Relations:-

A relation is mostly suitable in Comparing the objects which are related to one another.

Eg: father to Son, mother to son, less than, greater than, Parents and child.

Def. of Relation:

Let A, B are two sets, a subset of $A \times B$ is Called a relation from a set A to B.

If $(a,b) \in R$ and $R \subseteq A \times B$ then a is related to b by R i.e., aRb.

* If A and B are equal, then we say that $R \subseteq A \times A$ is a relation on A.

Eg: Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$.

Then $A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$.

Let $R = \{(a,1), (b,1), (b,2), (c,3)\} \subseteq A \times B$. is a relation from A to B.

Range and Domain of a Relation:-

Let R be a relation from A to B then

Domain of R:- It is ~~bu~~ denoted by DOM(R) and is defined as

$DOM(R) = \{ a / a \in A, (a, b) \in R \text{ for some } b \in B \}$

Range of R:- It is denoted by RAN(R) and is defined as

$RAN(R) = \{ b / b \in B, (a, b) \in R \text{ for some } a \in A \}$.

Eg. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and R is a relation from A to B then $DOM(R) = \{a, b, c\}$

$RAN(R) = \{1, 2, 3\}$.

Inverse of relation:-

Let R be a relation from A to B the inverse of relation R from B to A is denoted by R^{-1} and it is defined as $R^{-1} = \{ (b, a) / (a, b) \in R \}$.

Eg. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$.

$R = \{ (a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3) \}$.

Then $R^{-1} = \{ (1,a), (2,a), (3,a), (1,b), (2,b), (3,b), (1,c), (2,c), (3,c) \}$.

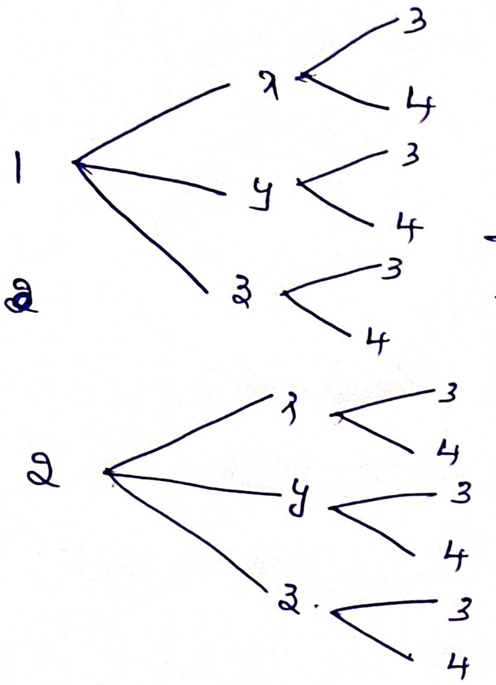
-x-

~~Proper~~

Recursive function: A function which calls itself is Called a recursive function (And the process of Calling a function by itself Called a recursion.)

Eg: Let $A=\{1,2\}$ $B=\{x,y,z\}$, $C=\{3,4\}$ then find $A\times B\times C$.

Sol:



$(1,x,3)$

$(1,x,4)$

$(1,y,3)$

$(1,y,4)$

$(1,z,3)$

$(1,z,4)$

$(2,x,3)$

$(2,x,4)$

$(2,y,3)$

$(2,y,4)$

$(2,z,3)$

$(2,z,4)$.

Asymmetric Relation:- A relation R defined on a set X. The relation R is said to be asymmetric if xRy, then $y \not R x$ (or) $(x,y) \in R \Rightarrow (y,x) \notin R$ for all $x,y \in X$.

Recursive function:- A function which calls itself is called a recursive function(And the process of calling a function by itself is called a recursion).

Universal relation:- A relation R on a set A is called universal if $R = A \times A$.

Eg: If $A = \{1,2,3\}$ then $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

Void Relation: A relation R on a set A is void relation if R is the null set ϕ.

Eg: If $A = \{3,4,5\}$ then $R = \{a+b>10\}$ is a null set.

Identity Relation: A relation R on a set A is identity relation if $R = \{(a,a) / a \in A\}$ and it is denoted by I_A.

Eg: If $A = \{1,2,3\}$ then $R = \{(1,1), (2,2), (3,3)\}$.

Inverse Relation:- If R is any relation from a set 'A' to set 'B' then the inverse of R is denoted by R^{-1}.

$R^{-1} = \{(y,x) / y \in B, x \in A, (x,y) \in R\}$.

Properties of Relation:-

① **Reflexive Relation:-** A relation R in a set X is reflexive if, for every $x \in X$, xRx, i.e., $(x,x) \in R$

Eg: The relation $R = \{(a,a), (b,b)\}$ on $X = \{a,b\}$ is reflexive.

② **Symmetric Relation:-** A relation R in a set X is symmetric if, for every a and b in X, whenever aRb then bRa, $\forall a, b \in X$.

Eg: $R = \{(1,2), (2,1), (3,2), (2,3)\}$ on $A = \{1,2,3\}$ is symmetric.

③ **Transitive Relation:-** A relation R in a set X is transitive if, for every a, b and c in X, whenever aRb and bRc then aRc.

Eg: $R = \{(1,2), (2,3), (1,3)\}$ on $A = \{1,2,3\}$ is transitive.

Irreflexive:- A relation R in a set X is irreflexive if, for every $a \in X$, $(a,a) \notin X$.

Antisymmetric: A relation R in a set X is antisymmetric, if, for every x and y in X, whenever xRy and yRx, then $x = y$.

Prob: Given $S = \{1,2,3, \ldots 10\}$, and a relation R on S. where $R = \{(x,y) \mid x+y = 10\}$ then find the property of relation R.

Sol: $R = \{(1,9), (2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2), (9,1)\}$.

$\therefore R$ satisfies symmetric property

Eg: If $A = \{2, 3, 5\}$, $B = \{6, 8, 10\}$ ~~then~~ and $R = \{(2,6), (2,8), (2,10), (3,6), (5,10)\}$.

Then $R^{-1} = \{(6,2), (8,2), (10,2), (6,3), (10,5)\}$.

Prob: If $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3)\}$ then find if relation is reflexive, Symmetric and Transitive.

Sol: i) Reflexive :- $R \cup \{(2,2), (4,4)\}$.

$\therefore R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3), (2,2), (4,4)\}$.

ii) Symmetric : ~~$R \cup \{(2,2), (4,4)\}$~~. $R \cup \{(4,2), (3,4)\}$.

