A note on no-homomorphism theorems and K_4-minor free graphs

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February 2, 2023

"I know what you're thinking about," said Tweedledum; "but it isn't so, nohow."
"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

—Lewis Carroll, Sylvie and Bruno Concluded, 1889

To doubt everything, or, to believe everything, are two equally convenient solutions; both dispense with the necessity of reflection.

-Henri Poincaré, La Science et l'Hypothèse (Preface), 1952

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Acknowledgements

I am extremely grateful to **Dr. K. Prakasan**, Principal in charge, PSG College of Technology, Coimbatore for giving me this opportunity to do my project at Institut de Rechearche En Fondamentale, Paris.

I am very much thankful to **Dr. R. Nadarajan**, Professor and Head, Department of Applied Mathematics and Computational Sciences, PSG College of Technology, Coimbatore, for his continual support and guidance throughout my project tenure.

I express my sincere thanks to my external guide **Dr. Reza Naserasr**, IRIF, Paris for his exemplary guidance and for helping me in various aspects to complete this project work.

I am deeply indebted to my internal guide **Dr. R. S. Lekshmi**, Professor, Department of Applied Mathematics and Computational Sciences, PSG College of Technology, Coimbatore, for her guidance, ardent motivation and constant support throughout this project.

Finally, I express my gratefulness to all the faculty members and staff of Department of Applied Mathematics and Computational Sciences, PSG College of Technology, Coimbatore, and my family and friends for their encouragement and support.

Synopsis

The notion of homomorphisms of signed graphs was first defined by B. Guenin in an unpublished manuscript in 2005. The development of the homomorphisms of signed graphs started when Naserasr, Rollová and Sopena [5] extended the classical notion of graph homomorphism to signed graphs, which integrated homomorphisms of signed graphs with the theory of homomorphisms of 2-edge-colored graphs. The important notion introduced in the paper [5] was the correlation of homomorphisms with the theory of graph minors. In this a report generalization of Four color theorem is considered, which leads to a number of simple no-homomorphism lemmas (which is related to graph minors), special subclasses and some characterization theorems.

A part of Hadwiger's Conjuncture (stronger statement than Four Color Theorem) is dealt in this research by considering homomorphisms of signed bipartite graphs of different girths, that are not homomorphic to C_{-4}. The proof of smallest signed bipartite graph with unbalanced girth 4 and its relation to the problem of K_4-minor free graphs are discussed.

CHAPTER 1

Introduction

1.1 Motivation

The motivation for Homomorphism of Signed graphs is primarily drawn form the most classical problem in Graph Theory - Four Color Theorem. The formulation of Four Color Problem date back to 1852 when Francis Guthrie, while trying to color the map of counties of England noticed that four colors sufficed. He asked his brother Frederick if it was true that any map can be colored using four colors in such a way that adjacent regions (i.e. those sharing a common boundary segment, not just a point) receive different colors. Frederick Guthrie then communicated the conjecture to Augustus DeMorgan. The first printed reference to Four color theorem is a short paper named "On the coloring of maps" by Arthur Cayley in 1879 where he explained some of the difficulties in attempting a proof and made some important contributions to the way the problem was approached. A year later the first proof by Kempe appeared using what was later known as Kempe chains, its incorrectness was pointed out by Heawood 11 years later. Another failed proof is by Tait in 1880 by modeling the problem as a 3-edge coloring problem. The gap in the argument was pointed out by Petersen in 1891. Although both were failed proofs, the structure gave certain discernments which had values. After this there were many formulations of the Four color problem, namely with knots, networks and topology. Finally the problem was proved in 1979 by Kenneth Appel and Wolfgang Haken.

The Four Color Theorem, henceforth known as FCT can be formulated using homomorphism of graphs, as a matter of fact a stronger statement than FCT can

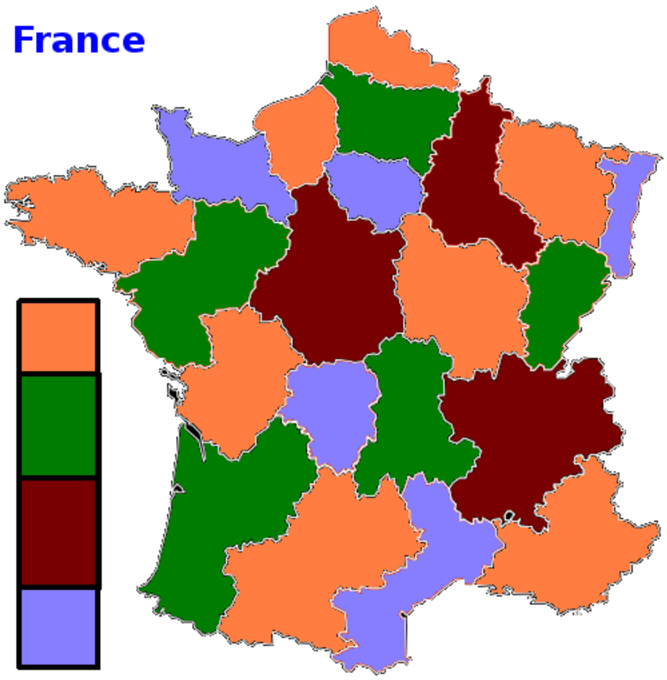


Figure 1.1.1: An Illustration of Four color theorem with provinces of France.

be modelled i.e. a statement, of which FCT is a special case. The details on the modelling of FCT based on homomorphisms (more precisely no-homomorphisms) is explained in the next chapter. To start with, anyone involved in some way with mathematics or computer science knows about integers modulo some n, which is based on homomorphisms. Homomorphisms provide a way of simplifying the structure of objects one wishes to study while preserving much of the structure that is of significance. In case of graphs, homomorphisms simplify the graphs by reducing the number of vertices and edges but still preserving the overall adjacency.

In this report, the study is specific to a special type of graphs - the signed graph. The notion of signed graph can be visualized and understood by the following example: Consider a group of people as nodes, if they are friends, then the edge between them is a positive edge and if they are enemies, edge between them is a

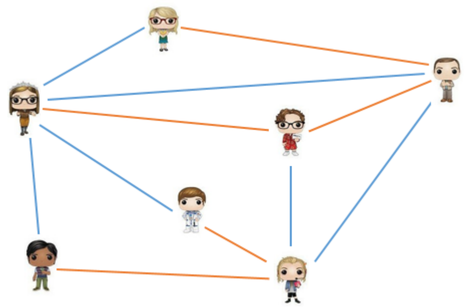


Figure 1.1.2: A Signed Graph, where red line (edge) denotes enmity and blue line (edge) denotes friendship between two individuals.

negative edge. For simplicity assume the friendship or enmity is mutual for persons' (nodes) at both the ends of an edge and exclude the possibility of neutral relationships.

The problems dealt in this report are an attempt to solve a subproblem of the generalized Four Color theorem (no-homomorphisms problems), by viewing it in the perspective of Signed graphs.

