INSTRUCTIONS

- **Due: Wednesday, 22 Feb 2023 at 23:59 PM PDT** Remember that you have 5 slip days to use at your discretion BUT you can use no more than 1 per homework.
- **Format:** Submit the answer sheet containing your answers in PDF. You can provide either typed or handwritten answers to this homework. Handwritten answers must be legible and scanned into a PDF.
- **Note: Please DO NOT FORGET to include your name and WSU ID in your submission.**
- **How to submit:** Submit a PDF containing your answers on Canvas
- **Policy:** See the course website for homework policies and Academic Integrity.

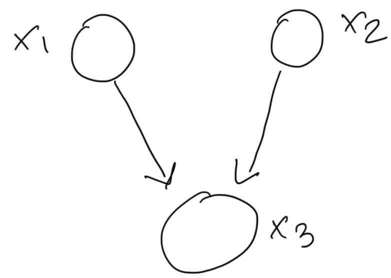
Last Name	Ghosh Dastidar
First Name	Priyanka
WSU ID	11804812

Q1. [30 pts] Bayesian Net: Conditional Independence

Consider the following random variables X_1 and X_2, which represent the outcomes of two independent coin tosses with bias (towards head, i.e. $X = 1$) 0.6. Let X_3 denotes the indicator function of the event that the outcomes are identical. That is, $X_3 = 1$ if $X_1 = X_2$ and $X_3 = 0$ otherwise.

(a) [15 pts] Specify a directed graphical model (Bayesian Network) that describes the joint probability distribution (i.e., draw the Bayesian Network and detail all conditional distributions)

Answer :



$$P(x_1, x_2, x_3) = P(x_1)\,P(x_2)\,P(x_3 | x_1, x_2)$$

The graphical model of Bayesian network is shown above. Conditional distributions are as follows:-

X1⊥X2|X3, i.e., from the graphical presentation above, we can see that X1 is conditionally independent on X2 , ⊥ given X3. It follows Common Effect where there is a head-to-head connection between nodes, connecting to X3.

Given P(x1=1)=0.6 , P(x1=0)=0.4 and P(x2=1)=0.6, P(x2=0)=0.4

Below are the conditional probability distributions for the given Bayesian network:

- P(X3=0 | X1=1, X2=1) = 0
- P(X3=0 | X1=0, X2=0) = 0
- P(X3=0 | X1=0, X2=1) = 1
- P(X3=0 | X1=1, X2=1) = 1
- P(X3=1 | X1=1, X2=1) = 1
- P(X3=1 | X1=0, X2=0) = 1
- P(X3=1 | X1=0, X2=1) = 0
- P(X3=1 | X1=1, X2=0) = 0

(b) [15 pts] We mentioned in class that a distribution that factorizes according to the Bayesian Network (BN) will exhibit all conditional independences (CI) the BN implies. But conversely, the BN does not necessarily capture all CI implied numerically by the distribution. Show that this is case when we set the bias of the coin to 0.5.

Answer :

If the bias of the coin is 0.5, then coin tosses will be independent and distributed identically. Joint distribution can be factored as :

P(x1,x2,x3) = P(x1)P(x2)P(x3|x1,x2)

We know, P(X1) = Bernoulli(0.5), P(X2) = Bernoulli(0.5), and P(X3 | X1, X2) = Bernoulli(1) if X1 = X2, Bernoulli(0) otherwise.

X1 and x2 are independent, so by product of their marginal distribution :

P(x1,x2) = P(x1)P(x2)

But since, x3 is a function of x1 and x2, X3 provides a direct probabilistic influence between X1 and X2, which is not captured by the factorization of the joint distribution.

Hence, Bayesian network does not capture conditional independence given by joint distribution when the bias is 0.5, so there happen to be a conditional independence which is not captured by Bayesian network, although it's present in joint distribution.

Q2. [20 pts] Bayesian Net: Conditional Independence

Consider the Bayesian Net shown in Fig 1, which depicts the relationships among variables associated with chest abnormality. Answer the following questions based on the graphical model.