$\therefore R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3), (4,2), (3,4)\}$

iii) Transitive : $R \cup \{(2,3), (4,1)\}$.

$\therefore R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3), (2,3), (4,1)\}$.

Relation Matrix and Diagraph:-

A relation can be represented in 2 ways.

① Relation Matrix ② Diagraph of a Relation.

① Relation Matrix:-

If $A=\{a_1, a_2, \ldots a_m\}$ and $B=\{b_1, b_2, \ldots b_n\}$ are finite sets containing m, n elements and 'R' is a relation from A to B, then we can represent R by $m \times n$ (read it as m by n) matrix called as "Relation Matrix" denoted by M_R.

i.e, $M_R = [M_{ij}]$, where $M_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$

Eg: If $A=\{1,2,3,4\}$ and $B=\{a,b,c,d\}$ then then relation 'R' from A to B is given by $R=\{(1,a),(1,b),(1,c),(2,b),(2,c),(2,d)\}$ then find the matrix relation of R.

Sol: Given $R=\{(1,a),(1,b),(1,c),(2,b),(2,c),(2,d)\}$.

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Eg: Let $A=\{1,2,3\}$, $B=\{x,y\}$ and also relation $R=\{(1,x),(1,y),(3,y)\}$ then find the matrix of relation.

Sol: $$M_R = \begin{array}{c|c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Eg: Let $A=\{1,2,3,4\}$ find relation R on A determine by matrix $M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

Sol: $R = \{(1,1), (1,3), (2,3), (3,1), (4,1), (4,2), (4,4)\}$.

Eg: Let $A = \{1,2,3\}$ and find R on A and determine the matrix and also find the properties of a Relation.

$$M_R = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol: $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (3,2)\}$.

The above relation is reflexive, symmetric and Transitive.

Note: Let R be a relation defined on $X = \{a_1, a_2, \ldots a_n\}$ be the matrix of the relation 'R' and $M_R = [a_{ij}]_{n \times n}$

Reflexive: A relation is reflexive if all the diagonal elements are one. (1)

Note: If all diagonal elements are zero then it is irreflexive.

Symmetric: A relation is symmetric if $a_{ij} = 1$, then $a_{ji} = 1$ for all i, j.

Note: In a matrix if $a_{ij} = 1$, $a_{ji} = 0$, then it is ~~and~~ not symmetric.

Transitive: A relation is transitive if $a_{ij} = 1$ and $a_{jk} = 1$ then $a_{ik} = 1$ for all i, j, k.

② **Diagraph of a Relation** :- A relation can be represented pictorially by drawing its diagraph as follows.

① A small circle is drawn for each element of set A, marked with corresponding element name. These circles are called vertices.

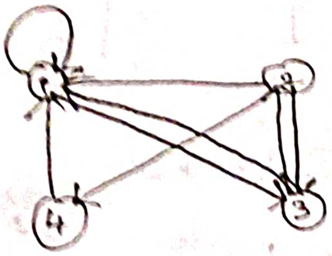
② An arrow is drawn from vertex a_i to vertex a_j if and only if $a_i R a_j$ called as an edge (directed).

③ An element of the form (a, a) in relation corresponds to a directed edge from 'a' to 'a' such an edge is called "loop".

4. ~~First~~ Pictorial representation of relation R is Called "Directed graph of Relation R" (or) "Diagraph of Relation R".

Eg: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$ be a relation defined on A. Draw the diagraph of relation R.

Sol: $A = \{1, 2, 3, 4\}$



Diagraph of Relation R.

② Draw the graph for following relation

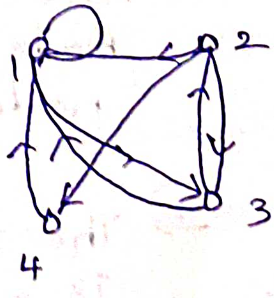
① $R = \{(1,1), (2,2), (1,2)\}$ on $X = \{1, 2\}$

② $S = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$ on $Y = \{1, 2, 3, 4\}$.

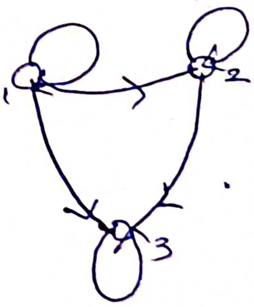
③ $R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$ on $Y = \{1, 2, 3\}$

Sol: ①

②



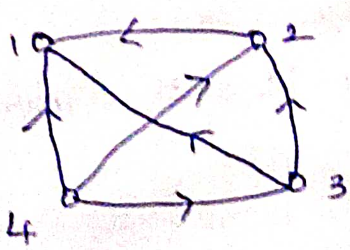
③



③ Let $X = \{1, 2, 3, 4\}$ and $R = \{(x,y) / x > y\}$. Draw the graph of R and also give its matrix on the relation.

Sol: $R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.

Sol:



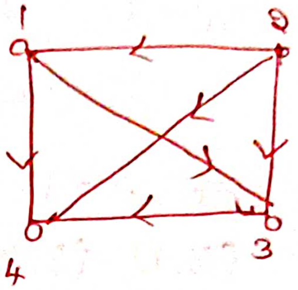
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Eg: Find the relation matrix of the following directed graph.

(a)

(b)

(c)



Sol: (a)
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

(c)
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

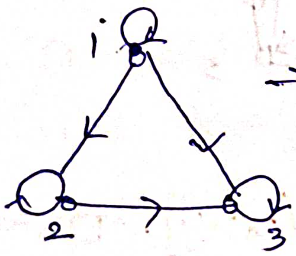
Properties of Diagraph

~~Note~~ **Reflexive:** A loop at each node.

Irreflexive: No loop at any node.

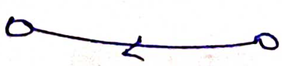
Symmetric: If one node is connected to another, then there is return arc from second node to first node.

Transitive: A relation is transitive iff for an edge from an node 'a' to another node 'b' and an edge from 'b' to node 'c', there is an edge from node 'a' to node 'c'.



→ The graph given in figure, is reflexive and transitive but not symmetric.

Antisymmetric: No direct path exists from one node to another node.



It is anti-symmetric.

Composition of two Relations:-

Let R be a relation from A to B and S be a relation from B to C. Then the Composite relation of R and S is the relation consisting of order pair (a, c), where $(a, c) \in ROS$, $a \in A$, $c \in C$ and for which there exist an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Eg: Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 9, 10\}$, $C = \{5, 6, 7, 8\}$.