1.2 Organization of the Report

Initially a brief introduction of Four Color theorem, Homomorphism and Signed graphs are presented. The literature review chapter has a total of five sections of which, the first introduces various graph theory and related mathematical terms required to proceed with the report, the next three chapters explains the most important topics dealt and final one to aggregate the notations used (that are not mentioned in the previous sections). Chapter 3 revolves around the current research in the domain with few other important theorems for better understanding. Finally the report ends with the summary of the work done and open problems that will be worked upon in the future.

CHAPTER 2

Literature Review

This chapter encompasses a basic definitions, brief history on each of the sections, introduces various definitions and relevant theorems that will facilitate insights of the report. Further information is enclosed in the Appendix, to not hinder the flow of the report.

2.1 Basic terminologies

Some basic graph theoretical terms used in the report are surmised.

Definition 2.1. ***Graph:*** *A graph G is an ordered pair (V, E), where V is a set of vertices and E is a set of edges, each of which connects two vertices.*

Definition 2.2. ***Undirected Graph:*** *An undirected graph is a graph where all the edges are bidirectional.*

Definition 2.3. ***Simple Graph:*** *A simple graph is a graph containing no loops and/or multiple edges.*

Definition 2.4. ***Subgraph:*** *A subgraph of G is a pair (V_1, E_1), with $V_1 \subseteq V$ and $E_1 \subseteq E$, that is itself a graph.*

Definition 2.5. ***Induced Subgraph:*** *An induced subgraph $G[S]$, $S \subseteq V$ is the subgraph of G whose vertex set is S and whose edge set consists of all of the edges in E that have both end vertices in S.*

Definition 2.6. ***Connected Graph:*** *A graph is said to be connected if there is a path from any vertex to any other vertex in the graph.*

Definition 2.7. ***Girth:*** *The length of the shortest graph cycle (if any) in a graph.*

Definition 2.8. ***Planar Graph:*** *A graph is planar if it can be drawn in a plane without graph edges crossing*

Definition 2.9. *k**-regular Graph:*** *A graph is called regular graph if degree of each vertex is equal. A graph is called k regular if degree of each vertex in the graph is k.*

Definition 2.10. ***Complete Graph:*** *A complete graph K_n is a graph with n vertices where $xy \in E(K_n)$ for all $x, y \in V(K_n)$, with $x \neq y$. A triangle is a K_3.*

Definition 2.11. ***Clique:*** *A clique of a graph G is a complete subgraph of G.*

Definition 2.12. ***Cycle:*** *A cycle in a graph is an ordered set of vertices $\{v_1, v_2, \cdots, v_j\}$ such that the graph contains edge (v_i, v_{i+1}), $\forall\ i \in \{1, \cdots, j-1\}$ and also an edge (v_j, v_1).*

Definition 2.13. ***Path:*** *A path P_n is a tree with two vertices of degree 1, and the other $n-2$ vertices of degree 2. A path graph is therefore a graph that can be drawn so that all of its vertices and edges lie on a single straight line*

Definition 2.14. ***Bipartite Graph:*** *A bipartite graph is a graph whose vertices can be partitioned into two nonempty sets S_1 and S_2 such that every edge in the graph connects a vertex in S_1 with a vertex in S_2.*

Definition 2.15. ***Equivalence Class:*** *An equivalence class is defined as a subset of the form $\{x \in X : xRa\}$, where a is an element of X and the notation "xRy" is used to mean that there is an equivalence relation between x and y.*

Definition 2.16. ***Group:*** *A group G is a finite or infinite set of elements together with a binary operation (called the group operation) that together satisfy the four* *fundamental properties of closure, associativity, the identity property, and the inverse property. The operation with respect to which a group is defined is often called the "group operation," and a set is said to be a group "under" this operation. Elements* $A, B, C, \cdots$ *with binary operation between* A *and* B *denoted* AB *form a group if the following properties are satisfied:*

1. Closure: If A *and* B *are two elements in* G*, then the product* AB *is also in* G.

2. Associativity: The defined multiplication is associative, i.e., for all A, B, C *in* G, $(AB)C = A(BC)$.

3. Identity: There is an identity element I *(a.k.a. 1,* E*, or* e*) such that* $IA = AI = A$ *for every element* A *in* G.

4. Inverse: There must be an inverse (a.k.a. reciprocal) of each element. Therefore, for each element A *of* G*, the set contains an element* $B = A^{-1}$ *such that* $AA^{-1} = A^{-1}A = I$.

Definition 2.17. ***Cayley graph:*** *Let* Γ *be a group,* S *be a set of elements of* Γ *not including the identity element. Suppose, furthermore, that the inverse of every element of* S *also belongs to* S*. The Cayley graph* $C(\Gamma, S)$ *is the graph with vertex set* Γ *in which two vertices* x *and* y *are adjacent if and only if* $xy^{-1} \in S$.

Definition 2.18. ***Hypercubes:*** *The hypercube of dimension* n*, denoted* $H(n)$*, is the graph whose vertex set is the set of all* n*-tuples of 0's and 1's, where two* n*-tuples are adjacent if and only if they differ in precisely one coordinate. It can be checked that,* $H(n)$ *is a Cayley graph* $(Z_n^2, \{e_1, e_2, \cdots, e_n\})$ *where* e_i*'s are the standard basis (n-tuple with 1 in* i^{th} *position and 0s in the others.) of* Z_n^2 *.* $H(n)$ *is also called n-cube*

Definition 2.19. ***Matching:*** *Given an undirected graph, a matching is a set of edges, such that no two edges share the same vertex. In other words, matching of a graph is a subgraph where each node of the subgraph has either zero or one edge* *incident to it.*

A perfect matching of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching.

A matching M is said to be maximal if M is not properly contained in any other matching. Formally, $M \not\subset M'$ for any matching M' of G. Intuitively, this is equivalent to saying that a matching is maximal if we cannot add any edge to the existing set.

Definition 2.20. ***Independent sets:*** *An independent set of a graph G is a subset of the vertices such that no two vertices in the subset induce an edge of G. The cardinality of a maximum independent set in a graph G is called the independence number of G, denoted by $\alpha(G)$.*

Definition 2.21. ***Kneser Graphs:*** *Given positive integers n and k such that $n \geq 2k$, the Kneser graph $K(n, k)$ is defined to be the graph whose vertices correspond to the k-element subsets of a set of n elements, where two vertices are adjacent if the two corresponding sets are disjoint.*

2.2 Homomorphism and Minors

Hadwiger's conjecture (will be stated later) when proved will provide a bound for the chromatic number of any graph (will be discussed in this section). His conjecture is based on the minors of the graph. This section furnishes the basics of homomorphism, minors and their relation.