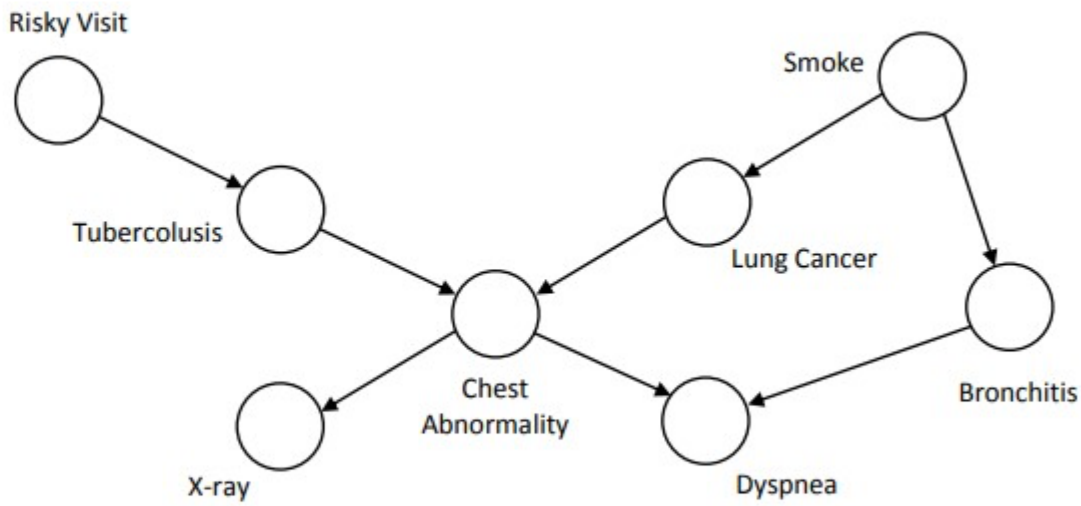
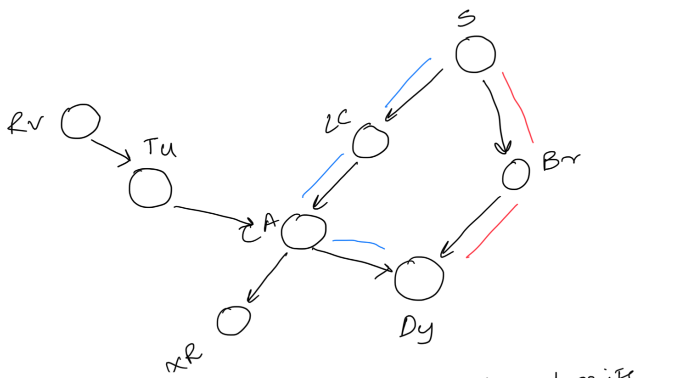


Figure 1: Bayesian Net for Chest Abnormality.

(a) [5 pts] Is (Smoke ⊥ Dyspnea | Bronchitis) True or False? Why?

Answer :

Considering smoke as S, Bronchitis as Br, Lung Cancer as Lc, Dyspnea as Dy, Chest Abnormality as CA, X-ray as Xr, Risky Visit as Rv, Tubercolusis Tu

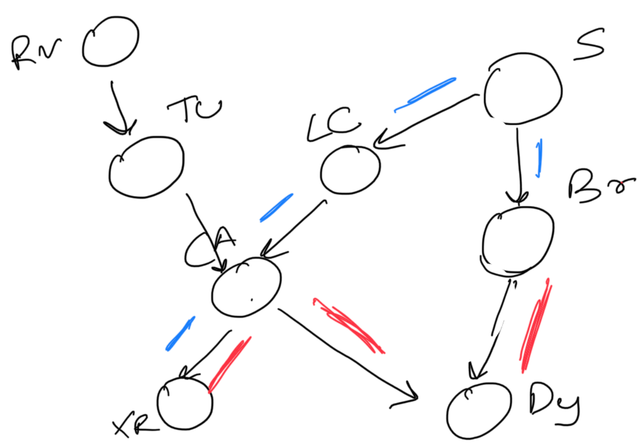


Path marked with Red is blocked as its
causal chain, with Br - dseparating S and
Dy
Path marked in Blue is not blocked,
Hence the statement is false

Here, Bronchitis blocks one path from smoke and is directed towards dyspnea. But the alternate path is not blocked. Hence, the statement is false.

(b) [5 pts] Is (Bronchitis ⊥ X − ray | Lung Cancer) True or False? Why?