$R = \{(1,1), (1,3), (2,9), (2,10), (3,3), (4,10)\}$.

$S = \{(1,5), (3,7), (9,7), (10,8)\}$. Find ROS and its relation matrix.

Sol: $ROS = \{(1,5), (1,7), (2,7), (2,8), (3,7), (4,8)\}$.

$$M_{ROS} = \begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Eg: Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 9, 10\}$, $C = \{5, 6, 7, 8\}$,

$R = \{(1,1), (1,3), (2,9), (2,10), (3,3), (4,10)\}$ and

$S = \{(1,5), (3,7), (9,7), (10,8)\}$. Find ~~ROR~~ SOR, SOS, ROS and ROR^T.

Sol: ~~ROR = {~~

$SOR = \{ \ \} = \phi$.

$SOS = \{ \ \} = \phi$

$ROS = \{(1,3), (1,3)\}\{ = \{1, 3\}$.

$ROR^T = \{(1,1), (1,3), (2,9), (2,10), (3,3), (4,10)\} \circ \{(1,1), (3,1), (9,2), (10,2), (3,3), (10,4)\}$
$= \{(1,1), (2,2), (3,3), (4,4)\}$.

Congruence Relation:- Congruence relation is an example of equivalence relation. Let a and b are two integers and r be the fixed positive integer then 'a' and 'b' are said to be congruent modulo 'r' is denoted by $a \equiv b \pmod{r}$. if r divides $(a-b)$.

Eg: $83 \equiv 13 \pmod{5}$.
Since $83-13=70$ is divisible by 5.

Eg. P.T the relation "congruence modulo m" over the set of positive integers is an equivalence relation.

Sol: Let $x, y \in N$, $x \equiv y \pmod{m}$ iff $x-y$ is divisible by m.

i) Let $x, y, z \in N$ then

$x - x = 0 \cdot m \Rightarrow x \equiv x \pmod{m}$. for all $x \in N$.

ii) Let $x \equiv y \pmod{m}$. Then $x-y$ is divisible by m.

$\Rightarrow -(x-y) = y-x$ is also divisible by m.

i.e., $y \equiv x \pmod{m}$.

$\therefore$ The relation $\equiv$ is symmetric.

iii) Let $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$.
i.e., $(x-y)$ and $(y-z)$ are divisible by 'm'

Now $(x-\cancel{y}) + (\cancel{y}-z)$ is divisible by m.

$\Rightarrow x-z$ is divisible by m.

$\Rightarrow x \equiv z \pmod{m}$.

$\Rightarrow$ The relation is transitive.

$\therefore \equiv$ is an equivalence relation.

Equivalence Relation:- A relation R in a set X is called an equivalence relation if it is reflexive, symmetric and transitive.

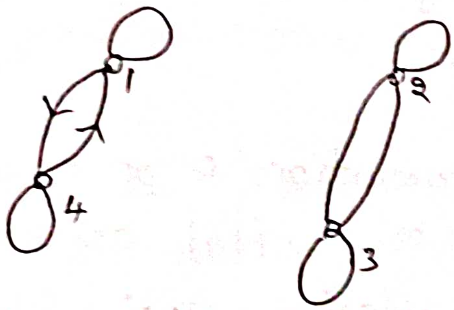
Eg: * Equality of subsets of a universal set

* Relation of lines being parallel on a set of lines in a plane.

* Similarity of triangles on the set of triangles.

Eg: Let $X = \{1,2,3,4\}$ and $R = \{(1,1),(1,4),(4,1),(4,4),(2,2),(2,3),(3,2),(3,3)\}$.
Write the matrix of R and sketch its graph.

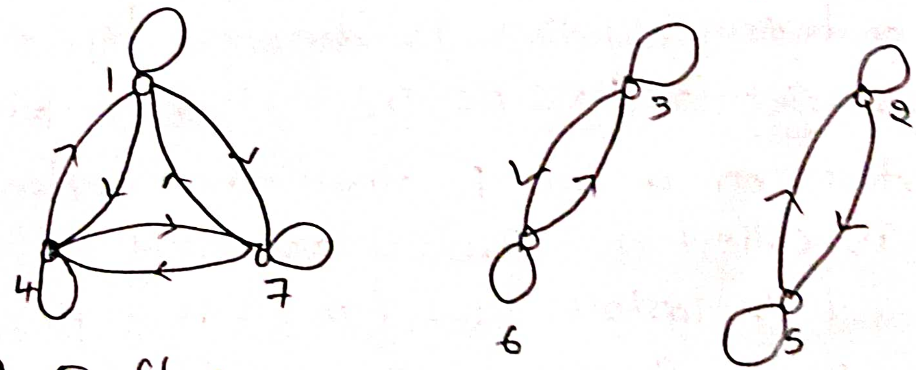
Sol: $X = \{1,2,3,4\}$. $R = \{(1,1),(1,4),(4,1),(4,4),(2,2),(2,3),(3,2),(3,3)\}$.
R is an equivalence relation.



$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

② Let $X = \{1,2,3,4,5,6,7\}$ and $R = \{(x,y) / x-y \text{ is divisible by } 3\}$
S.T. R is an equivalence relation. Draw the graph of R.

Sol:



① Reflexive: $\forall a \in X$, $a-a$ is divisible by 3.
Hence aRa.

② Symmetric: ~~for any~~ $a, b \in X$, if $a-b$ is divisible by 3, then $b-a$ is also divisible by 3.
Hence $aRb \Rightarrow bRa$.

③ Transitive: for any $a, b, c \in X$, if aRb, bRc, then both $a-b$ and $b-c$ are divisible by 3, so that $a-c = (a-b)+(b-c)$ is also divisible by 3,

Hence aRc.

Thus the relation is reflexive, Symmetric and Transitive and hence it is an equivalence relation.

Compatibility Relation: A relation R in a set X is called an Compatibility relation if it is reflexive, and Symmetric. It is denoted by $\approx$.

* All equivalence relations are compatibility relations.

Eg: Let $X = \{$ball, bed, dog, let, egg$\}$,

$R = \{ (x, y) / x, y \in X \wedge xRy$ if x and y contain some common letter.$\}$.

Partial ordering Relation:- A relation 'R' on a set 'p' is said to be partial ordering (or) partial ordering relation on if and only if R is Reflexive, Anti-Symmetric and Transitive.

* Partial ordering relation is denoted by "$\leq$".