2.2.1 Graph Homomorphism

Graph homomorphisms in the current sense were first studied by Sabidussi in the late fifties and early sixties, with results published in the paper on Graph derivatives [30]. Hedrlín, Pultr and their collaborators pursued homomorphisms of relational systems in general, and graph homomorphisms in particular, in the sixties and the seventies, with many of the results collected in Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories by Pultr and Trnková [27]. In computer science, graph homomorphisms have been studied as interpretations, especially in relation to grammars, and as "general colorings" [27]. Graph homomorphism is considered as tools, especially in connection with coloring problems. Homomorphism is also considered as generalization of Graph coloring which will be discussed in the later sections.

Definition 2.22 (Graph Homomorphism)**.** Let G and H be simple undirected graphs. A function $\phi : V(G) \to V(H)$ is a homomorphism from G to H if it preserves edges, that is, if for any edge (u, v) of G, $(\phi(u), \phi(v))$ is an edge of H. For simplicity it is denoted as $\phi : G \to H$.

Suppose that $\phi : G \to H$ is a graph homomorphism. Then ϕ is an isomorphism if and only if ϕ is bijective and if ϕ^{-1} is also a homomorphism. In particular, if $G = H$ then ϕ is an automorphism if and only if it is bijective.

Definition 2.23 (Core)**.** A graph G is a core if it has the minimum number of vertices of any graph in its equivalence class defined on Homomorphic relation.

Definition 2.24 (Bound)**.** Given a class C of graphs and a graph H, H bounds C if every graph in C admits a homomorphism to H. Also, it can be formulated as, C is bounded by H, or that H is a bound for C.

2.2.2 Minors

Before understanding the definition of minors, there are few other related concepts to be introduced, which will enable deeper knowledge of minors.

Definition 2.25 (Edge Subdivision)**.** Given a graph $G = (V, E)$ and an edge $e \in E$, subdividing e is the operation of replacing e with a path consisting of new vertices.

A graph G is a subdivision of H if G is obtained by subdividing all or some of the edges of H and the new vertices are called subdivision vertices.

Definition 2.26 (Topological minor)**.** A graph H is a topological minor of a graph G if G contains a subdivision of H as a subgraph.

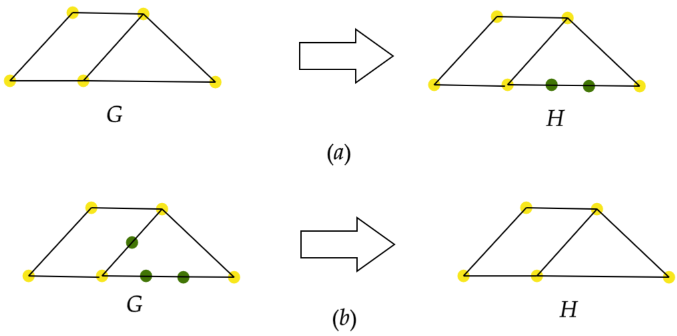


Figure 2.2.1: (a) An example edge subdivision, the green vertices are the vertices that were added (subdivision vertices)and (b) An example for topological minor,after performing vertex subdivision on H, G is obtained. (green vertices are subdivision vertices) H, is called topological minor G.

Definition 2.27 (Vertex Suppression)**.** Given a graph $G = (V, E)$, and a vertex $v \in V$ of degree 2, suppressing v is the operation of removing v and adding an edge between the two neighbors of V.

A graph H is a topological minor of a graph G if H can be obtained from G by suppressing vertices (of degree 2) and by removing edges and vertices.

Definition 2.28 (Edge Contraction)**.** Given a graph $G = (V, E)$ and an edge $e \in E$, contracting e is the operation of removing e and merging its two end vertices.

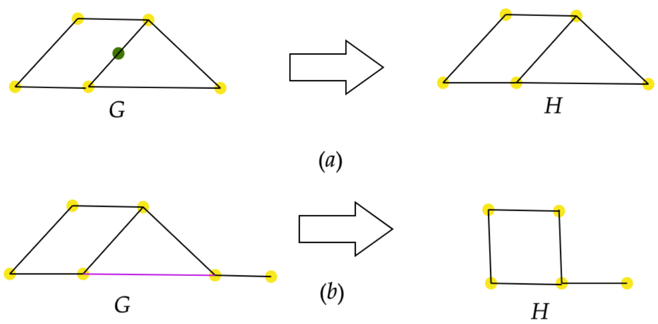


Figure 2.2.2: (a) An example of vertex suppression (green vertex is the suppressed vertex) and (b) An example for edge contraction (purple is the contracted edge).

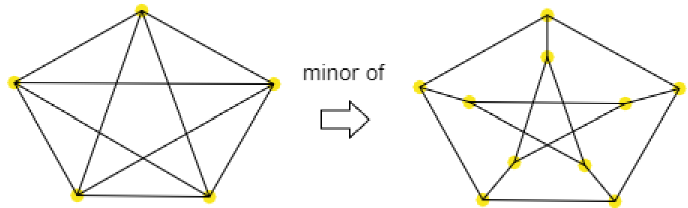


Figure 2.2.3: K_5 is a minor of Petersen graph.

Definition 2.29 (Minor)**.** A graph H is a minor of a graph G if H can be obtained from G by contracting edges and by removing edges and vertices.

The following diagram establishes the relation between the definitions discussed in this topic.

2.2.3 Relation between Homomorphism and Minors

Let $I = \{I_1, I_2, \cdots, I_i\}$ be a class of cores. $forb_h(I)$ is defined to be the class of all graphs to which no member of I admits a homomorphism. For example for $I = K_n$, $forb_h(I)$ is the class of K_n-free graphs and for $I = C_{2k-1}$ it is the class of graphs of odd-girth at least $2k+1$. Similarly, given $J = \{J_1, J_2, \cdots, J_j\}$, the notation $forb_m(J)$

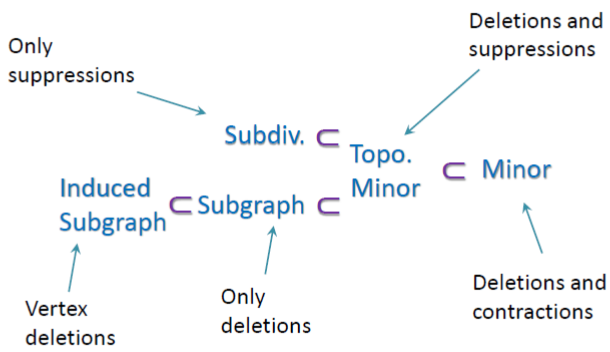


Figure 2.2.4: Relationship between Topological Minors, Minors, Subdivisions, Subgraph and induced subgraphs.

is used to denote the class of all graphs that have no member of J as a minor. The following is a fundamental theorem in the study of the relation between minors and homomorphisms.

Theorem 2.1. [22] Given a finite set I of connected graphs and a finite set J of graphs, there is a graph in $forb_h(I)$ which is a bound for the class $forb_h(I) \cap forb_m(J)$.

It is worthwhile to note that a bound for $forb_h(I) \cap forb_m(J)$ belonging itself to $forb_h(I) \cap forb_m(J)$ may not exist, for example there is no triangle-free planar graph bounding the class of triangle-free planar graphs.

