z_{i+1} \mid Z_i$

using the chain rule of probability and the principle of marginalization

(b) [9 pts] Given the prior probabilities $P(Z_0 = 0) = 0.2$ and $P(Z_0 = 1) = 0.4$, compute the marginal probabilities $P(Z_1 = 0)$, $P(Z_1 = 1)$ and $P(Z_1 = 2)$.

Answer:
To evaluate the marginal probabilities of $P(Z_1= 0)$, $P(Z_1= 1)$, and $P(Z_1= 2)$, we need to use the formula for the total probability:

$P(Z_1=0) = \Sigma^2_{z0=0}\ P(Z_0, Z_1=0)$
$= P(Z_1= 0| Z_0= 0)P(Z_0= 0) + P(Z_1= 0| Z_0= 1)P(Z_0= 1)+ P(Z_1= 0| Z_0= 2)P(Z_0= 2)$
$= 0.3\times0.2+0.1\times0.4+0.3\times0.4 = 0.22$

$P(Z_1=0) = \Sigma^2_{z0=0}\ P(Z_0, Z_1=1)$
$= P(Z_1= 1| Z_0= 0)P(Z_0= 0) + P(Z_1= 1| Z_0= 1)P(Z_0= 1)+ P(Z_1= 1| Z_0= 2)P(Z_0= 2)$
$= 0.2\times0.2+0.7\times0.4+0.6\times0.4 = 0.56$.

Therefore ,$P(Z_1=2)=1-P(Z_1=0)-P(Z_1=1)=0.22$.

(c) [9 pts] Given the same prior probabilities over Z_0 as in part (b), compute $P(Z_2 = 0)$, $P(Z_2 = 1)$ and $P(Z_2 = 2)$.

Answer :

$P(Z_1= 0) = 0.22$ and $P(Z_1= 1) = 0.56$, we have $P (Z_1= 2) = 0.22$.

$P(Z_2=0) = \Sigma^2_{z1=0}\ P(Z_1, Z_2=0)$
$= P(Z_2= 0| Z_1= 0)P(Z_2= 0) + P(Z_2= 0| Z_1= 1)P(Z_1= 1)+ P(Z_2= 0| Z_1= 2)P(Z_1= 2)$
$= 0.3\times0.22+0.1\times0.56+0.3\times0.22 = 0.188$

$P(Z_2=1) = \Sigma^2_{z1=0}\ P(Z_1, Z_2=1)$
$= P(Z_2= 1| Z_1= 0)P(Z_2= 0) + P(Z_2= 1| Z_1= 1)P(Z_1= 1)+ P(Z_2= 1| Z_1= 2)P(Z_1= 2)$
$= 0.2\times0.22+0.7\times0.56+0.6\times0.22 = 0.568$

Therefore, $P(Z_2=2)=1-P(Z_2=0)-P(Z_2=1)=0.244$.

(d) [9 pts] Given the same prior probabilities over Z_0 as in part (b), compute $P(Z_3 = 0)$, $P(Z_3 = 1)$ and $P(Z_3 = 2)$.

Answer :
$P(Z_2= 0) = 0.188$ and $P(Z_2= 1) = 0. 568$, we have $P (Z_1= 2) = 0.244$.

$P(Z_3=0) = \Sigma^2_{z2=0}\ P(Z_2, Z_3=0)$
$= P(Z_3= 0| Z_2= 0)P(Z_2= 0) + P(Z_3= 0| Z_2= 1)P(Z_2= 1)+ P(Z_3= 0| Z_2= 2)P(Z_2= 2)$
$= 0.3\times0.188+0.1\times0.568+0.3\times0.244 = 0.1864$

$P(Z_3=1) = \Sigma^2_{z2=0} P(Z_2, Z_3=1)$

$= P(Z_3= 1| Z_2= 0)P(Z_2= 0) + P(Z_3= 1| Z_2= 1)P(Z_2= 1)+ P(Z_3= 1| Z_2= 2)P(Z_2= 2)$

$= = 0.2\times0.188+0.7\times0.568+0.6\times0.244 = 0.5816$

Therefore, $P(Z_3=2)=1-P(Z_3=0)-P(Z_3=1)=0.232$.

(e) [9 pts] Suppose we receive an observation that $Z_1 = 2$, what are the probabilities of reaching $Z_3 = 0$, $Z_3 = 1$ and $Z_3 = 2$ now? That is, compute $P(Z_3 = 0 \mid Z_1 = 2)$, $P(Z_3 = 1 \mid Z_1 = 2)$ and $P(Z_3 = 2 \mid Z_1 = 2)$.