Partial ordered set (or) POSET:- If "$\leq$" is a partial ordering relation on a set 'p' then the ordered pair $(P, \leq)$ is called a "Partial ordered set" or "POSET."

Totally ordered Relation:- Let $(P, \leq)$ is a partially ordered set. If for every two elements $a, b \in P$, we have either $a \leq b$ or $b \leq a$ (comparable), then "$\leq$" is called a simple ordering (or) linear ordering on 'p' and $(P, \leq)$ is called a "Totally ordered set" (or) "Simply ordered set" (or) "a chain".

Eg: S.T. the relation "greater than or equal to" is a partially ordering relation on a set of integers.

Sol: Let $Z = \{ \cdots -3, -2, -1, 0, 1, 2, 3, \cdots \}$.

To prove "$\geq$" is an partially ordering relation

① Reflexive: $\forall a \in Z$, aRa

$\Rightarrow a \geq a$.

$\Rightarrow$ "$\geq$" is Reflexive.

② Anti-Symmetric:

for any $a, b \in Z$

$\boxed{\text{If } aRb,\ bRa \text{ Then } a=b}$

If $a \geq b$, $b \geq a$ Then $a=b$.

$\Rightarrow$ "$\geq$" is Anti-Symmetric.

(aRb, bRa Then a=b is anti-Symmetric)

③ Transitive: for any $a, b, c \in Z$. If aRb, bRc then aRc.

If $a \geq b$, $b \geq c$ then $a \geq c$.

$\therefore$ "$\geq$" is Transitive.

Hence the relation "$\geq$" is partially ordering relation.

Eg: S.T. "Inclusion" relation ($\subseteq$) is a partial ordering on the Powerset of a Set.

Sol: To prove $\subseteq$ is a Partial ordering on the Powerset $= P(S)$.

① Reflexive: for every set A, $A \subseteq A$.

Hence $\subseteq$ is reflexive.

② Anti-Symmetric:- for any two sets A and $B \in S$ if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Hence $\subseteq$ is anti-Symmetric.

③ Transitive: for any three sets A, B and $C \in S$ if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Hence "$\subseteq$" is Transitive.

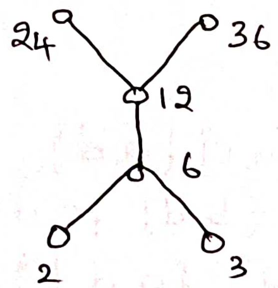
$\therefore (P(S), \subseteq)$ is a POSET.

Hasse Diagram:- A partial ordering ~~set~~ $\leq$ on a set P can be represented by means of a diagram is known as a Hasse diagram (or) a partially ordered set diagram of $(P, \leq)$.

* In Hasse diagram, each element is represented by a small circle or a dot.
* The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$, and a line is drawn between x and y if y covers x.
* If $x < y$ but y does not covers "x", then x and y are not connected directly by a single line. However, they are connected by one or more elements of P.

Eg: Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $(X, \leq)$.

Sol: Given $X = \{2, 3, 6, 12, 24, 36\}$.



② Let A be a given finite set and $P(A)$ its power set. Let $\subseteq$ be the inclusion ($\subseteq$) relation on the elements of $P(A)$. Draw Hasse diagram of $(P(A), \subseteq)$ for

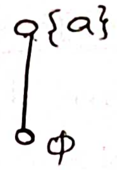
① $A = \{a\}$

② $A = \{a, b\}$

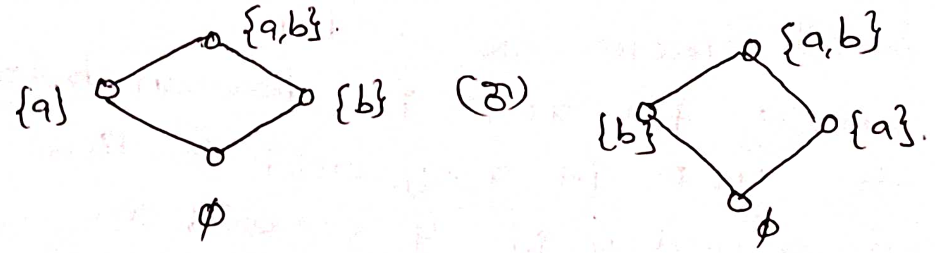
③ $A = \{a, b, c\}$.

④ $A = \{a, b, c, d\}$.

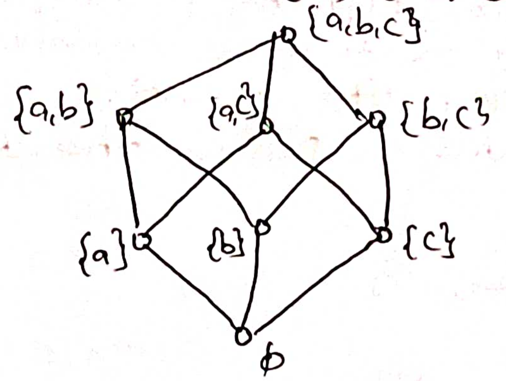
Sol: ① $A = \{a\}$.
Then $P(A) = \{\phi, \{a\}\}$.



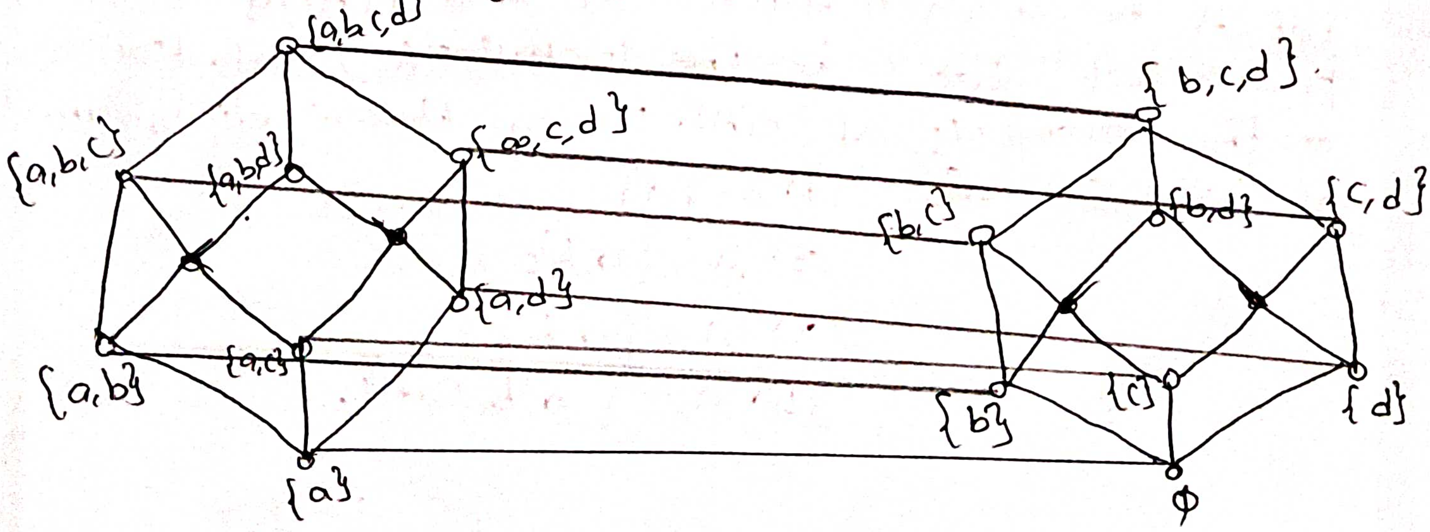
② $A = \{a, b\}$.
Then $P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$.