2.3 Graph Coloring

The famous four-color theorem was proved in 2005 by Benjamin Werner and Georges Gonthier using *Coq* proof assistant after numerous failed attempts by various mathematicians. Although the theorem has become well known after the computer-aided proof, graph coloring problems (GCP) date back to early 1970. By default graph coloring refers to vertex coloring but there are different types of GCP, for example vertex-coloring, edge-coloring and list coloring. In this report a generalization of vertex coloring is discussed.

The Vertex Coloring Problem is one of the classical problems, and it is well known not only for its theoretical aspects and for its difficulty from the computational point of view as it is NP-hard. It models many real world applications, including, among many others, scheduling, timetabling, register allocation, frequency assignment and communication networks.

Definition 2.30 (Proper Vertex Coloring)**.** Given an undirected graph $G = (V, E)$, the Vertex Coloring Problem (VCP) requires to assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized.

As mentioned earlier, homomorphism is a generalization of graph colorings. A homomorphism from the graph G to the complete graph K_r (with vertices numbered $1, 2, \cdots, r$) is exactly the same as an r-coloring of G (where the color of a vertex is its image under the homomorphism), since adjacent vertices map to distinct vertices of the complete graph.

Definition 2.31 (k-edge Coloring)**.** A k-edge-coloring of a graph G is a function $\phi : E(G) \longrightarrow [k]$ that assigns a number to each edge in G. G is monochromatic with respect to ϕ if $\phi(E(G)) = i,$ where $i \in [k]$ for some fixed $k \in \mathbb{N}$. A graph G is k-colorable if there exists a $\phi : E(G) \longrightarrow [k]$ and minimum k is called the edge chromatic number of the graph G.

Definition 2.32 (H-coloring)**.** A graph G is H-colorable if G admits homomorphism to H and is critically H-chromatic if $G \setminus e$ is not homomorphic to H.

Definition 2.33 (Fractional Chromatic Number)**.** The fractional chromatic number of a graph G can equivalently be defined as the smallest value of $\frac{p}{q}$ over all positive integers $p, q, 2q \leq p$, such that $G \to K(p,q)$, where $K(p,q)$ is the Kneser graph whose vertices are all q-subsets of a p-set and where two vertices are adjacent if they have no element in common. Fractional chromatic number is denoted as $\chi_f(G)$, and fractional edge-chromatic number of multigraphs is denoted $\chi'_f(G)$

Definition 2.34 (Circular Chromatic Number)**.** Given two integers p and q with $gcd(p,q) = 1$, the circular clique $C_{p,q}$ is the graph on vertex set $\{0, \cdots, p-1\}$ with i adjacent to j if and only if $q \leq i - j \leq p - q$. A homomorphism of a graph G to $C_{p,q}$ is called a (p,q)-coloring, and the circular chromatic number of G, denoted $\chi_c(G)$, is the smallest rational p/q such that G has a (p,q)-coloring.

2.4 Signed Graphs

The name "signed graph" and the notion of balance appeared first in a mathematical paper of Frank Harary in 1953. Dénes König had already studied equivalent notions in 1936 under a different terminology but without recognizing the relevance of the sign group. At the Center for Group Dynamics at the University of Michigan, Dorwin Cartwright and Harary generalized Fritz Heider's psychological theory of balance in triangles of sentiments to a psychological theory of balance in signed graphs.

A signed graph is a graph G together with an assignment $\Sigma : E \to \{+, -\}$ of a sign ($+$ or $-$) to each edge of G. G is called the underlying graph, and Σ is called the signature and the signed graph is denoted by (G, Σ). A signed graph where all edge are positive is denoted by $(G, +)$ and called all positive, and similarly if all edges are negative it will be denoted by $(G, -)$ and called all negative.

Signed graph (G, Σ) is called connected, bipartite, etc. when G is connected, bipartite, etc.

The signed graph (G, Σ) may be thought of as a 2-edge-colored graph (G, E^+, E^-), where E^+ and E^- denote the sets of positive and negative edges, respectively; but that does not express the fact that signs + and - are essentially different. The difference between a signed graph and a 2-edge-colored graph is on the notion of sign of a closed walk in (G, Σ). For any walk $W = e_1 e_2 \cdots e_l$, of a signed graph (G, Σ), the sign of W is $\Sigma(W) = \Sigma(e_1)\Sigma(e_2) \cdots \Sigma(e_l)$. Then W is said to be positive or negative depending on the value of $\Sigma(W)$. Since a cycle of a graph is also a closed walk naturally definition of positive cycles and negative cycles is derived. It is clear that the signs of cycles determine the signs of all closed walks. Signs of cycles determine many fundamental properties of a signed graph, which is captured by the balance of the signed graph. A signed graph (G, Σ) is said to be balanced if every cycle is positive, otherwise it is said to be unbalanced. These two subclasses of signed graphs together with the subclass of signed bipartite graphs form three subclasses of signed graphs. A closely related notion is the notion of ***switching***: to switch a vertex v of a signed graph (G, Σ) is to negate all signs in $[v, V(G) \setminus v]$. To switch a set X of vertices is to switch all the vertices of X in any sequence; it has the effect that it negates the edges in the cut $[X, Y]$, which is the same as switching the complementary set Y. A novel though obvious fact is that switching does not change the signs of closed walks. An equivalent property to balance is that, after a suitable switching, every edge is positive, equivalent to unbalance is that, after suitable switching, every edge is negative

Definition 2.35 (Homomorphism of signed graphs)**.** A homomorphism of a signed graph (G, Σ) to a signed graph (H, Π), written $(G, \Sigma) \to (H, \Pi)$, is a graph

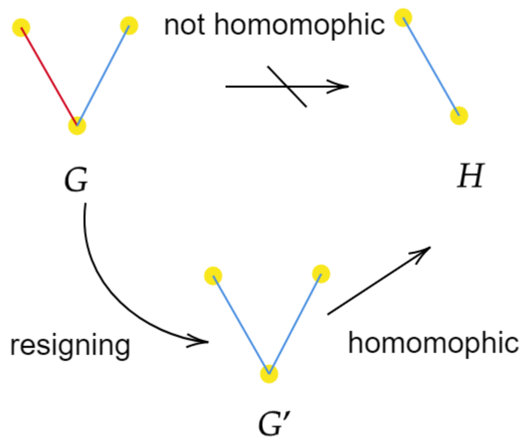


Figure 2.4.1: Graph G (which doesnot map to H) is converted to G' by switching (resigning) in the end vertex of the red edge in G which can then map to H.

homomorphism that preserves the signs of closed walks. More precisely, this is a switching homomorphism of signed graphs. The existence of a homomorphism is indicated by the notation $(G, \Sigma) \to (H, \Pi)$.