Answer :

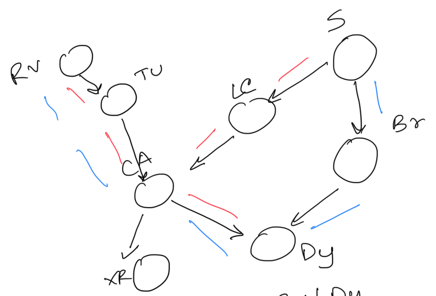


The statement is false because there are no blocked paths. From, bronchitis we go to x-ray via smoke, Lung cancer and chest abnormality, which has a blocked path. But the alternate path marked in red doesn’t have a blocked path, hence it’s not d-separated

(c) [5 pts] Is Smoke ⊥ RiskyVisit | Dyspnea True or False? Why?

Answer :

Given statement is false. Smoke and Risky Visit are dependent . From the Graph, it's clear that Chest Abnormality is descendants of smoke and Risky Visit. Chest Abnormality leads to X-Ray and there is Dyspnea in the path.

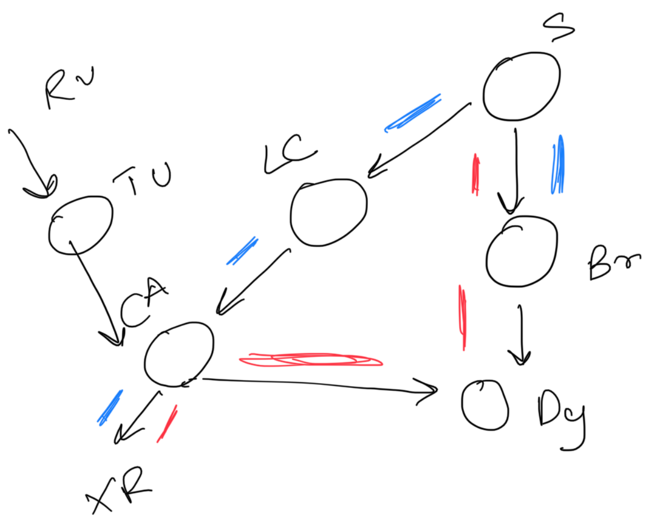


$S \perp RV \mid Dy$

Path marked in Red and Blue are not blocked, hence the statement is false

(d) [5 pts] Is X − ray ⊥ Smoke | {Cancer, Bronchitis} True or False? Why?

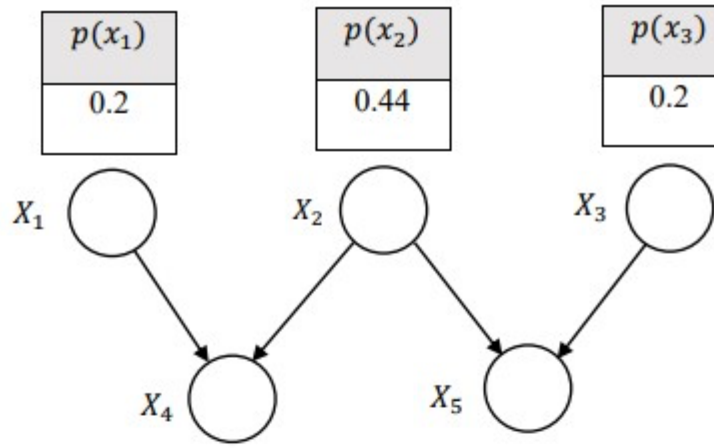
Answer :-



Statement is true. From the graph we can see, path through lung cancer and Bronchitis are blocked. This is evidential trail that caused it.

Q3. [25 pts] Bayesian Net: Inference

Given the Bayesian Net in Fig. 2, where each random variable is binary, i.e. $x_i \in$ {*True, False*}. Compute the distribution tables for the following probabilities. Show step-by-step derivations clearly in each case. Provide answers accurate to 4 decimal places.



X_1	X_2	$p(x_4 \mid x_1, x_2)$
T	*T*	0.35
T	*F*	0.6
F	*T*	0.01
F	*F*	0.95

X_2	X_3	$p(x_5 \mid x_2, x_3)$
T	*T*	0.35
T	*F*	0.6
F	*T*	0.01
F	*F*	0.95

Figure 2: Bayesian Net with Conditional Probability Tables. All probabilities in the above 5 tables are given with respect to the probability of being True given the corresponding configuration of its parents. For example, first row of the table for *X*4 reads *P*(*X*4 = *T* | *X*1 = *T*,*X*2 = *T*) = 0.35. First row of the table for *X*2 reads *P*(*X*2 = *T*) = 0.44. Note that, the probabilities of being False and being True always sum up to 1. So, knowing the probability of being True is sufficient to derive the probability of being False.