③ $A = \{a, b, c\}$.
Then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

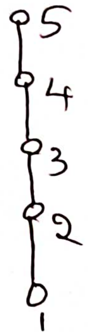


④ $A = \{a, b, c, d\}$.
$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,d\},$
$\{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\},$
$\{b,c,d\} \{a,b,c\}. \{a,c,d\}, \{a,d,b\}, \{a,b,c,d\}\}$



(3) Draw the Hasse diagram $S = \{1, 2, 3, 4, 5\}$ and $\leq$ be a relation on S.

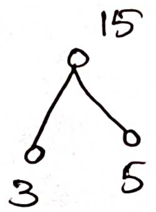
Sol:



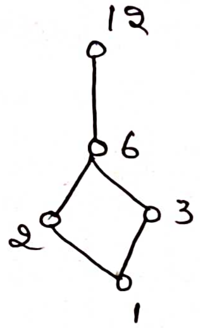
(4) Draw the Hasse Diagram for following set under the partial ordering relation "divides" and (1) $\{2, 6, 24\}$, (2) $\{3, 5, 15\}$ (3) $\{1, 2, 3, 6, 12\}$ (4) $\{2, 4, 8, 16\}$ (5) $\{3, 9, 27, 54\}$.

Sol: (1)

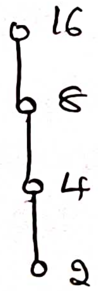
(2)



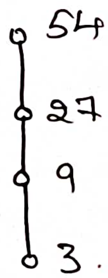
(3)



(4)



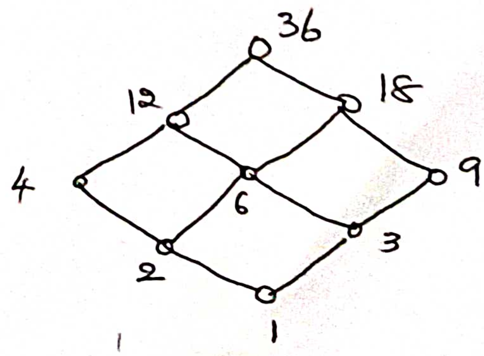
(5)



(5) Draw the Hasse Diagram representing the positive divisors diagram of 36.

The set of all +ve divisors of 36 is

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$.

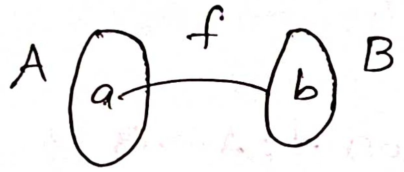


Functions:

Let A and B be any two sets. A relation "f" from A to B is said to be a function if for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$.

* The notations $f: A \rightarrow B$ (or) $A \xrightarrow{f} B$ are used to express f as a function from A to B.

* It can be represented as follows.



Note: ① Functions are called mappings or transformation or corresponding.

② If $f: A \rightarrow B$ be a function from A to B then A is called Domain of f, B is called Co-domain of f.

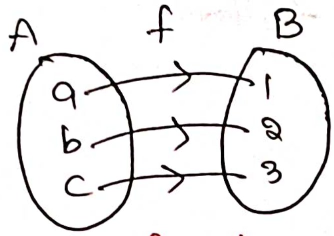
③ Every function from A to B is a relation. But every relation from A to B need not be a function from A to B.

④ One to one and many to one relations are functions but one to many and many to many relations are not functions.

Types of functions:

① **one-to-one function:-** A function $f: A \to B$ is said to be one-to-one function, if every element of A has a unique element in B. There may be some elements of B, which does not corresponds to any element of A.

* A one-to-one function is also called an "Injective function".

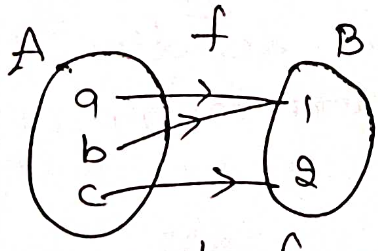


② **on to function:-** A function $f: A \to B$ is said to be an onto function, if every element of B has a preimage in A, under f.

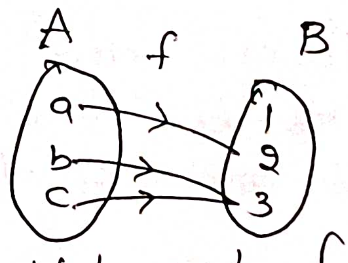
(OR)

f is an onto function from A to B if the range of f is equal to B. i.e., $f(A) = B$. [Range = Codomain].

* An onto function is also called a "Surjective function".

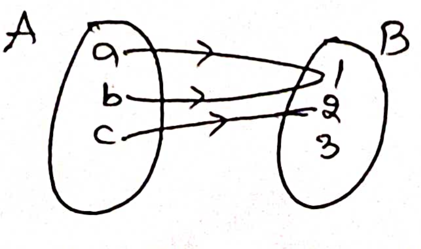


onto function



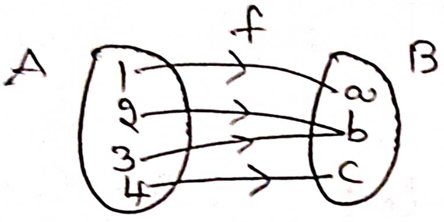
Not on-to function.