Given a class C of signed graphs, (H, Π) bounds C if every element of C maps to (H, Π). Observe that if uv is an edge of G, then the natural closed 2-walk $u - v - u$ is always balanced (independently of the sign of uv), thus its image must also be a balanced closed 2-walk, that is, an edge of H. Thus, a homomorphism of (G, Σ) to (H, Π) is always a homomorphism of G to H. In practice, the above property is used to determine whether there exists a homomorphism from (G, Σ) to (H, Π).

2.5 Notations

In this report finite, simple and undirected graphs are considered. Given a graph G and a set $A \subseteq V(G)$, $|G|$ is used to denote the number of vertices of G, and $G[A]$ to denote the subgraph of G obtained from G by deleting all vertices in $V(G) \setminus A$. A graph H is an induced subgraph of G, if $H = G[A]$ for some $A \subseteq V(G)$. P_n, C_n and

K_n are used to denote the path, cycle and complete graph on n vertices, respectively. For any positive integer k, $[k]$ is used instead of $\{1, 2, \ldots, k\}$. The graphs considered in this report are predominantly simple signed or unsigned graphs (no self loops and parallel edges), any other differences are mentioned as multigraphs or digraphs in the place of occurrence. Signed graphs are depicted in the figures with colors on the edge; blue for positive edge and red for negative edge. Unsigned graphs are depicted with black edges. Any other colors on the vertices or edges will be explained with the figure.

CHAPTER 3

Graph not Homomorphic to C_{-4}

This chapter explains few more relevant theorems and the graphs that are not homomorphic to C_{-4} with its relation to K_4 minor free graphs. The problem to be

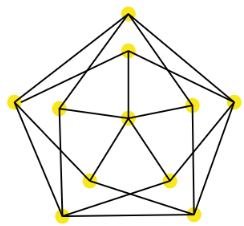


Figure 3.0.1: Grötzsch Graph.

discussed is similar to that of the problem to which Grötzsch presented a solution with Figure 3.0.1, which is a triangle free 4-chromatic graph. Grötzsch graph was introduced in the year 1959 and the research is still underway on the problem of H-minor free graphs from the point of algorithmic complexity. Harary proved that Grötzsch graph is the smallest graph (i.e. there is no triangle free 4-chromatic graph with less than 11 vertices) with above mentioned properties in 1969 [14] and further it's uniqueness on 11 vertices was proved by Chvátal in 1974 [13]. N. V. Ngoc and Z. Tuza extended to find 4-chromatic graphs of large girth by generalizing Mycielski graph [11]. The 4-coloring of a graph can be perceived in terms of homomorphism of a graph to the complete graph on four vertices, which led to generalized notion of coloring called H-coloring of a graph, where H is any graph. Motivated by the perspective of H-coloring of graph, this research extends H-coloring to signed graphs, specifically signed bipartite graphs. Every unsigned bipartite graphs is homomorphic to a single edge, which is evident as unsigned bipartite graphs are 2-colorable, which is not the case with signed bipartite graphs.

3.1 Theorems on Signed graphs

In order to interpret the significance of the forth coming sections, it is important to understand various characteristics of signed graphs, which are presented as theorems in this section.

Theorem 3.1. [29] Two signatures Σ_1 and Σ_2 on a graph G are equivalent if and only if they induce the same set of unbalanced cycles.

Loops and unbalanced digons (unbalanced cycles of length 2) play an important role in the study of coloring and homomorphisms of signed graphs (see the article by Brewster [4]). As previously mentioned, simple graphs are predominantly used and will specifically use the term multigraph when multiple edges are allowed. This restriction allows a smoother definition of signed graph homomorphisms.

Theorem 3.2. [3] Given two signed graphs (G, Σ) and (H, Π), a mapping $\phi : V(G) \to V(H)$ is a homomorphism of (G, Σ) to (H, Π) if there exists a signature Σ' of G equivalent to Σ such that:
(i) for every edge uv of G, $\phi(u)\phi(v)$ is an edge of H, and
(ii) uv is in Σ' if and only if $\phi(u)\phi(v)$ is in Π.

In a sense, a homomorphism from a signed graph to another is a homomorphism between two equivalence classes of signed graphs. The following lemma provides an easy "no-homomorphism" condition. Let UC_r be the signed graph (C_r, Σ) where Σ is a single edge (it is a representative for all cycles of order r having an odd number of negative edges)

Lemma 3.1. [3] Given two positive integers k and l, $UC_l \to UC_k$ if and only if l and k have the same parity and $l \geq k$.

This leads to the notion of unbalanced-girth of a signed graph, that is, the length of a shortest unbalanced cycle of the given signed graph (if no such cycle exists, it is defined to be infinite). In fact, above Lemma suggests to consider two variants of unbalanced-girth: the length of a shortest unbalanced even-cycle, and the length of a shortest unbalanced odd-cycle. However, in this research is concerned only with signed bipartite graphs, thus there is only one kind of unbalanced cycles: the even-length ones. It follows that if a signed bipartite graph (G, Σ) admits a homomorphism to a signed bipartite graph (B, Π), then the unbalanced-girth of (G, Σ) is at least the unbalanced-girth of (B, Π).

3.1.1 Projective Cubes

The projective cube of dimension d, denoted $PC(d)$, is the Cayley graph $(Z_2^d, S = \{e_1, e_2, \cdots, e_d, J\})$ where $\{e_1, e_2, \cdots, e_d\}$ is the standard basis and J is the all-1 tuple. Each edge of $PC(d)$ can be associated with one of these tuples (which is equal to the difference between the edge's end vertices). For $d = 1$, as $e_1 = J$, a multi-edge is allowed, and thus $PC(1)$ is a complete graph on two vertices with two parallel edges joining these two vertices. For $d \geq 2$, $PC(d)$ is a simple graph; $PC(2)$ is K_4 and $PC(3)$ is $K_{4,4}$.

3.1.2 Signed Projective Cubes

The signed projective cube of dimension d, denoted $SPC(d)$, is the signed graph $(PC(d), J)$ where J is the set of edges associated to the vector J. Thus, $SPC(1)$ is the unbalanced digon. Further, $SPC(2)$ is the signed K_4 where two parallel (non-incident) edges are negative. The graph $SPC(3)$ is the signed $K_{4,4}$ where the signature consists of the edges of a perfect matching. The following theorem by Naserasr, collects some of the key properties of the signed projective cubes $SPC(d)$.

Theorem 3.3. The signed projective cube $SPC(d)$ satisfies the following:

1. It is of unbalanced-girth $d+1$.

2. In every unbalanced (respectively balanced) cycle C of $SPC(d)$ and for each element $x \in \{e_1, e_2, \cdots, e_d, J\}$, an odd (respectively even) number of edges of C correspond to x.

3. Each pair of vertices belong to a common unbalanced cycle of length $d+1$.

Furthermore, it can be easily verified that when d is an odd number, the underlying graph $PC(d)$ of $SPC(d)$ is a bipartite graph. Thus, $SPC(2d-1)$ is a signed graph in which all cycles, balanced or unbalanced, are even. While for $d \geq 2$ the girth of the graph is always 4, the unbalanced-girth of $SPC(2d-1)$ is $2d$. On an attempt to check for homomorphism between signed graphs, theorems 3.1, 3.2 and lemma 3.1 are used.