(a) [5 pts] (a) $P(x_1 \mid x_5)$

We consider True as T and False as F

Answer :

Using Bayes theorem, we can state that:
P(x1|x5) = P(x5|x1) * P(x1) / P(x5)

Using the chain rule of conditional probability, we get:
P(x5|x1) = Σx2 Σx3 P(x5|x2,x3) * P(x2,x3|x1)

Simplifying the above equation as follows:

P(x5|x1) = Σx2 Σx3 P(x5|x2,x3) * P(x2|x1) * P(x3|x1)

Computing the probability distribution table for P(x2,x3|x1):
For, P(x2=T,x3=T|x1=T) = P(x2=T|x1=T) P(x3=T|x1=T) = 0.44 x 0.2 = 0.088
P(x2=T,x3=F|x1=T) = P(x2=T|x1=T) P(x3=F|x1=T) = 0.44 x 0.8 = 0.352
P(x2=F,x3=T|x1=T) = P(x2=F|x1=T) P(x3=T|x1=T) = 0.56 x 0.2 = 0.112
P(x2=F,x3=F|x1=T) = P(x2=F|x1=T) P(x3=F|x1=T) = 0.56 x 0.8 = 0.448
P(x2=T,x3=T|x1=F) = P(x2=T|x1=F) P(x3=T|x1=F) = 0.44 x 0.2 = 0.088
P(x2=T,x3=F|x1=F) = P(x2=T|x1=F) P(x3=F|x1=F) = 0.44 x 0.8 = 0.352
P(x2=F,x3=T|x1=F) = P(x2=F|x1=F) P(x3=T|x1=F) = 0.56 x 0.01 = 0.0056
P(x2=F,x3=F|x1=F) = P(x2=F|x1=F) P(x3=F|x1=F) = 0.56 x 0.99 = 0.5544

Computing the probability distribution table for P(x5|x2,x3):

X2	X3	P(x5=T|x2,x3)
T	T	0.35
T	F	0.6
F	T	0.01
F	F	0.95

The above tables are used to compute P(x1|x5) are as follows:
P(x5=T|x1=T) = 0.088 x 0.35 + 0.352 x 0.6 = 0.242
P(x5=T|x1=F) = 0.0056 x 0.01 + 0.5544 x 0.95 = 0.5267
P(x5=T) = P(x5=T|x1=T) P(x1=T) + P(x5=T|x1=F) x P(x1=F) = 0.242 x 0.2 + 0.5267 x 0.8 = 0.4698

P(x1| x5) is computed as follows:
P(x1=T|x5) = P(x5| x1=T) P(x1=T) / P(x5) = 0.242 x 0.2 / 0.4698 = 0.1031
P(x1=F|x5) = P(x5| x1=F) P(x1=F) / P(x5) = 0.5267 x 0.8 / 0.4698 = 0.8969

Therefore, the probability distribution table for P(x1|x5) is as follows:

x1	**P(x1| x5)**
True	0.1031
False	0.8969

(b) [5 pts] (b) $P(x_2 \mid x_4)$

Answer:

Using Bayes' rule as follows:
P(x2|x4) = P(x4|x2) * P(x2) / P(x4)

Computing P(x4), using the law of total probability:
P(x4) = P(x4|x1=T, x2=T) * P(x1=T) * P(x2=T) + P(x4|x1=T, x2=F) * P(x1=T) * P(x2=F) + P(x4|x1=F, x2=T) * P(x1=F) * P(x2=T) + P(x4|x1=F, x2=F) * P(x1=F) * P(x2=F)

P(x4) = (0.35 * 0.2 * 0.44) + (0.6 * 0.2 * (1-0.44)) + (0.01 * (1-0.2) * 0.44) + (0.95 * (1-0.2) * (1-0.44))
P(x4) = 0.0308 + 0.0672 + 0.00352 + 0.4256
P(x4) = 0.52712

Computing P(x4|x2) using the law of total probability:
P(x4|x2) = P(x4|x1=T, x2) * P(x1=T) + P(x4|x1=F, x2) * P(x1=F)
P(x4| x2=T) = 0.35 * 0.2 + 0.01 * (1-0.2) = 0.078
P(x4| x2=F) = 0.6 * 0.2 + 0.95 * (1-0.2) = 0.88

Computing P(x2|x4) using Bayes' rule:
P(x2=T|x4) = P(x4|x2=T) * P(x2=T) / P(x4)
= 0.078 * 0.44 / 0.52712 = 0.0651

P(x2=F|x4) = P(x4|x2=F) * P(x2=F) / P(x4)
= 0.88 * (1-0.44) / 0.52712 = 0.9349
Therefore, the distribution table for P(x2|x4) goes as follows :

X2	P(x2| x4)
True	0.0651
False	0.9349

(c) [5 pts] (c) $P(x_3 \mid x_2)$

Answer :
To compute P(x3 | x2), Bayes' theorem is used based on the conditional probability tables for x2, x3, and the dependencies between them.