③ **Many-to-one function:-** A function $f: A \to B$ is said to be many-to-one function if two or more elements of X are associated to the same element of set Y.



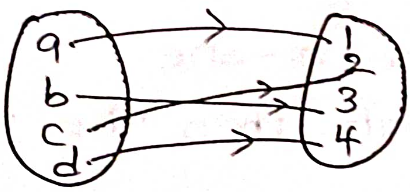
Many-one-onto function:-

A function f is said to be many-one-onto function from A to B if f is both many-one and onto.



Bijective:- A function $f: A \to B$ is said to be Bijective function if f is both one-to-one and onto.

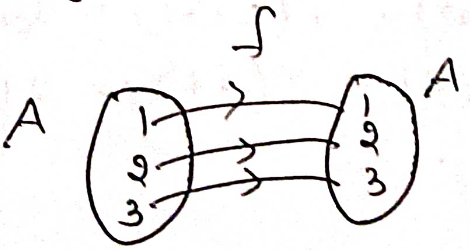
* Bijective function is also called as "one-to-one correspondence".



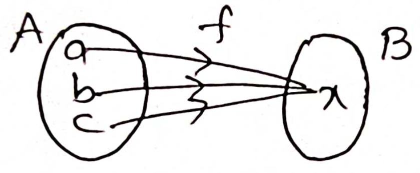
Identity function:- A function $f: A \to A$ is said to be identity function, if the image of every element of A under f is itself.

i.e., $f(A) = A$.

* The identity function is ~~said to be~~ denoted by I_A.



Constant function :- A function $f: A \to B$ is said to be constant function, if all the elements of set A have the same image in set B.



Composition of function :- Let $f: A \to B$ and $g: B \to C$ be two functions then the composition of f and g denoted by gof is a function from A to C and it is defined as

$$gof(x) = g(f(x)), \forall x \in X.$$

* If f and g are one-to-one then the function (gof) is also one-to-one.
* If f and g are onto then the function (gof) is also onto.
* Composition always holds associative property but does not hold commutative property.

Eg: Let $f(x) = x+2$, $g(x) = 2x+1$, find $(fog)(x)$ and $(gof)(x)$.

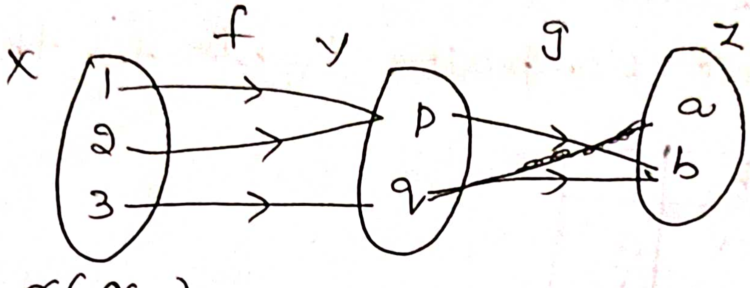
Sol: $(fog)(x) = f(g(x)) = f(2x+1) = (2x+1)+2 = 2x+3.$

$(gof)(x) = g(f(x)) = g(x+2) = 2(x+2)+1 = 2x+5.$

$(fog) \neq (gof).$

Eg: Let $X=\{1,2,3\}$, $Y=\{p,q\}$, $Z=\{a,b\}$, $f:X\to Y$ and $g:Y\to Z$ so that $f=\{(1,p),(2,p),(3,q)\}$, $g=\{(p,b),(q,b)\}$ then find (gof).

Sol:

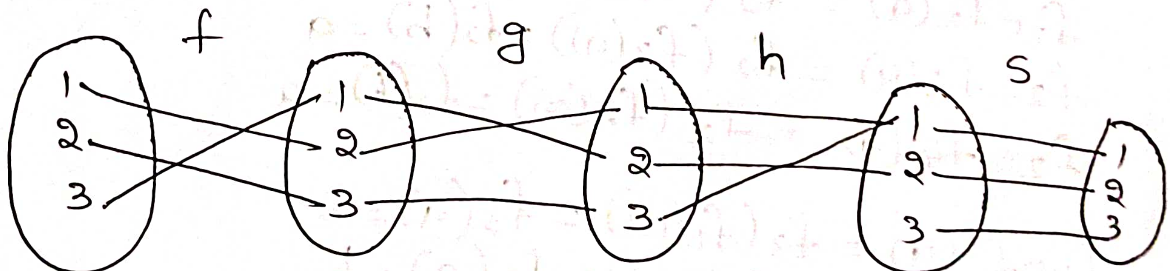


$(gof)(x)=g(f(x))$.

$\Rightarrow gof=\{(1,b),(2,b),(3,b)\}$.

Eg: Let $X=\{1,2,3\}$ and f, g, h and s are functions from X to X given by $f=\{(1,2),(2,3),(3,1)\}$, $g=\{(1,2),(2,1),(3,3)\}$, $h=\{(1,1),(2,2),(3,1)\}$, $s=\{(1,1),(2,2),(3,3)\}$. find ① fog ② gof ③ $fohog$ ④ sog ⑤ gos ⑥ sos ⑦ fos.

Sol:



① $fog(x)=f(g(x))=\{(1,3),(2,2),(3,1)\}$.

② $gof(x)=g(f(x))=\{(1,1),(2,3),(3,2)\}$.

③ $fohog=\{(1,3),(2,2),(3,2)\}=f(h(g(x)))$.

④ $sog=\{(1,2),(2,1),(3,3)\}$.

⑤ $gos=\{(1,2),(2,1),(3,3)\}$.

⑥ $sos=\{(1,1),(2,2),(3,3)\}$.

⑦ $fos=\{(1,2),(2,3),(3,1)\}$

1 Eg: Let $x = \{a, b\}$ and S denotes set of all f's: $x \to x$, where $S = \{f_1, f_2, f_3, f_4\}$
$f_1(a) = a$, $f_2(a) = a$, $f_1(b) = b$, $f_2(b) = a$, $f_3(a) = b$,
$f_3(b) = b$, $f_4(a) = b$, $f_4(b) = a$.
And Construct Composite table for the operation.

Sol:

a	a	b	b
a	a	a	a
b	b	b	b
b	b	a	a.