3.2 Graph Descriptions

Here are the graphs that are important and used in the later sections of the chapter.

A cycle with four vertices, one negative edge and with all other edges as positive edges is denoted C_{-4} (unbalanced C_4 with one negative edge).

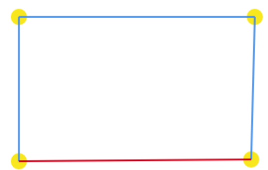


Figure 3.2.1: C_{-4}.

An unbalanced signed cube graph on seven vertices with three negative edges (refer Figure 3.2.2) is denoted as $UCube_7$ (unbalanced cube on seven vertices).

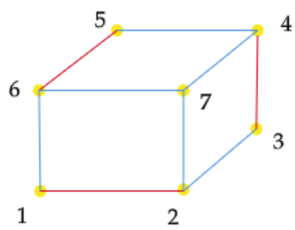


Figure 3.2.2: $UCube_7$.

In a complete bipartite graph with three vertices on each partitions, one edge removed and two negative edges is denoted as UG_4.

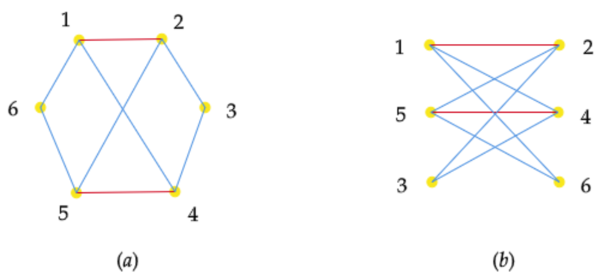


Figure 3.2.3: (a) and (b) UG_4.

A graph on twelve vertices and three red edges denoted as UG_6 (refer Figure 3.2.4) is the smallest graph which has no C_{-4} and does not map to no C_{-4}.

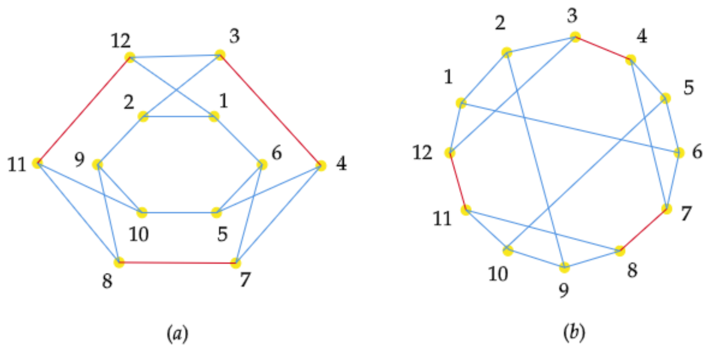


Figure 3.2.4: (a) and (b) UG_6.

3.3 Important Lemmas

In this section, the lemmas that were studied and proved during the internship are discussed. The proof of the lemmas try to force homomorphism of certain vertices by switching the signs of edges on some vertices.

Lemma 3.2. $UCube_7$ does not map to C_{-4}.

Proof. To prove this statement switch the signs of edges on the vertices 6, 4, 1 and 7. The resultant $UCube_7$ will be Figure 3.3.1. After switching, mapping of certain vertices of $UCube_7$ to C_{-4} are imposed. Consider the one negative edge in C_{-4} as ab, b is adjacent to c, c adjacent to d and d adjacent to a. This results in the following mapping, $5 \to a$, $4 \to b$, $6 \to d$, $3 \to c$, $2 \to b$ and $1 \to c$, which exhausts all the vertices in C_{-4} and 7 has to be mapped to a vertex adjacent to b and d. No such vertex exists in C_{-4} as b and d are diagonal vertices in C_{-4}, thus proving the lemma. □

Lemma 3.3. UG_4 is the smallest signed bipartite graph that has C_{-4} and is not homomorphic to C_{-4}.

Proof. This proof consists of two parts, one; to prove that UG_4 does not map to

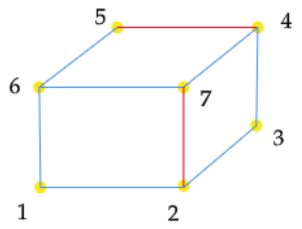


Figure 3.3.1: $UCube_7$ after switching.

C_{-4} and two; UG_4 is the smallest signed bipartite graph that contains C_{-4} and does not map to C_{-4}.

Part 1: Consider the structure of UG_4 as given in the Figure 3.2.3(a) and label C_{-4} as follows: one negative edge as (a, b), b is adjacent to c, c adjacent to d and d adjacent to a. Without loss of generality the following mappings can be certainly achieved: $1 \to a$, $2 \to b$, $3 \to c$ and $6 \to d$, that leaves 4 to be mapped to a vertex adjacent to a and c and 5 to be mapped to a vertex adjacent to b and d. a and c (b and d) are diagonal vertices and does not have a common vertex connecting them with both blue and red edges, thus 4 (5) cannot map to any vertex in C_{-4} which proves part one of the proof.

Part 2: To prove that UG_4 is the smallest graph, it is enough to give a homomorphism from $G \setminus e$ to C_{-4}. There are 2 cases to prove it is the smallest graph (with reference to figure Figure 3.2.3(a)), one; remove a red edge, two; remove a blue edge.

Case 1: Consider the edge $(5, 4)$ is removed. This case is trivial. The following will be the mapping; $1 \to a$, $2 \to b$, $3 \to c$, $4 \to d$, $5 \to c$ and $6 \to d$.

Case 2: Consider the edge removed is $(3, 2)$. Switch the signs on edges incident to vertex 4. Now the mapping will be $1 \to a$, $2 \to b$, $3 \to a$, $4 \to b$, $5 \to c$ and $6 \to d$, which proves that UG_4 is the smallest graph that does not map to C_{-4}. $\square$

The proof of lemmas mentioned in this section will enable the construction of UG_8 i.e., a graph that has no C_{-6} and does not map to C_{-4}.

3.4 Bound for K_4 minor free graphs

In order to establish relation between signed graphs that does not contain C_{-4} and K_4-minor free graphs of different odd girth, it is important to understand the latter problem also. This section attempts to explain the graphs that bounds K_4-minor free graphs.

Conjuncture 3.1 (Hadwiger's conjecture)**.** Any graph G which does not contain K_n as a minor is $(n-1)$-colorable.

In the language of graph homomorphisms, the Four-color Theorem states that: "Every planar graph admits a homomorphism to the complete graph K_4". If the planar graph is bipartite, it admits a homomorphism to complete graph K_2. Thus the non-bipartite planar graphs, i.e., planar graphs of odd-girth at least 3 are considered.