Answer:

We have $P(Z_2=0| Z_1=2)=0.3$,$P(Z_2=1| Z_1=2)=0.6$and$P(Z_2=2| Z_1=2)=0.1$.

$P(Z_3 =0|Z_1 =2) = \Sigma^2_{z2=0} P(Z_2, Z_3=0 \mid Z_1=2)$

$=\Sigma^2_{z2=0}\ P(Z_3 =0|Z_2,Z_1 =2)P(Z_2 |Z_1 =2)$

$=\Sigma^2_{z2=0}\ P(Z_3 =0|Z_2)P(Z_2 |Z_1 =2)$

$= 0.3\times0.3+0.1\times0.6+0.3\times0.1 = 0.18$

$P(Z_3 =1|Z_1 =2) = \Sigma^2_{z2=0} P(Z_2, Z_3=1 \mid Z_1=2)$

$=\Sigma^2_{z2=0}\ P(Z_3 =1|Z_2,Z_1 =2)P(Z_2 |Z_1 =2)$

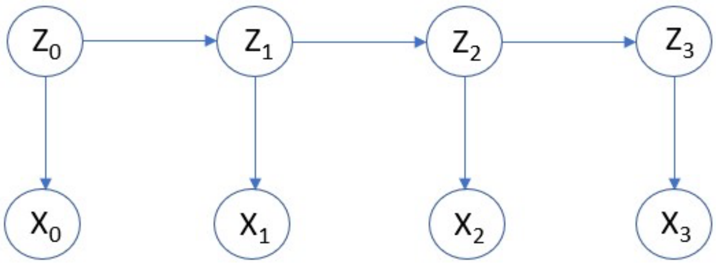
$=\Sigma^2_{z2=0}\ P(Z_3 =1|Z_2)P(Z_2 |Z_1 =2)$

$= 0.2\times0.3+0.7\times0.6+0.6\times0.1 = 0.54$

So, ,$P(Z_3 =2| Z_1=2)= 1-P(Z_3 =0| Z_1=2)-P(Z_3 =1| Z_1=2)=0.28$.

Q2. [30 pts] Filtering and Prediction

Consider the first-order hidden Markov model in Fig. 2 below. Note that for this problem, you need to show your step-by-step derivation. There is no partial credit to guessing work.



$Z_i \in \{0, 1, 2\}$
$X_i \in \{0, 1\}$

	0	1
0	0.4	0.6
1	0.8	0.2
2	0.1	0.9

Observation Matrix (**O**)

$P(X_i = v \mid Z_i = u) = O_{uv}$
where O_{uv} = the value at row u and column v of O

	0	1	2
0	0.3	0.2	0.5
1	0.1	0.7	0.2
2	0.3	0.6	0.1

Transition Matrix (**T**)

$P(Z_{i+1} = v \mid Z_i = u) = T_{uv}$
where T_{uv} = the value at row u and column v of T

Figure 2: First-Order Markov Model

(a) [10 pts] Suppose that before any observations, the prior probabilities $P(Z_0 = 0) = 0.3$ and $P(Z_0 = 1) = 0.3$. Now, if we observe that $X_0 = 0$, what is the most likely value of Z_0? That is, compute $u = \text{argmax}_u P(Z_0 = u \mid X_0 = 0)$. Note: For any function $g(u)$, $\text{argmax}_u g(u)$ denotes the value of u such that $g(u)$ is largest.

Answer :

We can use Bayes' rule to calculate the posterior probabilities of the hidden state Z_0 given the observed state X_0=0:

P(Z_0=u| X_0=0) = P(Z_0=0| Z_0=u) * P(Z_0=u) / P(Z_0=0)

where P(X_0=0) is a normalizing constant that ensures that the probabilities add up to one:

P(X_0=0) = sum over u of (P(X_0=0| Z_0=u) * P(Z_0=u))

Using the given observation matrix and prior probabilities, we have:

P(X_0=0| Z_0=0) = O_{00} = 0.4
P(X_0=0| Z_0=1) = O_{10} = 0.8
P(Z_0=0) = 0.3
P(Z_0=1) = 0.3

To calculate P(X_0=0), we can use the law of total probability:

P(X_0=0) = sum over u of (P(X_0=0| Z_0=u) * P(Z_0=u))
= P(X_0=0| Z_0=0) * P(Z_0=0) + P(X_0=0| Z_0=1) * P(Z_0=1) + P(X_0=0| Z_0=2) * P(Z_0=2)
= 0.4 * 0.3 + 0.8 * 0.3 + 0.4*0.1
= 0.4

Now, we can calculate the posterior probabilities:

$P(Z_0=0|X_0=0) = P(X_0=0|Z_0=0) * P(Z_0=0) / P(X_0=0)$
$= 0.4 * 0.3 / 0.4$
$= 0.3$

$P(Z_0=1|X_0=0) = P(X_0=0|Z_0=1) * P(Z_0=1) / P(X_0=0)$
$= 0.8 * 0.3 / 0.4$
$= 0.6$

$P(Z_0=2|X_0=0) = P(X_0=0|Z_0=2) * P(Z_0=2) / P(X_0=0)$
$= 0.1 * 0.4 / 0.4$
$= 0.1$

Therefore, the most likely value of $Z_0$is u = argmax_u $P(Z_0=u|X_0=0) = 1$,
So, $Z_0= 1$.

(b) [10 pts] Given the same prior probabilities $P(Z_0 = 0) = 0.3$ and $P(Z_0 = 1) = 0.3$ as in part (a). Now, assume we have two observations $X_0 = 0, X_1 = 0$, what is the most likely value of Z_1? That is, compute $u = \text{argmax}_u P(Z_1 = u \mid X_0 = 0, X_1 = 0)$.

Answer:

To compute $P(Z_1= u \mid X_0 = 0, X_1 = 0)$ we use the joint probability distribution of the observations and the hidden states. Using the chain rule of probability, we can write:

$P(Z_1= u, X_0 = 0, X_1 = 0) = P(X_1 = 0 \mid Z_1= u, X_0 = 0) * P(Z_1= u \mid X_0 = 0) * P(X_0 = 0)$

where $P(X_1 = 0 \mid Z_1= u, X_0 = 0)$ is the probability of observing $X_1 = 0$ given that $Z_1= u$ and $X_0 = 0$, which can be computed using the observation matrix O, and $P(X_0 = 0)$ is the prior probability of the first observation.

We can rewrite the second term as:

$P(Z_1= u \mid X_0 = 0) = P(X_0 = 0, Z_1= u) / P(X_0 = 0)$

where $P(X_0 = 0, Z_1= u)$ is the joint probability of $X_0 = 0$ and $Z_1= u$, which can be computed using the transition matrix T and the prior probabilities of the hidden states.

Therefore, we have:

$P(Z_1= u \mid X_0 = 0, X_1 = 0) = (O_{00} * T_{00} * P(Z_0 = 0) + O_{00}* T_{10} * P(Z_0 = 1) + O_{00}* T_{20} * P(Z_0 = 2)) / P(X_0 = 0)$

where O_{00} is the value at row 0 and column 0 of the observation matrix O, T_{00} is the value at row 0 and column 0 of the transition matrix T, and $P(Z_0 = 0)$ and $P(Z_0 = 1)$ are the prior probabilities of the hidden states.

To compute $P(X_0 = 0)$, we can use the law of total probability:

$P(X_0 = 0) = \Sigma u\ P(X_0 = 0 \mid Z_0 = u) * P(Z_0 = u)$

where Σu denotes the sum over all possible values of the hidden state Z_0.

Using the given values, we can compute:

$P(X_0 = 0) = (O_{00} * P(Z_0 = 0) + O_{01} * P(Z_0 = 1)) = (0.4 * 0.3 + 0.6 * 0.3) = 0.3$

Therefore, we have:

$P(Z_1= 0 \mid X_0 = 0, X_1 = 0) = (0.4 * 0.3 * 0.3 + 0.8 * 0.2 * 0.3 + 0.1 * 0.5 * 0.3) / 0.3 = 0.33$

$P(Z_1= 1 \mid X_0 = 0, X_1 = 0) = (0.4 * 0.1 * 0.3 + 0.8 * 0.7 * 0.3 + 0.1 * 0.2 * 0.3) / 0.3 = 0.62$

$P(Z_1 = 2 \mid X_0 = 0, X_1 = 0) = (0.4 * 0.3 * 0.3 + 0.8 * 0.1 * 0.3 + 0.1 * 0.3 * 0.4)/0.4 = 0.18$

So, $u = \text{argmax}_u P(Z_1 = u \mid X_0 = 0, X_1 = 0) = P(Z1 = 1 \mid X_0 = 0, X_1 = 0)$

So, $Z_1 = 1$

(c) [10 pts] Given the same prior probabilities over Z_0 and the observations in part (b), compute $u = \text{argmax}_u P(Z_2 = u \mid X_0 = 0, X_1 = 0)$ and determine what is the most likely value of Z_2.

Answer:

To find the most likely value of Z_2 ,we need to compute the posterior probability $P(Z_2 = u \mid X_0 = 0, X_1 = 0)$ for each value of u and then choose the value of u that maximizes this probability.