$f_1 o f_1(a) = f_1(f_1(a)) = f_1(a) = a$
$f_1 o f_2(a) = f_1(f_2(a)) = f_1(a) = a$
$f_1 o f_3(a) = f_1(f_3(a)) = f_1(b) = b$
$f_1 o f_4(a) = f_1(f_4(a)) = f_1(b) = b$

$f_2 o f_1(a) = f_2(f_1(a)) = f_2(a) = a$
$f_2 o f_2(a) = f_2(f_2(a)) = f_2(a) = a$
$f_2 o f_3(a) = f_2(f_3(a)) = f_2(b) = a$
$f_2 o f_4(a) = f_2(f_4(a)) = f_2(b) = a$

$f_3 o f_1(a) = f_3(f_1(a)) = f_3(a) = b$
$f_3 o f_2(a) = f_3(f_2(a)) = f_3(a) = b$
$f_3 o f_3(a) = f_3(f_3(a)) = f_3(b) = b$
$f_3 o f_4(a) = f_3(f_4(a)) = f_3(b) = b$

$f_4 o f_1(a) = f_4(f_1(a)) = f_4(a) = b$
$f_4 o f_2(a) = f_4(f_2(a)) = f_4(a) = b$
$f_4 o f_3(a) = f_4(f_3(a)) = f_4(b) = a$
$f_4 o f_4(a) = f_4(f_4(a)) = f_4(b) = a$

Eg: Let $f(x) = x+2$, $g(x) = x-2$ and $h(x) = 3x$ for $x \in R$, where R is the set of real numbers. Find gof, fog, fof, foh, hog, hof, and $fohog$.

Sol:

① $gof(x) = g(f(x)) = g(x+2) = x + \cancel{2} - \cancel{2} = x.$

② $fog(x) = f(g(x)) = f(x-2) = x - 2 + 2 = x.$

③ $fof(x) = f(f(x)) = f(f(x)) = f(x+2) = x+2+2 = x+4$

④ $foh(x) = f(h(x)) = f(3x) = 3x+2.$

⑤ $hog(x) = h(g(x)) = h(x-2) = 3(x-2) = 3x-6.$

⑥ $fohog(x) = f(h(g(x)))$

$= f(h(x-2))$

$= f(3(x-2))$

$= f(3x-6)$

$= 3x-6+2$

$= 3x-4.$

Inverse function:- Let A and B are two sets. If a function 'f' mapping from A to B i.e, $f: A \to B$, then its inverse f^{-1} mapping from $B \to A$ i.e., $f^{-1}: B \to A$ exists if and only if f is one-to-one and onto.

Method to find Inverse of a function:-

1. Let $f(x) = y$.
2. Interchange x's and y's.
3. Solve for y
4. Replace y with $f^{-1}(x)$.

Eg: find the inverse of the function $f(x) = \frac{3x+2}{2x+1}$.

Sol: $f(x) = \frac{3x+2}{2x+1} = y$. (say)

$$\Rightarrow y = \frac{3x+2}{2x+1}$$

$$x = \frac{3y+2}{2y+1}$$

$$\Rightarrow x(2y+1) = 3y+2$$

$$\Rightarrow 2xy + x = 3y + 2$$

$$\Rightarrow 2xy - 3y = 2 - x$$

$$\Rightarrow y(2x-3) = 2-x$$

$$\Rightarrow y = \frac{2-x}{2x-3}$$

$$\therefore f^{-1}(x) = \frac{2-x}{2x-3}$$

(2) Find the inverse of the function $f(x) = \sqrt{x+4} - 3$

Sol: Let $y = \sqrt{x+4} - 3$

$$\Rightarrow x = \sqrt{y+4} - 3$$

$$\Rightarrow x - 3 = \sqrt{y+4}$$

$$\Rightarrow (x-3)^2 = y+4.$$

$\Rightarrow x^2+6x+9 = y+4$

$\Rightarrow x^2+6x+5 = y.$

$\therefore y = x^2+6x+5$

$\therefore f^{-1}(x) = x^2+6x+5.$

(3) S.T. $f(x)=x^3$ and $g(x)=x^{1/3}$ for $x \in R$ are inverse of each other.

<u>Sol:</u> $f(x) = x^3$

Let $y=x^3$

$\Rightarrow x = y^3$ (Replace x by y)

$\Rightarrow y = x^{1/3}$ (Solve for y).

$\Rightarrow f^{-1}(x) = x^{1/3}$

$\Rightarrow g(x) = x^{1/3}$ $(f^{-1}(x) = g(x))$

(4) Let $f: R \to R$ be given by $f(x) = x^3-2$.
Find f^{-1}.

<u>Sol:</u> i) Let $x_1, x_2 \in R$ s.t. $f(x_1) = f(x_2)$.

$\Rightarrow x_1^3-2 = x_2^3-2.$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2.$

$\therefore f$ is one-to-one.

ii) If $f(x)=y$ then $y=x^3-2 \Rightarrow x^3 = y+2.$

$\Rightarrow x = \sqrt[3]{y+2}.$

$\therefore f$ is onto.

$\Rightarrow f^{-1}$ is invertible.

To find f^{-1}.

$f(x) = x^3-2.$

$\Rightarrow y = x^3-2.$

$\Rightarrow x^3 = y+2$

$\Rightarrow x = \sqrt[3]{y+2}.$ (replace x by y)

$y = \sqrt[3]{x+2}.$

$f^{-1}(x) = \sqrt[3]{x+2}$

~~Let $f(x)=y$~~ $\Rightarrow g(x) = \sqrt[3]{x+2}$

Floor and Ceiling functions:- Let 'x' be a real number then the least integer that is not less than 'x' is called the ceiling of 'x'. The ceiling of 'x' is denoted by $\lceil x \rceil$.

Eg: $\lceil 2.15 \rceil = 3$, $\lceil \sqrt{5} \rceil = 3$, $\lceil -7.4 \rceil = -7$, $\lceil -2 \rceil = -2$.

Let 'x' be a real number, then the greatest integer that does not exceed 'x' is called the 'Floor of x'. The floor of 'x' is denoted by $\lfloor x \rfloor$.

Eg: $\lfloor 5.14 \rfloor = 5$ $\lfloor \sqrt{5} \rfloor = 2$, $\lfloor -7.6 \rfloor = 8$.

Recursive Function:- A recursive function is a function which is defined in terms of itself. we can also define functions respectively.

* A recursive definition has two parts

1. Definition of the smallest argument i.e., $f(0)$ or $f(1)$.

2. Definition of $f(n)$, given $f(n-1)$, $f(n-2)$, etc...

Eg: An example of a recursively defined function.