Conjuncture 3.2. [16] Given an integer $k \geq 1$, every planar graph of odd-girth at least $2k+1$ admits a homomorphism to $PC(2k)$.

Since $PC(2)$ is isomorphic to K_4, the first case of this conjecture is the Four-color Theorem. The conjecture is related to determining the edge-chromatic number of a class of planar multigraphs.

Conjuncture 3.3. [23] [24] For each planar multigraph G, $\chi'(G) = \lceil \chi'_f \rceil$

The restriction of Conjecture 3.2 to planar 3-regular multigraphs corresponds to a claim of Tait from the late 19th century. Conjecture 3.3 has been studied extensively for the special case of planar r-graphs, for which the fractional edge-chromatic number is known to be exactly r [23] [25]. An r-regular multigraph is an r graph if for each set X of an odd number of vertices, the number of edges leaving X is at least r. Hence, Conjecture 3.3 restricted to this class is stated as follows.

Conjuncture 3.4. [23] [24] Every planar r-graph is r-edge-colorable.

In the paper by Laurent Beaudou, Florent Foucaud, Reza Naserasr [2] have studied K_4 the case $I = C_{2k-1}$ and $J = K_4$ (Refer theorem 2.1), that is, the case of K_4-minor-free graphs (also known as series-parallel graphs) of odd-girth at least $2k+1$, denoted by $SP(2k+1)$. The main tool was to prove necessary and sufficient conditions for a graph B of odd-girth $2k+1$ to be a bound for $SP(2k+1)$. These conditions are given in terms of the existence of a certain weighted graph containing B as a subgraph, and that satisfies certain properties. The main idea of the proof is based on homomorphisms of weighted graphs.

From this, the authors were able to deduce a polynomial-time algorithm to decide whether a graph of odd-girth $2k+1$ is a bound for $SP(2k+1)$. This algorithm is then used in the main theorem, to prove that the projective hypercube $PC(2k)$ bounds $SP(2k+1)$, showing that Conjecture 3.2 holds when restricted to K_4-minor-free graphs. In fact, they also showed that this is far from being optimal (with respect to the order), by exhibiting two families of subgraphs of the projective hypercubes that are an answer: the Kneser graphs $K(2k+1,k)$, and a family of order $4k^2$ (which was called augmented square toroidal grids). Note that the order $\mathcal{O}(k^2)$ is optimal, as shown by He, Sun and Naserasr [19], while for planar graphs it is known that any answer must have order at least 2^{2k} (see Sen, Sun and Naserasr [19]).

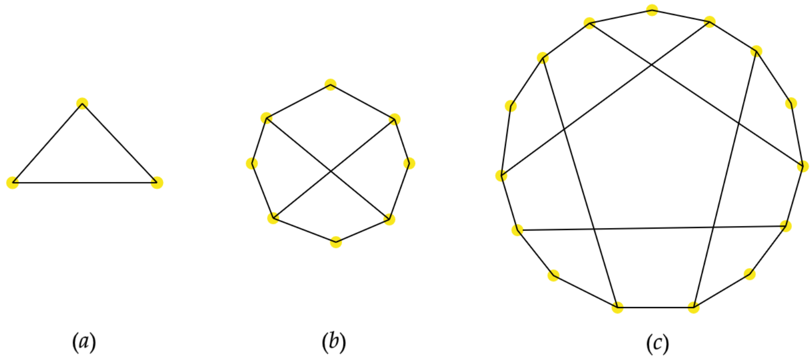


Figure 3.4.1: Graph that bounds K_4 minor free graphs of odd girth (a) three ,(b) five and (c) seven.

The paper also determined optimal answers to the problem for $k \leq 3$. For $k = 1$, it is well-known that K_3 is a bound; K_3 being a K_4-minor-free graph, it is the optimal bound in many senses (in terms of order, size, and homomorphism order). The authors proved that the smallest triangle-free graph bounding SPC_5 has order 8, and the smallest graph of odd-girth 7 bounding SPC_7 has order 15 (and determined concrete bounds of these orders). These graphs are not K_4-minor-free, and, therefore, these two bounds are not optimal in the sense of the homomorphism order. All bounds mentioned are subgraphs of the corresponding projective hypercubes. These optimal bounds for $k \leq 3$ yield a strengthening of other results about both the fractional and the circular chromatic numbers of graphs in SPC_5 and SPC_7.

3.5 Relation between K_4 minor free graphs and Bipartite graphs not homomorphic to C_{-4}

Given a signed bipartite graph (G, Σ), where G is a K_4-minor-free graph, which can found in polynomial time, a homomorphism of (G, Σ) to $(K_{3,3}, M)$. Furthermore, that the following stronger statement should also be true:

Conjuncture 3.5. [5] If G is bipartite and (G, Σ) has no $(K_4, E(K_4))$ as a signed minor, then $(G, \Sigma) \to (K_{3,3}, M)$, where M is a perfect maximal matching of $K_{3,3}$.

For for unbalanced girth four it is proved in the paper [5], by Reza Naserasr, Edita Rollová, Eric Sopena. $(K_{3,3}, M)$ does not map to C_{-4} which means any graph that maps to $(K_{3,3}, M)$ will not map to C_{-4} and by the above Conjuncture it does not contain a K_4-minor. Based on the Conjuncture 3.5, the solution to the problem: The graph that does not map to C_{-4}, and has unbalanced girth 8 will narrow down the solution space of K_4-minor free graphs of unbalanced girth 8. The graphs UG_4 and UG_6 are induced subgraph of $SPC(3)$ and $SPC(5)$ respectively, which can be at simply proved by labeling the edges of UG_4 and UG_6 with the edges of corresponding signed projective cubes. The graph of mystery sequence (refer Figure 3.5.1) will be a signed graph that bounds Signed graphs of unbalanced girth 8 that does not have K_4 as it's minor.

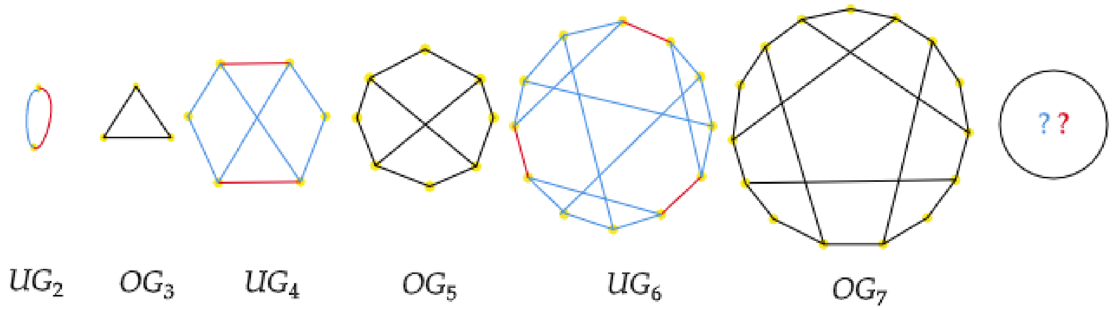


Figure 3.5.1: A bridge of K_4 minor free graph with odd girth $2k+1$ and Signed bipartite graphs of Unbalanced girth $2k$. (The Mystery Sequence).