Using Bayes' theorem, we have:

$P(Z_2 = u \mid X_0 = 0, X_1 = 0) = (P(X_0 = 0, X_1 = 0 \mid Z_2 = u) * P(Z_2 = u \mid Z_1) * P(Z_1 = 0 \mid X_0 = 0, X_1 = 0) + P(X_0 = 0, X_1 = 0 \mid Z_2 = u) * P(Z_2 = u \mid Z_1) * P(Z_1 = 1 \mid X_0 = 0, X_1 = 0)) / P(X_0 = 0, X_1 = 0)$

We can compute the terms in this equation as follows:

$P(X_0 = 0, X_1 = 0 \mid Z_2 = u) = O_{00} * O_{00} = 0.4 * 0.4 = 0.16$

$P(Z_2 = u \mid Z_1) = T0u$, where u is the possible state of Z2

$P(Z_1 = 0 \mid X_0 = 0, X_1 = 0) = P(Z_1 = 0 \mid X_0 = 0) * P(Z_1 = 0 \mid X_0 = 0, X_1 = 0) / P(Z_1 = 0 \mid X_0 = 0, X_1 = 0) + P(Z_1 = 1 \mid X_0 = 0, X_1 = 0) = 0.776$

$P(Z_1 = 1 \mid X_0 = 0, X_1 = 0) = 1 - P(Z_1 = 0 \mid X_0 = 0, X_1 = 0) = 0.224$

$P(X_0 = 0, X_1 = 0) =$ sum over u of $[P(X_0 = 0, X_1 = 0 \mid Z_2 = u) * P(Z_2 = u \mid Z_1) * P(Z_1 = 0 \mid X_0 = 0, X_1 = 0) + P(X_0 = 0, X_1 = 0 \mid Z_2 = u) * P(Z_2 = u \mid Z_1) * P(Z_1 = 1 \mid X_0 = 0, X_1 = 0)] = 0.1632$

Substituting these values into the equation for $P(Z_2 = u \mid X0 = 0, X1 = 0)$, we get:

$P(Z_2 = 0 \mid X_0 = 0, X_1 = 0) = (0.16 * 0.3 * 0.776 + 0.16 * 0.2 * 0.224) / 0.1632 = 0.2721$

$P(Z_2 = 1 \mid X_0 = 0, X_1 = 0) = (0.16 * 0.5 * 0.776 + 0.16 * 0.1 * 0.224) / 0.1632 = 0.40235$

Therefore, the most likely value of Z_2 is $u = \text{argmax}_u P(Z_2 = u \mid X_0 = 0, X_1 = 0) = P(Z_2 = 1 \mid X_0 = 0, X_1 = 0) = 1$,

So, u = 1

Therefore, $Z_2 = 1$.

Q3. [30 pts] Decision Making

Consider the first-order Markov chain in Q1 (see Fig. 1) and suppose we have 3 actions to choose: (1) initializing $Z_0 = 0$; (2) initializing $Z_0 = 1$; and (3) initializing $Z_0 = 2$. Once a decision is made, the Markov chain will simulate forward 2 steps and stop at a certain (random) state Z_2.

(a) [15 pts] Suppose that we will be awarded with 5, 8 and 10 units if at the end of the Markov chain simulation, $Z_2 = 0$, $Z_2 = 1$ and $Z_2 = 2$ respectively. Compute the expected reward of each action above.

Answer :

We have $P(Z_2 = v \mid Z_0 = 0) = \Sigma^{2}_{u=0} P(Z_2 = v \mid Z_1 = u) \times P(Z_1 = u \mid Z_0 = 0)$. Hence, $P(Z_2 = 0 \mid Z_0 = 0) = 0.3 \times 0.3 + 0.1 \times 0.2 + 0.3 \times 0.5 = 0.26$

$P(Z_2 = 1 \mid Z_0 = 0) = 0.2 \times 0.3 + 0.7 \times 0.2 + 0.6 \times 0.5 = 0.5$
Thus, $P(Z_2 = 2 \mid Z_0 = 0) = 1 - 0.5 - 0.26 = 0.24$.