Computing the joint probability distribution for x2 and x3 using the probabilities given:
x2=T,x3=T => P(x2, x3)=0.088
x2=T,x3=F => P(x2,x3)=0.352
x2=F,x3=T => P(x2,x3)=0.032
x2=F,x3=F => P(x2,x3)=0.528

Using the given values, and Bayes' theorem to compute P(x3 | x2) by multiplying the joint probability table by the conditional probabilities and normalizing:

P(x2, x3) * P(x5=T|x2,x3) / P(x2) * P(x4=T|x1,x2) * P(x5=T|x2,x3) * P(x3)

P(x2, x3) * P(x5=T|x2,x3):

X2	X3	P(x2, x3) * P(x5=T|x2,x3))
T	T	0.088 * 0.35 =0.0308
T	F	0.352 * 0.6 = 0.2112
F	T	0.032 * 0.01 = 0.00032
F	F	0.528*0.95 = 0.5016

P(x2) * P(x4=T|x1,x2) * P(x5=T|x2,x3) * P(x3):

(0.44 * 0.35 * 0.35 * 0.088) + (0.44 * 0.6 * 0.35 * 0.352) + (0.56 * 0.01 * 0.01 * 0.032) + (0.56 * 0.95 * 0.01 * 0.528) = 0.0735
=0.0047432 + 0.0325248 + 0.000001792 + 0.00280896

=0.040078752
=0.0401

Finally, we can use Bayes' theorem to compute P(x3 | x2) by multiplying the joint probability table by the conditional probabilities and normalizing it:
P(x2, x3) * P(x5=T|x2,x3) / P(x2) * P(x4=T|x1,x2) * P(x5=T|x2,x3) * P(x3)

To obtain it, we need to use Bayes' theorem and the probabilities given in the problem. Recall that Bayes' theorem states:

P(A | B) = P(B | A) * P(A) / P(B)

To calculate P(x3 | x2), which is the probability of x3 being true given that x2 is true. Using Bayes' theorem, it is given as:
P(x3 | x2) = P(x2 | x3) * P(x3) / P(x2)

Given the values of P(x2), P(x3), and the conditional probabilities of x2 given x3. We can substitute these values into the formula to obtain the conditional probabilities of x3 given x2.

X2	X3	P(x3 | x2)
T	T	P(x2=T | x3=T) * P(x3=T) / P(x2=T)= 0.088*0.2/0.44 =0.04
T	F	P(x2=T | x3=F) * P(x3=F) / P(x2=T)= 0.352*0.8/0.44=0.64
F	T	P(x2=F | x3=T) * P(x3=T) / P(x2=F)= 0.032*0.2/0.56= 0.0114
F	F	P(x2=F | x3=F) * P(x3=F) / P(x2=F)= 0.528*0.8/0.56=0.7542

Therefore, P(x3 | x2) is:
P(x3=T | x2=T) = 0.04 / (0.04 + 0.64)=0.0588
P(x3=F | x2=T) = 0.64 / (0.04 + 0.64)=0.9412
P(x3=T | x2=F) = 0.0114 / (0.0114 + 0.7542)=0.0149
P(x3=F | x2=F) = 0.7542 / (0.0114 + 0.7542)=0.9851

(d) [5 pts] (d) $P(x_4 \mid x_3)$

Answer:

To compute the distribution table for P(x4|x3), using the Bayes' rule:
P(x4|x3) = P(x3|x4)P(x4) / P(x3)

We can compute the joint probability P(x3,x4) using the chain rule for Bayesian networks:
P(x3,x4) = P(x4|x1,x2) P(x1) P(x2) P(x3|x2)

Since x1, x2, and x3 are independent, we have:
P(x1) P(x2) P(x3|x2) = P(x1) P(x2) P(x3)