$f(0) = 3$, $f(n+1) = 2f(n) + 3$. Then

$f(1) = 2f(0) + 3 = 2(3) + 3 = 9$.

$f(2) = 2f(1) + 3 = 2(9) + 3 = 21$.

$f(3) = 2f(2) + 3 = 2(21) + 3 = 45$

$f(4) = 2f(3) + 3 = 2(45) + 3 = 93 \ldots$

Eg: Calculate the values of the following recursively defined function $f(0)=1$ and $f(n)=n\,f(n-1)$

Eg: $f(1)=1$, $f(2)=1$ and $f(n)=f(n-2)+f(n-1)$.

Eg: Consider the following recursive function. If $x<y$ then $f(x,y)=0$. If $y\le x$ then $f(x,y)=f(x-y,\,y)+1$. Find the value of $f(4,7)$ and $f(19,6)$

Sol: Given $f(x,y)=\begin{cases}0 & \text{if } x<y\\ f(x-y,\,y)+1 & \text{if } y\le x.\end{cases}$

i) $f(4,7)=0 \quad (\because 4<7)$.

ii) $f(19,6) = f(19-6,\,6)+1$

$= f(13,6)+1$

$= f(13-6,\,6)+1$

$= f(7,6)+1$

$= f(7-6,\,6)+1$

$= f(1,6)+1$

$= 0+1$

$= 1.$

Factorial Function:-

ⓐ If $n=0$ then $n! = 1$

ⓑ If $n>0$, then $n! = n(n-1)!$.

Permutation function:

A Bijection from a set A to itself is called a Permutation of A.

If $A = \{a_1, a_2, \ldots a_n\}$ and $p(a_1), p(a_2), \ldots p(a_n)$ are images then $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ p(a_1) & p(a_2) & \cdots & p(a_n) \end{pmatrix}$. ~~is a perm~~

<u>Eg</u>: Let $A = \{1, 2, 3\}$. Then all the permutations of A are

$$I_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

<u>Eg</u>: In above example find P_4^{-1}, $P_3 \circ P_2$.

$$P_4^{-1} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

$$P_3 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Cycle Permutation :- Let $b_1, b_2, \ldots b_r$ be r distinct elements of the set $A = \{a_1, a_2, \ldots a_n\}$. The permutation $p: A \to A$ defined by.

$$p(b_1) = b_2$$
$$p(b_2) = b_3$$
$$\vdots$$
$$p(b_{r-1}) = \cancel{p(b_r)}.\, b_r$$
$$p(b_r) = \cancel{p} \cdot b_1.$$

is called cyclic permutation of length "r".

Eg: Let $A = \{1, 2, 3, 4, 5\}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$ is a permutation.

Let $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$

$= (1\ 3\ 5)(2)(4)$

P is a cyclic permutation of length 3.

Equivalence relations:- A relation R in a set "A" is called an equivalence relation if it is
① Reflexive ② Symmetric ③ Transitive.

If $a, b, c \in A$ then "$a \sim b$" (or) "$a \equiv b$" denotes a is equivalent to b.

i) Reflexive : $a R a$ for $a \in A$.

ii) Symmetric : If $a R b$ & $b R a$ then $a = b$ for $a, b \in A$.

iii) Transitive : If $a R b$, $b R c$ then $a R c$.

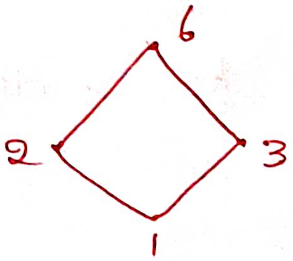
Lattice: A POSET $(L, \leq)$ is said to be lattice if for every pair of elements in L, has least upper bound (lub) and greatest lower bound (glb).

→ The lub (Supremum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \vee b$ (or) $a \oplus$ and is called the join (or) sum of 'a' and 'b'.

→ The glb (Infimum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \wedge b$ (or) $a * b$ and is called the meet (or) product of 'a' and 'b'.

$GLB\{a, b\} = a \wedge b$ (or) $a * b$ (Meet)
$LUB\{a, b\} = a \vee b$ (or) $a \oplus b$ (Join).

Eg:



LUB Table

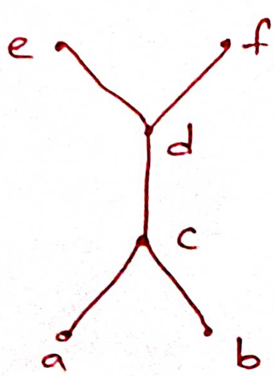
$\vee$	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

GLB Table

$\wedge$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

Here for every pair of elements in the given poset, both LUB & GLB exists.
Hence given poset is a lattice.

② Check the following Hasse diagram is Lattice or not.



Sol: Consider the incomparable pairs (e, f), (f, e).

$GLB\{e, f\} = d$ $\qquad$ $GLB\{f, e\} = d$
$LUB\{e, f\} = \phi$ $\qquad$ $LUB\{f, e\} = \phi$.

Also for (a, b)

$GLB\{a, b\} = \phi$ $\quad$ $GLB\{b, a\} = \phi$

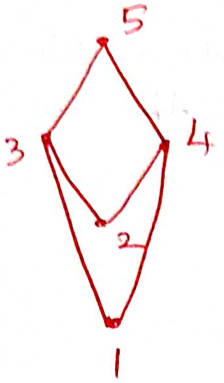
$LUB\{a, b\} = c$ $\quad$ $LUB\{a, b\} = c$

for e & f, f & e LUB does not exist.

for a & b, b & a GLB " " "

Hence given Hasse diagram is not a lattice.

(3)



Here the GLB (3,4) is not exist.
Hence it is not a Lattice. i.e, there are 2 & 1 both Glb's

Properties of Lattices:-

Let $(L, \leq)$ be a Lattice. L satisfies the following properties of the two binary operation of meet and join denoted by $\wedge$ and $\vee$.

For every $a, b, c \in L$, we have

① Idempotent Property:

(i) $a \vee a = a$ (ii) $a \wedge a = a$

② Commutative Property:

(i) $a \vee b = b \vee a$ (ii) $a \wedge b = b \wedge a$

③ Associative Property:

(i) $a \vee (b \vee c) = (a \vee b) \vee c$ (ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

④ Absorption Property:

(i) $a \vee (b \wedge c) = a$ (ii) $a \wedge (b \vee c) = a$

⑤ Distributive:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$