Expected reward of setting $Z_0 = 0$ is therefore $5 \times 0.26 + 8 \times 0.5 + 10 \times 0.24 = 7.7$
Similarly ,we have $P(Z_2 = v \mid Z_0 = 1) = \Sigma^{2}_{u=0} P(Z_2 = v \mid Z_1 = u) \times P(Z_1 = u \mid Z_0 = 1)$. Hence,

$P(Z_2 = 0 \mid Z_0 = 1) = 0.3 \times 0.1 + 0.1 \times 0.7 + 0.3 \times 0.2 = 0.16$ $P(Z_2 = 1 \mid Z_0 = 1) = 0.2 \times 0.1 + 0.7 \times 0.7 + 0.6 \times 0.2 = 0.63$

Thus, $P(Z_2 = 2 \mid Z_0 = 1) = 1 - 0.16 - 0.63 = 0.21$. Expected reward of setting $Z_0 = 1$ is therefore $5 \times 0.16 + 8 \times 0.63 + 10 \times 0.21 = 7.94$

Last, we have $P(Z_2 = v \mid Z_0 = 2) = \Sigma^{2}_{u=0} P(Z_2 = v \mid Z_1 = u) \times P(Z_1 = u \mid Z_0 = 2)$. Hence, $P(Z_2 = 0 \mid Z_0 = 2) = 0.3 \times 0.3 + 0.1 \times 0.6 + 0.3 \times 0.1 = 0.18$

$P(Z_2 = 1 \mid Z_0 = 2) = 0.2 \times 0.3 + 0.7 \times 0.6 + 0.6 \times 0.1 = 0.54$
Thus, $P(Z_2 = 2 \mid Z_0 = 2) = 1 - 0.18 - 0.54 = 0.28$. Expected reward of setting $Z_0 = 2$ is therefore

$5 \times 0.18 + 8 \times 0.54 + 10 \times 0.28 = 8.02$. Therefore, the best action here is setting $Z_0 = 2$.

(b) [15 pts] Instead of the above 3 actions, suppose we are only given 2 actions **(A)** and **(B)**.

Executing **(A)** changes the prior probabilities over Z_0 to $(P(Z_0 = 0) = 0.1, P(Z_0 = 1) = 0.9)$.

Executing **(B)** changes the prior probabilities over Z_0 to $(P(Z_0 = 0) = 0.3, P(Z_0 = 2) = 0.7)$.

Given the above, compute the expected reward of **(A)** and **(B)**. Which one is better?

Answer :

For action (A) : If we execute action (A), the prior probabilities over Z0 will be (P(Z0 = 0) = 0.1, P(Z0 = 1) = 0.9). The probability of staying in state 0 after one step is 0.3, and the probability of transitioning to state 1 or 2 is 0.2 and 0.5 respectively. Therefore, the probability of ending up in state 0 after two steps is:
P(Z2 = 0 | A) = P(Z1 = 0) * P(Z2 = 0 | Z1 = 0)
= 0.1 * 0.3 = 0.03
The probability of ending up in state 1 after two steps is:

P(Z2 = 1 | A) = P(Z1 = 0) * P(Z2 = 1 | Z1 = 0) + P(Z1 = 1) * P(Z2 = 1 | Z1 = 1)
= 0.1 * 0.2 + 0.9 * 0.3

= 0.29

The probability of ending up in state 2 after two steps is:

P(Z2 = 2 | A) = P(Z1 = 0) * P(Z2 = 2 | Z1 = 0) + P(Z1 = 1) * P(Z2 = 2 | Z1 = 1)
= 0.1 * 0.5 + 0.9 * 0.5 = 0.5

Therefore, the expected reward for action (A) is:

Expected reward = 5 * 0.03 + 8 * 0.29 + 10 * 0.5
= 7.47

For action (B). If we execute action (B), the prior probabilities over Z0 will be (P(Z0 = 0) = 0.3, P(Z0 = 2) = 0.7). The probability of staying in state 0 after one step is 0.3, and the probability of transitioning to state 1 or 2 is 0.2 and 0.5 respectively. Therefore, the probability of ending up in state 0 after two steps is:

P(Z2 = 0 | B) = P(Z1 = 0) * P(Z2 = 0 | Z1 = 0)
= 0.3 * 0.3 = 0.09

The probability of ending up in state 1 after two steps is:

P(Z2 = 1 | B) = P(Z1 = 0) * P(Z2 = 1 | Z1 = 0) + P(Z1 = 1) * P(Z2 = 1 | Z1 = 1)
= 0.3 * 0.2 + 0.7 * 0.3
= 0.27

The probability of ending up in state 2 after two steps is:

P(Z2 = 2 | B) = P(Z1 = 0) * P(Z2 = 2 | Z1 = 0) + P(Z1 = 1) * P(Z2 = 2 | Z1 = 1)
= 0.3 * 0.5 + 0.7 * 0.5
= 0.5

Therefore, the expected reward for action (B) is:

Expected reward = 5 * 0.09 + 8 * 0.27 + 10 * 0.5 = 7.61

Therefore, action (B) is better than action (A) in terms of expected reward.