Therefore, we simplify the expression for P(x3,x4) to:
P(x3,x4) = P(x4|x1,x2) P(x1) P(x2) P(x3)

Now we substitute this expression into the equation for P(x4|x3) to get:
P(x4|x3) = P(x4|x1,x2) P(x1) P(x2) / P(x3)

We can compute P(x4|x1,x2) using the conditional probabilities given in the problem:
P(x4=T|x1=T,x2=T) = 0.35
P(x4=T|x1=T,x2=F) = 0.6
P(x4=T|x1=F,x2=T) = 0.01
P(x4=T|x1=F,x2=F) = 0.95

To compute P(x3), we use the law of total probability:
P(x3) = P(x3|x2=T) P(x2=T) + P(x3|x2=F) P(x2=F)

We can compute P(x3|x2=T) and P(x3|x2=F) using the conditional probabilities given in the problem:

P(x3=T|x2=T) = 0.2
P(x3=T|x2=F) = 0.2

The above values are since P(x3) is independent on x2, the values are similar.

We also know that:
P(x2=T) = 0.44
P(x2=F) = 1 - P(x2=T) = 0.56

Putting it all together, we get:
P(x4=T|x3=T) = (0.35)(0.2)(0.44) / [(0.2)(0.44) + (0.2)(0.56)] = 0.154
P(x4=F|x3=T) = 1 - P(x4=T|x3=T) = 0.846
P(x4=T|x3=F) = (0.6)(0.2)(0.56) / [(0.2)(0.44) + (0.2)(0.56)] = 0.336
P(x4=F|x3=F) = 1 - P(x4=T|x3=F) = 0.664

Therefore, the table for P(x4|x3) is:

X3	X4=T	X4=F
T	0.154	0.846
F	0.336	0.664

(e) [5 pts] (e) $P(x_5)$

Answer:

The total probability rule, which asserts that the probability of an event may be estimated by adding the probabilities of all conceivable events that lead to that event, must be used to generate the distribution table for P(x5).

With all potential values for x2 and x3, we can determine the likelihood that x5 is true in this situation by using the conditional probabilities provided in the problem. Namely, we have
P(x5=T|x2=T,x3=T) = 0.35

P(x5=T|x2=T,x3=F) = 0.6
P(x5=T|x2=F,x3=T) = 0.01
P(x5=T|x2=F,x3=F) = 0.95

We also know the probabilities of x2 and x3 being true, which are given as:
P(x2=T) = 0.44
P(x3=T) = 0.2

Using the total probability rule, we can calculate the probability of x5 being true as follows:
P(x5=T) = P(x5=T|x2=T,x3=T) * P(x2=T) * P(x3=T) + P(x5=T|x2=T,x3=F) * P(x2=T) * P(x3=F) + P(x5=T|x2=F,x3=T) * P(x2=F) * P(x3=T) + P(x5=T|x2=F,x3=F) * P(x2=F) * P(x3=F)

Substituting the values given in the problem, we get:
P(x5=T) = (0.35 * 0.44 * 0.2) + (0.6 * 0.44 * 0.8) + (0.01 * 0.56 * 0.2) + (0.95 * 0.56 * 0.8)
=0.0308 + 0.2112 + 0.00112 + 0.4256 =0.66872

Simplifying, we get:
P(x5=T) = 0.6687

Therefore, the table for P(x5) is:

X5	Value
T	0.6687
F	0.3313

Q4. [25 pts] Bayesian Net: Inference

Fig. 3 shows a Bayesian Net characterizing the causal relationship between 5 random variables x_1, x_2, x_3, x_4 and x_5 where $x_1, x_2, x_4 \in \{0,1\}$ while $x_3, x_5 \in \{0,1,2\}$. Provide answers accurate to 4 decimal places.