CHAPTER 4

Further Discussion and Future Work

Although, The scope of the problem is very wide, the immediate focus is to find and prove the next element in the Sequence (Refer Figure [3.5.1]) of graphs. For which the following Conjuncture is a generalization and a stronger statement.

Conjuncture 4.1. [2] Every planar signed bipartite graph of unbalanced girth $2k$ admits a homomorphism to $SPC(2k-1)$.

The case $k = 2$ of the above Conjecture is shown to be stronger than the four-color theorem in [5]. Regarding edge-coloring, Seymour, strongly extending Tait's reformulation of the four-color theorem, conjectured that the edge-chromatic number of any planar multigraph is determined by its maximum degree and its fractional edge-chromatic number (which was later proved to be computable in polynomial time). Using the above Conjuncture will give raise to more problems on Computing the complexity to arrive at K_4-minor free graph. The graph to be next in the Sequence is expected to have a construction like Myceilski graphs.

For every $k \geq 1$, $M_k(C_{2k+1})$ has odd-girth $2k+1$, is 4-chromatic and is a subgraph of $PC(2k)$. Note that $M_2(C_5)$ contains OG_5 as a subgraph and therefore bounds SPC_5. Similarly, $M_3(C_7)$ contains OG_7 as a subgraph and hence bounds SPC_7. Thus, there is a conjecture that $M_k(C_{2k+1})$ bounds SPC_{2k+1}. (In the paper [2], the authors have verified this conjecture by a computer check for $k \leq 10$.) Since $M_k(C_{2k+1})$ has order $2k^2+k+1$, a confirmation of this conjecture would provide a family of smaller bounds than the augmented square toroidal grids (Refer Section 5.3) (that have order $4k^2$).

CHAPTER 5

Appendices

5.1 Proof of Grötzsch Theorem

There has been various different types of proof for Grötzsch theorem, one of them proves that all triangle-free planar graphs are three colorable in an algorithmic perspective. The following is a simple proof by contradiction.

Theorem 5.1. *Grötzsch Graph is the smallest triangle free planar graph that is 4 colorable.*

Proof. Consider the contradiction of the above statement (i.e. Grötzsch graph is 3-colorable). Grötzsch graph has 2 layers (l_1 is the outermost layer, l_2 is the inner layer), has five vertices in each and has a center vertex. Let ϕ be the 3-coloring of the graph. Without loss of generality assume that the center most vertex has color 1, since all the vertices in the inner layer can be colored with one color (but not color 1), say 2. Now the outer cycle has to be colored with just 1 and 3, but the an odd cycle cannot be colored with just two colors, which is a contradiction. To prove that this is the smallest graph, remove an edge from the outer most 5-cycle that consists of all vertices in l_1. Now the graph is 3-colorable as the outer layer can be colored with two colors, inner layer with the third color and center vertex with one of the two colors used in the outer layers. Similarly, the proof can be proceeded removing an edge that joins a vertex in the inner layer and outer layer and an edge that joins the center vertex. □

5.2 Generalized Myceilski Graph

The graph $G = G_{s,t} = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$ has $2st + t + 1$ vertices and $4st + 2t$ edges. Let

$$V = V_1 \cup \cdots \cup V_t \cup \{w\}$$

$$V_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,2s+1}\}$$

For convenience view the second subscripts modulo $2s + 1$, i.e., $x_{i,j} = x_{i,2s+1+j}$ for all values of i and j.

$$E = E_1 \cup E_2 \cup \cdots \cup E_t \cup E_w,$$

$$E_1 = \{x_{1,j}, x_{1,j+l} : 1 \leq j \leq 2s + 1\},$$

$$E_i = \{x_{i,j}x_{i-l,j-l}, x_{i,j}x_{i-l,j+l} : 1 \leq j \leq 2s + 1\} \text{ for } 2 \leq i \leq t,$$

$$E_w = \{wx_{i,j} : 1 \leq j \leq 2s + l\}.$$

The graph $G_{s,t}$ is a generalization of Mycielski's graph ($t = 2$ and $s \geq 2$).

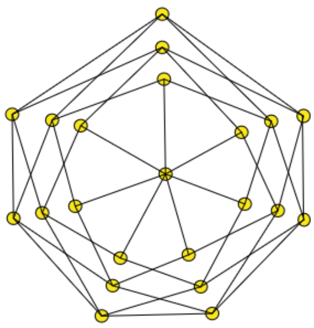


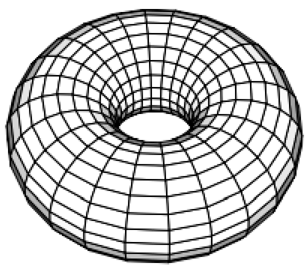
Figure 5.2.1: $M_3(C_7)$ or $G_{3,3}$.

Theorem 5.2. [11] For every $s \geq 1$ and $t \geq 1$, the graphs $G_{s,t}$, are critically 4-chromatic.

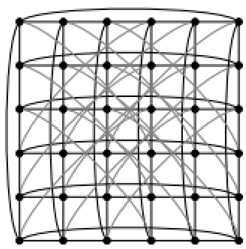
Generalized Mycielski graphs can also be represented as $M_t(G)$ where t denotes number of layers and G is the graph in the outer layer, adjacency of which will be copied in inner layers.

5.3 Augmented Toroidal Grids

For any pair of integers (a, b), let $T(a, b)$ denote the cartesian product $C_a \square C_b$. This graph can be seen as the toroidal grid of dimension $a \times b$. Figure 5.3.1a depicts a representation of $T(24, 24)$. The augmented toroidal grid of dimensions $2a$ and $2b$, denoted $AT(2a, 2b)$ is the graph obtained from $T(2a, 2b)$ by adding an edge between v and v' for each vertex v, here restricting to augmented toroidal grids of equal dimensions. More formally, for any positive integer k, let $AT(2k, 2k)$ be the graph defined on the vertex set $\{0, 1, ..., 2k-1\}^2$ such that a pair $\{(i_1, j_1), (i_2, j_2)\}$ is an edge if $i_1 = i_2$ and $|j_1 - j_2| \in \{1, 2k-1\}$ (vertical edges), or $j_1 = j_2$ and $|i_1 - i_2| \in \{1, 2k-1\}$ (horizontal edges), or $i_2 - i_1 + k \in \{0, 2k\}$ and $j_2 - j_1 + k \in 0, 2k$ (antipodal edges). Figure 5.3.1b gives a representation of the graph $AT(6, 6)$.



(a)



(b)

Figure 5.3.1: (a) A representation of the 24×24 toroidal grid. (b) The augmented toroidal grid $AT(6, 6)$. Gray edges belong to $AT(6, 6)$ but not to $T(6, 6)$.

List of Figures

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