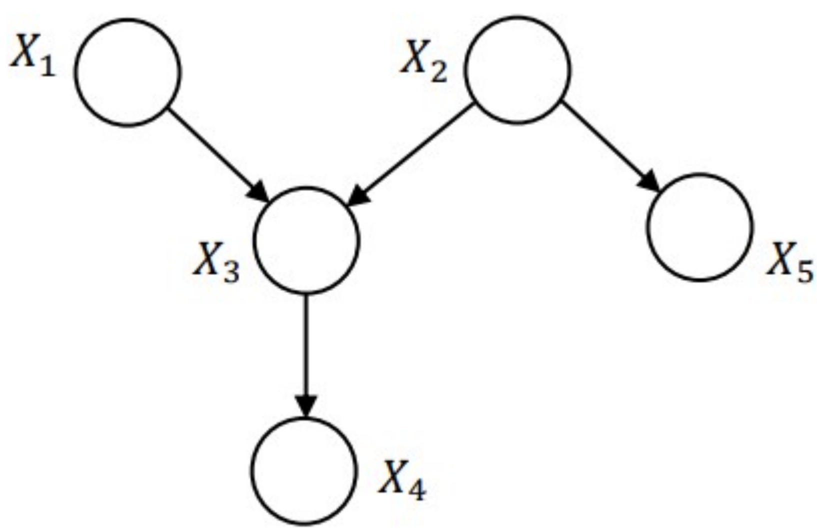


Figure 3: Bayesian Net.

X_1	X_2	X_3	$p(x_3 \vert x_1, x_2)$
0	0	0	0.3
0	0	1	0.4
0	1	0	0.9
0	1	1	0.08
1	0	0	0.05
1	0	1	0.25
1	1	0	0.5
1	1	1	0.3

X_1	$p(x_1)$
0	0.6

X_2	$p(x_2)$
0	0.7

X_3	X_4	$p(x_4 \vert x_3)$
0	0	0.1
1	0	0.4
2	0	0.99

Figure 4: Conditional Probability Tables.

(a) [5 pts] Write down all the conditional independences given by the Bayesian Net.

Answer :

The conditional independence provided in the Bayesian net are as follows:

- x1 ⊥ x4 | x3
- x4 ⊥ x5 | x3
- x3 ⊥ x5 | x2
- x2 ⊥ x4 | x3
- x1 ⊥ x5 | x2
- x4 ⊥ x5 | x2
- x3 ⊥ x5 | {x2, x1}
- x3 ⊥ x5 | {x2, x4}
- x3 ⊥ x5 | {x2, x1, x4}
- x4 ⊥ x5 | {x2, x1}
- x4 ⊥ x5 | {x2, x1, x3}
- x1 ⊥ x5 | {x2, x4}
- x1 ⊥ x4 | {x3, x5}
- x2 ⊥ x4 | {x3, x5}

(b) Write down the factorized expression of the joint probability given by the Bayesian Net.

Answer :

P(x1, x2, x3, x4, x5) = P(x1) * P(x2) * P(x3|x1, x2) * P(x4|x3) * P(x5|x2)

(c) [15 pts] Fig. 4 gives the probability tables of the Bayesian Net. Find the conditional probability table for $p(x_1 \mid x_3 = 1, x_2)$. Show step-by-step derivation clearly.

Answer:

Conditional probability table is given as follows:

p(x1 | x3 = 1, x2) = p(x1,x2,x3=1)/p(x3=1,x2)

p(x1,x2,x3=1) = Σx4 Σx5 p(x1) p(x2) p(x3=1|x1,x2) p(x4|x3=1) p(x5|x2)

= p(x1) p(x2) p(x3=1|x1,x2) Σx4 p(x4|x3=1) Σx5 p(x5|x2)

= p(x1)p(x2)p(x3=1|x1,x2) Σx4 p(x1|x3=1)

= p(x1) p(x2) p(x3=1|x1,x2)

X1	X2	X3	p(x1) p(x2) p(x3=1|x1,x2)
0	0	1	0.6*0.7*0.4 = 0.168
0	1	1	0.6*0.3*0.08 = 0.0144
1	0	1	0.4*0.7*0.25 = 0.07
1	1	1	0.4*0.3*0.3=0.036

P(x2,x3=1)= Σx1 P(x1,x2,x3=1)

=p(x1=0,x3,x3=1) + p(x1=1,x3,x3=1)

Now we have, P(x2=0, x3=1)=0.238 and

P(x2=1,x3=1)=0.0504

$P(x1=0|x3=1,x2=0) = P(x1=0,x2=0,x3=1)/P(x2=0,x3=1)$

$= 0.168/0.238=0.7059$

$P(x1=1|x3=1,x2=0) = 1- P(x1=0|x3=1,x2=0)$

$= 1- 0.7059 =0.2941$

$P(x1=0|x3=1,x2=1) = p(x1=0,x2=1,x3=1) /p(x2=1,x3=1)=0.0144/0.0504$

$=0.2857$

$P(x1=1|x3=1,x2=1)= 1- P(x1=0|x3=1,x2=1)$

$= 1-0.2857$

$=0.7143$